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RESEARCH ARTICLE

# Modeling and pricing European-style continuous-installment option under the Heston' stochastic volatility model: A PDE approach

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## Abstract

Installment options, as path-dependent contingent claims, involve paying the premium discretely or continuously in installments, rather than as a lump sum at the time of purchase. In this paper, we applied the PDE approach to price European continuous-installment option and consider Heston stochastic volatility model for the dynamics of the underlying asset. We proved the existence and uniqueness of the weak solution for our pricing problem based on the two-dimensional finite element method. Due to the flexibility to continue or stop paying installments, installment options pricing can be modeled as an optimal stopping time problem. This problem is formulated as an equivalent free boundary problem and then as a complementarity linear problem (LCP). We wrote the resulted LCP in the form of a variational inequality and used the finite element method for the discretization. Then the resulting time-dependent LCPs are solved by using a projected successive over relaxation iteration method. Finally, we implemented our numerical method. The numerical results are verified the efficiency and usefulness of the suggested method.

**KEYWORDS**
Option pricing, Installment option, stochastic volatility model, Free boundary value problem, Variational inequality, linear complementarity problem, Finite element method

## 1 | INTRODUCTION

Pricing of derivative securities is a challenging and important issue in mathematical finance, given their widespread use of this instrument particularly in option contracts for controlling and managing risk in financial markets. A novel approach to solving the problem of pricing financial derivatives has a history of just over a century and its roots can be traced back to the works of French mathematician Louis Bachelier. In his doctoral thesis in 1900, he examined fundamental topics such as Brownian motion and mathematical modeling of financial markets.

The pricing discussion requires modeling, solution methods and implementation of the model in a studied market. In continuous-time models, there are two main approaches to obtain the derivative pricing formula: the martingale approach and the PDE approach. In this paper, we are going to price European-style installment option by using the PDE approach. As we know, in conventional options such as vanilla, barrier and Asian options the buyer pays the premium up front. In a pay-later options the buyer pays the premium on the maturity if the option is in-the-money on the expiration date. Unlike these options in Installment options an initial premium is paid upon purchase, and then a sequence of installments are paid up to a fixed maturity. In discrete-installment options the installment premiums are paid on predetermined dates and in continue-installment option the installment premiums will be paid per time unit.

The installment options gives the holder the right to cancel the contract before maturity. In other word,the holder can be allowed to continue the contract by paying the installment premium or cancel it by not paying the installment premium on any payment date before maturity.The holder decides to continue or terminate the option contract based on the net present value (NPV) of the contract: if the option has a value greater than the NPV of the remaining payments then the holder continues to pay installment premiums. Otherwise, the holder allowed the contract to be canceled. Therefore, the right to cancel the option contract transforms the installment options pricing problem into a optimal time stopping problem or a free boundary value problem.

In the real market, installment options are actively traded, e.g., On the Australian Stock Exchange (ASX), Installment warrants written on Australian stocks are listed [1,2] and Deutsche Bank supplied a 10-year warrant with 9 annual payments [3]. Many contracts of the life insurance and capital are also considered installment options. Majd and Pindyck [4] developed an model of the optimal sequential investment in which firms invest continuously until a project is completed. One can stop investing and restart it later without incurring additional costs.

For the first time, Geske [5] introduced the compound option issue which is equivalent to tow installments case. He evaluated the compound option in conditions that the stock price follows the geometric Brownian motion. Geske showed that if the company does not have a debt or the time ($t_2$) of selling the stock is too long the Black-Scholes valuation formula is correct. Otherwise, one must use the valuation of the compound option. Geske also proved that the assumption of risk stability is not valid in this case.

There are few research article in the literature that focus on the pricing of installment options. The framework of Black-Scholes is used for discrete type of European installment options: Davise et al. [3] obtained no-arbitrage bound of the installment option price under the Black-Scholes model with stochastic volatility and then to examined dynamic and static hedging strategies. Griebsch et al. [6] derived a closed form solution for the initial premium of European discrete installment options and investigated the limit of the installment option price with continuous payment.

For the continuous type of European installment option, Alobaidi et al. [7,8] used a partial Laplace transform to solve the free boundary problem that arises pricing European continuous-installment options. They also analyzed the behavior of the optimal stopping boundary as the option approaches its expiration date. Kimura [9] used Laplace transform to price European-style continuous-installment option and obtained the initial premium and integral representations for optimal boundaries. Ciurlia [10] provided the Monte Carlo method to price European-style continuous installment options under the Black-Scholes model. Beiranvand et al. [11] used the finite element method to solve the linear complementary problem resulting from the pricing problem of the European-style continuous installment option under the Black-Scholes model. Jeon et al. [12] applied the PDE approach to price the European-style continuous-installment currency option under the mean reverting log-normal model. They further obtained the integral equation representation for the optimal stopping boundary using the Mellin transform techniques.

Most of the research conducted in this field is based on the incorrect assumption that stock prices follow geometric Brownian motion, which academic research has decisively rejected. Therefor, we need to use a more accurate assumption for stock price movements to value options. One of this incorrect assumption is constant volatility, which causes calculated prices to differ from market prices. We have considered the Heston model as the underlying asset price model for the valuation of the European-style continuous-installment call option to make the calculated prices more realistic. Valuation of European-style continuous-installment options under Heston's model cannot be obtained accurately and analytically and numerical methods must be used.

The following is the organization of the paper: In section 2, we presented the risk-neutral pricing of European-style continuous-installment call option and obtained the PDE option price valuation formula equivalent to it by using result of the Feynman-kac type in a general multidimensional setting. Finally, in order to eliminate the unknown free boundary, the pricing problem is formulated as a linear complementary problem. In section 3, we solved this linear complementary problem by using a finite element method and proved the existence and uniqueness of the weak solution for our problem. The projected successive over relaxation (PSOR) iteration method is used for complementary linear problem (LCP) solution. In Section 4, numerical results are presented to illustrate the efficiency and usefulness of the proposed method. We briefly conclude in Section 5.

## 2 | FORMULATION OF PRICING MODEL FOR EUROPEAN-STYLE CONTINUOUS-INSTALLMENT CALL OPTION

In this section,a pricing model is formulated for European-style continuous-installment call option prices under the Heston volatility model. Assuming the standard assumptions of a frictionless market: we permit short selling, allow holding assets in arbitrary amounts, have no transaction costs and conduct borrowing and lending at the same intrest rate. We consider a financial market consist of two assets B and S over a fix time period [0,T]. The risk-free money account B is given by $B_t = e^{rt}$ where r is the instantaneous riskless interest rate. The risky asset $S = \{S_t\}_{t\geq 0}$ solution of the following stochastic differential equation

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t \tag{1}$$

where according to Heston's framework (1993), the squared volatility $V = \{V_t\}_{t\geq 0}$ is stochastic and is defined as follows:

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^1 \tag{2}$$

for nonnegative constant $\kappa, \theta, \sigma$. $W_t, W_t^1$ are Brownian motions with instantaneous correlation $\rho$ , i.e. , $dW_t dW_t^1 = \rho dt$ for some constant $\rho \in (-1, 1)$. The process $W_t$ can be set as a linear combination of independent Brownian motions $W_t^1, W_t^2$.

$$W_t := \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$$

The processes S and V are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by two independent Brownian motions $W_t^1, W_t^2$.

By choosing the volatility risk market price $\lambda\sqrt{V}$ for some constant $\lambda$ in Heston (1993) and Using Girsanov's theorem, a risk-neutral probability measure $Q$ can be found that is equivalent to the original probability measure $P$. The $Q$-dynamics of the model Under the risk-neutral measure $Q$ are expressed as follows

$$dS_t = (r - \delta)S_t dt + \sqrt{V_t} S_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2) \tag{3}$$
$$dV_t = [\kappa(\theta - V_t) - \sigma\lambda V_t]dt + \sigma\sqrt{V_t}dW_t^1 \tag{4}$$

where $\delta$ is the dividend yield. Since the discounted price process $B^{-1}S$ is a $Q$-martingale and the two-assets market model $(B, S)$ is free arbitrage, each contingent claim will be priced uniquely.

Consider an European-style continuous-installment call option on S with the strike price K, maturity date T, installment rate q and the payoff $(S_T - K, 0)^+$, where $(z)^+ = z \vee 0 = max(z, 0)$. For this option, the holder has the right to end the option contract by stopping the payments at any time prior to the maturity date. Therefor, the pricing of European-style installment call option can be formulated as an optimal time stopping problem. Let $C(t, S_t, V_t; q)$ be the price of this option at time $t \in [0, T]$, $C(t, S_t, V_t; q)$ is a solution of the following optimal time stopping problem:

$$\begin{aligned} C(t, S, V; q) &= \sup_{\tau\in\mathcal{T}} E^Q[\mathbf{1}_{\{\tau\geq T\}} e^{-r(\tau\wedge T-t)}(S_T - K)^+ - \int_t^{\tau\wedge T} q e^{-r(u-t)} du \mid \mathcal{F}_t] \\ &= \sup_{\tau\in\mathcal{T}} E^Q[\mathbf{1}_{\{\tau\geq T\}} e^{-r(\tau\wedge T-t)}(S_T - K)^+ - \frac{q}{r}(1 - e^{-r(\tau\wedge T-t)}) \mid \mathcal{F}_t] \end{aligned} \tag{5}$$

where $\mathcal{T}$ is the set of all $\mathcal{F}_t$-stopping times $\tau$ in the time interval [t,T], $\mathbf{1}_A$ denotes the indicator function of the set A and $\tau \wedge T = min(\tau, T)$. Finding the points $(t, S_t, V_t) \in \mathcal{D}$ where stopping the option contract before maturity is optimal is equivalent to solving the optimal stopping problem (5). Let $\mathcal{D}_\infty = [0, T] \times [0, +\infty) \times [0, +\infty)$ and define $\mathcal{C}, \mathcal{S}$ as the continuation and stopping regions,respectively, such that $\mathcal{D} = \mathcal{C} \cup \mathcal{S}$. The stopping region is defined by

$$\mathcal{S} = \{(t, S_t, V_t) \in \mathcal{D}_\infty \mid C(t, S_t, V_t; q) = 0\}$$

for which the stopping time $\tau_{\mathcal{S}}^* = \inf\{u \in [t, T] \mid (u, S_u) \in \mathcal{S}\}$. The complement of $\mathcal{S}$ in $\mathcal{D}$ is the continuation $\mathcal{C}$ and defined by

$$\mathcal{C} = \{(t, S_t, V_t) \in \mathcal{D}_\infty \mid C(t, S_t, V_t; q) > 0\}$$

The boundary that separates $\mathcal{S}$ from $\mathcal{C}$ is termed as a stopping boundary

$$\underline{S}_t = \inf\{S_t \in [0, +\infty] \mid C(t, S_t, V_t; q) > 0\}, \qquad 0 < \underline{S}_t \leq K$$

$C(t, S_t, V_t; q)$ is non-decreasing in $S_t$, so the stopping boundary is a lower critical asset price, below which it is beneficial to end the option contract.

Now we find the PDE option price $C(t, S_t, V_t; q)$ valuation formula corresponding to the problem (5). By using the Ito's lemma, We can show the function $C \equiv C(t, s, v; q)$ where $S_t \equiv s$, $V_t \equiv v$ satisfies the time-dependent partial differential equation

$$\frac{\partial C}{\partial t} + \mathcal{L}C = q \qquad on\, \mathcal{C} \tag{6}$$

where $\mathcal{L}$ is the parabolic differential operator

$$\mathcal{L}C = (r-\delta)s\frac{\partial C}{\partial s} + [\kappa(\theta - v) - \sigma\lambda v]\frac{\partial C}{\partial v} + \sigma s v\rho\frac{\partial^2 C}{\partial s\partial v} + \frac{1}{2}vS^2\frac{\partial^2 C}{\partial s^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} - rC$$

with the free boundary conditions

$$\lim_{s\downarrow \underline{S}_t} C(t, s, v; q) = 0 \tag{7}$$

$$\lim_{s\downarrow \underline{S}_t} \frac{\partial C}{\partial s} = 0 \tag{8}$$

and the terminal condition is given by

$$C(T, s, v; q) = (S_T - K)^+$$

Respectively, the value-matching condition (7) and smooth-pasting condition (8) imply that initial premium function $C(t, s, v; q)$ and the $\frac{\partial C}{\partial s}$ are continuous across the respective optimal stopping boundary. The smooth fit conditions (7), (8) are collectively known as ensuring the optimality of the early stopping strategy.

Given that the $q > 0$, we have

$$\frac{\partial C}{\partial t} + \mathcal{L}C < q \qquad on\, \mathcal{S} \tag{9}$$

It is clear that $C = 0$ in stopping region $\mathcal{S}$ and $C > 0$ in continuous region. Therefor in all the two region, we see that the following equality holds

$$(\frac{\partial C}{\partial t} + \mathcal{L}C - q)(C) = 0 \tag{10}$$

Hence, from (6), (9), (10) and given that $C \geq 0$ in $\mathcal{D}$, we see that $C$ satisfies the following linear complementary problem (LCP)

$$\begin{cases} \frac{\partial C}{\partial t} + \mathcal{L}C - q \leq 0 \\ (\frac{\partial C}{\partial t} + \mathcal{L}C - q)(C) = 0 \\ C \geq 0 \end{cases} \tag{11}$$

in $\mathcal{D}$, where the terminal condition is $C(T, s, v; q) = (S_T - K)^+$.
We use the transformation $T - t \to t$, meaning that we work with time to maturity. So we get

$$\begin{cases} \frac{\partial C}{\partial t} - \mathcal{L}C + q \geq 0 \\ (\frac{\partial C}{\partial t} - \mathcal{L}C + q)(C) = 0 \\ C \geq 0 \end{cases} \tag{12}$$

with the initial condition $C(0, s, v; q) = (s - K)^+$.
The problem (12) can be rewritten with the gradient and divergence operator as follows

$$\begin{cases} \frac{\partial C}{\partial t} - \nabla \cdot A\nabla C + b \cdot \nabla C + rC + q \geq 0 \\ (\frac{\partial C}{\partial t} - \nabla \cdot A\nabla C + b \cdot \nabla C + rC + q)(C) = 0 \\ C \geq 0 \end{cases} \tag{13}$$

where the diffusion matrix and convection field are given by

$$A = \frac{1}{2}v\begin{bmatrix} s^2 & \rho\sigma s \\ \rho\sigma s & \sigma^2 \end{bmatrix}, \qquad b = v\begin{bmatrix} -\kappa - \lambda - \frac{\rho\sigma}{2} \\ -s \end{bmatrix} + \begin{bmatrix} \kappa\theta - \frac{\sigma^2}{2} \\ (r-\delta)s - \frac{\rho\sigma s}{2} \end{bmatrix}$$

## 3 | THE NUMERICAL METHOD

The linear complementary problem obtained in the previous section is solved using a finite element method. The problem is defined on the computational domain $\mathcal{D}_\infty = [0,T] \times [0,+\infty) \times [0,+\infty)$. To obtain the solution using the finite element method: in the first time, we must limit the domain $\mathcal{D}_\infty$ with a finite domain $\mathcal{D} = [0,T] \times \underbrace{(0,s_{max}) \times (0,v_{max})}_{\Omega}$. The second step is formulation of the linear complementary obtained from the European-style installment option pricing problem as a variational inequality. The third step is to prove the existence and uniqueness of the variational inequality solution which depends on the space in which we search the solution in. The last step is discretization.

### 3.1 | Formulation as a variational inequality

To recast the problem (13) in a weak form or variational form, we define Hilbert spaces $X, V, H$ as follows [13]

$$X = H^1_v(\Omega) = \left\{ u : \left( u, \frac{u}{\sqrt{v}}, \sqrt{v}\frac{\partial u}{\partial v}, \sqrt{v}s\frac{\partial u}{\partial s} \right) \in (L^2(\Omega))^4 \right\}$$

with the norm

$$\|u\|_{H^1_v} = \left( \int_\Omega vs^2 \left(\frac{\partial u}{\partial s}\right)^2 + v\left(\frac{\partial u}{\partial v}\right)^2 + (1+\frac{1}{v})u^2 \right)^{\frac{1}{2}}$$

equivalent to

$$\|u\|^2_{H^1_v} = \|\sqrt{v}s\frac{\partial u}{\partial s}\|^2_{L^2} + \|\sqrt{v}\frac{\partial u}{\partial v}\|^2_{L^2} + \|\frac{u}{\sqrt{v}}\|^2_{L^2} + \|u\|^2_{L^2}$$

and

$$V = \{\psi \in X : \psi \mid_{\Gamma_D} = 0, \psi > o\}, \qquad H = L^2_v(\Omega)$$

weak formulation of (13) with the non-homogeneous Dirichlet boundary conditions is obtained as follows

$$\int_\Omega \left( \frac{\partial C}{\partial t} + \mathcal{L}C + q \right)(\psi - C)dz \geq 0 \qquad \forall \psi \in V \tag{14}$$
$$C(0,z;q) = (s-K)^+$$

where $z = (s,v)$, $\mathcal{L}C = (-\nabla \cdot A\nabla C) + b \cdot \nabla C + rC$ and

$$(\mathcal{L}C, \psi)_{L^2} = \int_\Omega -(\nabla \cdot A\nabla C)\psi + \int_\Omega (b \cdot \nabla C)\psi + \int_\Omega rC\psi \tag{15}$$

by using Green's theorem, we have

$$(\mathcal{L}C, \psi)_{L^2} = \int_\Omega A\nabla C \cdot \nabla\psi - \int_{\partial\Omega} (A\nabla C \cdot \vec{n})\psi - \int_\Omega (b \cdot \nabla C)\psi + \int_\Omega rC\psi \tag{16}$$

$\partial\Omega = \Gamma_D \cup \Gamma_N$. We assume g is Dirichlet boundary data; $C = g$ on $\Gamma_D$. For convenience, we want to use the homogeneous Dirichlet boundary conditions. The Dirichlet lift function $u_L \in W(0,T;X)$ is defined as $u_L = \mathcal{R}g$ where $\mathcal{R}$ is the linear continuous extension operator that extends non-homogeneous Dirichlet boundary conditions to the interior of the domain. We introduce the function u as $u(t,z;q) := C(t,z;q) - u_L(t,z;q)$, For all $t \in (0,T)$ and define the closed convex subset $\mathcal{K}$ of $V$ as

$$\mathcal{K} = \{\psi \in V : \quad \psi \geq -u_L \quad in\Omega\}, \qquad a.e.\, t \in [0,T] \tag{17}$$

then define a bilinear form $a = V \times V \to \mathbb{R}$ as

$$a(u,\psi) = \int_\Omega (A\nabla u \cdot \nabla\psi + (b \cdot \nabla u)\psi + ru\psi)dz \tag{18}$$

Now we can get the weak formulation of (13) with the homogeneous Dirichlet boundary conditions: find $u \in W(0,T;V)$, such that it holds

$$\frac{d}{dt}(u(t,z;q), \psi - u(t,z;q))_{L^2} + a(u(t,z;q), \psi - u(t,z;q)) \geq (f, \psi - u(t,z;q))_{L^2} \qquad \forall \psi \in \mathcal{K} \tag{19}$$

$$u(0,z;q) = (s-K)^+ - u_L(0,z;q) \tag{20}$$

where

$$(f,\psi)_{L^2} = \int_{\Gamma_N} h\psi d\Gamma_N - \frac{d}{dt}(u_L(t,z;q),\psi)_{L^2} - a(u_L(t,z;q),\psi) - \int_\Omega q\psi \qquad \forall \psi \in \mathcal{K} \tag{21}$$

and $h$ is Neumann boundary data.

## 3.2 | Existence and uniqueness

The Gel'fand triple $V \subseteq H \subseteq V'$ is utilized by us, i.e., first, we apply the Riesz representation theorem to identify the Hilbert space $H$ with its dual space $H'$. Then, we utilize the dense and continuous imbedding $V \hookrightarrow H$ to construct the adjoint mapping $V' \hookrightarrow H'$, which is also a dense and continuous imbedding of $H'$ into the dual space $V'$ of $V$.

To prove existence and uniqueness, we examine the properties of the linear form $a(\cdot,\cdot)$ in weighted Sobolev space.

**Theorem 1.** [14] *Given $V, H$ form the Gel'fand triple. Assume that the bilinear form $a(\cdot,\cdot)$ is continuous and satisfies the Garding's inequality on $V$. Also, consider $f, f' \in L^2(0,T;V')$ , where $f' = \frac{df}{dt}$. Moreover, assume, $u(0,z;q) \in \mathcal{K}(0), \mathcal{K} \neq \emptyset, t \in [0,T]$, and $f(0) - a(u(0,z;q),\psi) \in H$. Then the problem* (19) *have a unique solution u*

*Proof.* We rewrite the complete expression of the bilinear form (18)

$$\begin{aligned} a(u,\psi) = &\frac{1}{2}\int_\Omega vs^2\frac{\partial u}{\partial s}\frac{\partial \psi}{\partial s} + \frac{1}{2}\sigma^2\int_\Omega v\frac{\partial u}{\partial v}\frac{\partial \psi}{\partial v} + \frac{\rho\sigma}{2}\int_\Omega vs\left(\frac{\partial u}{\partial v}\frac{\partial \psi}{\partial s} + \frac{\partial u}{\partial s}\frac{\partial \psi}{\partial v}\right) \\ &- \int_\Omega \left((r-\delta) - v - \frac{\rho\sigma}{2}\right)s\frac{\partial u}{\partial s}\psi - \int_\Omega \left(\{\kappa[\theta - v] - \lambda v\} - \frac{\rho\sigma v}{2} - \frac{\sigma}{2}\right)\frac{\partial u}{\partial v}\psi \\ &+ r\int_\Omega u\psi \end{aligned} \tag{22}$$

By using Cauchy-Schwartz inequality on the above expression, the continuity of the bilinear form can shown

$$\begin{aligned} |a(u,\psi)| \leq &\frac{1}{2}\|\sqrt{v}s\frac{\partial u}{\partial s}\|_{L^2}\|\sqrt{v}s\frac{\partial \psi}{\partial s}\|_{L^2} + \frac{1}{2}\sigma^2\|\sqrt{v}\frac{\partial u}{\partial v}\|_{L^2}\|\sqrt{v}\frac{\partial \psi}{\partial v}\|_{L^2} \\ &+ \frac{\rho\sigma}{2}\|\sqrt{v}\frac{\partial u}{\partial v}\|_{L^2}\|\sqrt{v}s\frac{\partial \psi}{\partial s}\|_{L^2} + \frac{\rho\sigma}{2}\|\sqrt{v}s\frac{\partial u}{\partial s}\|_{L^2}\|\sqrt{v}\frac{\partial \psi}{\partial s}\|_{L^2} \\ &+ ((r-\delta) + \frac{\rho\sigma}{2})\|\sqrt{v}s\frac{\partial u}{\partial s}\|_{L^2}\|\frac{\psi}{\sqrt{v}}\|_{L^2} + \|\sqrt{v}s\frac{\partial u}{\partial s}\|_{L^2}\underbrace{\|\sqrt{v}\psi\|_{L^2}}_{\leq \max_\Omega(\sqrt{v})\|\psi\|_{L^2}} \\ &+ (\kappa + \lambda + \frac{\rho\sigma}{2})\|\sqrt{v}\frac{\partial u}{\partial v}\|_{L^2}\underbrace{\|\sqrt{v}\psi\|_{L^2}}_{\leq \max_\Omega(\sqrt{v})\|\psi\|_{L^2}} \\ &+ (\kappa\theta + \frac{\sigma}{2})\|\sqrt{v}\frac{\partial u}{\partial v}\|_{L^2}\|\frac{\psi}{\sqrt{v}}\|_{L^2} \\ &+ r\|u\|_{L^2}\|\psi\|_{L^2} \\ \leq &\, c\|u\|_{H_v^1}\|\psi\|_{H_v^1} \end{aligned}$$

where $c = max\left\{\frac{1}{2}, \frac{1}{2}\sigma^2, \frac{\rho\sigma}{2}, \left(\frac{\rho\sigma}{2} + (r-\delta)\right)\max_\Omega(\sqrt{v}), \left(\kappa + \lambda + \frac{\rho\sigma}{2}\right), \left(\kappa\theta + \frac{\sigma}{2}\right), r\right\}$. since all norms in the above expression are include in $\|\cdot\|_{H_v^1}$, any multiplication of $L^2$-norms is smaller or equal to $\|u\|_{H_v^1}\|\psi\|_{H_v^1}$. Now we show the bilinear form $a$ satisfies the Garding inequality, which is defined as follows

$$a(u,u) \geq K\|u\|^2_{H_v^2} - k\|u\|^2_{L^2} \qquad \forall u \in H_v^1$$

We write the bilinear form for $\psi = u$

$$
\begin{aligned}
a(u,u) &= \frac{1}{2}\int_\Omega vs^2\left(\frac{\partial u^2}{\partial s}\right)dz + \frac{1}{2}\sigma^2\int_\Omega v\left(\frac{\partial u}{\partial v}\right)^2 dz + \rho\sigma\int_\Omega vs\left(\frac{\partial u}{\partial v}\frac{\partial u}{\partial s}\right)dz \\
&- \int_\Omega\left((r-\delta) - v - \frac{\rho\sigma}{2}\right)s\frac{\partial u}{\partial s}udz - \int_\Omega\left(\{\kappa[\theta - v] - \lambda v\} - \frac{\rho\sigma v}{2} - \frac{\sigma}{2}\right)\frac{\partial u}{\partial v}udz \\
&+ r\int_\Omega u^2 dz \\
&= \frac{1}{2}\int_\Omega vs^2\left(\frac{\partial u^2}{\partial s}\right)dz + \frac{1}{2}\sigma^2\int_\Omega v\left(\frac{\partial u}{\partial v}\right)^2 dz + \rho\sigma\int_\Omega vs\left(\frac{\partial u}{\partial v}\frac{\partial u}{\partial s}\right)dz \\
&- \int_\Omega -vs\frac{\partial u}{\partial s}udz - \int_\Omega -\left\{(\kappa + \lambda + \frac{\rho\sigma}{2})v\right\}\frac{\partial u}{\partial v}udz + (\frac{\rho\sigma}{2} - (r-\delta))\int_\Omega s\frac{\partial u}{\partial s}udz \\
&+ (\frac{\sigma}{2} - \kappa\theta)\int_\Omega \frac{\partial u}{\partial v}udz \\
&+ r\int_\Omega u^2 dz
\end{aligned}
$$

By applying integration by parts, the convection terms in the above expression can be simplified, while the boundary terms are disregarded since they amount to zero.

$$
\begin{aligned}
&-\int_\Omega -vs\frac{\partial u}{\partial s}udz = \int_\Omega -v\frac{\partial}{\partial s}(su)dz = \int_\Omega -vu(u + s\frac{\partial u}{\partial s}) \\
&\Rightarrow -\int_\Omega -vs\frac{\partial u}{\partial s}u = \frac{1}{2}\int_\Omega -vu^2
\end{aligned}
\tag{23}
$$

to perform integration by parts on the second term of the convection, we set

$$
m(v) = -(\kappa + \lambda + \frac{\rho\sigma}{2})v
$$

and then we integrate

$$
\begin{aligned}
&-\int_\Omega m(v)\frac{\partial u}{\partial v}udz = \int_\Omega u\frac{\partial}{\partial v}(m(v)u)dz = \int_\Omega u\left(m(v)'u + m(v)\frac{\partial u}{\partial v}\right) \\
&\Rightarrow -\int_\Omega m(v)\frac{\partial u}{\partial v}u = \frac{1}{2}\int_\Omega m(v)'u^2
\end{aligned}
\tag{24}
$$

by placing (23) , (24) in the bilinear form a

$$
\begin{aligned}
a(u,u) &= \frac{1}{2}\int_\Omega vs^2\left(\frac{\partial u^2}{\partial s}\right)dz + \frac{1}{2}\sigma^2\int_\Omega v\left(\frac{\partial u}{\partial v}\right)^2 dz + \rho\sigma\int_\Omega vs\left(\frac{\partial u}{\partial v}\frac{\partial u}{\partial s}\right)dz \\
&- \frac{1}{2}\int_\Omega vu^2dz - \frac{1}{2}\int_\Omega(\kappa + \lambda + \frac{\rho\sigma}{2})u^2dz + (\frac{\rho\sigma}{2} - (r-\delta))\int_\Omega s\frac{\partial u}{\partial s}udz \\
&+ (\frac{\sigma}{2} - \kappa\theta)\int_\Omega \frac{\partial u}{\partial v}udz \\
&+ r\int_\Omega u^2 dz
\end{aligned}
\tag{25}
$$

for the third and forth terms of convection and the combined term $\rho\sigma\int_\Omega vs\left(\frac{\partial u}{\partial v}\frac{\partial u}{\partial s}\right)dz$, we use the Young's inequality

$$
\begin{aligned}
\left(\frac{\sigma}{2} - \kappa\theta\right)\frac{\partial u}{\partial v}u = \left(\sqrt{v}\frac{\partial u}{\partial v}\right)\left(\left(\frac{\sigma}{2} - \kappa\theta\right)\frac{u}{\sqrt{v}}\right) &\geq -\left|\left(\sqrt{v}\frac{\partial u}{\partial v}\right)\left(\left(\frac{\sigma}{2} - \kappa\theta\right)\frac{u}{\sqrt{v}}\right)\right| \\
&\geq -\frac{1}{2}v\left(\frac{\partial u}{\partial v}\right)^2 - \frac{1}{2}\left(\frac{\sigma}{2} - \kappa\theta\right)^2\frac{u^2}{v}
\end{aligned}
$$

and

$$vs\left(\frac{\partial u}{\partial v}\frac{\partial u}{\partial s}\right)=\left(\sqrt{v}s\frac{\partial u}{\partial s}\right)\left(\sqrt{v}\frac{\partial u}{\partial v}\right)\geq-\left|\left(\sqrt{v}s\frac{\partial u}{\partial s}\right)\left(\sqrt{v}\frac{\partial u}{\partial v}\right)\right|$$
$$\geq-\frac{1}{2}vs^2\left(\frac{\partial u}{\partial s}\right)^2-\frac{1}{2}v\left(\frac{\partial u}{\partial v}\right)^2$$

by substituting the above relationship in (25), we get

$$\begin{aligned}a(u,u)\geq&\frac{1}{2}\left(1-\rho\sigma-\left(\frac{\rho\sigma}{2}\right)^2\right)\int_\Omega vs^2\left(\frac{\partial u}{\partial s}\right)^2dz+\frac{1}{2}(\sigma^2-\rho\sigma-1)\int_\Omega v\left(\frac{\partial u}{\partial v}\right)^2dz\\&+\frac{1}{2}\left(-1-\left(\frac{\sigma}{2}-\kappa\theta\right)^2\right)\int_\Omega\frac{u^2}{v}dz+\left((r-\delta)-\frac{1}{2}\left(\kappa+\lambda+\frac{\rho\sigma}{2}\right)\right)\int_\Omega u^2dz\\&-\frac{1}{2}\int_\Omega vu^2dz\end{aligned}$$

Due to the inequality $\int_\Omega vu^2dz\leq\max_\Omega(v)\int_\Omega u^2dz$, we can write the above expression as a Garding's inequality

$$a(u,u)\geq K\|u\|_{H_v^1}-k\|u\|_{L^2}^2$$

□

## 3.3 | Discretization

To discretize time, we divide the interval $[0,T]$ into $N$ uniform time intervals $I_k=[t^k,t^{k+1}]$ with a length of $\Delta t=t^{k+1}-t^k$, $k=1,2,\dots,N$. we set discrete times $t^k=k\Delta t, k=0,1,\dots,N$ and denote by $u^k=u(t^k,z;q)$ and $u_L^k=u_L(t^k,z;q)$. we obtain the semi-discrete formulation of (19) in the time direction as follow

$$\frac{1}{\Delta t}(u^{k+1}-u^k,\psi-u^{k+1})_{L^2}+a(u^{k+1},\psi-u^{k+1})\geq f^{k+1}(\psi-u^{k+1})\qquad\forall\psi\in\mathcal{K}^{k+1}$$

where $f$ defined for $h\equiv 0$, i.e.,

$$f^{k+1}(\psi)=-\frac{1}{\Delta t}(u_L^{k+1}-u_L^k,\psi)_{L^2}-a(u_L^{k+1},\psi)-(q,\psi)_{L^2}$$

and

$$\mathcal{K}^{k+1}=\{\psi\in V:\quad\psi\geq-u_L^{k+1},\ in\Omega\}$$

we use the weighted $\theta$-scheme for full-discretization

$$\frac{1}{\Delta t}(u^{k+1}-u^k,\psi-u^{k+1})_{L^2}+a(\theta u^{k+1}+(1-\theta)u^k,\psi-u^{k+1})\geq f^{k+\theta}(\psi-u^{k+1})\qquad\forall\psi\in\mathcal{K}^{k+\theta},\theta\in[0,1]\tag{26}$$

where

$$f^{k+\theta}(\psi)=-\frac{1}{\Delta t}(u_L^{k+1}-u_k^k,\psi)_{L^2}-a(\theta u_L^{k+1}+(1-\theta)u_L^k,\psi)-(q,\psi)_{L^2}$$

and

$$\mathcal{K}^{k+\theta}=\{\psi\in V:\quad\psi\geq-(\theta u_L^{k+1}+(1-\theta)u_L^k),\ in\Omega\}$$

we write the values of $u_L(t,z;q)=\mathcal{R}g$ based on section 2:

$$\begin{aligned}&s=0 && u_L(t,0,v;q)=0\\&s=s_{max} && u_L(t,s_{max},v;q)=\frac{\partial C}{\partial s}(t,s_{max},v;q)\triangleq A\nabla C\cdot\overrightarrow{n}=\frac{1}{2}vse^{-\delta t},\overrightarrow{n}=\begin{bmatrix}1\\0\end{bmatrix}\\&v=0 && u_L(t,s,0;q)=se^{-\delta t}\Phi(d_+)-Ke^{-rt}\Phi(d_-)-q\int_0^T e^{-rt}\Phi(d_-)\\&v=v_{max} && u_L(t,s,v_{max};q)=se^{-\delta t}\Phi(d_+)\end{aligned}$$

The values of $u_L(t,z;q) = \mathcal{L}g$ for $v = 0$ and $v = v_{max}$ follow from the equation (26). The values of $d_+$ and $d_-$ are computed for $\theta = 0$ and $\theta = v_{max}$, respectively. The equation (27) is a European-style installment option value function under the Black-Scholes framework, which is a special case of the Heston model for $\sigma = 0$ and initial variance $v(0) = \theta$. we have [10]:

$$C(t,s;q) = se^{-\delta t}\Phi(d_+) - Ke^{-rt}\Phi(d_-) - q\int_0^t e^{-ru}\Phi(d_-)du \tag{27}$$

where $\Psi(\cdot)$ is the standard normal cumulative distribution function

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy, \quad x \in \mathbb{R}$$

$$d_{\pm}(x,y,t) = \frac{log\left(\frac{x}{y}\right) + \left(r - \delta \pm \frac{1}{2}\theta^2\right)t}{\sqrt{\theta t}}$$

## 3.4 | Numerical solution

We will now implement the finite element method for the variational problem (26) with $\theta = \frac{1}{2}$. In each time step, we need to solve the problem in the following form

$$\frac{1}{\Delta t}(u^{k+1} - u^k, \psi - u^{k+1})_{L^2} + a(\frac{1}{2}u^{k+1} + \frac{1}{2}u^k, \psi - u^{k+1}) \geq f^{k+\frac{1}{2}}(\psi - u^{k+1}) \qquad \forall \psi \in \mathcal{K}^{k+\frac{1}{2}} \tag{28}$$

where

$$f^{k+\frac{1}{2}}(\psi) = -\frac{1}{\Delta t}(u_L^{k+1} - u_k^k, \psi)_{L^2} - a(\frac{1}{2}u_L^{k+1} + \frac{1}{2}u_L^k, \psi) - (q,\psi)_{L^2}$$

and

$$\mathcal{K}^{k+\frac{1}{2}} = \{\psi \in V : \quad \psi \geq -(\frac{1}{2}u_L^{k+1} + \frac{1}{2}u_L^k),\ in\Omega\}$$

For problem (28), we consider approximate spaces with a finite dimension $X_h \subset X, V_h \subset V$, given in terms of the basis functions $\varphi_i$

$$V_h = X_h \cap V = span\{\varphi_i, i = 1,\ldots,h_V\} \qquad X_h = span\{\varphi_i, i = 1,\ldots,h_X\}$$
$$dim(V_h) = h_V \qquad dim(X_h) = h_X$$

The spaces of $X_h, V_h$ inherit the inner products and norms of the exact spaces, i.e.,

$$(\cdot,\cdot)_{X_h} = (\cdot,\cdot)_X\ ,\ \|\cdot\|_{X_h} = \|\cdot\|_X$$

$$(\cdot,\cdot)_{V_h} = (\cdot,\cdot)_V\ ,\ \|\cdot\|_{V_h} = \|\cdot\|_V$$

We decompose the close region $\overline{\Omega} = \Omega + \partial\Omega = [0, s_{max}] \times [0, v_{max}]$ into a mesh $\mathcal{T}_h$ consisting of triangles $T_i$, $i = 1,\ldots,N^T$ and their nodes $D_i$, $i = 1,\ldots,N^D$. Then we consider the basis functions $\varphi_i$, $i = 1,\ldots,N^D$, which are piecewise linear, take the value 1 at node $D_i$ and 0 at all other nodes. To simplify the computation, we state a transformation $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ on the unit triangle $\hat{T} = \{(\hat{s},\hat{v}) \in \mathbb{R}^2 : 0 \leq \hat{s} \leq 1, 0 \leq \hat{v} \leq 1 - \hat{s}\}$ for each triangle $T_i$ in the mesh $\mathcal{T}_h$ with three nodes $D_1 = (s_1,v_1), D_2 = (s_2,v_2), D_3 = (s_3,v_3)$. The affine transformation $F$ is defined as follows

$$F(s,v) = M\begin{bmatrix}\hat{s}\\ \hat{v}\end{bmatrix} + \begin{bmatrix}s_1\\ v_1\end{bmatrix}$$
$$M = \begin{bmatrix}s_2 - s_1 & s_3 - s_1\\ v_2 - v_1 & v_3 - v_1\end{bmatrix}$$

The tree nodes in the unit triangles are $(0,0), (1,0), (0,1)$, respectively and three basis functions on a unit triangle are as follows

$$\hat{\varphi}_1(\hat{s},\hat{v}) = 1 - \hat{s} - \hat{v},\ \hat{\varphi}_2(\hat{s},\hat{v}) = \hat{s},\ \hat{\varphi}_3(\hat{s},\hat{v}) = \hat{v}$$

After inserting the basis functions $\varphi_i$ into the variational inequality (28), we proceed to solve the resulting system of linear equations. We find a function $u_h : D \longrightarrow \mathbb{R}$

$$\frac{1}{\Delta t}\langle u_h^{k+1} - u_h^k, \psi - u_h^{k+1}\rangle + a(\frac{1}{2}u_h^{k+1} + \frac{1}{2}u_h^k, \psi - u_h^{k+1}) \geq f_h^{k+\frac{1}{2}}(\psi - u^{k+1}) \qquad \forall \psi \in \mathcal{K}_h^{k+\frac{1}{2}} \tag{29}$$

where

$$f_h^{k+\frac{1}{2}}(\psi) = -\frac{1}{\Delta t}\langle u_{Lh}^{k+1} - u_k^k, \psi\rangle - a(\frac{1}{2}u_{Lh}^{k+1} + \frac{1}{2}u_{Lh}^k, \psi) - \langle q, \psi\rangle$$

and

$$\mathcal{K}_h^{k+\frac{1}{2}} = \{\psi \in V_h : \quad \psi \geq -(\frac{1}{2}u_{Lh}^{k+1} + \frac{1}{2}u_{Lh}^k),\ in\Omega\}$$

We write

$$\begin{aligned}
u_h^k &= \sum_{D_i \in \Omega} u_i^k \varphi_i \qquad i = 1, \ldots, N^D \\
\psi &= \sum_{D_i \in \Omega} \psi_i^k \varphi_i \qquad i = 1, \ldots, N^D \\
u_{L,h}^k &= \sum_{D_i \in \partial\Omega} u_{L,i}^k \varphi_i \quad i = 1, \ldots, N^D
\end{aligned}$$

The matrix form of the above problem is obtained as follows

$$(\psi^{k+1} - U^{k+1})^{tr}\left[\frac{1}{\Delta}B_1(U^{k+1} - U^k) + \frac{1}{2}A_1(U^{k+1} + U^k)\right] \geq (\psi^{k+1} - U^{k+1})^{tr}\left[-C_1\frac{1}{\Delta}(U_L^{k+1} - U_L^k) + \frac{1}{2}D_1(U_L^{k+1} + U_L^k) + E_1 q\right] \tag{30}$$

where

$$\begin{aligned}
&U^k = (u_1^k, \ldots, u_f^k)^T, && \psi^k = (\psi_1, \ldots, \psi_f) \\
&U_L^k = (u_{L,1}^k, \ldots, u_{L,f}^k)^T, && u_L^k = \begin{cases} 0 & D_i \notin \partial\Omega \\ u_{L,i}^k\varphi_i & D_i \in \partial\Omega \end{cases}, \qquad u_{L,i} = u_L(D_i)
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \left[A_{1_{ij}}\right]_{f\times f}, A_{1_{ij}} = \int_\Omega \varphi_i \mathcal{L}\varphi_j d\Omega && \forall D_i, D_j \in \Omega \\
B_1 &= \left[B_{1_{ij}}\right]_{f\times f}, B_{1_{ij}} = \int_\Omega \varphi_i \varphi_j d\Omega && \forall D_i, D_j \in \Omega \\
C_1 &= \left[C_{1_{ij}}\right]_{f\times c}, C_{1_{ij}} = \int_\Omega \varphi_i \varphi_j d\Omega && \forall D_i \in \Omega, D_j \in \partial\Omega \\
D_1 &= \left[D_{1_{ij}}\right]_{f\times c}, D_{1_{ij}} = \int_\Omega \varphi_i \mathcal{L}\varphi_j d\Omega && \forall D_i \in \Omega, D_j \in \partial\Omega \\
E_1 &= \left[E_{1_{ij}}\right]_{f\times 1}, E_{1_{ij}} = \int_\Omega \varphi_i d\Omega && \forall D_i \in \Omega
\end{aligned}$$

$f$ and $c$ represent the number of internal nodes and the number of boundary nodes, respectively. we rewrite equation (30) in a simpler form

$$\begin{cases} (\psi^{k+1} - U^{k+1})^{tr}(AU^{k+1} - B) \geq 0 \\ U^{k+1} \geq 0 \end{cases} \tag{31}$$

where

$$\begin{aligned}
A &= B_1\frac{1}{\Delta t} + \frac{1}{2}A_1 \\
B &= \left(B_1\frac{1}{\Delta t} - \frac{1}{2}A_1\right)U^k - \left(C_1\frac{1}{\Delta t} + \frac{1}{2}D_1\right)U_L^{k+1} + \left(C_1\frac{1}{\Delta t} - \frac{1}{2}D_1\right)U_L^k - E_1 q
\end{aligned}$$

The first condition (31) can be transformed into the form of finite different method. Therefor, the inequality can be rewritten as

$$\begin{cases} AU^{k+1} - B \geq 0 \\ (AU^{k+1} - B)(U^{k+1})^{tr} = 0 \\ U^{k+1} \geq 0 \end{cases} \tag{32}$$

### 3.4.1 | PSOR iteration method

Problem (32) is a sequence of LCPs, which we denote briefly as LCP(B,A) and we solve it using the PSOR iteration method.

The fixed-point problem is equivalent to LCP(B,A), as stated by the following theorem.

**Theorem 2.** [15] *Let $A \in R^{m\times m}$. Then, the LCP(B,A) is equivalent to the following fixed-point equation*

$$(U - (AU - B))^{+} - U = 0 \tag{33}$$

In addition, $A$ can be expressed as the difference between the diagonal matrix $D$ and the sum of the strictly lower-triangular matrix (-M) and the strictly upper-triangular matrix (-N). Based on the equivalence (33) mentioned above, we can introduce the PSOR iteration method for solving LCP(B,A) in the following manner.

**Definition 1.** [15] (PSOR Iteration Method). Given $U^0 \geq 0$,

$$U^{k+1} = (U^k - wD^{-1}(AU^k - B - M(U^{k+1} - U^k)))^{+}, \quad k = 0, 1, 2, \ldots, \tag{34}$$

where 0<w<2.

If $w$ substitute $w = 1$ in equation (34), the resulting method (1) is known as the Projected Gauss-Seidel Iteration method. Algorithm (1) describes the PSOR iteration method for LCP(B,A).

**Algorithm 1** PSOR Iteration Method

```
Choose m, n, w, ϵ, maxit;
for it = 1, 2, ..., maxit do
    for i = 1, 2, ..., mn do
        k = [1, 2, ..., mn];
        U(i) = U(i) + w(B(i) − A(i, k)U(k)/A(i, i));
        U(i) = max(U(i), 0);
    end for
        Res = ||min(AU − B, U)||;
    if Res < ϵ then
        Break;
    end if
end for
```

## 4 | NUMERICAL RESULTS

In this section, we present numerical results for pricing of the European-style continuous-installment call option under the Heston model to show the accuracy and efficiency of our numerical method. All calculations have been implemented on a windows 11 machine with a processor 2.60 GHz and 12 GB RAM using MATLAB R 2014a. The numerical results for this option are presented, with the model parameters given by

$$\kappa = 5.0 \quad \theta = 0.16 \quad \sigma = 0.9 \quad \rho = 0.1 \quad r = 0.1 \quad \delta = 0.02 \quad \lambda = 0.1 \quad T = 0.25 \quad q = 1, 3, 6$$

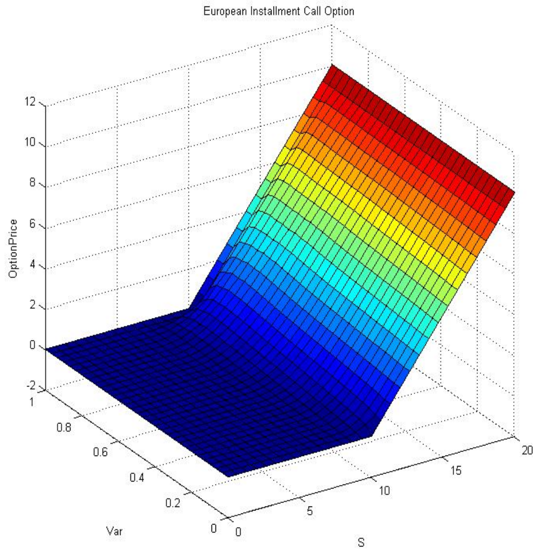


**FIGURE 1** The surface plot of the European installment-continuous call option under the Heston model

in the domain $[0, S] \times [0, V] \times [0, T] = [0, 20] \times [0, 1] \times [0, 0.25]$, where the Feller condition $2\kappa\theta \geq \sigma^2$ is satisfied. The uniform grids are defined by the triplets $(M, N, T_N)$, where $M, N$ and $T_N$ represent the number of partitions in $s$-direction, $v$-direction and $t$-direction, respectively.

We choose the following stopping criterion

$$\|min(AU^k + B, U^k)\|_{L_2} < 10^{-6}$$

and where $U^k$ represents the $k$-th numerical solution of the LCP (32).

The relative error is defined as []:

$$Error = \frac{\|Price - Price^*\|_{L_2}}{\|Price^*\|_{L_2}} \tag{35}$$

where $Price$ and $Price^*$ represent the numerical solution and reference solution, respectively. the reference numerical solution $Price^*$ is computed by a fine grid $(M, N, T_N)$.

The surface plot of the European installment-continuous call option are depicted in figure 1. From this figure, we can see that the quality of the solutions is excellent. We calculate the reference numerical solutions of the European continuous-installment call option for $q = 1, 3, 6$ with a good grid, (320,128,128). In table 1, we presented the solutions, relative error and CPU time (in seconds) on different grids. The reference solutions $Price^* = 0.4917, 0.2178, 0.1049$ are obtained for $q = 1, 3, 6$, respectively. The table shows that the obtained solutions are accurate and efficient.

## 5 | CONCLUSION

In this paper, we have examined the value of the European-style continuous-installment option. To price the installment option, we need to use more accurate assumptions on the stock price movement. One of these assumptions is non-constant volatility, which causes the calculated prices to be closer to the market prices. So far, many studies and researches have been done in the field of modeling and predicting the volatility of financial returns, but still only a few theoretical models are used to calculate the amount and how financial volatility occur. We have used Heston's stochastic volatility model as the underlying asset price model for modeling and pricing installment option in the European style. We wrote the obtained free boundary problem as a linear complementary problem and used a two-dimensional finite element method to solve it. We have proved The existence and uniqueness of solution to this pricing problem. Finally, our results demonstrated that our numerical method for European-style continuous-installment option has suitable accuracy and efficiency.

**TABLE 1** European installment-continuous call option values for q=1,2,3 at $(s_0, v_0) = (10, 0.09)$.

| $(M, N, T_N)$ | q | **Price** | **Error** | **CPU** |
|---|---|---|---|---|
| (20,16,16) | | 0.4456 | 9.37e-02 | 3.813576 |
| (30,24,24) | | 0.4594 | 6.56e-02 | 7.012673 |
| (40,32,32) | 1 | 0.4702 | 4.37e-02 | 12.152738 |
| (60,48,48) | | 0.4762 | 3.15e-02 | 33.006972 |
| (80,64,64) | | 0.4608 | 6.28e-02 | 82.621707 |
| (120,96,96) | | 0.4757 | 3.25e-02 | 318.304815 |
| *Price** | | 0.4917 | - | 4540.863038 |
| (20,16,16) | | 0.1974 | 9.36e-02 | 2.143585 |
| (30,24,24) | | 0.2214 | 1.65e-02 | 3.898188 |
| (40,32,32) | 3 | 0.2199 | 9.64e-03 | 6.691886 |
| (60,48,48) | | 0.2192 | 6.42e-03 | 18.837798 |
| (80,64,64) | | 0.2189 | 5.05e-03 | 48.740799 |
| (120,96,96) | | 0.2186 | 3.67e-03 | 245.633789 |
| *Price** | | 0.2178 | - | 3976.046975 |
| (20,16,16) | | 0.0271 | 7.41e-01 | 3.730712 |
| (30,24,24) | | 0.0693 | 3.39e-01 | 7.257341 |
| (40,32,32) | 6 | 0.0913 | 1.29e-01 | 12.108995 |
| (60,48,48) | | 0.0998 | 4.86e-02 | 33.048788 |
| (80,64,64) | | 0.1014 | 3.33e-02 | 81.981186 |
| (120,96,96) | | 0.1032 | 1.62e-02 | 389.953850 |
| *Price** | | 0.1049 | - | 4657.387346 |

## AUTHOR CONTRIBUTIONS

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## FINANCIAL DISCLOSURE

None reported.


## CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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