Optimum SIC Order in Downlink NOMA with Proportional Fairness

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***Abstract*—Successive Interference Cancellation (SIC) is a key element in a Non-Orthogonal Multiple-Access (NOMA) receiving terminal and must be properly tuned in order to achieve the optimal performance. To this purpose, receivers have to be informed about which signal components have to be decoded before decoding their own. In this framework, most literature refers to a proposition reported in [1, Summary 6.1] for broadcast Gaussian channels implying that the optimal decoding order consists of decoding the weaker user signals first and then their own. In this work, the blurred contours of the proposition are brought into focus by considering the widespread Proportional Fairness (PF) criterion to harmonize the user rates. Theorem 1 is the main contribution assessing the optimal SIC order of a NOMA channel under a PF-optimal criterion. It is observed in the paper that the optimum SIC order may change with different utility functions which are concave and monotonically increasing with the users' individual SINRs. Numerical results are also presented to confirm the validity in a fading environment.**

***Index Terms*—Non-Orthogonal Multiple Access, Successive Interference Cancellation, Proportional Fairness.**

I. Introduction

NON-ORTHOGONAL Multiple Access (NOMA) is one of the key technologies for future 5G and 6G wireless systems that has emerged to enhance the efficiency, the fairness, and the capacity of communication networks [2]–[6]. This work focuses on the downlink broadcast NOMA channel, which is based on superposition encoding at the transmitting base station and Successive Interference Cancellation (SIC) at the user terminals. This theoretical analysis is based on the achievable rates derived in [7]–[9] for the broadcast channel, and its main goal is the optimization of a Key Performance Indicator (KPI) characterizing the fairness of the communication network. The KPI considered is based on the Proportional Fairness (PF) criterion proposed in [10]. The PF criterion consists of the maximization of the minimum ratio of the achievable rates of the different users to certain user target rates. Since the target rates can be all equal to each other, PF encompasses as a special case the Maximum-Minimum (MAX-MIN) fairness criterion, consisting of the maximization of the minimum user achievable rate. The advantage of PF stands in the fact that MAX-MIN may lead to an excessive penalty on the stronger users (experiencing better channel conditions) in order to cope with the presence of very weak users (experiencing bad channel conditions). If that penalty is not acceptable, choosing target rates depending on the channel conditions (higher target rates for stronger users and lower for weaker users) allows to limit the penalty itself.

The implementation of a NOMA communication system is based on several requirements: i) the base station must know the Channel State Information at the Transmitter (CSIT) in oreder to establish a dynamic user hierarchy which allows to allocate the transmitted resources in an optimum way according to the KPI selected; ii) the base station adopts a superposition encoding technique based on linear weights characterizing the power allocation of the encoded components targeted to the different users in the transmitted signal; this power allocation is determined in order to satisfy an optimality criterion; iii) the user terminals must know the signal gains from the base station over the wireless channel, which is referred to as Channel State Information at the Receiver (CSIR), and the transmitted power allocation for themselves and for the weaker users in order to implement correctly the SIC. The knowledge of the weaker users' power allocation is related with the SIC ordering property investigated in this work. According to [1, Summary 6.1], *the cancellation order at every receiver is always to decode the weaker users before decoding its own data*. This statement is referred to in most of the NOMA literature without critical assessment. The proposition in the original source is not stated as a theorem and misses a specification of the KPI. Moreover, the analytic development refers exclusively to the two-user broadcast channel and its direct applicability to a multiuser NOMA system is questionable.

In the framework of broadcast channels, the maximum sum-rate is achieved when transmission occurs to only one of the strongest user by greedy resource allocation. However, this approach is deemed to be *unfair* since the users would all strive to reach the proximity of the base station to take a chance at receiving some information. This has been recognized and rationalized by defining fairness criteria, like PF, without detailed investigation on the optimization of the SIC order. Thus, the purpose of this work is filling the gap and shedding a light on this issue to resolve it by either confirming the classic proposition of [1, Summary 6.1] or proposing a different ordering scheme.

This work relies on earlier literature results for the definition of the PF KPI [10] and of the basic condition allowing to optimize the power allocation to maximize PF [11, Th. 1]. Building on these results, the optimum SIC order is derived for any set of user SNR's and target rates with respect to the PF criterion. The organization is summarized briefly: Section II characterizes the NOMA channel and introduces the relevant notation. Section III introduces the framework of SIC order optimization and the main result, Theorem 1. Section IV provides numerical results to confirm the assessment and Section V reports some concluding remarks.

II. System Model

Consider a Base Station (BS) transmitting to K users the signal $X = \sum_{k=1}^{K} X_k$ corresponding to the superposition of K independent component signals X_k, representing the information to be conveyed to the k-th user. The average transmitted power is P_x and the power of the k-th component signal X_k is $\alpha_k P_x$. The power allocation coefficients are collected in the vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K) \in \mathcal{S}_K$, where $\mathcal{S}_K$ is the K-dimensional simplex:

$$\mathcal{S}_K \triangleq \left\{ \boldsymbol{\alpha} : \alpha_k \geq 0, k = 1, \ldots, K, \sum_{k=1}^{K} \alpha_k = 1 \right\}. \quad (1)$$

The received signal at the k-th user is:

$$Y_k = H_k X + Z_k, \quad (2)$$

for $k = 1, \ldots, K$, where H_k is the channel gain to the k-th user and $Z_k \sim \mathcal{CN}(0,1)$ is the additive noise. Assuming that the k-th user decodes the signals of users 1 to $k-1$, the achievable rates are [11]:

$$R_k = \log_2\left(1 + \frac{\rho_k \alpha_k}{1 + \rho_k \beta_k}\right) = \log_2\left(\frac{1 + \rho_k \beta_{k+1}}{1 + \rho_k \beta_k}\right), \quad (3)$$

for $k = 1, \ldots, K$, where $\rho_k \triangleq |H_k|^2 P_x$ is the k-th user SNR and

$$\beta_k \triangleq \sum_{\ell=1}^{k-1} \alpha_\ell. \quad (4)$$

For convenience, the following vectors are defined:

$$\boldsymbol{\rho} \triangleq (\rho_1, \ldots, \rho_K) \quad (5a)$$

$$\boldsymbol{T} \triangleq (T_1, \ldots, T_K) \quad (5b)$$

$$\boldsymbol{\alpha} \triangleq (\alpha_1, \ldots, \alpha_K) \quad (5c)$$

According to the notations introduced, PF consists of the maximization (over the power allocation) of the minimum (over the K users) ratio of the achievable rates R_k to the target rates T_k. This ratio is referred to as Achievable to Target rate Ratio (ATR) in the following and is defined by

$$\Phi_{\text{min}}(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}) \triangleq \min_{1 \leq k \leq K} \Phi_k(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}), \quad (6)$$

where

$$\Phi_k(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}) \triangleq \frac{1}{T_k} \log_2\left(\frac{1 + \rho_k \beta_{k+1}}{1 + \rho_k \beta_k}\right). \quad (7)$$

The PF-optimal power allocation vector $\boldsymbol{\alpha}^*(\boldsymbol{\rho}, \boldsymbol{T})$ maximizes the ATR $\Phi_{\text{min}}(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha})$ over the K-dimensional simplex:

$$\boldsymbol{\alpha}^*(\boldsymbol{\rho}, \boldsymbol{T}) \triangleq \arg\max_{\boldsymbol{\alpha} \in \mathcal{S}_K} \Phi_{\text{min}}(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}). \quad (8)$$

The PF criterion reduces to MAX-MIN by setting $T_k = 1$ for all $k = 1, \ldots, K$.

Given $\boldsymbol{\rho}$ and $\boldsymbol{T}$, it is proved in [11, Th. 1] that the resulting non-convex optimization problem leading to the PF optimum power allocation is equivalent to the solution wrt $\boldsymbol{\alpha}$ of the following equations:

$$\lambda = \Phi_k(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}), \quad k = 1, \ldots, K. \quad (9)$$

π			α_1	α_2	α_3	$\Phi_{\text{max-min}}(\boldsymbol{\rho}, \boldsymbol{T})$
1	2	3	0.2134	0.2583	0.5283	0.2791
1	3	2	0.2200	0.3426	0.4375	0.2868
2	1	3	0.1720	0.2710	0.5571	0.3001
2	3	1	0.1951	0.3934	0.4115	0.3323
3	1	2	0.1987	0.3099	0.4915	0.3317
3	2	1	0.2172	0.3482	0.4347	**0.3535**

TABLE I

Example of PF power allocation optimization with different permutations of $\boldsymbol{\rho} = (1, 3, 5)$ and $\boldsymbol{T} = (1, 2, 3)$.

The unknown λ is independent of k and coincides with the maximum of $\Phi_{\text{min}}(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha})$ over $\boldsymbol{\alpha} \in \mathcal{S}_K$:

$$\begin{aligned} \lambda = \Phi_{\text{max-min}}(\boldsymbol{\rho}, \boldsymbol{T}) &\triangleq \max_{\boldsymbol{\alpha} \in \mathcal{S}_K} \Phi_{\text{min}}(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}) \\ &= \Phi_{\text{min}}(\boldsymbol{\rho}, \boldsymbol{T}, \boldsymbol{\alpha}^*(\boldsymbol{\rho}, \boldsymbol{T})). \end{aligned} \quad (10)$$

III. Optimal interference cancellation order

The ordering of the vector $\boldsymbol{\rho}$ determines how SIC is implemented at the receivers. Specifically, it is assumed that the k-th user terminal decodes, from its own received signal:

$$Y_k = H_k(X_1 + \ldots + X_K) + Z_k, \quad (11)$$

the signals $X_{k+1}, \ldots, X_K$ (if $k = K$ it decodes no other signals), removes their interference by subtracting $H_k(X_{k+1} + \ldots + X_K)$ from Y_k to obtain:

$$\begin{aligned} Y_k &- H_k(X_{k+1} + \ldots + X_K) \\ &= H_k X_k + H_k(X_1 + \ldots + X_{k-1}) + Z_k. \end{aligned} \quad (12)$$

If $k = 1$, there is no interfering term $H_k(X_1 + \ldots + X_{k-1})$ (it is equal to 0). In general, the interfering term is considered as noise and the achievable rates (3) are derived. Thus, the ordering of the elements in the SNR vector $\boldsymbol{\rho}$ determines the SIC order.

The PF criterion leads to the optimum ATR $\Phi_{\text{max-min}}(\boldsymbol{\rho}, \boldsymbol{T})$ depending on the given SNR and target rate vectors, namely, $\boldsymbol{\rho}, \boldsymbol{T}$. Any permutation $\pi \in S_K$ (the permutation group of K elements) determines a different value of the ATR $\Phi_{\text{max-min}}(\pi\boldsymbol{\rho}, \pi\boldsymbol{T})$ and the optimum permutation maximizes this value. Notice that the permutation applies to both the SNR and the target rate vectors since the target rates are associated to the users as the SNR's. A simple numerical example is reported in Table I. This example shows that the maximum optimum ATR is attained when the permutation π sorts the SNR's in decreasing order, agreeing with the proposition reported from [1, Summary 6.1] in the Introduction.[1] However, the question remains whether or not this is a general rule or it occurs to be true in specific cases that the optimum SIC order consists of decoding the weaker user signals first and then the own signal. An answer is provided by the following theorem.

Theorem 1 *The optimum SIC order with a PF criterion characterized by the SNR vector $\boldsymbol{\rho}$ and the target rate vector*

[1]Notice that this property does not hold when the power allocation is not the optimum one, i.e., comparing $\Phi_{\text{min}}(\pi\boldsymbol{\rho}, \pi\boldsymbol{T}, \pi\boldsymbol{\alpha})$ for different permutations $\pi \in S_K$ does not necessarily lead to a maximum when the permuted SNR vector $\pi\boldsymbol{\rho}$ is sorted in noncincreasing order.

$\boldsymbol{T}$ consists of decoding the weaker user signals first (the ones with lower SNR) and then the own signal.

Proof: Any permutation $\pi \in S_K$ is the composition of several transpositions (2-cycles) [12]. Therefore, to prove the theorem, it is sufficient to consider the effect of arbitrary transpositions $(m, m+1)$ for $m = 1, \ldots, K-1$. To simplify the notation, define the inverse SNR's $\sigma_k \triangleq \rho_k^{-1}$ and the corresponding vector as $\boldsymbol{\sigma} \triangleq (\sigma_1, \ldots, \sigma_K)$. The PF-optimal ATR is obtained by solving the equations:

$$\lambda = \frac{1}{T_k} \log_2 \left(1 + \frac{\alpha_k}{\sigma_k + \beta_k}\right) \tag{13}$$

for $k = 1, \ldots, K$. Then,

$$\alpha_k = (2^{\lambda T_k} - 1)(\sigma_k + \beta_k). \tag{14}$$

As a consequence,

$$\beta_{k+1} = \beta_k + \alpha_k = (2^{\lambda T_k} - 1)\sigma_k + 2^{\lambda T_k} \beta_k. \tag{15}$$

Then,

$$\beta_2 = (2^{\lambda T_1} - 1)\sigma_1, \tag{16a}$$

$$\beta_3 = (2^{\lambda T_2} - 1)\sigma_2 + 2^{\lambda T_2}(2^{\lambda T_1} - 1)\sigma_1. \tag{16b}$$

Iterating the previous equation yields:

$$\beta_{K+1} = \sum_{k=1}^{K} \sigma_k (2^{\lambda T_k} - 1) 2^{\lambda(T_{k+1} + \ldots + T_K)}. \tag{17}$$

The PF-optimum ATR is obtained by solving the equation:

$$\phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T}) \triangleq \sum_{k=1}^{K} \sigma_k (2^{\lambda T_k} - 1) 2^{\lambda(T_{k+1} + \ldots + T_K)} = 1. \tag{18}$$

The function $\phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T})$ is monotonically increasing wrt λ. Moreover, $\phi(0, \boldsymbol{\sigma}, \boldsymbol{T}) = 0$ and

$$\phi\left(\lambda_{\max}(\boldsymbol{\sigma}, \boldsymbol{T}) \triangleq \max_{1 \le k \le K} \frac{\log_2(1 + \rho_k)}{T_k}\right) > 1. \tag{19}$$

Therefore, a solution of $\phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T}) = 0$ always exists, is unique, and lies in the interval $(0, \lambda_{\max}(\boldsymbol{\sigma}, \boldsymbol{T}))$.

Now, consider the transposition $\tau = (m, m+1)$ for some $m = 1, \ldots, K-1$, and the difference $\Delta\phi = \phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T}) - \phi(\lambda, \tau\boldsymbol{\sigma}, \tau\boldsymbol{T})$. The transposition τ does not affect the terms in the definition of $\phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T})$ corresponding to the indexes k such that $1 \le k < m$ or $m+1 < k \le K$. Then, the difference can be calculated as follows:

$$\begin{aligned}
\Delta\phi &= \sigma_m (2^{\lambda T_m} - 1) 2^{\lambda(T_{m+1} + \ldots + T_K)} \\
&\quad + \sigma_{m+1} (2^{\lambda T_{m+1}} - 1) 2^{\lambda(T_{m+2} + \ldots + T_K)} \\
&\quad - \sigma_{m+1} (2^{\lambda T_{m+1}} - 1) 2^{\lambda(T_m + T_{m+2} + \ldots + T_K)} \\
&\quad - \sigma_m (2^{\lambda T_m} - 1) 2^{\lambda(T_{m+2} + \ldots + T_K)} \\
&= (\sigma_m - \sigma_{m+1})(2^{\lambda T_m} - 1)(2^{\lambda T_{m+1}} - 1) 2^{\lambda(T_{m+2} + \ldots + T_K)} \\
&< 0,
\end{aligned} \tag{20}$$

since $\sigma_m < \sigma_{m+1}$ and the other terms in the product are positive for $\lambda > 0$. Therefore, $\phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T}) < \phi(\lambda, \tau\boldsymbol{\sigma}, \tau\boldsymbol{T})$ for

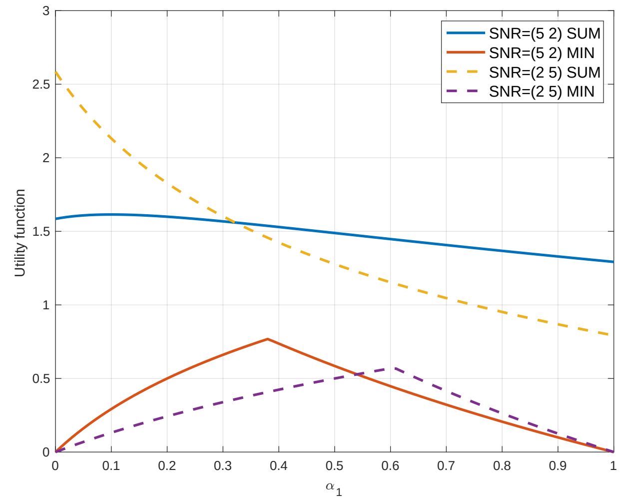

Fig. 1. Counterexample for Remark 2.

every transposition $\tau = (m, m+1)$. Denoting $\lambda^*(\boldsymbol{\sigma}, \boldsymbol{T})$ as the solution of $\phi(\lambda, \boldsymbol{\sigma}, \boldsymbol{T}) = 1$ leads to the inequality:

$$\lambda^*(\tau\boldsymbol{\sigma}, \tau\boldsymbol{T}) < \lambda^*(\boldsymbol{\sigma}, \boldsymbol{T}). \tag{21}$$

Therefore, any diversion from the monotonically increasing order of $\boldsymbol{\sigma}$ reduces the PF-optimal ATR, so that the maximum is attained when the inverse SNR vector is monotonically non-decreasing or the SNR vector is monotonically nonincreasing, which concludes the proof of the theorem. ■

Remark 1 The conclusion by Theorem 1 is expected in view of the literature results, mostly referencing to the proposition of [1, Summary 6.1] mentioned in the Introduction. However, the result is not trivial and holds specifically in the case of PF-optimal ATR, which is one of the most interesting fairness criteria for practical applications. This fills a conceptual gap which, to the author's knowledge, has been overlooked in the current technical literature for the last twenty years.

Remark 2 One of the anonymous Reviewers suggested that the fact that the optimum SIC order consists of canceling the weaker user signals first and then the own signal should apply to any utility function that is concave and monotonically increasing with each user's individual SINR. However, this generalization is not true, as illustrated by the following example.

Consider the maximization of $\sum_{k=1}^{K} R_k/T_k$ instead of $\min_{1 \le k \le K} R_k/T_k$. This sum utility function is concave and monotonically increasing with each user's individual SINR in both cases, as the Reviewer proposed.

To assess the claim, Fig. 1 illustrates the cases corresponding to

$$K = 2, \boldsymbol{T} = (2, 1), \boldsymbol{\rho} = (5, 2) \text{ or } (2, 5). \tag{22}$$

The top curves report the target utility functions $\sum_{k=1}^{K} R_k/T_k$ and $\min_{1 \le k \le K} R_k/T_k$ (labeled as "SUM" and "MIN" in the legend, respectively). The curves evidence the different behavior of the "SUM" and "MIN" utility functions. In the "SUM" case, the optimum SIC order is canceling the stronger

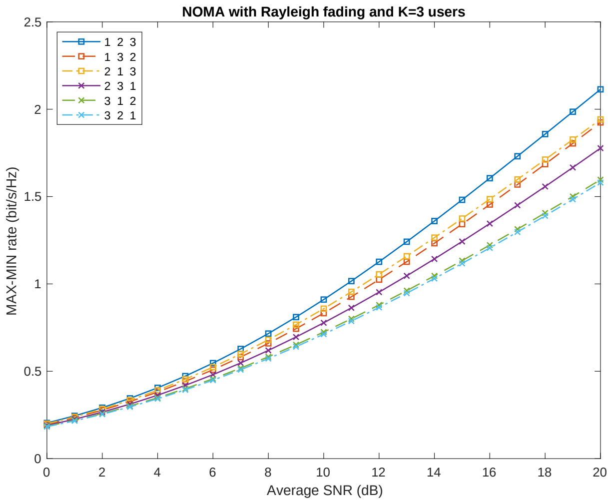

Fig. 2. Plot of the average MAX-MIN rate versus the average SNR for a NOMA broadcast channel with $K = 3$ users and Rayleigh fading, in correspondence of different permutations of the SIC order with respect to the optimum nonincreasing order.

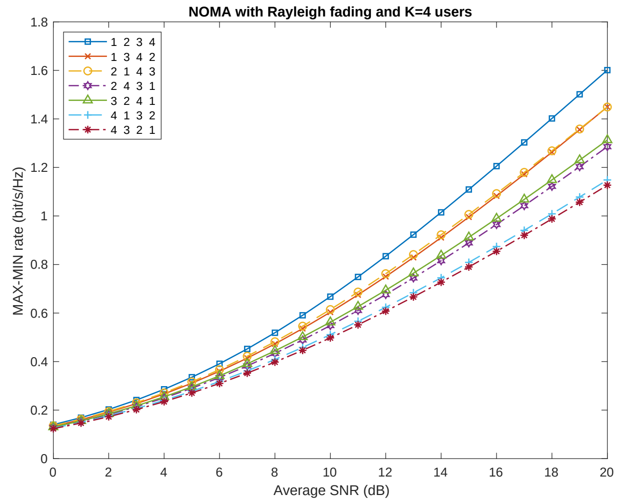

Fig. 3. Same as Fig. 2 but with $K = 4$ users.

user signal first and then the own signal, contrary to the "MIN" case, where the standard rule must be followed.

IV. Numerical results with Rayleigh-distributed channel gains

This section provides a numerical illustration of the impact of SIC ordering by considering the average MAX-MIN rate following the application of all possible permutations to a nonincreasingly ordered SNR vector. Specifically, considering Rayleigh-distributed channel gains, the SNRs are generated as iid random variables with the following common cumulative distribution function:

$$P(\rho_k < \rho) = 1 - \mathrm{e}^{-\rho}. \tag{23}$$

Fig. 2 refers to the case of $K = 3$ users. The diagrams report the average (based on $N_s = 10^3$ pseudo-random SNR sample vectors sorted in nonincreasing order) of the MAX-MIN rate obtained after the application of all the possible permutations to the sorted SNR sample vectors. They show that the optimum MAX-MIN rate is attained when the SNR vector is nonincreasingly order (corresponding to the identical permutation), i.e., when every user decodes the weaker user signals and cancels them before decoding its own. They also illustrate the loss entailed by the other permutations, confirming that the pessimum approach is having the SNRs nondecreasingly ordered (i.e., when each user decodes the stronger user signals before its own). The numerical values show that the impact of the SIC order increases as the average SNR grows larger. Similar results are reported in Fig. 3 for $K = 4$ users (again, $N_s = 10^3$ pseudo-random realizations sorted in nonincreasing order are considered) and a subset of the 24 possible permutations, including the best and the worst ones. The diagrams confirm, also in this case, that the best and worst SIC orders agree with the predictions of Theorem 1.

V. Conclusions

The main goal of this work is providing a rigorous proof of the well known proposition stated in [1, Summary 6.1] that the optimum SIC ordering for NOMA consists in having every receiver decode all the weaker user signals first and then their own. Though every user can autonomously decide the SIC order, the PF-optimal approach requires them to follow the rule. The base station confides in their compliance and uses it to optimize the power allocation in the superposition encoded transmitted signal. Though the proposition has been considered common knowledge for twenty years now, and it is often dismissed as a trivial fact, there is no evidence, to the author's knowledge, of any proof of its optimality in the literature, at least under the PF criterion.

Appendix A
MAX-MIN optimum SIC ordering

The theorem in this appendix addresses the special case of MAX-MIN fairness. Though this is a special case of PF, it provides a direct illustration of the applicability of the proof concept in this simpler case.

Theorem 2 *The optimum SIC order with a MAX-MIN fairness criterion characterized by the SNR vector $\boldsymbol{\rho}$ consists of decoding the weaker user signals first (the ones with lower SNR) and then the own signal.*

Proof: Any permutation $\pi \in S_K$ is the composition of several transpositions (2-cycles) [12]. Therefore, to prove the theorem, it is sufficient to consider the effect of arbitrary transpositions $(m, m+1)$ for $m = 1, \ldots, K-1$. For convenience, define the inverse SNR's $\sigma_k \triangleq \rho_k^{-1}$ and $\boldsymbol{\sigma} \triangleq (\sigma_1, \ldots, \sigma_K)$. The MAX-MIN ATR is obtained by solving the following equations wrt ξ:

$$\alpha_k = (\sigma_k + \beta_k)\xi, \quad k = 1, \ldots, K \tag{24}$$

$$\sum_{k=1}^{K} \alpha_k = 1. \tag{25}$$

The first equations yield:

$$\beta_2 = \alpha_1 = \sigma_1 \xi \tag{26a}$$
$$\beta_3 = \beta_2 + \alpha_2 = \sigma_2 \xi + \sigma_1 \xi(\xi + 1). \tag{26b}$$

Iterating this approach yields the following K-th degree polynomial equation:

$$\beta_{K+1} = \xi \sum_{k=1}^{K} \sigma_k (\xi + 1)^{K-k} = 1. \tag{27}$$

Now, define the polynomial function

$$\phi(\xi, \boldsymbol{\sigma}) \triangleq \xi \sum_{k=1}^{K} \sigma_k (\xi + 1)^{K-k}. \tag{28}$$

It can be checked that $\phi(\xi, \boldsymbol{\sigma})$ is monotonically increasing for $\xi \geq 0$, $\phi(0, \boldsymbol{\sigma}) = 0$, and

$$\begin{aligned}
&\phi\left(\frac{1}{\sum_{m=1}^{K} \sigma_m}, \boldsymbol{\sigma}\right) \\
&= \frac{1}{\sum_{m=1}^{K} \sigma_m} \sum_{k=1}^{K} \sigma_k \left(1 + \frac{1}{\sum_{m=1}^{K} \sigma_m}\right)^{K-k} \\
&> \frac{1}{\sum_{m=1}^{K} \sigma_m} \sum_{k=1}^{K} \sigma_k \\
&= 1.
\end{aligned} \tag{29}$$

Therefore, a solution of $\phi(\xi, \boldsymbol{\sigma}) = 1$ exists and is unique in the interval $(0, 1/\sum_{k=1}^{K} \sigma_k)$. Now, let $\sigma_m < \sigma_{m+1}$ (or equivalently $\rho_m > \rho_{m+1}$) and let τ denote the transposition $(m, m + 1)$ for some $1 \leq m < K$. Since, for any $\xi > 0$,

$$\begin{aligned}
&\sigma_m (\xi + 1)^{K-m} + \sigma_{m+1} (\xi + 1)^{K-m-1} \\
&\quad < \sigma_{m+1} (\xi + 1)^{K-m} + \sigma_m (\xi + 1)^{K-m-1},
\end{aligned} \tag{30}$$

equivalent to the readily checked inequality

$$(\sigma_m - \sigma_{m+1})\xi < 0, \tag{31}$$

the following holds:

$$\phi(\xi, \boldsymbol{\sigma}) < \phi(\xi, \tau\boldsymbol{\sigma}). \tag{32}$$

Thus, if $\xi^*(\boldsymbol{\sigma})$ denotes the solution of $\phi(\xi, \boldsymbol{\sigma}) = 1$,

$$\phi(\xi^*(\boldsymbol{\sigma}), \tau\boldsymbol{\sigma}) > \phi(\xi^*(\boldsymbol{\sigma}), \boldsymbol{\sigma}) = 1. \tag{33}$$

Hence, since $\phi(\xi, \boldsymbol{\sigma})$ is a monotonically increasing function of ξ for $\xi \geq 0$, the solution of $\phi(\xi, \tau\boldsymbol{\sigma}) = 1$ must be lower than the solution of $\phi(\xi, \boldsymbol{\sigma}) = 1$, namely,

$$\xi^*(\tau\boldsymbol{\sigma}) < \xi^*(\boldsymbol{\sigma}). \tag{34}$$

This means that the maximum solution of $\phi(\xi, \pi\boldsymbol{\sigma}) = 1$ among all the possible permutations $\pi \in S_K$, corresponds to the permutation that sorts the vector $\boldsymbol{\sigma}$ in nondecreasing order. ∎

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