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RESEARCH ARTICLE

New moment formulas for moments and characteristic function of the geometric distribution in terms of Apostol-Bernoulli polynomials and numbers

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Abstract

Although it is very easy to calculate the 1st moment and 2nd moment values of the geometric distribution with the methods available in existing books and other articles, it is quite difficult to calculate moment values larger than the 3rd order. Because in order to find these moment values, many higher order derivatives of the geometric series and convergence properties of the series are needed. The aim of this article is to find new formulas for characteristic function of the geometric random variable (with parameter p) in terms of the Apostol-Bernoulli polynomials and numbers, and the Stirling numbers. This characteristic function characterizes the geometric distribution. Using the Euler's identity, we give relations among theis characteristic function, the Apostol-Bernoulli polynomials and numbers, and also trigonometric functions including $\sin w$ and $\cos w$. A relations between the characteristic function and the moment generating function is also given. By using these relations, we derive new moments formulas in terms of the Apostol-Bernoulli polynomials and numbers. Moreover, we give some applications of our new formulas.

KEYWORDS:
Generating function, Moments, Characteristic function, Geometric distribution, Apostol-Bernoulli polynomials and numbers, Eulerian and Stirling numbers.
Mathematical Subject Classifications: 05A15, 60E10, 60E05, 11B68, 11B73.

1 | INTRODUCTION

The characteristic function is defined by

$$K_X(u) = \mathbf{E}(\exp(iXu))$$

where $i^2 = -1$ and $\exp(iXu) = e^{iXu}$ (*cf.*[9]).

The characteristic function is a function of u. It entirely establishes the behavior and properties of the probability distribution of the random variable X. That is, it is not only a way to describe a random variable, but also characterizes the probability distribution. It is the Fourier transform of the probability density function.

Originally, characteristic function and generating functions was used to solve certain problems in probability theory. Characteristic function for random variables of any probability distribution is defined on more general cases. It is used in statistics, probability and mathematical statistics in applications of a wide variety of problems including variance, expected value and other statistical parameters that can be approached by the method of characteristic functions. For this reason, it has now been recognized that characteristic function provides various different and important applications involving real world problems in applied sciences (see[9], [10]). Unlike the moment generating functions of any probability distribution, characteristic function always exists when treated as a function of a real-valued argument. It is well known that interesting, applicable relations exist between the behavior of the characteristic function and properties of the distribution functions, such as the existence of moments and the density functions, see[3]. These important properties gives us the existence of relationships between the behavior of a distribution's characteristic function and its properties, such as the existence of moments and the existence of the probability function (*cf.*[4]-[19]; see also the references cited in each of these earlier works). Thus, it is also well known that the important applications of characteristic functions have been studied with many probabilistic, combinatorial concepts, mathematical and statistical techniques. In both probability theory and related fields, the characteristic functions of any real-valued random variable thoroughly characterizes the probability distribution of the related random variable (*cf.*[4],[6],[5],[7],[9]; see also the references cited in each of these earlier works).

Regarding probability and statistics, characteristic functions have been used the process of fitting probability distributions to data samples. Thus, there are prediction processes that match the theoretical characteristic functions calculated from very large data with the empirical characteristic functions. Ultimately, this describes the application of empirical characteristic functions to fit time series models in situations where probability operations cannot be easily applied. Therefore, the characteristic functions always remain up-to-date for all relevant events and data samples (see, for detail,[2],[8]).

Motivation of this article is to give relations and formulas among characteristic function for the geometric random variable with parameter p, expected value, variance, higher-order moments, moment generating function, the Apostol- Bernoulli numbers and polynomials, and the Stirling numbers.

With the aid of the Bernoulli trials, the geometric distribution, which is a member of discrete probability distributions, is defined as follows:

Let X be a geometric random variable with parameter p. The geometric distribution gives the probability that the first occurrence of success requires m independent trials, each with success probability p. If the probability of success on each trial is p, then the probability that the mth trial is the first success is

$$P(X = m) = pq^{m-1}, \tag{1}$$

where $m \in \mathbb{N} = \{1, 2, \dots\}$, $p + q = 1$ (see[11] 312, see also[4] 98 and[5]).

The characteristic function, which characterizes the geometric distribution, is given by

$$K_X(u) = p\exp(iu)\sum_{k=1}^{\infty} q^{k-1}\exp(i(k-1)u) \tag{2}$$

(*cf.*[4] p.236).

A relation between moment generating function and characteristic function of the geometric distribution is given as follows:

$$K_X(-iu) = M_X(u) \tag{3}$$

(*cf.*[9]).

To reach the main results of this article, the following definitions and notations are needed.

Let $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Stirling numbers of the second kind, which is the number of partitions of a set of n elements into j nonempty subsets, are defined by the following generating function:

$$(\exp(w) - 1)^d = \sum_{m=0}^{\infty} S_2(m, d)\frac{d!w^m}{m!}, \tag{4}$$

so that

$$S_2(m, d) = \sum_{v=0}^{d}(-1)^{d-v}\frac{v^m}{d!(v-d)!}, \tag{5}$$

where

$$S_2(m, d) = 0$$

if $d > m$ or $d < 0$ (*cf.*[6],[7],[17]). Few values of these numbers can be given by the formula in equation (5) as follows:

$m\backslash d$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

and so on (*cf.*[7],[17]).

In order to give main theorems and formulas of this article, we need so-called Apostol-Bernoulli polynomials and numbers, which are given below. These polynomials and numbers have many applications in mathematics and other sciences.

Generating function for the Apostol-Bernoulli polynomials, which were found by Apostol-[1], is given by

$$\sum_{m=0}^{\infty} \frac{w^m \mathcal{B}_m(\alpha; t)}{m!} = \frac{w \exp(w\alpha)}{t \exp(w) - 1}, \tag{6}$$

where $|w| < 2\pi$ when $t = 1$ and $|w| < |\log t|$ when $t \neq 1$.

When $t = 1$, $B_m(\alpha) = \mathcal{B}_m(\alpha; 1)$ denotes the classical Bernoulli polynomials.

From (6), the Apostol-Bernoulli numbers are given by

$$\mathcal{B}_m(t) = \mathcal{B}_m(0; t), \tag{7}$$

which satisfies the following properties:

$$t\mathcal{B}_1(1; t) - \mathcal{B}_1(t) = 1,$$
$$\mathcal{B}_0(t) = 0$$

and for $m \geq 2$,

$$t\mathcal{B}_m(1; t) - \mathcal{B}_m(t) = 0, \tag{8}$$

(*cf.*[1]). Applying equation (6), Apostol-[1] gave two interesting formulas for his polynomials and also numbers: for $m \in \mathbb{N}_0$ and $t \neq 1$,

$$\mathcal{B}_m(\alpha; t) = \sum_{c=0}^{m} \binom{n}{c} \alpha^{m-c} \mathcal{B}_j(t). \tag{9}$$

Apostol-[1] also gave the following formula for the numbers $\mathcal{B}_m(t)$:

$$\mathcal{B}_m(t) = \frac{mt}{(t-1)^m} \sum_{c=1}^{m-1} (-1)^c c! t^{c-1} (t-1)^{m-1-c} S_2(m-1, c), \tag{10}$$

where $m \in \mathbb{N}$ and $t \neq 1$.

We now give a relation between (6) and (2).

For $|u| < 1$, joining the following geometric series expansion

$$\sum_{k=0}^{\infty} u^k = \frac{1}{1-u}$$

with (2), we have the following relations between characteristic function of the geometric random variable and generating function of the Apostol-Bernoulli polynomials, which is given by (6) for $\alpha = 1$ and $t = 1 - p = q$ with $0 < p < 1$:

$$K_x(u) = -\frac{p \exp(iu)}{q \exp(iu) - 1} = -\frac{p}{iu} \sum_{m=0}^{\infty} \frac{(iu)^m \mathcal{B}_m(1; q)}{m!}. \tag{11}$$

Note that the following relation

$$K_x(u) = -\frac{p \exp(iu)}{q \exp(iu) - 1}$$

was given by Bertsekas and Tsitsiklis[4] p.236. However, the relationship of this function to any family special polynomial or family of special numbers has not never given.

The following sections of this article cover the sections, which are briefly summarized below:

In Section 2, we give computational formulas for characteristic function of the geometric distribution in terms of the Apostol-Bernoulli numbers. We take higher derivative of this characteristic function at the origin, we derive some computational formulas for the mth the moments of the geometric random variable with parameter p.

In Section 3, with the aid of series of the trigonometric functions involving $\sin w$ and $\cos w$, the Euler formula and also the Stirling numbers of the second kind, we also derive other formulas for the mth the moments of the geometric random variable with parameter p. We give some numerical examples for the moments with the aid of the Apostol-Bernoulli polynomials and numbers.

In Section 4, we finish this article by the conclusion section.

2 | CHARACTERISTIC FUNCTION AND MOMENTS OF THE GEOMETRIC IN TERMS OF THE APOSTOL-BERNOULLI NUMBERS AND POLYNOMIALS

In Section, using the definition and properties of the characteristic function with the help of the geometric random variable with parameter p, we derive some new formulas of characteristic function of the geometric random variable (with parameter p) in terms of the Apostol-Bernoulli numbers and polynomials. Using these formulas, we give new formulas for moments of the geometric random variable (with parameter p). Moreover, we give a relations between the characteristic function and the moment generating function. This relation gives us to analyze and interpret statistical parameters such as variance and expected value.

We give some new and interesting formulas for characteristic function of the geometric random variable with parameter p in terms of the Apostol-Bernoulli numbers and polynomials and other special functions. By the help of this function, we find new formulas for jth moment of the geometric random variable with parameter p. We also give, direct computation formula for these moments.

With the aid of (11), the characteristic function of the geometric random variable with parameter p in terms of the Apostol-Bernoulli numbers is given by the following theorem.

Theorem 1. (Characteristic function of the geometric random variable X with parameter p) Let $p + q = 1$ with $0 < p < 1$. Then we have

$$K_X(u) = q - \frac{p}{q}\sum_{n=1}^{\infty}\frac{i^n \mathcal{B}_{n+1}(q)}{n+1}\frac{u^n}{n!}.$$

Proof. Combining (11) with (3) and (6), after some calculations, we get

$$K_X(u) = -p\sum_{n=0}^{\infty} i^{n-1}\mathcal{B}_n(1;q)\frac{u^{n-1}}{n!}. \tag{12}$$

Hence

$$K_X(u) = -p\sum_{n=-1}^{\infty} i^{n}\mathcal{B}_{n+1}(1;q)\frac{u^{n}}{(n+1)!}.$$

Putting $\mathcal{B}_0(1;q) = 0$ in the previous equation, we have

$$K_X(u) = -p\sum_{n=0}^{\infty}\frac{i^n \mathcal{B}_{n+1}(1;q)}{n+1}\frac{u^n}{n!}. \tag{13}$$

Combining (13) with (8) yields

$$K_x(u;p,q) = -p\mathcal{B}_1(1;q) - \frac{p}{q}\sum_{n=1}^{\infty}\frac{i^n \mathcal{B}_{n+1}(q)}{n+1}\frac{u^n}{n!}.$$

We get

$$K_X(u) = -p\left(\mathcal{B}_1(q) + 1\right) - \frac{p}{q}\sum_{n=1}^{\infty}\frac{i^{n+1} \mathcal{B}_{n+1}(q)}{n+1}\frac{u^n}{n!}.$$

Substituting

$$\mathcal{B}_1(q) = \frac{1}{q-1}$$

into the previous equation, we get

$$K_X(u) = \frac{pq}{1-q} - \frac{p}{q}\sum_{n=1}^{\infty}\frac{i^n \mathcal{B}_{n+1}(q)}{n+1}\frac{u^n}{n!}.$$

Therefore, the previous equation gives us proof of theorem after some calculations. □

From (13), we have the following jth moments of the geometric random variable with parameter p)

$$\mu_j = (i)^{-n} \frac{d^j}{du^j} \{\mathbf{E}(\exp(iXu))\} |_{u=0}$$
$$= (i)^{-n} \frac{d^j}{du^j} \left\{K_X(u; p, q)\right\} |_{u=0} ,$$

where $p + q = 1$ with $0 < p < 1$, see also[4] p.236.

3 | FORMULAS FOR CHARACTERISTIC FUNCTION AND MOMENTS OF THE GEOMETRIC IN TERMS OF THE FUNCTIONS $\cos T$ AND $\sin T$

In Section, we give explicit formulas in terms of the Apostol-Bernoulli numbers with the aid of the series expansions of the $\sin x$ and $\cos x$ functions. We give relations between higher order derivative of the characteristic function and the jth moments of the geometric random variable with parameter p), these relations is analyzed and interpreted in statistical parameters such as variance, expected value and other parameters.

We now give decomposition of the characteristic function of the geometric random variable (with parameter p) in terms of $\sin(xu)$ and $\cos(xu)$ by the following theorem:

Theorem 2. Let $p + q = 1$ with $0 < p < 1$. Then we have

$$\mathbf{E}(\cos(Xu)) = \mathrm{Re}\left\{-p\sum_{n=0}^{\infty}\frac{i^n}{n+1}\mathcal{B}_{n+1}(1;q)\frac{u^n}{n!}\right\} \tag{14}$$

and

$$\mathbf{E}(\sin(Xu)) = \mathrm{Im}\left\{-p\sum_{n=0}^{\infty}\frac{i^n}{n+1}\mathcal{B}_{n+1}(1;q)\frac{u^n}{n!}\right\}, \tag{15}$$

were

$$\mathrm{Re}\{a + ib\} = a$$

and

$$\mathrm{Im}\{a + ib\} = b.$$

Proof. Using (13) and

$$\exp(it) = \cos t + i \sin t,$$

which is well-known the Euler's identity, we rewrite the function $K_x(u; p, q)$ as follows:

$$K_X(u) = \mathbf{E}(\exp(ixu))$$
$$= \mathbf{E}(\cos(xu) + i\sin(ux)).$$

Since $\mathbf{E}$ have a linear property, we have

$$\mathbf{E}(\cos(xu)) + i\mathbf{E}(\sin(ux)) = -p\sum_{n=0}^{\infty}\frac{i^n}{n+1}\mathcal{B}_{n+1}(1;q)\frac{u^n}{n!}.$$

By using the previous equation, we give decomposition of the characteristic function of the geometric random variable (with parameter p). Thus we complete proof of theorem. □

By using decomposition of the characteristic function of the geometric random variable (with parameter p) in terms of $\sin(xu)$ and $\cos(xu)$, we now give explicit formulas for the jth moment of the geometric random variable (with parameter p) in terms of the Apostol-Bernoulli numbers by the following theorems:

Theorem 3. Let $p + q = 1$ with $0 < p < 1$ and $j \in \mathbb{N}_0$. Let $\mu_j := \mu_j(p, q)$ be jth moment of the geometric random variable with parameter p. Then we have

$$\mu_{2j} = -\frac{p\mathcal{B}_{2j+1}(1;q)}{2j+1}. \tag{16}$$

Proof. Joining the following well known formulas, see [19],

$$\cos(xu) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}u^{2n}}{(2n)!}$$

with (14) and (15), we have the following new formulas for the moments of geometric random variable with parameter p:

$$\mathbf{E}(\cos(xu)) = \sum_{n=0}^{\infty}(-1)^n \mathbf{E}(x^{2n}) \frac{u^{2n}}{(2n)!}.$$

With the aid of the definition of the jth moments, the above equations gives us the following series

$$\sum_{n=0}^{\infty}(-1)^{j+1}\left(\frac{p\mathcal{B}_{2j+1}(1;q)}{2j+1}\right)\frac{u^{2j}}{(2j)!} = \sum_{j=0}^{\infty}(-1)^j \mu_{2j}\frac{u^{2j}}{(2j)!}.$$

If we compare $\frac{u^{2j}}{(2j)!}$ the coefficients on both sides of the above equation, respectively, then we complete proof of theorem. □

Theorem 4. Let $p + q = 1$ with $0 < p < 1$ and $j \in \mathbb{N}_0$. Let $\mu_j := \mu_j(p,q)$ be jth moment of the geometric random variable with parameter p. Then we have

$$\mu_{2j+1} = -\frac{p\mathcal{B}_{2j+2}(1;q)}{2(j+1)}. \tag{17}$$

Proof. Joining the following well known formulas, see [19],

$$\sin(xu) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}u^{2n+1}}{(2n)!}$$

with (15), we have the following new formulas for the moments of geometric random variable with parameter p:

$$\mathbf{E}(\sin(xu)) = \sum_{n=0}^{\infty}(-1)^n \mathbf{E}(x^{2n+1}) \frac{u^{2n}}{(2n)!}.$$

With the aid of the definition of the jth moments, the above equation gives us the following series

$$\sum_{n=0}^{\infty}(-1)^{j+1}\left(\frac{p\mathcal{B}_{2j+2}(1;q)}{2(j+1)}\right)\frac{u^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty}(-1)^j \mu_{2j+1}\frac{u^{2j+1}}{(2j+1)!}.$$

If we compare $\frac{u^{2j+1}}{(2j+1)!}$ the coefficients on both sides of the above equation, then we complete proof of theorem. □

3.1 | Applications of the formulas in (16) and (17)

Thanks to the formulas in (16), (17), and (10), it is no longer difficult to calculate the values of moments μ_j. That is, with the aid of these formulas and the Apostol-Bernoulli numbers and polynomials, the following values can be given very easily without calculating the convergence of any series:

Substituting $j = 0$ into (16) and (17), we get the following values of μ_0 and μ_1, respectively

$$\begin{aligned}\mu_0 &= -p\mathcal{B}_1(1;q) = -\frac{p}{q}(\mathcal{B}_1(q)+1)\\ &= -\frac{p}{q}\left(\frac{1}{q-1}+1\right) = 1\end{aligned}$$

and

$$\begin{aligned}\mu_1 &= -\frac{p\mathcal{B}_2(1;q)}{2} = -\frac{p}{2q}\mathcal{B}_2(q)\\ &= -\frac{p}{2q}\left(-\frac{2q}{(q-1)^2}\right) = \frac{1}{p}.\end{aligned}$$

Similarly, with the help of Apostol-Bernoulli numbers and polynomials, some of moment values are obtained from the following table:

j	$\mathcal{B}_j(q)$	$\mathcal{B}_j(1,q)$	μ_j
0	0	0	1
1	$\frac{1}{q-1}$	$\frac{\mathcal{B}_1(q)+1}{q}$	$\mu_1 = -\frac{p\mathcal{B}_2(1;q)}{2} = \frac{1}{p}$
2	$-2\frac{q}{(q-1)^2}$	$\mathcal{B}_2(1;q) = \frac{\mathcal{B}_2(q)}{q}$	$\mu_2 = -\frac{p\mathcal{B}_3(1;q)}{3} = \frac{q+1}{p^2}$
3	$3\frac{q^2+q}{(q-1)^3}$	$\mathcal{B}_3(1;q) = \frac{\mathcal{B}_3(q)}{q}$	$\mu_3 = -\frac{p\mathcal{B}_4(1;q)}{4} = \frac{q^2+4q+1}{p^3}$
4	$-4\frac{q^3+4q^2+q}{(q-1)^4}$	$\mathcal{B}_4(1;q) = \frac{\mathcal{B}_4(q)}{q}$	$\mu_4 = -\frac{p\mathcal{B}_5(1;q)}{5} = \frac{q^3+11q^2+11q+1}{p^4}$
5	$5\frac{q^4+11q^3+11q^2+q}{(q-1)^5}$	$\mathcal{B}_5(1;q) = \frac{1}{q}\mathcal{B}_5(q)$	$\mu_5 = -\frac{p\mathcal{B}_6(1;q)}{6} = \frac{q^4+26q^3+66q^2+26q+1}{p^5}$
6	$-6\frac{q^5+26q^4+66q^3+26q^2+q}{(q-1)^6}$	$\mathcal{B}_6(1;q) = \frac{1}{q}\mathcal{B}_6(q)$	$\vdots$

As can be easily seen in the first column of above table, Apostol- Bernoulli numbers are rational functions of the variable q with $q \neq 1$.

In the first column of the above table, when the coefficients of each of the Apostol- Bernoulli numbers are carefully examined respectively, we observe that the following sequence can be easily noticed:

$$\{1, -2, 3, -4, 5, -6, \dots\} = \left\{\left((-1)^{j+1} j\right)\right\}_{j=1}^{\infty},$$

where positive integer j is an index of the Apostol- Bernoulli numbers $\mathcal{B}_j(q)$. That is, each of member of this sequence corresponds to the index of the relevant Apostol- Bernoulli numbers. It can also be observed that if this index is even and different 0, the number takes the $-$ sign, and if this index is odd, it takes the $+$ sign.

By using μ_1 and μ_2, the variance formula for the geometric random variable with parameter p is given by

$$\begin{aligned}\sigma^2 &= \mu_2 - \mu_1^2 \\ &= \frac{q+1}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}\end{aligned}$$

or

$$\sigma^2 = \frac{1-p}{p^2}.$$

Here we note that the expected value and variance calculation given above was obtained by using the geometric series formula and its derivative in almost all probability and statistics books (*cf.* [2]-[19]).

As can be easily seen in above table, unlike conventional moment calculations, our new formulas containing Apostol-Bernoulli polynomials and numbers can easily calculate higher order moments.

Inspired by the above calculations of moments in above table, we noticed the following interesting problem.

We observed that the coefficients in the numerator of each moment correspond to the triangle of Eulerian numbers, which are positive integers given in *A008292 in oeis.org*. There are many notations of these numbers. Here, we can use $T(n,k)$ for these numbers. Some of the array of the triangle of Eulerian numbers are given the following triangle matrix:

$$\begin{array}{l}1\\ 1,1\\ 1,4,1\\ 1,4,1\\ 1,11,11,1\\ 1,26,66,26,1\\ 1,57,302,302,57,1\\ 1,120,1191,2416,1191,120,1\end{array}$$

and so on ([13]).

Coefficients of $T(n,k)$ = number of permutations of $\{1,2,\dots,n\} = [n]$ with k runs.

Therefore the following **open problems** arise:

Is there any relationship between moments and the triangle of Eulerian numbers? How can the polynomial of q in the numerators of the moments be expressed in terms of the triangle of Eulerian numbers?

It is well-known that the Eulerian polynomials are also related to the distribution of other descent statistics: permutation statistics that depend only on the descent set and length of a permutation. There are many useful applications of the Eulerian polynomials in various descent statistics associated with the number of peaks and left peaks (*cf.* [18]).

Remark 1. In [15], for $j \in \mathbb{N}$, using jth derivatives of the moment generating function $M_X(u)$ for $u = 0$, we showed that the following new formulas for the jth moments of geometric random variable with parameter p:

$$\mu_j = p \sum_{v=1}^{j} \frac{v! q^{v-1}}{(1-q)^{v+1}} S_2(j, v) \tag{18}$$

and

$$\mu_j = \frac{1}{q} \sum_{v=1}^{j} \sum_{d=0}^{v} (-1)^{v-d} \binom{v}{d} \left(\frac{q}{p}\right)^{v} d^j. \tag{19}$$

4 | CONCLUSION

The purpose of this paper is to investigate and survey properties of moments of geometric distribution using characteristic function and the Apostol-Bernoulli numbers and polynomials and also the Stirling numbers.

By using the definition and properties of the characteristic function with the help of geometric distribution, we gave some new formulas for moments and characteristic function. These formulas were given by combining both the serial expansions of the $\sin x$ and $\cos x$ functions and the generating functions of the Apostol-Bernoulli polynomials and polynomials. A the relation between the characteristic function and the moment generator function was given. Finally, we gave some new computational formulas for characteristic function of the geometric distribution in terms of the Apostol-Bernoulli numbers and polynomials and other special functions. Using these formulas, we also gave some new formulas for jth moment of the geometric random variable with parameter p. Our future project is to investigate applications of the Eulerian polynomials with our new formulas in probability, in statistics theory, and also in other applied sciences.

vspace0.75cm **Author contributions:** This paper has only one author.

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