

ARTICLE TYPE

Some new post-quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable convex functions

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ABSTRACT

In this paper, we establish a new (p, q) -integral identity using twice (p, q) -differentiable convex functions. Then, we use this result to derive some new post-quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable convex functions. The newly established results are also proven to generalize some existing results in the area of integral inequalities of already published ones.

MSC: 05A30; 26D10; 26A51; 26D15

KEYWORDS:

Ostrowski-type inequality, (p, q) -differentiable function, (p, q) -integral inequalities, (p, q) -calculus, Convex function

1 | INTRODUCTION

Integral inequalities are a very necessary tool in the study of applied and pure mathematics. One of the integral inequalities that many researchers have focused on significant attention is Ostrowski-type integral inequalities because it can be applied in statistics, quadrature, stochastic, probability and optimization theory, integral operator theory, and information. The classical integral inequality for the differentiable function is as follows:

Theorem 1. ¹ Let $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2) whose the derivative function $\varphi' : (a_1, a_2) \rightarrow \mathbb{R}$ is bounded on (a_1, a_2) and $\|\varphi'\|_\infty = \sup_{t \in (a_1, a_2)} |\varphi'(t)| < \infty$. Then

$$\left| \varphi(x) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varphi(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a_1 + a_2}{2} \right)^2}{(a_2 - a_1)^2} \right] (a_2 - a_1) \|\varphi'\|_\infty$$

for all $x \in [a_1, a_2]$.

In recent years, many researchers have focused on the Ostrowski-type integral inequalities and their applications, see^{2,3,4,5,6,7,8,9,10,11,12} and the references cited therein for more details. Specifically, many researchers have worked on the Ostrowski-type integral inequalities and their applications using quantum calculus, some results can be found in^{13,14,15,16,17,18,19,20,21} and the references cited therein.

Quantum calculus, also known as q -calculus, is the study of calculus without limits. The concept was revealed by renowned mathematician Euler (1707-1783), who introduced the number in q -infinite series defined by Newton. Then, in 1910, Jackson²² defined q_{a_1} -integrals and q_{a_1} -derivatives on the interval $(0, \infty)$ extending the concept of Euler. Later, in 1966, Al-Salam²³ introduced fractional q_{a_1} -integrals and fractional q_{a_1} -derivatives. In q -calculus, we obtain q -analogues of mathematical objects that can be recaptured by taking $q \rightarrow 1$. The topic of q -calculus has been received outstanding attention from many scientists because it has numerous applications in various fields of physics and mathematics, for example, the theory of relativity, combinatorics, hypergeometric functions, orthogonal polynomials, mechanics, and number theory, see^{24,25,26,27,28,29,30,31,32,33,34} and the references cited therein for more details.

In 2013, Tariboon and Ntouyas³⁵ presented the q_{a_1} -integrals and the q_{a_1} -derivatives on finite intervals and addressed numerous problems on q_{a_1} -analogues of classical inequalities. Recently, in 2020, Bermudo et al.³⁶ presented q^{a_2} -integrals and q^{a_2} -derivatives on finite intervals and also proved their some basic properties. Currently, these topics of q -calculus have been studied in various integral inequalities such as Hanh, Hermite-Hadamard, Hermite-Hadamard-like, Newton-type, Simpson-type, Fejér-type, and Ostrowski-type integral inequalities, see^{37,38,39,40,41,42,43} and the references cited therein for more details.

The q -calculus generalization is called (p, q) -calculus, also known as post-quantum calculus. The (p, q) -calculus has two independent parameters that are p -number and q -number. Apparently, the q -calculus cannot be directly obtained by substituting q by q/p in q -calculus, but it can be directly obtained by taking $p = 1$ in (p, q) -calculus. Then, the classical inequalities can be gained by taking $q \rightarrow 1$. The concept of $(p, q)_{a_1}$ -integrals and $(p, q)_{a_1}$ -derivatives on the interval $(0, \infty)$ was first presented by Chakrabarti and Jagannathan⁴⁴ in 1991. Later, the concept of $(p, q)_{a_1}$ -integrals and $(p, q)_{a_1}$ -derivatives on finite intervals was presented by Tunç, and Göv^{45,46} in 2016. Recently, the concept of the $(p, q)^{a_2}$ -integrals and $(p, q)^{a_2}$ -derivatives on the finite intervals has been presented by Vivas-Cortez et al.⁴⁷ in 2021. Currently, the topic of (p, q) -calculus has been received outstanding attention from many scientists, some new results can be found in^{48,49,50,51,52,53,54,55,56} and the references cited therein.

In 2021, Ali et al.⁴³ introduced quantum Ostrowski-type integral inequalities for twice q -differentiable convex functions. By taking $q \rightarrow 1$, they obtain classical results on some Ostrowski-type integral inequalities for functions, whose second derivatives are h -convex functions⁶⁰. Inspired by the above mentioned reports, we establish some new post quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable convex functions to extend and generalize the results given in previous reports.

The rest of the paper is organized as follows: In Section 2, we provide some basic knowledge and definitions of (p, q) -calculus. In Section 3, post quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable convex functions are presented. In Section 4, we summarize our results.

2 | PRELIMINARIES

In this section, we discuss some basic knowledge and definitions of (p, q) -calculus which will be used in our work. Throughout this paper, we assume that $0 < q < p \leq 1$ is constants and $[a_1, a_2] \subseteq \mathbb{R}$ is an interval with $a_1 < a_2$. The (p, q) -number of η is given by

$$[\eta]_{p,q} = \frac{p^\eta - q^\eta}{p - q} = p^{\eta-1} + p^{\eta-2}q + \cdots + pq^{\eta-2} + q^{\eta-1}, \quad \eta \in \mathbb{N},$$

which is a generalization of the q -analogue or q -number of η such that

$$[\eta]_q = \frac{1 - q^\eta}{1 - q} = 1 + q + \cdots + q^{\eta-2} + q^{\eta-1}, \quad \eta \in \mathbb{N},$$

see³⁴ for more details.

Definition 1.^{45,46} For a continuous function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$, then the $(p, q)_{a_1}$ -derivative on $[a_1, a_2]$ of function φ at t is defined by

$$\begin{aligned} {}_{a_1}D_{p,q}\varphi(t) &= \frac{\varphi(pt + (1-p)a_1) - \varphi(qt + (1-q)a_1)}{(p-q)(t-a_1)}, \quad t \neq a_1, \\ {}_{a_1}D_{p,q}\varphi(a_1) &= \lim_{t \rightarrow a_1} {}_{a_1}D_{p,q}\varphi(t). \end{aligned} \quad (1)$$

The function φ is called $(p, q)_{a_1}$ -differentiable function on $[a_1, a_2]$ if ${}_{a_1}D_{p,q}\varphi(t)$ exists for all $t \in [a_1, (a_2 - a_1)/p + a_1]$.

Note that if $p = 1$ and ${}_{a_1}D_{1,q}\varphi(t) = {}_{a_1}D_q\varphi(t)$ in (1), then (1) reduces to

$$\begin{aligned} {}_{a_1}D_q\varphi(t) &= \frac{\varphi(t) - \varphi(qt + (1-q)a_1)}{(1-q)(t-a_1)}, \quad t \neq a_1, \\ {}_{a_1}D_q\varphi(a_1) &= \lim_{t \rightarrow a_1} {}_{a_1}D_q\varphi(t), \end{aligned} \quad (2)$$

which is the well-known q_{a_1} -derivative of function φ on $[a_1, a_2]$, see^{57,58} for more details.

Moreover, if $a_1 = 0$ and ${}_0D_q\varphi(t) = D_q\varphi(t)$ in (2), then (2) reduces to

$$\begin{aligned} D_q\varphi(t) &= \frac{\varphi(t) - \varphi(qt)}{(1-q)t}, \quad t \neq a_1, \\ D_q\varphi(a_1) &= \lim_{t \rightarrow 0} D_q\varphi(t), \end{aligned}$$

which is the well-known q -derivative of function φ on $[0, a_2]$, also called q -Jackson derivative, see³⁴ for more details.

Example 1. Define function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ by $\varphi(t) = t^2 + C$, where C is constant. Applying Definition 1 for $t \neq a_1$, we have

$$\begin{aligned} {}_{a_1}D_{p,q}(t^2 + C) &= \frac{[(pt + (1-p)a_1)^2 + C] - [qt + (1-q)a_1]^2 + C}{(p-q)(t-a_1)} \\ &= \frac{(p+q)t^2 + 2a_1t[1-(p+q)] + a_1^2[(p+q)-2]}{(t-a_1)} \\ &= \frac{(p+q)(t-a_1)^2 + 2a_1(t-a_1)}{(t-a_1)} \\ &= [2]_{p,q}(t-a_1) + 2a_1. \end{aligned}$$

Definition 2.⁴⁷ For a continuous function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$, then the $(p, q)^{a_2}$ -derivative on $[a_1, a_2]$ of function φ at t is defined by

$$\begin{aligned} {}^{a_2}D_{p,q}\varphi(t) &= \frac{\varphi(qt + (1-q)a_2) - \varphi(pt + (1-p)a_2)}{(p-q)(a_2-t)}, \quad t \neq a_2, \\ {}^{a_2}D_{p,q}\varphi(a_2) &= \lim_{t \rightarrow a_2} {}^{a_2}D_{p,q}\varphi(t). \end{aligned} \quad (3)$$

The function φ is called $(p, q)^{a_2}$ -differentiable function on $[a_1, a_2]$ if ${}^{a_2}D_{p,q}\varphi(t)$ exists for all $t \in [a_2 - (a_2 - a_1)/p, a_2]$.

Note that if $p = 1$ and ${}^{a_2}D_{1,q}\varphi(t) = {}^{a_2}D_q\varphi(t)$ in (3), then (3) reduces to

$$\begin{aligned} {}^{a_2}D_q\varphi(t) &= \frac{\varphi(qt + (1-q)a_2) - \varphi(t)}{(1-q)(a_2-t)}, \quad t \neq a_2, \\ {}^{a_2}D_q\varphi(a_2) &= \lim_{t \rightarrow a_2} {}^{a_2}D_q\varphi(t), \end{aligned}$$

which is the well-known q^{a_2} -derivative of function φ on $[a_1, a_2]$, see^{36,59} for more details.

Example 2. Define function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ by $\varphi(t) = t^2 + C$, where C is constant. Applying Definition 2 for $t \neq a_2$, we have

$$\begin{aligned} {}^{a_2}D_{p,q}(t^2 + C) &= \frac{[(qt + (1-q)a_2)^2 + C] - [pt + (1-p)a_2]^2 + C}{(p-q)(a_2-t)} \\ &= \frac{-(p+q)t^2 + 2a_2t[(p+q)-1] + a_2^2[2-(p+q)]}{(a_2-t)} \\ &= \frac{-(p+q)(a_2-t)^2 + 2a_2(a_2-t)}{(a_2-t)} \\ &= [2]_{p,q}(t-a_2) + 2a_2. \end{aligned}$$

Definition 3. ⁴⁵ For a continuous function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$, then the $(p, q)_{a_1}$ -integral on $[a_1, a_2]$ of function φ at t is defined by

$$\int_{a_1}^x \varphi(t) {}_{a_1}d_{p,q}t = (p - q)(x - a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi \left(\frac{q^\eta}{p^{\eta+1}}x + \left(1 - \frac{q^\eta}{p^{\eta+1}}\right)a_1 \right) \quad (4)$$

for $x \in [a_1, a_2]$.

The function φ is called $(p, q)_{a_1}$ -integrable function on $[a_1, a_2]$ if $\int_{a_1}^x \varphi(t) {}_{a_1}d_{p,q}t$ exists for all $t \in [a_1, a_1 + p(x - a_1)]$.

Example 3. Define function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ by $\varphi(t) = At + B$, where A and B are constants. Applying Definition 3, we have

$$\begin{aligned} \int_{a_1}^{a_2} \varphi(t) {}_{a_1}d_{p,q}t &= \int_{a_1}^{a_2} (At + B) {}_{a_1}d_{p,q}t \\ &= A(p - q)(a_2 - a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \left(\frac{q^\eta}{p^{\eta+1}}a_2 + \left(1 - \frac{q^\eta}{p^{\eta+1}}\right)a_1 \right) \\ &\quad + B(p - q)(a_2 - a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \\ &= \frac{A(a_2 - a_1)(a_2 - a_1(1 - p - q))}{[2]_{p,q}} + B(a_2 - a_1). \end{aligned}$$

Definition 4. ⁴⁷ For a continuous function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$, then the $(p, q)^{a_2}$ -integral on $[a_1, a_2]$ of function φ at t is defined by

$$\int_x^{a_2} \varphi(t) {}^{a_2}d_{p,q}t = (p - q)(a_2 - x) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi \left(\frac{q^\eta}{p^{\eta+1}}x + \left(1 - \frac{q^\eta}{p^{\eta+1}}\right)a_2 \right) \quad (5)$$

for $x \in [a_1, a_2]$.

The function φ is called $(p, q)^{a_2}$ -integrable function on $[a_1, a_2]$ if $\int_x^{a_2} \varphi(t) {}^{a_2}d_{p,q}t$ exists for all $t \in [a_2 - p(a_2 - x), a_2]$.

Example 4. Define function $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ by $\varphi(t) = At + B$, where A and B are constants. Applying Definition 4, we have

$$\begin{aligned} \int_{a_1}^{a_2} \varphi(t) {}^{a_2}d_{p,q}t &= \int_{a_1}^{a_2} (At + B) {}^{a_2}d_{p,q}t \\ &= A(p - q)(a_2 - a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \left(\frac{q^\eta}{p^{\eta+1}}a_1 + \left(1 - \frac{q^\eta}{p^{\eta+1}}\right)a_2 \right) \\ &\quad + B(p - q)(a_2 - a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \\ &= \frac{A(a_2 - a_1)(a_1 - a_2(1 - p - q))}{[2]_{p,q}} + B(a_2 - a_1). \end{aligned}$$

Lemma 1. ⁴⁵ For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following inequality holds:

$$\int_{a_1}^{a_2} (t - a_1)^\alpha {}_{a_1}d_{p,q}t = \frac{(a_2 - a_1)^{\alpha+1}}{[\alpha + 1]_{p,q}} \quad (6)$$

for $t \in [a_1, a_2]$.

Theorem 2. ⁴⁶ Let $\varphi, \psi : [a_1, a_2] \rightarrow \mathbb{R}$ be continuous functions and $r > 1$ with $1/s + 1/r = 1$, then

$$\int_{a_1}^{a_2} |\varphi(t)\psi(t)| {}_{a_1}d_{p,q}t \leq \left(\int_{a_1}^{a_2} |\varphi(t)|^s {}_{a_1}d_{p,q}t \right)^{1/s} \left(\int_{a_1}^{a_2} |\psi(t)|^r {}_{a_1}d_{p,q}t \right)^{1/r} \quad (7)$$

for $t \in [a_1, a_2]$.

3 | MAIN RESULTS

In this section, we give some new estimates of post quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable functions involving $(p, q)_{a_1}$ - and $(p, q)^{a_2}$ -integrals. Let $J_1 = [a_2 - p(a_2 - x), a_2]$ and $J_2 = [a_1, a_1 + p(x - a_1)]$. We start with the following (p, q) -integral identities.

Theorem 3. If $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ is a twice (p, q) -differentiable function such that ${}^{a_2}D_{p,q}^2 \varphi$ and ${}_{a_1}D_{p,q}^2 \varphi$ are continuous and integrable functions on J_1 and J_2 , respectively. Then

$$(x - a_1)^2(a_2 - x)^2 \left[(a_1 - x) \int_0^1 t^2 {}_{a_1}D_{p,q}^2 \varphi(tx + (1 - t)a_1) d_{p,q}t + (x - a_2) \int_0^1 t^2 {}^{a_2}D_{p,q}^2 \varphi(tx + (1 - t)a_2) d_{p,q}t \right] = {}^{a_2}L_{p,q}(x), \quad (8)$$

where

$$\begin{aligned} & {}^{a_2}L_{p,q}(x) \\ &= \frac{(x - a_1)(a_2 - x)}{pq^3(p - q)} \left[(x - a_1)pq\varphi(qx + (1 - q)a_2) + (a_2 - x)pq\varphi(qx + (1 - q)a_1) - (x - a_1)(q^2 + pq - p^2)\varphi(px + (1 - p)a_2) \right. \\ & \quad \left. - (a_2 - x)(q^2 + pq - p^2)\varphi(px + (1 - p)a_1) \right] - \frac{[2]_{p,q}}{p^3q^3} \left[(x - a_1)^2 \int_{p^2x+(1-p^2)a_2}^{a_2} \varphi(t) {}^{a_2}d_{p,q}t + (a_2 - x)^2 \int_{a_1}^{p^2x+(1-p^2)a_1} \varphi(t) {}_{a_1}d_{p,q}t \right]. \end{aligned}$$

Proof. Using Definition 1, we have

$$\begin{aligned} & {}_{a_1}D_{p,q}^2 \varphi(ta_2 + (1 - t)a_1) \\ &= {}_{a_1}D_{p,q}({}_{a_1}D_{p,q} \varphi(ta_2 + (1 - t)a_1)) \\ &= {}_{a_1}D_{p,q} \left(\frac{\varphi(p ta_2 + (1 - pt)a_1) - \varphi(q ta_2 + (1 - qt)a_1)}{(p - q)(a_2 - a_1)t} \right) \\ &= \frac{1}{(p - q)(a_2 - a_1)t} \left[\frac{\varphi(p^2 ta_2 + (1 - p^2 t)a_1) - \varphi(pqt a_2 + (1 - pqt)a_1)}{pt(p - q)(a_2 - a_1)} - \frac{\varphi(pqt a_2 + (1 - pqt)a_1) - \varphi(q^2 ta_2 + (1 - q^2 t)a_1)}{qt(p - q)(a_2 - a_1)} \right] \\ &= \frac{q\varphi(p^2 ta_2 + (1 - p^2 t)a_1) - [2]_{p,q}\varphi(pqt a_2 + (1 - pqt)a_1) + p\varphi(q^2 ta_2 + (1 - q^2 t)a_1)}{pq t^2 (p - q)^2 (a_2 - a_1)^2}. \end{aligned} \quad (9)$$

Applying (9) and Definition 3, we obtain

$$\begin{aligned} & \int_0^1 t^2 {}_{a_1}D_{p,q}^2 \varphi(tx + (1 - t)a_1) d_{p,q}t \\ &= \int_0^1 \frac{q\varphi(p^2 tx + (1 - p^2 t)a_1) - [2]_{p,q}\varphi(pqt x + (1 - pqt)a_1) + p\varphi(q^2 tx + (1 - q^2 t)a_1)}{pq(p - q)^2(x - a_1)^2} d_{p,q}t \\ &= \frac{q(p - q)(x - a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi \left(p^2 \frac{q^\eta}{p^{\eta+1}} x + \left(1 - p^2 \frac{q^\eta}{p^{\eta+1}} \right) a_1 \right)}{pq(p - q)^2(x - a_1)^3} \\ & \quad - \frac{[2]_{p,q}(p - q)(x - a_1) \sum_{\eta=0}^{\infty} \frac{q^{\eta+1}}{p^{\eta+1}} \varphi \left(p \frac{q^{\eta+1}}{p^{\eta+1}} x + \left(1 - p \frac{q^{\eta+1}}{p^{\eta+1}} \right) a_1 \right)}{pq^2(p - q)^2(x - a_1)^3} \\ & \quad + \frac{p(p - q)(x - a_1) \sum_{\eta=0}^{\infty} \frac{q^{\eta+2}}{p^{\eta+1}} \varphi \left(\frac{q^{\eta+2}}{p^{\eta+1}} x + \left(1 - \frac{q^{\eta+2}}{p^{\eta+1}} \right) a_1 \right)}{pq^3(p - q)^2(x - a_1)^3} \end{aligned}$$

$$\begin{aligned}
& (p-q)(x-a_1) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi \left(p^2 \frac{q^\eta}{p^{\eta+1}} x + \left(1 - p^2 \frac{q^\eta}{p^{\eta+1}} \right) a_1 \right) \\
&= \frac{p(p-q)^2(x-a_1)^3}{p[2]_{p,q}(p-q)(x-a_1) \sum_{\eta=0}^{\infty} \frac{q^{\eta+1}}{p^{\eta+2}} \varphi \left(p^2 \frac{q^{\eta+1}}{p^{\eta+2}} x + \left(1 - p^2 \frac{q^{\eta+1}}{p^{\eta+2}} \right) a_1 \right)} \\
&\quad - \frac{pq^2(p-q)^2(x-a_1)^3}{p^3(p-q)(x-a_1) \sum_{\eta=0}^{\infty} \frac{q^{\eta+2}}{p^{\eta+3}} \varphi \left(p^2 \frac{q^{\eta+2}}{p^{\eta+3}} x + \left(1 - p^2 \frac{q^{\eta+2}}{p^{\eta+3}} \right) a_1 \right)} \\
&\quad + \frac{pq^3(p-q)^2(x-a_1)^3}{p^3 q^3 (x-a_1)^3} \\
&= \frac{[2]_{p,q}}{p^3 q^3 (x-a_1)^3} \int_{a_1}^{p^2 x + (1-p^2)a_1} \varphi(t) {}_{a_1} d_{p,q} t + \frac{(q^2 + pq - p^2)\varphi(px + (1-p)a_1)}{pq^3(p-q)(x-a_1)^2} - \frac{\varphi(qx + (1-q)a_1)}{q^2(p-q)(x-a_1)^2}. \tag{10}
\end{aligned}$$

Using Definition 2, we have

$$\begin{aligned}
& {}^{a_2} D_{p,q}^2 \varphi(ta_1 + (1-t)a_2) \\
&= {}^{a_2} D_{p,q} ({}^{a_2} D_{p,q} \varphi(ta_1 + (1-t)a_2)) \\
&= {}^{a_2} D_{p,q} \left(\frac{\varphi(qta_1 + (1-qt)a_2) - \varphi(pta_1 + (1-pt)a_2)}{(p-q)(a_2-a_1)t} \right) \\
&= \frac{1}{(p-q)(a_2-a_1)t} \left[\frac{\varphi(q^2ta_1 + (1-q^2t)a_2) - \varphi(pqta_1 + (1-pqt)a_2)}{qt(p-q)(a_2-a_1)} - \frac{\varphi(pqta_1 + (1-pqt)a_2) - \varphi(p^2ta_1 + (1-p^2t)a_2)}{pt(p-q)(a_2-a_1)} \right] \\
&= \frac{p\varphi(q^2ta_1 + (1-q^2t)a_2) - [2]_{p,q}\varphi(pqta_1 + (1-pqt)a_2) + q\varphi(p^2ta_1 + (1-p^2t)a_2)}{pqt^2(p-q)^2(a_2-a_1)^2}. \tag{11}
\end{aligned}$$

Applying (11) and Definition 4, we obtain

$$\begin{aligned}
& \int_0^1 t^2 {}^{a_2} D_{p,q}^2 \varphi(tx + (1-t)a_2) d_{p,q} t \\
&= \int_0^1 \frac{p\varphi(q^2tx + (1-q^2t)a_2) - [2]_{p,q}\varphi(pqtx + (1-pqt)a_2) + q\varphi(p^2tx + (1-p^2t)a_2)}{pq(p-q)^2(a_2-x)^2} d_{p,q} t \\
&= \frac{p(p-q)(a_2-x) \sum_{\eta=0}^{\infty} \frac{q^{\eta+2}}{p^{\eta+1}} \varphi \left(\frac{q^{\eta+2}}{p^{\eta+1}} x + \left(1 - \frac{q^{\eta+2}}{p^{\eta+1}} \right) a_2 \right)}{pq^3(p-q)^2(a_2-x)^3} \\
&\quad - \frac{[2]_{p,q}(p-q)(a_2-x) \sum_{\eta=0}^{\infty} \frac{q^{\eta+1}}{p^{\eta+1}} \varphi \left(p \frac{q^{\eta+1}}{p^{\eta+1}} x + \left(1 - p \frac{q^{\eta+1}}{p^{\eta+1}} \right) a_2 \right)}{pq^2(p-q)^2(a_2-x)^3} \\
&\quad + \frac{q(p-q)(a_2-x) \sum_{\eta=0}^{\infty} \frac{q^\eta}{p^{\eta+1}} \varphi \left(p^2 \frac{q^{\eta+2}}{p^{\eta+1}} x + \left(1 - p^2 \frac{q^{\eta+2}}{p^{\eta+1}} \right) a_2 \right)}{pq(p-q)^2(a_2-x)^3} \\
&\quad + \frac{p^3(p-q)(a_2-x) \sum_{\eta=0}^{\infty} \frac{q^{\eta+2}}{p^{\eta+3}} \varphi \left(p^2 \frac{q^{\eta+2}}{p^{\eta+3}} x + \left(1 - p^2 \frac{q^{\eta+2}}{p^{\eta+3}} \right) a_2 \right)}{pq^3(p-q)^2(a_2-x)^3}
\end{aligned}$$

$$\begin{aligned}
 & - \frac{p[2]_{p,q}(p-q)(a_2-x) \sum_{\eta=0}^{\infty} \frac{q^{\eta+1}}{p^{\eta+2}} \varphi \left(p^2 \frac{q^{\eta+1}}{p^{\eta+1}} x + \left(1 - p^2 \frac{q^{\eta+1}}{p^{\eta+2}} \right) a_2 \right)}{pq^2(p-q)^2(a_2-x)^3} \\
 & + \frac{q(p-q)(a_2-x) \sum_{\eta=0}^{\infty} \frac{q^{\eta}}{p^{\eta+1}} \varphi \left(p^2 \frac{q^{\eta+2}}{p^{\eta+1}} x + \left(1 - p^2 \frac{q^{\eta+2}}{p^{\eta+1}} \right) a_2 \right)}{pq(p-q)^2(a_2-x)^3} \\
 & = \frac{[2]_{p,q}}{p^3 q^3 (a_2 - x)^3} \int_{p^2 x + (1-p^2)a_2}^{a_2} \varphi(t) {}^{a_2}d_{p,q}t + \frac{(q^2 + pq - p^2)\varphi(px + (1-p)a_2)}{pq^3(p-q)(a_2-x)^2} - \frac{\varphi(qx + (1-q)a_2)}{q^2(p-q)(a_2-x)^2}. \quad (12)
 \end{aligned}$$

By multiplying (10) and (12) by $(x-a_1)^2(a_2-x)^2(a_1-x)$ and $(x-a_1)^2(a_2-x)^2(x-a_2)$, respectively, and adding the resultant inequalities, we obtain the required identity (8). Therefore, the proof is completed. \square

Remark 1. If $p = 1$ in (8), then we have the following identity:

$$(x-a_1)^2(a_2-x)^2 \left[(a_1-x) \int_0^1 t^2 {}^{a_1}D_q^2 \varphi(tx + (1-t)a_1) d_q t + (x-a_2) \int_0^1 t^2 {}^{a_2}D_q^2 \varphi(tx + (1-t)a_2) d_q t \right] = {}^{a_2}L_q(x),$$

where

$$\begin{aligned}
 {}^{a_2}L_q(x) &= \frac{(x-a_1)(a_2-x)}{q^3(p-q)} \left[(x-a_1)q\varphi(qx + (1-q)a_2) + (a_2-x)q\varphi(qx + (1-q)a_1) - (q^2 + q - 1)(a_2-a_1)\varphi(x) \right] \\
 &\quad - \frac{[2]_q}{q^3} \left[(x-a_1)^2 \int_x^{a_2} \varphi(t) {}^{a_2}d_q t + (a_2-x)^2 \int_{a_1}^x \varphi(t) {}^{a_1}d_q t \right],
 \end{aligned}$$

which appeared in⁴³.

Theorem 4. Let $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ be a twice (p, q) -differentiable function such that ${}^{a_2}D_{p,q}^2 \varphi$ and ${}^{a_1}D_{p,q}^2 \varphi$ are continuous and integrable functions on J_1 and J_2 , respectively. If $|{}^{a_2}D_{p,q}^2 \varphi|$ and $|{}^{a_1}D_{p,q}^2 \varphi|$ are convex functions, then

$$\begin{aligned}
 |{}^{a_2}L_{p,q}(x)| &\leq (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \left(\frac{1}{[4]_{p,q}} |{}^{a_1}D_{p,q}^2 \varphi(x)| + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} |{}^{a_1}D_{p,q}^2 \varphi(a_1)| \right) \right. \\
 &\quad \left. + (a_2-x) \left(\frac{1}{[4]_{p,q}} |{}^{a_2}D_{p,q}^2 \varphi(x)| + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} |{}^{a_2}D_{p,q}^2 \varphi(a_2)| \right) \right]. \quad (13)
 \end{aligned}$$

Proof. Taking modulus of 8, applying the convexity of $|{}^{a_2}D_{p,q}^2 \varphi|$ and $|{}^{a_1}D_{p,q}^2 \varphi|$, and by using Lemma 1, we obtain

$$\begin{aligned}
 |{}^{a_2}L_{p,q}(x)| &\leq (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \int_0^1 t^2 |{}^{a_1}D_{p,q}^2 \varphi(tx + (1-t)a_1)| d_{p,q}t + (a_2-x) \int_0^1 t^2 |{}^{a_2}D_{p,q}^2 \varphi(tx + (1-t)a_2)| d_{p,q}t \right] \\
 &\leq (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \int_0^1 t^2 \left(t |{}^{a_1}D_{p,q}^2 \varphi(x)| + (1-t) |{}^{a_1}D_{p,q}^2 \varphi(a_1)| \right) d_{p,q}t \right. \\
 &\quad \left. + (a_2-x) \int_0^1 t^2 \left(t |{}^{a_2}D_{p,q}^2 \varphi(x)| + (1-t) |{}^{a_2}D_{p,q}^2 \varphi(a_2)| \right) d_{p,q}t \right] \\
 &= (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \left(\frac{1}{[4]_{p,q}} |{}^{a_1}D_{p,q}^2 \varphi(x)| + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} |{}^{a_1}D_{p,q}^2 \varphi(a_1)| \right) \right. \\
 &\quad \left. + (a_2-x) \left(\frac{1}{[4]_{p,q}} |{}^{a_2}D_{p,q}^2 \varphi(x)| + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} |{}^{a_2}D_{p,q}^2 \varphi(a_2)| \right) \right],
 \end{aligned}$$

which completes the proof. \square

Corollary 1. With the assumptions of Theorem 4, if $|{}^{a_2}D_{p,q}^2\varphi|$ and $|{}_{a_1}D_{p,q}^2\varphi| \leq M$, then the following inequality holds:

$$\left| {}_{a_1}^{a_2}L_{p,q}(x) \right| \leq \frac{M(x-a_1)^2(a_2-x)^2(a_2-a_1)}{[3]_{p,q}}, \quad (14)$$

where M is constant.

Remark 2. If $p = 1$ in (13), then we have the following inequality:

$$\begin{aligned} \left| {}_{a_1}^{a_2}L_q(x) \right| &\leq (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \left(\frac{1}{[4]_q} \left| {}_{a_1}D_q^2\varphi(x) \right| + \frac{q^3}{[3]_q[4]_q} \left| {}_{a_1}D_q^2\varphi(a_1) \right| \right) \right. \\ &\quad \left. + (a_2-x) \left(\frac{1}{[4]_q} \left| {}^{a_2}D_q^2\varphi(x) \right| + \frac{q^3}{[3]_q[4]_q} \left| {}^{a_2}D_q^2\varphi(a_2) \right| \right) \right], \end{aligned}$$

which appeared in⁴³.

Remark 3. If $p = 1$ in (14), then we have the following inequality:

$$\left| {}_{a_1}^{a_2}L_q(x) \right| \leq \frac{M(x-a_1)^2(a_2-x)^2(a_2-a_1)}{[3]_q}, \quad (15)$$

which appeared in⁴³. Moreover, if $q \rightarrow 1$ and $x = (a_1 + a_2)/2$ in (15), then we obtain the following inequality:

$$\left| \varphi\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1} \int_{a_1}^{a_2} \varphi(t)dt \right| \leq \frac{M(a_2-a_1)^2}{24},$$

which appeared in⁶⁰ and it can be found in⁸.

Theorem 5. Let $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ be a twice (p, q) -differentiable function on (a_1, a_2) such that ${}^{a_2}D_{p,q}^2\varphi$ and ${}_{a_1}D_{p,q}^2\varphi$ are continuous and integrable functions on J_1 and J_2 , respectively. If $|{}^{a_2}D_{p,q}^2\varphi|^r$ and $|{}_{a_1}D_{p,q}^2\varphi|^r$ are convex functions for $r \geq 1$, then

$$\begin{aligned} \left| {}_{a_1}^{a_2}L_{p,q}(x) \right| &\leq (x-a_1)^2(a_2-x)^2 \left(\frac{1}{[3]_{p,q}} \right)^{1-1/r} \left[(x-a_1) \left(\frac{1}{[4]_{p,q}} \left| {}_{a_1}D_{p,q}^2\varphi(x) \right|^r + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} \left| {}_{a_1}D_{p,q}^2\varphi(a_1) \right|^r \right)^{1/r} \right. \\ &\quad \left. + (a_2-x) \left(\frac{1}{[4]_{p,q}} \left| {}^{a_2}D_{p,q}^2\varphi(x) \right|^r + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} \left| {}^{a_2}D_{p,q}^2\varphi(a_2) \right|^r \right)^{1/r} \right]. \end{aligned} \quad (16)$$

Proof. Applying Theorem 3 and the power mean inequality, we obtain

$$\begin{aligned} \left| {}_{a_1}^{a_2}L_{p,q}(x) \right| &\leq (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \int_0^1 t^2 \left| {}_{a_1}D_{p,q}^2\varphi(tx + (1-t)a_1) \right| d_{p,q}t + (a_2-x) \int_0^1 t^2 \left| {}^{a_2}D_{p,q}^2\varphi(tx + (1-t)a_2) \right| d_{p,q}t \right] \\ &\leq (x-a_1)^2(a_2-x)^2 \left[(x-a_1) \left(\int_0^1 t^2 d_{p,q}t \right)^{1-1/r} \left(\int_0^1 t^2 \left| {}_{a_1}D_{p,q}^2\varphi(tx + (1-t)a_1) \right|^r d_{p,q}t \right)^{1/r} \right. \\ &\quad \left. + (a_2-x) \left(\int_0^1 t^2 d_{p,q}t \right)^{1-1/r} \left(\int_0^1 t^2 \left| {}^{a_2}D_{p,q}^2\varphi(tx + (1-t)a_2) \right|^r d_{p,q}t \right)^{1/r} \right]. \end{aligned}$$

Using Lemma 1 and applying the convexity of $|{}^{a_2}D_{p,q}^2\varphi|^r$ and $|{}_{a_1}D_{p,q}^2\varphi|^r$, we have

$$\begin{aligned}
 \left| {}_{a_1}^{a_2} L_{p,q}(x) \right| &\leq (x - a_1)^2 (a_2 - x)^2 \left[(x - a_1) \left(\int_0^1 t^2 d_{p,q} t \right)^{1-1/r} \left(\int_0^1 t^2 \left(t \left| {}_{a_1} D_{p,q}^2 \varphi(x) \right|^r + (1-t) \left| {}_{a_1} D_{p,q}^2 \varphi(a_1) \right|^r \right) d_{p,q} t \right)^{1/r} \right. \\
 &\quad \left. + (a_2 - x) \left(\int_0^1 t^2 d_{p,q} t \right)^{1-1/r} \left(\int_0^1 t^2 \left(t \left| {}_{a_2} D_{p,q}^2 \varphi(x) \right|^r + (1-t) \left| {}_{a_2} D_{p,q}^2 \varphi(a_2) \right|^r \right) d_{p,q} t \right)^{1/r} \right] \\
 &= (x - a_1)^2 (a_2 - x)^2 \left[(x - a_1) \left(\frac{1}{[3]_{p,q}} \right)^{1-1/r} \left(\frac{1}{[4]_{p,q}} \left| {}_{a_1} D_{p,q}^2 \varphi(x) \right|^r + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} \left| {}_{a_1} D_{p,q}^2 \varphi(a_1) \right|^r \right)^{1/r} \right. \\
 &\quad \left. + (a_2 - x) \left(\frac{1}{[3]_{p,q}} \right)^{1-1/r} \left(\frac{1}{[4]_{p,q}} \left| {}_{a_2} D_{p,q}^2 \varphi(x) \right|^r + \frac{[4]_{p,q} - [3]_{p,q}}{[3]_{p,q}[4]_{p,q}} \left| {}_{a_2} D_{p,q}^2 \varphi(a_2) \right|^r \right)^{1/r} \right],
 \end{aligned}$$

which completes the proof. \square

Remark 4. If $p = 1$ in (16), then we have the following inequality:

$$\begin{aligned}
 \left| {}_{a_1}^{a_2} L_q(x) \right| &\leq (x - a_1)^2 (a_2 - x)^2 \left(\frac{1}{[3]_{p,q}} \right)^{1-1/r} \left[(x - a_1) \left(\frac{1}{[4]_q} \left| {}_{a_1} D_q^2 \varphi(x) \right|^r + \frac{q^3}{[3]_q[4]_q} \left| {}_{a_1} D_q^2 \varphi(a_1) \right|^r \right)^{1/r} \right. \\
 &\quad \left. + (a_2 - x) \left(\frac{1}{[4]_q} \left| {}_{a_2} D_q^2 \varphi(x) \right|^r + \frac{q^3}{[3]_q[4]_q} \left| {}_{a_2} D_q^2 \varphi(a_2) \right|^r \right)^{1/r} \right],
 \end{aligned}$$

which appeared in⁴³.

Theorem 6. Let $\varphi : [a_1, a_2] \rightarrow \mathbb{R}$ be a twice (p, q) -differentiable function such that ${}_{a_2} D_{p,q}^2 \varphi$ and ${}_{a_1} D_{p,q}^2 \varphi$ are continuous and integrable functions on J_1 and J_2 , respectively. If $|{}_{a_2} D_{p,q}^2 \varphi|^r$ and $|{}_{a_1} D_{p,q}^2 \varphi|^r$ are convex functions for $r > 1$ and $1/s + 1/r = 1$, then

$$\begin{aligned}
 \left| {}_{a_1}^{a_2} L_{p,q}(x) \right| &\leq (x - a_1)^2 (a_2 - x)^2 \left(\frac{1}{[2s + 1]_{p,q}} \right)^{1/s} \left[(x - a_1) \left(\frac{\left| {}_{a_1} D_q^2 \varphi(x) \right|^r + (p + q - 1) \left| {}_{a_1} D_q^2 \varphi(a_1) \right|^r}{[2]_{p,q}} \right)^{1/r} \right. \\
 &\quad \left. + (a_2 - x) \left(\frac{\left| {}_{a_2} D_q^2 \varphi(x) \right|^r + (p + q - 1) \left| {}_{a_2} D_q^2 \varphi(a_2) \right|^r}{[2]_{p,q}} \right)^{1/r} \right]. \tag{17}
 \end{aligned}$$

Proof. Applying Theorem 3 and the Hölders inequality, we have

$$\begin{aligned}
 \left| {}_{a_1}^{a_2} L_{p,q}(x) \right| &\leq (x - a_1)^2 (a_2 - x)^2 \left[(x - a_1) \int_0^1 t^2 \left| {}_{a_1} D_{p,q}^2 \varphi(tx + (1-t)a_1) \right| d_{p,q} t + (a_2 - x) \int_0^1 t^2 \left| {}_{a_2} D_{p,q}^2 \varphi(tx + (1-t)a_2) \right| d_{p,q} t \right] \\
 &\leq (x - a_1)^2 (a_2 - x)^2 \left[(x - a_1) \left(\int_0^1 t^{2s} d_{p,q} t \right)^{1/s} \left(\int_0^1 \left| {}_{a_1} D_{p,q}^2 \varphi(tx + (1-t)a_1) \right|^r d_{p,q} t \right)^{1/r} \right. \\
 &\quad \left. + (a_2 - x) \left(\int_0^1 t^{2s} d_{p,q} t \right)^{1/s} \left(\int_0^1 \left| {}_{a_2} D_{p,q}^2 \varphi(tx + (1-t)a_2) \right|^r d_{p,q} t \right)^{1/r} \right].
 \end{aligned}$$

Using Lemma 1 and applying the convexity of $|{}_{a_2} D_{p,q}^2 \varphi|^r$ and $|{}_{a_1} D_{p,q}^2 \varphi|^r$, we obtain

$$\begin{aligned} \left| {}_{a_1}^{a_2} L_{p,q}(x) \right| &\leq (x - a_1)^2 (a_2 - x)^2 \left[(x - a_1) \left(\int_0^1 t^{2s} d_{p,q} t \right)^{1/s} \left(\int_0^1 \left(t \left| {}_{a_1} D_{p,q}^2 \varphi(x) \right|^r + (1 - t) \left| {}_{a_1} D_{p,q}^2 \varphi(a_1) \right|^r \right) d_{p,q} t \right)^{1/r} \right. \\ &\quad \left. + (a_2 - x) \left(\int_0^1 t^{2s} d_{p,q} t \right)^{1/s} \left(\int_0^1 \left(t \left| {}_{a_2} D_{p,q}^2 \varphi(x) \right|^r + (1 - t) \left| {}_{a_2} D_{p,q}^2 \varphi(a_2) \right|^r \right) d_{p,q} t \right)^{1/r} \right] \\ &= (x - a_1)^2 (a_2 - x)^2 \left[(x - a_1) \left(\frac{1}{[2s + 1]_{p,q}} \right)^{1/s} \left(\frac{\left| {}_{a_1} D_q^2 \varphi(x) \right|^r + (p + q - 1) \left| {}_{a_1} D_q^2 \varphi(a_1) \right|^r}{[2]_{p,q}} \right)^{1/r} \right. \\ &\quad \left. + (a_2 - x) \left(\frac{1}{[2s + 1]_{p,q}} \right)^{1/s} \left(\frac{\left| {}_{a_2} D_q^2 \varphi(x) \right|^r + (p + q - 1) \left| {}_{a_2} D_q^2 \varphi(a_2) \right|^r}{[2]_{p,q}} \right)^{1/r} \right], \end{aligned}$$

which completes the proof. □

Remark 5. If $p = 1$ in (17), then we have the following inequality:

$$\begin{aligned} \left| {}_{a_1}^{a_2} L_q(x) \right| &\leq (x - a_1)^2 (a_2 - x)^2 \left(\frac{1}{[2s + 1]_q} \right)^{1/s} \\ &\quad \times \left[(x - a_1) \left(\frac{\left| {}_{a_1} D_q^2 \varphi(x) \right|^r + q \left| {}_{a_1} D_q^2 \varphi(a_1) \right|^r}{[2]_q} \right)^{1/r} + (a_2 - x) \left(\frac{\left| {}_{a_2} D_q^2 \varphi(x) \right|^r + q \left| {}_{a_2} D_q^2 \varphi(a_2) \right|^r}{[2]_q} \right)^{1/r} \right], \end{aligned}$$

which appeared in⁴³.

4 | CONCLUSIONS

In this work, we established a new (p, q) -integral identity using the second $(p, q)_{a_1}$ - and $(p, q)^{a_2}$ -derivatives. Then, we used this result to derive some new post quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable convex functions. The main results in this study were proven to be generalizations of some previously proved results of quantum Ostrowski-type integral inequalities for twice q -differentiable convex functions. Researchers can obtain similar inequalities in future works by using (p, q) -fractional calculus.

ACKNOWLEDGMENTS

The first author is supported by Development and Promotion of Science and Technology talents project (DPST), Thailand.

Author contributions

All authors contributed equally to this article. They read and approved the final manuscript.

Financial disclosure

Not applicable.

Conflict of interest

The authors declare that they have no competing interests.

SUPPORTING INFORMATION

Not applicable.

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How to cite this article: Luangboon W., Nonlaopon K., Tariboon J., Ntouyas S.K., and Budak H. (2021), Some new post quantum Ostrowski-type integral inequalities for twice (p, q) -differentiable convex functions, -, -.