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Post quantum Ostrowski-type inequalities for twice (p, q)-differentiable convex functions

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ABSTRACT

In this paper, we establish a new post quantum integral identity for twice (p, q)-differentiable convex functions. Then, we use this result to derive some new post quantum Ostrowski-type inequalities for twice (p, q)-differentiable convex functions involving $(p, q)_a$- and $(p, q)^b$-integrals. The newly established results are also proven to be generalizations of some existing results in the field of integral inequalities of already published ones.

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KEYWORDS:

Ostrowski-type inequality, (p, q)-differentiable function, (p, q)-integral inequalities, (p, q)-calculus, Convex function

1 | INTRODUCTION

Mathematical inequalities are a necessary tool in the study of pure and applied mathematics. One of the type inequalities that have been the focus of significant attention from many researchers is Ostrowski-type inequalities because it can be applied in quadrature, stochastic, probability and optimization theory, statistics, integral operator theory, and information. The classical integral inequality for the differentiable function is as follows:

Theorem 1. [1] Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) whose the derivative function $f : (a, b) \to \mathbb{R}$ is bounded on (a, b) and $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left((x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty$$

for all $x \in [a, b]$.

In recent years, many researchers focused on the Ostrowski-type inequalities and their applications, see[2,3,4,5,6,7,8,9,10,11,12] and the references cited therein for more details. Specifically, many researchers worked on the Ostrowski-type inequalities and their applications using quantum calculus, some results can be found in[13,14,15,16,17,18,19,20,21] and the references cited therein.

Quantum calculus, also known as q-calculus, is the study of calculus without limits. The concept was revealed by renowned mathematician Euler (1707-1783), who introduced the number in q-infinite series defined by Newton. Then, in 1910, Jackson[22] defined q-integral and q-derivative of a continuous function on the interval $(0, \infty)$ extending the concept of Euler. In this, the main objective is to obtain the q-analoques of mathematical objects that can be recaptured as $q \to 1$. The topic of q-calculus has received outstanding attention from many scientists because it has numerous applications in various fields of mathematics and physics such as hypergeometric functions, orthogonal polynomials, mechanics, number theory, combinatorics, and the theory of relativity, see[23,24,25,26,27,28,29,30,31,32,33] and the references cited therein for more details.

In 2013, Tariboon and Ntouyas[34] presented the q_a-derivative and q_a-integral of a continuous function on a finite interval and addressed numerous problems on q_a-analogues of classical inequalities. Recently, the q^b-derivative and q^b-integral of a continuous function on a finite interval and proved some basic properties was presented by Bermudo et al.[35] in 2020 and some basic properties were proved. Currently, these topics of q-calculus have been studied in various inequalities, for example, Hanh integral inequalities, Hermite-Hadamard inequalities, Hermite-Hadamard-like inequalities, Newton-type inequalities, Simpson-type inequalities, Fejér-type inequalities, and Ostrowski-type inequalities, see[36,37,38,39,40,41,42] and the references cited therein for more details.

Post quantum calculus, also known as (p, q)-calculus, is a generalization of q-calculus. The (p, q)-calculus has two independent parameters that are p-number and q-number. Apparently, q-calculus cannot be obtained directly by substituting q by q/p in q-calculus, but it can be obtained directly by taking $p = 1$ in (p, q)-calculus. Then, the classical inequalities can be gain by taking $q \to 1$. The concept of (p, q)-calculus of a continuous function on the interval $(0, \infty)$ was first presented by Chakrabarti and Jagannathan[43] in 1991. Then, the concept of the $(p, q)_a$-derivative and $(p, q)_a$-integral of a continuous function on a finite interval was presented by Tunç, and Göv[44,45] in 2016. Recently, the concept of the $(p, q)^b$-derivative and $(p, q)^b$-integral of a continuous function on a finite interval was presented by Vivas-Cortez et al.[46] in 2021. Currently, the topic of (p, q)-calculus is being investigated extensively by many scientists, and some new results can be found in[47,48,49,50,51,52,53,54,55] and the references cited therein.

In 2021, Ali et al.[42] presented quantum Ostrowski-type inequalities for twice q-differentiable convex functions. By taking $q \to 1$, they obtain classical results on some Ostrowski-type inequalities for functions, whose second derivatives are h-convex functions[59]. Inspired by the above mentioned reports, we establish some new post quantum Ostrowski-type inequalities for twice (p, q)-differentiable convex functions to extend and generalize the results given in previous reports.

The rest of the paper is organized as follows: In Section 2, we give some basic knowledge and notation. In Section 3, we establish some new post quantum Ostrowski-type inequalities for twice (p, q)-differentiable convex functions. In Section 4, we summarize our results.

2 | PRELIMINARIES

In this section, we give basic knowledge and notation that will be used in our work. Throughout this paper, let $[a, b] \subseteq \mathbb{R}$ be an interval with $a < b$ and $0 < q < p \leq 1$ be constants. The (p, q)-number of n is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}, \quad n \in \mathbb{N}, \tag{1}$$

which is a generalization of the q-number or q-analogue of n such that

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-2} + q^{n-1}, \quad n \in \mathbb{N}, \tag{2}$$

see[33] for more details.

Definition 1. [44,45] For a continuous function $f : [a, b] \to \mathbb{R}$, then the $(p, q)_a$-derivative of function f at $t \in [a, b]$ is defined by

$$\begin{aligned} {}_aD_{p,q} f(t) &= \frac{f(pt + (1 - p)a) - f(qt + (1 - q)a)}{(p - q)(t - a)}, \quad t \neq a, \\ {}_aD_{p,q} f(a) &= \lim_{t \to a} {}_aD_{p,q} f(t). \end{aligned} \tag{3}$$

The function f is said to be $(p,q)_a$-differentiable function on $[a,b]$ if ${}_aD_{p,q}f(t)$ exists for all $t \in [a,(b-a)/p+a]$.

In Definition 1, if $p=1$ and ${}_aD_{1,q}f(t) = {}_aD_qf(t)$, then (3) reduces to

$$\begin{aligned} {}_aD_qf(t) &= \frac{f(t)-f(qt+(1-q)a)}{(1-q)(t-a)}, \quad t \neq a, \\ {}_aD_qf(a) &= \lim_{t\to a} {}_aD_qf(t), \end{aligned} \tag{4}$$

which is the q_a-derivative of function f defined on $[a,b]$, see[56,57] for more details. In addition, if $a=0$ and ${}_0D_qf(t) = D_qf(t)$, then (4) reduces to

$$\begin{aligned} D_qf(t) &= \frac{f(t)-f(qt)}{(1-q)t}, \quad t \neq a, \\ D_qf(a) &= \lim_{t\to 0} D_qf(t), \end{aligned} \tag{5}$$

which is the q-derivative of function f defined on $[0,b]$, also called q-Jackson derivative, see[33] for more details.

Example 1. Define function $f : [a,b] \to \mathbb{R}$ by $f(t) = t^2 + C$, where C is constant. Applying Definition 1 for $t \neq a$, we have

$$\begin{aligned} {}_aD_{p,q}(t^2+C) &= \frac{\left[(pt+(1-p)a)^2+C\right]-\left[(qt+(1-q)a)^2+C\right]}{(p-q)(t-a)} \\ &= \frac{(p+q)t^2+2at[1-(p+q)]+a^2[(p+q)-2]}{(t-a)} \\ &= \frac{(p+q)(t-a)^2+2a(t-a)}{(t-a)} \\ &= [2]_{p,q}(t-a)+2a. \end{aligned} \tag{6}$$

Definition 2. [46] For a continuous function $f : [a,b] \to \mathbb{R}$, then the $(p,q)^b$-derivative of function f at $t \in [a,b]$ is defined by

$$\begin{aligned} {}^bD_{p,q}f(t) &= \frac{f(qt+(1-q)b)-f(pt+(1-p)b)}{(p-q)(b-t)}, \quad t \neq b, \\ {}^bD_{p,q}f(b) &= \lim_{t\to b} {}^bD_{p,q}f(t). \end{aligned} \tag{7}$$

The function f is said to be $(p,q)^b$-differentiable function on $[a,b]$ if ${}^bD_{p,q}f(t)$ exists for all $t \in [b-(b-a)/p, b]$.

In Definition 2, if $p=1$ and ${}^bD_{1,q}f(t) = {}^bD_qf(t)$, then (7) reduces to

$$\begin{aligned} {}^bD_qf(t) &= \frac{f(qt+(1-q)b)-f(t)}{(1-q)(b-t)}, \quad t \neq b, \\ {}^bD_qf(b) &= \lim_{t\to b} {}^bD_qf(t), \end{aligned} \tag{8}$$

which is the q^b-derivative of function f defined on $[a,b]$, see[35,58] for more details.

Example 2. Define function $f : [a,b] \to \mathbb{R}$ by $f(t) = t^2 + C$, where C is constant. Applying Definition 2 for $t \neq a$, we have

$$\begin{aligned} {}^bD_{p,q}(t^2+C) &= \frac{\left[(qt+(1-q)b)^2+C\right]-\left[(pt+(1-p)b)^2+C\right]}{(p-q)(b-t)} \\ &= \frac{-(p+q)t^2+2bt[(p+q)-1]+b^2[2-(p+q)]}{(b-t)} \\ &= \frac{-(p+q)(b-t)^2+2b(b-t)}{(b-t)} \\ &= [2]_{p,q}(t-b)+2b. \end{aligned} \tag{9}$$

Definition 3. [44] For a continuous function $f : [a,b] \to \mathbb{R}$, then the $(p,q)_a$-integral of function f at $t \in [a,x]$ is defined by

$$\int_a^x f(t)\, {}_ad_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)a\right) \tag{10}$$

for $x \in [a,b]$.

The function f is said to be $(p,q)_a$-integrable function on $[a,x]$ if $\int_a^x f(t)\, {}_ad_{p,q}t$ exists for all $t \in [a, a+p(x-a)]$.

Example 3. Define function $f : [a,b] \to \mathbb{R}$ by $f(t) = At + B$, where A and B are constants. Applying Definition 3, we have

$$\begin{aligned}\int_a^b f(t)\ {}_a d_{p,q} t &= \int_a^b (At+B)\ {}_a d_{p,q} t \\ &= A(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}b + \left(1-\frac{q^j}{p^{j+1}}\right)a\right) \\ &\quad + B(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}} \\ &= \frac{A(b-a)(b-a(1-p-q))}{[2]_{p,q}} + B(b-a). \end{aligned} \tag{11}$$

Definition 4. [46] For a continuous function $f : [a,b] \to \mathbb{R}$, then the $(p,q)^b$-integral of function f at $t \in [x,b]$ is defined by

$$\int_x^b f(t)\ {}^b d_{p,q} t = (p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)b\right) \tag{12}$$

for $x \in [a,b]$.

The function f is said to be $(p,q)^b$-integrable function on $[x,b]$ if $\int_a^b f(t)\ {}^b d_{p,q} t$ exists for all $t \in [b-p(b-x), b]$.

Example 4. Define function $f : [a,b] \to \mathbb{R}$ by $f(t) = At + B$, where A and B are constants. Applying Definition 4, we have

$$\begin{aligned}\int_a^b f(t)\ {}^b d_{p,q} t &= \int_a^b (At+B)\ {}^b d_{p,q} t \\ &= A(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}a + \left(1-\frac{q^j}{p^{j+1}}\right)b\right) \\ &\quad + B(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}} \\ &= \frac{A(b-a)(a-b(1-p-q))}{[2]_{p,q}} + B(b-a). \end{aligned} \tag{13}$$

Lemma 1. [44] For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following expression holds:

$$\int_a^b (t-a)^{\alpha}\ {}_a d_{p,q} t = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_{p,q}}. \tag{14}$$

Theorem 2. [45] If $f, g : [a,b] \to \mathbb{R}$ are continuous functions and $r > 0$ with $1/r + 1/s = 1$, then

$$\int_a^b |f(t)g(t)|\ {}_a d_{p,q} t \le \left(\int_a^b |f(t)|^r\ {}_a d_{p,q} t\right)^{1/r}\left(\int_a^b |g(t)|^s\ {}_a d_{p,q} t\right)^{1/s} \tag{15}$$

for $t \in [a,b]$.

3 | MAIN RESULTS

In this section, we give some new estimates of post quantum Ostrowski-type inequalities for twice (p,q)-differentiable functions involving $(p,q)_a$- and $(p,q)^b$-integrals. Let $J_1 = [b-p(b-x), b]$ and $J_2 = [a, a+p(x-a)]$. We start with the following (p,q)-integral identities.

Theorem 3. If $f : [a,b] \to \mathbb{R}$ is a twice (p,q)-differentiable function such that ${}^{b}D^2_{p,q}f$ and ${}_{a}D^2_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. Then

$$(x-a)^2(b-x)^2\left[(a-x)\int_0^1 t^2\,{}_{a}D^2_{p,q}f(tx+(1-t)a)\,d_{p,q}t+(x-b)\int_0^1 t^2\,{}^{b}D^2_{p,q}f(tx+(1-t)b)\,d_{p,q}t\right]={}^{b}_{a}L_{p,q}(x), \tag{16}$$

where

$$\begin{aligned}{}^{b}_{a}L_{p,q}(x)=&\frac{(x-a)(b-x)}{pq^3(p-q)}\Big[(x-a)pqf(qx+(1-q)b)+(b-x)pqf(qx+(1-q)a)-(x-a)(q^2+pq-p^2)f(px+(1-p)b)\\&-(b-x)(q^2+pq-p^2)f(px+(1-p)a)\Big]-\frac{[2]_{p,q}}{p^3q^3}\left[(x-a)^2\int_{p^2x+(1-p^2)b}^{b}f(t)\,{}^{b}d_{p,q}t+(b-x)^2\int_{a}^{p^2x+(1-p^2)a}f(t)\,{}_{a}d_{p,q}t\right].\end{aligned}$$

Proof. Using Definition 1, we have

$$\begin{aligned}&{}_{a}D^2_{p,q}f(tb+(1-t)a)\\&={}_{a}D_{p,q}({}_{a}D_{p,q}f(tb+(1-t)a))\\&={}_{a}D_{p,q}\left(\frac{f(ptb+(1-pt)a)-f(qtb+(1-qt)a)}{(p-q)(b-a)t}\right)\\&=\frac{1}{(p-q)(b-a)t}\left[\frac{f(p^2tb+(1-p^2t)a)-f(pqtb+(1-pqt)a)}{pt(p-q)(b-a)}-\frac{f(pqtb+(1-pqt)a)-f(q^2tb+(1-q^2t)a)}{qt(p-q)(b-a)}\right]\\&=\frac{qf(p^2tb+(1-p^2t)a)-[2]_{p,q}f(pqtb+(1-pqt)a)+pf(q^2tb+(1-q^2t)a)}{pqt^2(p-q)^2(b-a)^2}.\end{aligned} \tag{17}$$

Applying (17) and Definition 3, we obtain

$$\begin{aligned}&\int_0^1 t^2\,{}_{a}D^2_{p,q}f(tx+(1-t)a)\,d_{p,q}t\\&=\int_0^1\frac{qf(p^2tx+(1-p^2t)a)-[2]_{p,q}f(pqtx+(1-pqt)a)+pf(q^2tx+(1-q^2t)a)}{pq(p-q)^2(x-a)^2}\,d_{p,q}t\\&=\frac{q(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(p^2\frac{q^n}{p^{n+1}}x+\left(1-p^2\frac{q^n}{p^{n+1}}\right)a\right)}{pq(p-q)^2(b-x)^3}\\&\quad-\frac{[2]_{p,q}(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}f\left(p\frac{q^{n+1}}{p^{n+1}}x+\left(1-p\frac{q^{n+1}}{p^{n+1}}\right)a\right)}{pq^2(p-q)^2(b-x)^3}\\&\quad+\frac{p(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}f\left(\frac{q^{n+2}}{p^{n+1}}x+\left(1-\frac{q^{n+2}}{p^{n+1}}\right)a\right)}{pq^3(p-q)^2(b-x)^3}\\&=\frac{(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(p^2\frac{q^n}{p^{n+1}}x+\left(1-p^2\frac{q^n}{p^{n+1}}\right)a\right)}{p(p-q)^2(b-x)^3}\\&\quad-\frac{p[2]_{p,q}(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f\left(p^2\frac{q^{n+1}}{p^{n+2}}x+\left(1-p^2\frac{q^{n+1}}{p^{n+2}}\right)a\right)}{pq^2(p-q)^2(b-x)^3}\end{aligned}$$

$$+\frac{p^3(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f\left(p^2\frac{q^{n+2}}{p^{n+3}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+3}}\right)a\right)}{pq^3(p-q)^2(b-x)^3}$$

$$=\frac{[2]_{p,q}}{p^3q^3(b-x)^3}\int_a^{p^2x+(1-p^2)a}f(t)\ {}_ad_{p,q}t+\frac{(q^2+pq-p^2)f(px+(1-p)a)}{pq^3(p-q)(x-a)^2}-\frac{f(qx+(1-q)a)}{q^2(p-q)(x-a)^2}. \tag{18}$$

Using Definition 2, we have

$$\begin{aligned}
&{}^bD^2_{p,q}f(ta+(1-t)b)\\
&={}^bD_{p,q}({}^bD_{p,q}f(ta+(1-t)b))\\
&={}^bD_{p,q}\left(\frac{f(qta+(1-qt)b)-f(pta+(1-pt)b)}{(p-q)(b-a)t}\right)\\
&=\frac{1}{(p-q)(b-a)t}\left[\frac{f(q^2ta+(1-q^2t)b)-f(pqta+(1-pqt)b)}{qt(p-q)(b-a)}-\frac{f(pqta+(1-pqt)b)-f(p^2ta+(1-p^2t)b)}{pt(p-q)(b-a)}\right]\\
&=\frac{pf(q^2ta+(1-q^2t)b)-[2]_{p,q}f(pqta+(1-pqt)b)+qf(p^2ta+(1-p^2t)b)}{pqt^2(p-q)^2(b-a)^2}.
\end{aligned} \tag{19}$$

Applying (19) and Definition 4, we obtain

$$\begin{aligned}
&\int_0^1 t^2\ {}^bD^2_{p,q}f(tx+(1-t)b)\ d_{p,q}t\\
&=\int_0^1\frac{pf(q^2tx+(1-q^2t)b)-[2]_{p,q}f(pqtx+(1-pqt)b)+qf(p^2tx+(1-p^2t)b)}{pq(p-q)^2(b-x)^2}\ d_{p,q}t\\
&=\frac{p(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}f\left(\frac{q^{n+2}}{p^{n+1}}x+\left(1-\frac{q^{n+2}}{p^{n+1}}\right)b\right)}{pq^3(p-q)^2(b-x)^3}\\
&\quad-\frac{[2]_{p,q}(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}f\left(p\frac{q^{n+1}}{p^{n+1}}x+\left(1-p\frac{q^{n+1}}{p^{n+1}}\right)b\right)}{pq^2(p-q)^2(b-x)^3}\\
&\quad+\frac{q(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(p^2\frac{q^{n+2}}{p^{n+1}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+1}}\right)b\right)}{pq(p-q)^2(b-x)^3}\\
&=\frac{p^3(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f\left(p^2\frac{q^{n+2}}{p^{n+3}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+3}}\right)b\right)}{pq^3(p-q)^2(b-x)^3}\\
&\quad-\frac{p[2]_{p,q}(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f\left(p^2\frac{q^{n+1}}{p^{n+1}}x+\left(1-p^2\frac{q^{n+1}}{p^{n+2}}\right)b\right)}{pq^2(p-q)^2(b-x)^3}\\
&\quad+\frac{q(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(p^2\frac{q^{n+2}}{p^{n+1}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+1}}\right)b\right)}{pq(p-q)^2(b-x)^3}\\
&=\frac{[2]_{p,q}}{p^3q^3(b-x)^3}\int_{p^2x+(1-p^2)b}^{b}f(t)\ {}^bd_{p,q}t+\frac{(q^2+pq-p^2)f(px+(1-p)b)}{pq^3(p-q)(b-x)^2}-\frac{f(qx+(1-q)b)}{q^2(p-q)(b-x)^2}.
\end{aligned} \tag{20}$$

By multiplying (18) and (20) by $(x-a^2)(b-x)^2(a-x)$ and $(x-a^2)(b-x)^2(x-b)$, respectively, and adding the resultant inequalities, we obtain the required identity (16). Therefore, the proof is completed. □

Remark 1. If $p=1$, then (16) reduces to

$$(x-a)^2(b-x)^2\left[(a-x)\int_0^1 t^2\,{}_aD_q^2 f(tx+(1-t)a)\,d_qt+(x-b)\int_0^1 t^2\,{}^bD_q^2 f(tx+(1-t)b)\,d_qt\right]={}_a^bL_q(x), \tag{21}$$

where

$$\begin{aligned}{}_a^bL_q(x)=&\frac{(x-a)(b-x)}{q^3(p-q)}\left[(x-a)qf(qx+(1-q)b)+(b-x)qf(qx+(1-q)a)-(q^2+q-1)(b-a)f(x)\right]\\&-\frac{[2]_q}{q^3}\left[(x-a)^2\int_x^b f(t)\,{}^bd_qt+(b-x)^2\int_a^x f(t)\,{}_ad_qt\right],\end{aligned}$$

which appeared in [42].

Theorem 4. Let $f:[a,b]\to\mathbb{R}$ be a twice (p,q)-differentiable function such that ${}^bD_{p,q}^2f$ and ${}_aD_{p,q}^2f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD_{p,q}^2f|$ and $|{}_aD_{p,q}^2f|$ are convex functions, then

$$\begin{aligned}\left|{}_a^bL_{p,q}(x)\right|\le(x-a)^2(b-x)^2&\left[(x-a)\left(\frac{1}{[4]_{p,q}}\left|{}_aD_{p,q}^2f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD_{p,q}^2f(a)\right|\right)\right.\\&\left.+(b-x)\left(\frac{1}{[4]_{p,q}}\left|{}^bD_{p,q}^2f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD_{p,q}^2f(b)\right|\right)\right].\end{aligned} \tag{22}$$

Proof. Taking the modulus in Theorem 3, using Lemma 1, and applying the convexity of $|{}^bD_{p,q}^2f|$ and $|{}_aD_{p,q}^2f|$, we obtain

$$\begin{aligned}\left|{}_a^bL_{p,q}(x)\right|&\le(x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left|{}_aD_{p,q}^2f(tx+(1-t)a)\right|\,d_{p,q}t+(b-x)\int_0^1 t^2\left|{}^bD_{p,q}^2f(tx+(1-t)b)\right|\,d_{p,q}t\right]\\&\le(x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left(t\left|{}_aD_{p,q}^2f(x)\right|+(1-t)\left|{}_aD_{p,q}^2f(a)\right|\right)\,d_{p,q}t\right.\\&\left.\quad+(b-x)\int_0^1 t^2\left(t\left|{}^bD_{p,q}^2f(x)\right|+(1-t)\left|{}^bD_{p,q}^2f(b)\right|\right)\,d_{p,q}t\right]\\&=(x-a)^2(b-x)^2\left[(x-a)\left(\frac{1}{[4]_{p,q}}\left|{}_aD_{p,q}^2f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD_{p,q}^2f(a)\right|\right)\right.\\&\left.\quad+(b-x)\left(\frac{1}{[4]_{p,q}}\left|{}^bD_{p,q}^2f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD_{p,q}^2f(b)\right|\right)\right],\end{aligned}$$

which completes the proof. □

Corollary 1. With the assumptions of Theorem 4, if $|{}^bD_{p,q}^2f|$ and $|{}_aD_{p,q}^2f|\le M$, where M is constant, then we obtain the following post quantum Ostrowski-type inequality for twice (p,q)-differentiable functions as follows

$$\left|{}_a^bL_{p,q}(x)\right|\le\frac{M(x-a)^2(b-x)^2(b-a)}{[3]_{p,q}}. \tag{23}$$

Remark 2. If $p=1$, then (22) reduces to

$$\begin{aligned}\left|{}_a^bL_q(x)\right|\le(x-a)^2(b-x)^2&\left[(x-a)\left(\frac{1}{[4]_q}\left|{}_aD_q^2f(x)\right|+\frac{q^3}{[3]_q[4]_q}\left|{}_aD_q^2f(a)\right|\right)\right.\\&\left.+(b-x)\left(\frac{1}{[4]_q}\left|{}^bD_q^2f(x)\right|+\frac{q^3}{[3]_q[4]_q}\left|{}^bD_q^2f(b)\right|\right)\right],\end{aligned}$$

which appeared in [42].

Remark 3. If $p = 1$, then (23) reduces to quantum Ostrowski-type inequality for twice q-differentiable functions as follows

$$\left|{}_a^b L_q(x)\right| \leq \frac{M(x-a)^2(b-x)^2(b-a)}{[3]_q}, \tag{24}$$

which appeared in [42].

In addition, if $q \to 1$ and $x = (a+b)/2$ in (24), then we obtain the following

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M(b-a)^2}{24},$$

which appeared in [59] and it can be found in [8].

Theorem 5. Let $f : [a,b] \to \mathbb{R}$ be a twice (p,q)-differentiable function on (a,b) such that ${}^bD^2_{p,q}f$ and ${}_aD^2_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD^2_{p,q}f|^r$ and $|{}_aD^2_{p,q}f|^r$ are convex functions for $r \geq 1$, then

$$\begin{aligned}\left|{}_a^b L_{p,q}(x)\right| \leq{}& (x-a)^2(b-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left[(x-a)\left(\frac{1}{[4]_{p,q}}\left|{}_aD^2_{p,q}f(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD^2_{p,q}f(a)\right|^r\right)^{1/r}\right.\\ &\left.+ (b-x)\left(\frac{1}{[4]_{p,q}}\left|{}^bD^2_{p,q}f(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD^2_{p,q}f(b)\right|^r\right)^{1/r}\right].\end{aligned} \tag{25}$$

Proof. Taking the modulus in Theorem 3 and applying the power mean inequality, we have

$$\begin{aligned}\left|{}_a^b L_{p,q}(x)\right| \leq{}& (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left|{}_aD^2_{p,q}f(tx+(1-t)a)\right| d_{p,q}t + (b-x)\int_0^1 t^2\left|{}^bD^2_{p,q}f(tx+(1-t)b)\right| d_{p,q}t\right]\\ \leq{}& (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^2\, d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left|{}_aD^2_{p,q}f(tx+(1-t)a)\right|^r d_{p,q}t\right)^{1/r}\right.\\ &\left.+(b-x)\left(\int_0^1 t^2\, d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left|{}^bD^2_{p,q}f(tx+(1-t)b)\right|^r d_{p,q}t\right)^{1/r}\right].\end{aligned}$$

Using Lemma 1 and applying the convexity of $|{}^bD^2_{p,q}f|^r$ and $|{}_aD^2_{p,q}f|^r$, we obtain

$$\begin{aligned}\left|{}_a^b L_{p,q}(x)\right| \leq{}& (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^2\, d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left(t\left|{}_aD^2_{p,q}f(x)\right|^r + (1-t)\left|{}_aD^2_{p,q}f(a)\right|^r\right) d_{p,q}t\right)^{1/r}\right.\\ &\left.+(b-x)\left(\int_0^1 t^2\, d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left(t\left|{}^bD^2_{p,q}f(x)\right|^r + (1-t)\left|{}^bD^2_{p,q}f(b)\right|^r\right) d_{p,q}t\right)^{1/r}\right]\\ ={}& (x-a)^2(b-x)^2\left[(x-a)\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left(\frac{1}{[4]_{p,q}}\left|{}_aD^2_{p,q}f(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD^2_{p,q}f(a)\right|^r\right)^{1/r}\right.\\ &\left.+ (b-x)\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left(\frac{1}{[4]_{p,q}}\left|{}^bD^2_{p,q}f(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD^2_{p,q}f(b)\right|^r\right)^{1/r}\right],\end{aligned}$$

which completes the proof. □

Remark 4. If $p = 1$, then (25) reduces to

$$\left|{}_a^bL_q(x)\right| \leq (x-a)^2(b-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left[(x-a)\left(\frac{1}{[4]_q}\left|{}_aD_q^2f(x)\right|^r+\frac{q^3}{[3]_q[4]_q}\left|{}_aD_q^2f(a)\right|^r\right)^{1/r}\right.$$
$$\left.+\,(b-x)\left(\frac{1}{[4]_q}\left|{}^bD_q^2f(x)\right|^r+\frac{q^3}{[3]_q[4]_q}\left|{}^bD_q^2f(b)\right|^r\right)^{1/r}\right],$$

which appeared in[42].

Theorem 6. Let $f : [a,b] \to \mathbb{R}$ be a twice (p,q)-differentiable function such that ${}^bD_{p,q}^2f$ and ${}_aD_{p,q}^2f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD_{p,q}^2f|^r$ and $|{}_aD_{p,q}^2f|^r$ are convex functions for $r>1$ and $1/s+1/r=1$, then

$$\left|{}_a^bL_{p,q}(x)\right| \leq (x-a)^2(b-x)^2\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left[(x-a)\left(\frac{\left|{}_aD_q^2f(x)\right|^r+(p+q-1)\left|{}_aD_q^2f(a)\right|^r}{[2]_{p,q}}\right)^{1/r}\right.$$
$$\left.+(b-x)\left(\frac{\left|{}^bD_q^2f(x)\right|^r+(p+q-1)\left|{}^bD_q^2f(b)\right|^r}{[2]_{p,q}}\right)^{1/r}\right]. \tag{26}$$

Proof. Taking the modulus in Theorem 3 and applying the Hölders inequality, we have

$$\left|{}_a^bL_{p,q}(x)\right| \leq (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left|{}_aD_{p,q}^2f(tx+(1-t)a)\right|\,d_{p,q}t+(b-x)\int_0^1 t^2\left|{}^bD_{p,q}^2f(tx+(1-t)b)\right|\,d_{p,q}t\right]$$
$$\leq (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1\left|{}_aD_{p,q}^2f(tx+(1-t)a)\right|^r\,d_{p,q}t\right)^{1/r}\right.$$
$$\left.+(b-x)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1\left|{}^bD_{p,q}^2f(tx+(1-t)b)\right|^r\,d_{p,q}t\right)^{1/r}\right].$$

Using Lemma 1 and applying the convexity of $|{}^bD_{p,q}^2f|^r$ and $|{}_aD_{p,q}^2f|^r$, we obtain

$$\left|{}_a^bL_{p,q}(x)\right| \leq (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1\left(t\left|{}_aD_{p,q}^2f(x)\right|^r+(1-t)\left|{}_aD_{p,q}^2f(a)\right|^r\right)\,d_{p,q}t\right)^{1/r}\right.$$
$$\left.+(b-x)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1\left(t\left|{}^bD_{p,q}^2f(x)\right|^r+(1-t)\left|{}^bD_{p,q}^2f(b)\right|^r\right)\,d_{p,q}t\right)^{1/r}\right]$$
$$=(x-a)^2(b-x)^2\left[(x-a)\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left(\frac{\left|{}_aD_q^2f(x)\right|^r+(p+q-1)\left|{}_aD_q^2f(a)\right|^r}{[2]_{p,q}}\right)^{1/r}\right.$$
$$\left.+\,(b-x)\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s}\left(\frac{\left|{}^bD_q^2f(x)\right|^r+(p+q-1)\left|{}^bD_q^2f(b)\right|^r}{[2]_{p,q}}\right)^{1/r}\right],$$

which completes the proof. □

Remark 5. If $p = 1$, then (22) reduces to

$$\left| {}_a^b L_q(x) \right| \leq (x-a)^2(b-x)^2 \left(\frac{1}{[2s+1]_q} \right)^{1/s} \times \left[(x-a) \left(\frac{\left| {}_aD_q^2 f(x) \right|^r + q \left| {}_aD_q^2 f(a) \right|^r}{[2]_q} \right)^{1/r} + (b-x) \left(\frac{\left| {}^bD_q^2 f(x) \right|^r + q \left| {}^bD_q^2 f(b) \right|^r}{[2]_q} \right)^{1/r} \right], \tag{27}$$

which appeared in [42].

4 | CONCLUSIONS

In this work, we established some new integral inequalities of post quantum Ostrowski-type inequalities for twice (p, q)-differentiable convex functions by using the definition of (p, q)-derivatives and (p, q)-integrals. The main results in this study were the extension and generalization of some previously proved research in the literature of quantum Ostrowski-type inequalities for twice q-differentiable convex functions. Authors can obtain similar inequalities in future works by using (p, q)-fractional calculus.

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Author contributions

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