# Effective optimized decomposition algorithms for solving nonlinear fractional differential equations

**Marwa Laoubi**[a], **Zaid Odibat**[b*], **Banan Maayah**[a]

[a] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan

[b] Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan

**Abstract**

In this paper, the optimized decomposition method, which was developed to solve integer-order differential equations, will be modified and extended to handle nonlinear fractional differential equations. Fractional derivatives will be considered in terms of Caputo sense. The suggested modifications design new optimized decompositions for the series solutions depending on linear approximations of the nonlinear equations. Two optimized decomposition algorithms have been introduced to obtain approximate solutions of broad classes of IVPs consisting of nonlinear fractional ODEs and PDEs. A comparative study of the suggested algorithms with the Adomian decomposition method was performed by means of some test illustration problems. The executed numerical simulation results demonstrated that the proposed algorithms give better accuracy and convergence compared with Adomian's approach. This confirms the belief that the optimized decomposition method will be effectively and widely used in solving various functional equations.

**Keywords:** Optimized decomposition method; Linear approximation; Adomian decomposition method; Caputo derivative; Fractional differential equation.

## 1 Introduction

The topic of fractional calculus has received great popularity and importance over the past few decades as a powerful tool to model complex systems with long term memory and nonlinear behavior. The main reason is due to its multiple applications in the mathematical, chemical, physical, biological, medical, financial and environmental sciences, in control theory, as well as in signal and image processing, and other areas [1-10].

In recent years, scientists have devoted great efforts to find and develop stable and robust numerical methods as well as analytical methods for solving nonlinear differential equations involving fractional derivatives of physical interest. One of these methods is the Adomian decomposition method (ADM) [11, 12]. It has been observed that the ADM has demonstrated to be efficient in solving various functional equation types including nonlinear ODEs and PDEs (deterministic and stochastic). Significantly, the method has been employed successfully to provide approximate solutions to numerous classes of nonlinear ODEs [13-20] and nonlinear PDEs [21-32]. This method turns a nonlinear problem into a sequence of linear equations that can be straightforwardly and directly solved. Furthermore, the method has been modified to handle nonlinear ODEs and PDEs of fractional order, where the fractional derivative is taken

*Corresponding author. E-mail addresses: z.odibat@gmail.com, odibat@bau.edu.jo (Z. Odibat)

into consideration in Caputo sense [3, 4]. However, the series solution provided using this method often converges in a very small region and has a slow convergent rate or divergent in wider regions.

Recently, Odibat in [33] established a new decomposition method, the so-called optimized decomposition method (ODM), to handel nonlinear ODEs, and developed an extension of it to deal with nonlinear PDEs. The suggested ODM employs Taylor series linearization of the considered nonlinear equation and designs an optimal formulation of the series solution. The proposed ODM offers a powerful technique for producing precise approximate analytical solutions to nonlinear differential equations. The results obtained in [33] confirm that the developed ODM provides better convergence and accuracy compared to ADM.

In this work, we aim to modify and expand the applications of the ODM for use in the treatment of nonlinear fractional differential equations. We will present two modified algorithms of the ODM as new tools for obtaining series solutions to broad classes of IVPs consisting of fractional ODEs and time-fractional PDEs. Several examples will be examined, and the obtained results will indicate that the proposed modifications accelerate the convergence and enhance the accuracy of the series solutions compared to ADM. The proposed algorithms of the ODM will be highly accurate and efficient in handling nonlinear fractional differential equations.

## 2 The ADM for fractional differential equations

The ADM, presented by Adomian in 1980, has been used to solve accurately and effectively large categories of differential equations. The method provides a decomposition solution in the form of an infinite series for nonlinear differential equations that rapidly converge to an accurate solution. This method has been adapted and modified to provide analytical and numerical solutions for nonlinear ODEs and PDEs of fractional order. To describe the basic ideas of the ADM to handle nonlinear fractional differential equations, we consider the IVP consists of the fractional differential equation

$$D_t^\alpha u(t) = N[u(t)] + g(t), \qquad t > 0, \tag{1}$$

and the initial conditions

$$u^{(k)}(0) = c_k, \qquad k = 0, 1, ..., m-1, \tag{2}$$

where $m-1 < \alpha \leq m$, $D_t^\alpha$ is the Caputo fractional derivative operator of order $\alpha > 0$, $N$ is a nonlinear function of $u$ and $g$ is a given function of $t$. The Caputo fractional derivative operator, $D_t^\alpha$, of order $\alpha > 0$ is defined by the composition form

$$D_t^\alpha f(t) = (I_t^{m-\alpha} D^m f)(t). \tag{3}$$

Here $D^m$ denotes the ordinary derivative operator of order $m$ and $I_t^\beta$ indicates the Riemann-Liouville fractional integral operator of order $\beta > 0$ which is defined by the formula

$$I_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau, \qquad t > 0. \tag{4}$$

The Caputo operator, which is an adjustment of the Riemann-Liouville operator, is often one of the most frequently employed definitions of fractional derivatives in fractional calculus applications. Fractional order systems can serve as a useful tool in the modeling of many complex physical systems. The advantages of fractional order models are that they provide more degrees of freedom to include "memory" in the model. Fractional calculus models are characterized by having long-term memory and long-term spatial interactions. Applications and properties of the Caputo fractional derivative operator can be obtained from the literature [3, 4].

Now, the main step of the ADM is to apply the integral operator $I_t^\alpha$, the inverse of the derivative operator $D_t^\alpha$, to both sides of Eq. (1). This gives, using the initial conditions given in Eq. (2),

$$u(t) = \sum_{i=0}^{m-1} c_i \frac{t^i}{i!} + I_t^\alpha g(t) + I_t^\alpha N[u(t)]. \tag{5}$$

The standard ADM offers the solution $u(t)$ be formulated by the infinite series

$$u(t) = \sum_{n=0}^{\infty} \phi_n(t), \qquad n \geq 0, \tag{6}$$

and the nonlinear function $N[u(t)]$ be decomposed as

$$N[u(t)] = \sum_{n=0}^{\infty} A_n(t), \qquad n \geq 0, \tag{7}$$

where the Adomian polynomials $A_n(t)$ can be evaluated using the relation

$$A_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \phi_k(t)\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{8}$$

Inserting the series given in Eqs. (6) and (7) into the integral Eq. (5) produces the ADM series solution $u(t) = \sum_{n=0}^{\infty} \phi_n(t)$, where the term function $\phi_n(t)$, $n = 0, 1, \cdots$, can be obtained recursively by using the formula

$$\begin{cases} \phi_0(t) &= \sum_{i=0}^{m-1} c_i \frac{t^i}{i!} + I_t^\alpha g(t), \\ \\ \phi_n(t) &= I_t^\alpha A_{n-1}(t), \qquad n \geq 1. \end{cases} \tag{9}$$

The convergence of the ADM has been discussed by Cherruault in [34, 35]. However, the obtained series solutions for the most studied nonlinear fractional differential equations converge in a very small region or completely diverge in the wider regions. This highlights the need to develop new techniques that enhance the convergence and the accuracy of the decomposition series solutions.

## 3 The ODM for fractional differential equations

The ODM has been worked successfully in handling IVPs for ODEs and PDEs of integer-order and provided better convergence and accuracy compared to ADM [33]. The method proposes an optimal framework for series solutions depending on employing a linear approximation of the nonlinear equation. In this section, we will modify the ODM to become an effective tool for producing approximate solutions of nonlinear fractional order ordinary and partial differential equations.

### 3.1 Nonlinear fractional ODEs

Now, we will introduce the basic steps of the ODM to deal with the IVP (1)-(2). In the first place, we will derive Taylor series linearizaion of the nonlinear equation $D_t^\alpha u - N[u]$. To achieve this goal, under the consideration that the nonlinear function $F(D_t^\alpha u, u) = D_t^\alpha u - N[u]$ can be approximated by the first-order Taylor polynomial at $t = 0$, we solve the algebraic equation $F(D_t^\alpha u(0), u(0)) = 0$ for $D_t^\alpha u(0)$, say $u(0) = k_0$ and $D_t^\alpha u(0) = k_0^*$. Hence, the linear approximation to the function $F(D_t^\alpha u, u)$ near the point $(k_0^*, k_0)$ can be obtained as

$$F(D_t^\alpha u, u) \approx D_t^\alpha u + \frac{\partial F}{\partial u}(k_0^*, k_0)u, \tag{10}$$

since $\frac{\partial F}{\partial D_t^\alpha u}(k_0^*, k_0) = 1$. In view of the previous approximation, given in Eq.(10), Eq.(1) can be reformulated as

$$L[u(t)] = C_0[N]u(t) + N[u(t)] + g(t), \tag{11}$$

where

$$L[u(t)] = D_t^\alpha u(t) + C_0[N]u(t), \tag{12}$$

and the constant $C_0[N]$ can be computed as

$$C_0[N] = \frac{\partial F}{\partial u}(k_0^*, k_0) = -\frac{\partial N}{\partial u}\Big|_{t=0}. \tag{13}$$

According to the constant $C_0[N]$ in the derived linear operator $L$, given in Eq. (13), our modified ODM will be established. Since the linear operator $L$ can not be easily inverted, the decomposition series of our approach will be designed differently from the ADM. The modified ODM proposes the decomposition series solution $u(t) = \sum_{n=0}^{\infty} \psi_n(t)$ for the IVP (1)-(2), where the components $\{\psi_n(t)\}_{n=0}^{\infty}$, $n = 0, 1, \cdots$, can be determined recursively by the formula

$$\begin{cases} \psi_0(t) &= \sum_{i=0}^{m-1} c_i \frac{t^i}{i!} + I_t^\alpha g(t), \\ \psi_1(t) &= I_t^\alpha \big[P_0(t)\big], \\ \psi_2(t) &= I_t^\alpha \big[P_1(t) + C_0[N]\psi_1(t)\big], \\ \psi_{n+1}(t) &= I_t^\alpha \big[P_n(t) + C_0[N](\psi_n(t) - \psi_{n-1}(t))\big], \ n \geq 2, \end{cases} \tag{14}$$

where

$$P_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \psi_k(t)\right)\right]_{\lambda=0}, \quad n \geq 0, \tag{15}$$

and $N(\sum_{n=0}^{\infty} \psi_n(t)) = \sum_{n=0}^{\infty} P_n(t)$. Obviously, if our decomposition series $\sum_{n=0}^{\infty} \psi_n(t)$ converges then $\lim_{n\to\infty} \psi_n = 0$, and so by cancelling the adjacent component we get

$$\sum_{n=0}^{\infty} \psi_n(t) = \sum_{i=0}^{m-1} c_i \frac{t^i}{i!} + I_t^\alpha g(t) + \sum_{n=0}^{\infty} I_t^\alpha [P_n(t)]. \tag{16}$$

Hence

$$u(t) = \sum_{n=0}^{\infty} \psi_n(t) = \sum_{i=0}^{m-1} c_i \frac{t^i}{i!} + I_t^\alpha g(t) + \sum_{n=0}^{\infty} I_t^\alpha [P_n(t)] = \sum_{i=0}^{m-1} c_i \frac{t^i}{i!} + I_t^\alpha g(t) + I_t^\alpha N[u(t)]. \tag{17}$$

Therefore $D_t^\alpha u(t) = N[u(t)] + g(t)$. So, $u(t) = \sum_{n=0}^{\infty} \psi_n(t)$ is exactly a solution of the IVP (1)-(2).

## 3.2 Nonlinear fractional PDEs

In [33] it has been demonstrated that the ODM reduces the volume of computations and accelerates the convergence for the series solutions of nonlinear PDEs compared to ADM. Here, we extend the modified ODM, introduced in the previous subsection, to solve analytically the IVP consists of the fractional partial differential equation

$$D_t^{\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + M[u(x,t)], \qquad t > 0, \tag{18}$$

where $0 < \alpha \leq 2$, $M$ is a nonlinear given function of $u$, subject to the initial conditions

$$\begin{cases} u(x,0) = f_0(x), & 0 < \alpha \leq 1, \\ u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), & 1 < \alpha \leq 2. \end{cases} \tag{19}$$

Now, to outline the main steps of the ODM in order to provide an analytical solution for the IVP (18)-(19), let's consider the claim that the nonlinear function $F(D_t^{\alpha} u, u_{xx}, u) = D_t^{\alpha} u - \frac{\partial^2}{\partial x^2} u - M[u]$, where $0 < \alpha \leq 2$, can be linearized by the first-order Taylor polynomial at $t = 0$. On this basis, the lineariation of the function $F(D_t^{\alpha} u, u_{xx}, u)$ at $t = 0$ can be simply derived as

$$F(D_t^{\alpha} u, u_{xx}, u) \approx D_t^{\alpha} u - \frac{\partial^2}{\partial x^2} u - C_0[M](x) u, \tag{20}$$

where

$$C_0[M](x) = \frac{\partial M}{\partial u}\Big|_{t=0}. \tag{21}$$

Hence, using Eq. (20), the nonlinear fractional PDE given in Eq. (18) can be reformulated as follows

$$L[u(x,t)] = M[u(x,t)] - C_0[M](x) u(x,t), \quad t > 0, \tag{22}$$

where

$$L[u(x,t)] = D_t^{\alpha} u - \frac{\partial^2}{\partial x^2} u - C_0[M](x) u. \tag{23}$$

As discussed previously, the ODM will be extended based on the engagement of the function $C_0[M](x)$, the coefficient of $u(x,t)$ in the derived linear operator $L$, within our construction. In this case, the extended ODM suggests the series solution $u(x,t)$ of the IVP (18)-(19) be expressed by the series $u(x,t) = \sum_{n=0}^{\infty} \varphi_n(x,t)$, such that the component functions $\{\varphi_n(x,t)\}_{n=0}^{\infty}$, $n = 0, 1, \cdots$, are evaluated recursively using the formula

$$\begin{cases} \varphi_0(x,t) = \sum_{i=0}^{m-1} f_i(x) \frac{t^i}{i!}, \\ \varphi_1(x,t) = I_t^{\alpha}\big[Q_0(x,t) + \frac{\partial^2}{\partial x^2}\varphi_0(x,t)\big], \\ \varphi_2(x,t) = I_t^{\alpha}\big[Q_1(x,t) + \frac{\partial^2}{\partial x^2}\varphi_1(x,t) - (\frac{\partial^2}{\partial x^2} + C_0[M](x))\varphi_1(x,t)\big], \\ \varphi_{n+1}(x,t) = I_t^{\alpha}\big[Q_n(x,t) + \frac{\partial^2}{\partial x^2}\varphi_n(x,t) - (\frac{\partial^2}{\partial x^2} + C_0[M](x))(\varphi_n(x,t) - \varphi_{n-1}(x,t))\big], \quad n \geq 2, \end{cases} \tag{24}$$

where

$$Q_n(x,t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} M\left(\sum_{k=0}^{n} \lambda^k \varphi_k(x,t)\right)\right]_{\lambda=0}, \quad n \geq 0, \tag{25}$$

and $M[\sum_{n=0}^{\infty} \varphi_n(x,t)] = \sum_{n=0}^{\infty} Q_n(x,t)$. Obviously, if the decomposition series $\sum_{n=0}^{\infty} \varphi_n(t)$ converges, that is $\lim_{n\to\infty} \varphi_n = 0$, then, using the deformation Eq. (24), we get

$$\sum_{n=0}^{\infty} \varphi_n(x,t) = \sum_{i=0}^{m-1} f_i(x)\frac{t^i}{i!} + \sum_{n=0}^{\infty} I_t^{\alpha}\left[\frac{\partial^2}{\partial x^2}\varphi_n(x,t) + Q_n(x,t)\right]. \tag{26}$$

This due to the elimination of the adjacent component functions. Thus

$$\begin{aligned} u(x,t) &= \textstyle\sum_{n=0}^{\infty} \varphi_n(x,t), \\ &= \textstyle\sum_{i=0}^{m-1} f_i(x)\frac{t^i}{i!} + I_t^{\alpha} \sum_{n=0}^{\infty}\left[\frac{\partial^2}{\partial x^2}\varphi_n(x,t) + Q_n(x,t)\right], \\ &= \textstyle\sum_{i=0}^{m-1} f_i(x)\frac{t^i}{i!} + I_t^{\alpha}\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}\varphi_n(x,t) + M[\sum_{n=0}^{\infty}\varphi_n(x,t)]\right], \end{aligned} \tag{27}$$

and so $D_t^{\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + M[u(x,t)]$. Therefore, $u(x,t) = \sum_{n=0}^{\infty} \varphi_n(x,t)$ is exactly a solution of the IVP (18)-(19).

**Remark 1** *The modified ODM algorithms, where the series solution components are described by formulas (14) and (24), are optimal algorithms in the direction that the generated linear operator, given in Eq. (12) or (23), is the best linear interpolation to estimate the nonlinear function $F$.*

**Remark 2** *In the two previous subsections, if we replace $C_0[N]$ by $0$ in (14) or $C_0[M](x)$ by $0$ in (24) then the ODM turns down to the ADM.*

**Remark 3** *For the application purposes, we may truncate the infinite series $\sum_{n=0}^{\infty} \psi_n(t)$ and $\sum_{n=0}^{\infty} \varphi_n(x,t)$ at the $(K+1)$-th term and use the resulting partial sum $\sum_{n=0}^{K} \psi_n(t)$ and $\sum_{n=0}^{K} \varphi_n(x,t)$ as approximations to the solutions $u(t)$ and $u(x,t)$, respectively.*

# 4 Application

This section explores the efficiency, accuracy and performance of the modified ODM algorithms, presented in section 3, for solving nonlinear fractional ODEs and PDEs. Some nonlinear fractional ODEs and PDEs have been analytically solved by using the ADM and the proposed algorithms of the ODM. Actual numerical comparisons between the proposed algorithms of ODM approximate solutions and the ADM approximate solutions are performed using the Mathematica software. Here, to give a clear comparison between the ADM and the proposed algorithms of the ODM and to examine the accuracy of the suggested algorithms, we evaluated approximate solutions in the integer order case in order to compare them with exact solutions. Next, some numerical simulations are performed in the fractional order case.

## 4.1 Nonlinear fractional ODEs

**Example 1**. We first consider the nonlinear fractional Riccati differential equation

$$D_t^\alpha u(t) = 1 - u^2(t), \qquad t > 0, \qquad 0 < \alpha \le 1, \tag{28}$$

with the initial condition

$$u(0) = u_0, \tag{29}$$

where $u_0 \in I\!R$. The exact solution of the Riccati Eq. (28) together with the condition (29), when $\alpha = 1$, is

$$u(t) = \frac{\tanh(t) + u_0}{u_0 \tanh(t) + 1}. \tag{30}$$

The first-order Taylor series approximation of the function $F(D_t^\alpha u, u) = D_t^\alpha u + u^2 - 1$ near the point $(k_0^*, k_0)$ can be obtained as

$$F(D_t^\alpha u, u) \approx D_t^\alpha u + C_0[N]u, \tag{31}$$

where $N[u(t)] = -u^2(t)$ and

$$C_0[N] = -\frac{\partial N}{\partial u}\bigg|_{t=0} = 2u_0. \tag{32}$$

In view of the ADM, the solution of Eq. (28) can be given by the infinite series $u(t) = \sum_{n=0}^{\infty} \phi_n(t)$, such that the components $\{\phi_n(t)\}_{n=0}^{\infty}$ are determined recursively by using the formula

$$\begin{cases} \phi_0(t) & = & u_0, \\ \phi_1(t) & = & \frac{t^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha A_0(t), \\ \phi_n(t) & = & I_t^\alpha A_{n-1}(t), \qquad n \ge 2, \end{cases} \tag{33}$$

and the term $A_n(t)$ is represented by

$$A_n(t) = \frac{1}{n!}\left[-\frac{d^n}{d\lambda^n}\left(\sum_{k=0}^{n} \lambda^k \phi_k(t)\right)^2\right]_{\lambda=0}, \quad n \ge 0. \tag{34}$$

According to our proposed algorithm, presented in subsection 3.1, the ODM suggests the decomposition series solution $u(t) = \sum_{n=0}^{\infty} \psi_n(t)$, where the components $\{\psi_n(t)\}_{n=0}^{\infty}$ are derived recursively by using the representation

$$\begin{cases} \psi_0(t) & = & u_0, \\ \psi_1(t) & = & \frac{t^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha \big[P_0(t)\big], \\ \psi_2(t) & = & I_t^\alpha \big[P_1(t) + 2u_0\psi_1(t)\big], \\ \psi_{n+1}(t) & = & I_t^\alpha \big[P_n(t) + 2u_0(\psi_n(t) - \psi_{n-1}(t))\big], \ n \ge 2, \end{cases} \tag{35}$$

and the term $P_n(t)$ is represented by

$$P_n(t) = \frac{1}{n!}\left[-\frac{d^n}{d\lambda^n}\left(\sum_{k=0}^{n} \lambda^k \psi_k(t)\right)^2\right]_{\lambda=0}, \quad n \ge 0. \tag{36}$$

Fig. 1 pictures ADM approximate solutions $\sum_{n=0}^{K}\phi_n(t)$, ODM approximate solutions $\sum_{n=0}^{K}\psi_n(t)$ and the exact solutions for the Riccati Eq. (28), when $\alpha=1$, for some certain values of $u_0$ and $K$. We can notice, from the numerical results shown in Fig. 1, that the approximate solutions displayed using our ODM algorithm are more accurate than the ones produced by the ADM. Fig. 2 draws ADM approximate solutions $\sum_{n=0}^{K}\phi_n(t)$ and ODM approximate solutions $\sum_{n=0}^{K}\psi_n(t)$ for the fractional Riccati Eq. (28) for some fractional orders $\alpha$, when $u_0=-0.4$ and $K=10$. From the numerical results appeared in Fig. 2, we can observe that our ODM algorithm enhances the interval of convergence for the series solutions. Of course, the efficiency of both approaches can be improved by adding more terms to the truncated series approximate solutions.

**Example 2**. Now, consider the nonlinear fractional differential equation

$$D_t^{\alpha}u(t)=-u(t)-au^3(t),\qquad t>0,\quad 0<\alpha\leq 1, \tag{37}$$

with the initial condition

$$u(0)=u_0, \tag{38}$$

where $a,u_0\in I\!R$. The exact solution of fractional differential Eq. (37) together with the condition (38), when $\alpha=1$, is

$$u(t)=u_0\frac{\exp(-t)}{\sqrt{1+au_0^2(1-\exp(-2t)}}. \tag{39}$$

The first-order Taylor series approximation of the function $F(D_t^{\alpha}u,u)=D_t^{\alpha}u+u(t)+au^3(t)$ near the point $(k_0^*,k_0)$ can be derived as

$$F(D_t^{\alpha}u,u)\approx D_t^{\alpha}u+C_0[N]u, \tag{40}$$

where $N[u(t)]=-u(t)-au^3(t)$, and

$$C_0[N]=-\frac{\partial N}{\partial u}\Big|_{t=0}=1+3au_0^2. \tag{41}$$

Now, in view of the ADM, the solution of Eq. (37) can be given by the infinite series $u(t)=\sum_{n=0}^{\infty}\phi_n(t)$, such that the components $\{\phi_n(t)\}_{n=0}^{\infty}$ are determined recursively by using the formula

$$\begin{cases}\phi_0(t) &= u_0\\[2ex] \phi_n(t) &= I_t^{\alpha}A_{n-1}(t),\qquad n\geq 1,\end{cases} \tag{42}$$

where the term $A_n(t)$ is represented by

$$A_n(t)=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(-\sum_{k=0}^{n}\lambda^k\phi_k(t)-a\left(\sum_{k=0}^{n}\lambda^k\phi_k(t)\right)^3\right)\right]_{\lambda=0},\quad n\geq 0. \tag{43}$$

According to our proposed algorithm, presented in subsection 3.1, the ODM assumes the decomposition series solution $u(t)=\sum_{n=0}^{\infty}\psi_n(t)$, where the components $\{\psi_n(t)\}_{n=0}^{\infty}$ are derived recursively by using the representation

$$\begin{cases}\psi_0(t) &= u_0,\\[2ex] \psi_1(t) &= I_t^{\alpha}\big[P_0(t)\big],\\[2ex] \psi_2(t) &= I_t^{\alpha}\big[P_1(t)+\left(1+3au_0^2\right)\psi_1(t)\big],\\[2ex] \psi_{n+1}(t) &= I_t^{\alpha}\big[P_n(t)+\left(1+3au_0^2\right)\left(\psi_n(t)-\psi_{n-1}(t)\right)\big],\ n\geq 2,\end{cases} \tag{44}$$

and the term $P_n(t)$ is defined by

$$P_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(-\sum_{k=0}^{n}\lambda^k\psi_k(t) - a\left(\sum_{k=0}^{n}\lambda^k\psi_k(t)\right)^3\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{45}$$

Fig. 3 pictures ADM approximate solutions $\sum_{n=0}^{K}\phi_n(t)$, ODM approximate solutions $\sum_{n=0}^{K}\psi_n(t)$ and the exact solutions for the fractional differential Eq. (37), when $\alpha = 1$, for some certain values of $a$, $u_0$ and $K$. As observed in Example 1, from the numerical results shown in Fig. 3, we can notice that the approximate solutions displayed using our ODM algorithm are better than the approximate solutions produced by the ADM. Fig. 4 draws ADM approximate solutions $\sum_{n=0}^{K}\phi_n(t)$ and ODM approximate solutions $\sum_{n=0}^{K}\psi_n(t)$ for the fractional differential Eq. (37) for some fractional orders $\alpha$, when $a = 0.1$, $u_0 = 3$ and $K = 12$. From the numerical results appeared in Fig. 4, we can observe that our ODM algorithm improves the convergence and accuracy of the series solutions to Eq. (37) compared with ADM.

## 4.2 Nonlinear fractional PDEs

Here, to display a clear comparison between the ADM and the ODM approaches and to examine the accuracy of the suggested algorithm, we evaluated approximate solutions for the studied fractional PDEs in the case of fixation of the space variable $x$.

**Example 3**. Next, we study the nonlinear time-fractional Fisher equation

$$D_t^\alpha u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + au(x,t)\left(1 - u(x,t)\right), \quad t > 0, \quad 0 < \alpha \leq 1, \tag{46}$$

where $a > 0$, subject to the initial condition

$$u(x,0) = \frac{1}{\left(1+\exp\left[\sqrt{\frac{a}{6}}x\right]\right)^2}. \tag{47}$$

The exact solution of Fisher Eq. (46) together with the condition (47), when $\alpha = 1$, is

$$u(x,t) = \frac{1}{\left(1+\exp\left[-\frac{5}{6}at+\sqrt{\frac{a}{6}}x\right]\right)^2}. \tag{48}$$

Fisher equation, which is used to describe the spreading of biological populations, is commonly arises in chemistry, heat and mass transfer, ecology and biology [36,37]. Now, the first-order Taylor series approximation of the function $F(D_t^\alpha u, u_{xx}, u) = D_t^\alpha u - \frac{\partial^2}{\partial x^2}u - au\,(1-u)$, at $t = 0$, can be obtained as

$$F(D_t^\alpha u, u_{xx}, u) \approx D_t^\alpha u - \frac{\partial^2}{\partial x^2}u - C_0[M](x)u, \tag{49}$$

where $M[u(t)] = au(x,t)\left(1 - u(x,t)\right)$ and

$$C_0[M](x) = \frac{\partial M}{\partial u}\Big|_{t=0} = a - 2au(x,0). \tag{50}$$

The ADM gives the decomposition series solution $u(t) = \sum_{n=0}^{\infty}\phi_n(t)$, such that the components $\{\phi_n(t)\}_{n=0}^{\infty}$ are calculated by the recurrence relation

$$\begin{cases} \phi_0(x,t) = \frac{1}{\left(1+\exp\left(\sqrt{\frac{a}{6}}x\right)\right)^2}, \\ \phi_n(x,t) = I_t^\alpha\left[A_{n-1}(x,t) + \frac{\partial^2}{\partial x^2}\phi_{n-1}(x,t)\right], \qquad n \geq 1, \end{cases} \tag{51}$$

and the term $A_n(x,t)$ is defined by

$$A_n(x,t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(a\sum_{k=0}^{n}\lambda^k\phi_k(t) - a\left(\sum_{k=0}^{n}\lambda^k\phi_k(t)\right)^2\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{52}$$

In view of our proposed algorithm, presented in subsection 3.2, the ODM assumes the decomposition series solution $u(x,t) = \sum_{n=0}^{\infty}\varphi_n(x,t)$, where the components $\{\varphi_n(x,t)\}_{n=0}^{\infty}$ are derived recursively by the formula

$$\begin{cases} \varphi_0(x,t) &= \frac{1}{\left(1+\exp\left[\sqrt{\frac{a}{6}}x\right]\right)^2}, \\ \varphi_1(x,t) &= I_t^{\alpha}\left[Q_0(x,t) + \frac{\partial^2}{\partial x^2}\varphi_0(x,t)\right], \\ \varphi_2(x,t) &= I_t^{\alpha}\left[Q_1(x,t) + \frac{\partial^2}{\partial x^2}\varphi_1(x,t) - (\frac{\partial^2}{\partial x^2} + C_0[M](x))\varphi_1(x,t)\right], \\ \varphi_{n+1}(x,t) &= I_t^{\alpha}\left[Q_n(x,t) + \frac{\partial^2}{\partial x^2}\varphi_n(x,t) - (\frac{\partial^2}{\partial x^2} + C_0[M](x))(\varphi_n(t) - \varphi_{n-1}(t))\right], \quad n \geq 2, \end{cases} \tag{53}$$

where the term $Q_n(x,t)$ is represented by the formula

$$Q_n(x,t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(a\sum_{k=0}^{n}\lambda^k\varphi_k(x,t) - a\left(\sum_{k=0}^{n}\lambda^k\varphi_k(x,t)\right)^2\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{54}$$

For $n \geq 1$, simplifying the polynomial (54), the relation (53) can be reduced to

$$\begin{aligned} \varphi_n(x,t) &= I_t^{\alpha}\left[\frac{\partial^2}{\partial x^2}\varphi_{n-1}(x,t) + a\varphi_{n-1}(x,t) - a\sum_{i=0}^{n-1}\varphi_i(x,t)\varphi_{n-i-1}(x,t)\right. \\ &\quad \left. -(\frac{\partial^2}{\partial x^2} + C_0[M](x))(\chi_n\varphi_{n-1}(x,t) - \chi_{n-1}\varphi_{n-2}(x,t))\right], \end{aligned} \tag{55}$$

where

$$\chi_n = \begin{cases} 1, & n > 1, \\ 0, & n \leq 1. \end{cases} \tag{56}$$

Consequently, the first few components of the ODM solution can be derived as

$$\begin{cases} \varphi_0(x,t) &= \frac{1}{\left(1+\exp\left[\sqrt{\frac{a}{6}}x\right]\right)^2}, \\ \varphi_1(x,t) &= I_t^{\alpha}\left[\frac{\partial^2}{\partial x^2}\varphi_0(x,t) + a\varphi_0(x,t) - a\varphi_0^2(x,t)\right], \\ \varphi_2(x,t) &= 0, \\ \varphi_n(x,t) &= I_t^{\alpha}\left[\frac{\partial^2}{\partial x^2}\varphi_{n-1}(x,t) + a\varphi_{n-1}(x,t) - a\sum_{i=0}^{n-1}\varphi_i(x,t)\varphi_{n-i-1}(x,t)\right. \\ & \quad \left. -C_0[M](x))(\varphi_{n-1}(x,t) - \varphi_{n-2}(x,t))\right], \quad n \geq 3. \end{cases} \tag{57}$$

Fig. 5 pictures ADM approximate solutions $\sum_{n=0}^{K}\phi_n(x,t)$, ODM approximate solutions $\sum_{n=0}^{K}\varphi_n(x,t)$ and the exact solutions for the Fisher Eq. (46) when $\alpha = 1$ and $x = 1$ for some specific values of $a$ and $K$. It is evident, from the numerical results shown in Fig. 5, that approximate solutions displayed using our

ODM algorithm are more accurate than those found using the ADM. Fig. 6 draws ADM approximate solutions $\sum_{n=0}^{K}\phi_n(x,t)$ and ODM approximate solutions $\sum_{n=0}^{K}\varphi_n(x,t)$ for the fractional Fisher Eq. (46) for some fractional orders $\alpha$, when $x=1$, $a=1$ and $K=5$. From the numerical results appeared in Fig. 6, we can observe that our ODM algorithm enhances the interval of convergence for the series solutions.

**Example 4**. Next, we consider the nonlinear time-fractional Klein-Gordon equation

$$D_t^{\alpha}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)+au(x,t)+bu^2(x,t),\quad t>0,\quad 0<\alpha\le 2, \tag{58}$$

where $a>0$ and $b<0$, subject to the initial conditions

$$\begin{cases} u(x,0) &= \frac{-3a}{2b\cosh^2[\sqrt{a}x/2]}, \\ \\ u_t(x,0) &= \frac{3a}{b}\sqrt{\frac{a}{2}}\frac{\sinh[\sqrt{a}x/2]}{\cosh^3[\sqrt{a}x/2]}. \end{cases} \tag{59}$$

The exact solution of the Klein-Gordon Eq. (58) together with the conditions given in Eq. (59), when $\alpha=1$, is

$$u(x,t) = \frac{-3a}{2b\cosh^2[\sqrt{a}/2(x+\sqrt{2}t)]}. \tag{60}$$

Now, the first-order Taylor series approximation of the function $F(D_t^{\alpha}u,u_{xx},u)=D_t^{\alpha}u-\frac{\partial^2}{\partial x^2}u-au-bu^2$, at $t=0$, can be derived as

$$F(D_t^{\alpha}u,u_{xx},u)\approx D_t^{\alpha}u-\frac{\partial^2}{\partial x^2}u-C_0[M](x)u, \tag{61}$$

where $M[u(t)]=au(x,t)+bu^2(x,t)$ and

$$C_0[M](x)=\frac{\partial M}{\partial u}\Big|_{t=0}=a+2bu(x,0). \tag{62}$$

The ADM gives the decomposition series solution $u(t)=\sum_{n=0}^{\infty}\phi_n(t)$, such that the components $\{\phi_n(t)\}_{n=0}^{\infty}$ are calculated by the recurrence relation

$$\begin{cases} \phi_0(x,t) &= \frac{-3a}{2b\cosh^2[\sqrt{a}x/2]}+\frac{3a}{b}\sqrt{\frac{a}{2}}\frac{\sinh[\sqrt{a}x/2]}{\cosh^3[\sqrt{a}x/2]}t, \\ \\ \phi_n(x,t) &= I_t^{\alpha}\Big[A_{n-1}(x,t)+\frac{\partial^2}{\partial x^2}\phi_{n-1}(x,t)\Big], \qquad n\ge 1, \end{cases} \tag{63}$$

and the term $A_n(x,t)$ is defined by

$$A_n(x,t)=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(a\sum_{k=0}^{n}\lambda^k\phi_k(t)+b\left(\sum_{k=0}^{n}\lambda^k\phi_k(t)\right)^2\right)\right]_{\lambda=0},\quad n\ge 0. \tag{64}$$

In view of our proposed algorithm, presented in subsection 3.2, the ODM suggests the decomposition series solution $u(x,t)=\sum_{n=0}^{\infty}\varphi_n(x,t)$, where the components $\{\varphi_n(x,t)\}_{n=0}^{\infty}$ are determined recursively by using the formula

$$\begin{cases} \varphi_0(x,t) &= \frac{-3a}{2b\cosh^2[\sqrt{a}x/2]}+\frac{3a}{b}\sqrt{\frac{a}{2}}\frac{\sinh[\sqrt{a}x/2]}{\cosh^3[\sqrt{a}x/2]}t, \\ \\ \varphi_1(x,t) &= I_t^{\alpha}\Big[Q_0(x,t)+\frac{\partial^2}{\partial x^2}\varphi_0(x,t)\Big], \\ \\ \varphi_2(x,t) &= I_t^{\alpha}\big[Q_1(x,t)+\frac{\partial^2}{\partial x^2}\varphi_1(x,t)-(\frac{\partial^2}{\partial x^2}+C_0[M](x))\varphi_1(x,t)\big], \\ \\ \varphi_{n+1}(x,t) &= I_t^{\alpha}\big[Q_n(x,t)+\frac{\partial^2}{\partial x^2}\varphi_n(x,t)-(\frac{\partial^2}{\partial x^2}+C_0[M](x))(\varphi_n(t)-\varphi_{n-1}(t))\big],\quad n\ge 2, \end{cases} \tag{65}$$

where the term $Q_n(x,t)$ is represented by the formula

$$Q_n(x,t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(a\sum_{k=0}^{n}\lambda^k\varphi_k(x,t) + b\left(\sum_{k=0}^{n}\lambda^k\varphi_k(x,t)\right)^2\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{66}$$

For $n \geq 1$, simplifying the polynomial (66), the relation (65) can be reduced to

$$\begin{aligned} \varphi_n(x,t) \quad = \quad & I_t^{\alpha}\big[\tfrac{\partial^2}{\partial x^2}\varphi_{n-1}(x,t) + a\varphi_{n-1}(x,t) + b\textstyle\sum_{i=0}^{n-1}\varphi_i(x,t)\varphi_{n-i-1}(x,t) \\ & -(\tfrac{\partial^2}{\partial x^2} + C_0[M](x))(\chi_n\varphi_{n-1}(x,t) - \chi_{n-1}\varphi_{n-2}(x,t))\big]. \end{aligned} \tag{67}$$

Fig. 7 pictures ADM approximate solutions $\sum_{n=0}^{K}\phi_n(x,t)$, ODM approximate solutions $\sum_{n=0}^{K}\varphi_n(x,t)$ and the exact solutions for the Klein-Gordon Eq. (58) when $\alpha = 1$ and $x = 1$ for some specific values of $a, b$ and $K$. It is evident, from the numerical results shown in Fig. 7, that the approximate solutions displayed by the ODM are better than those found by the ADM. Fig. 8 draws ADM approximate solutions $\sum_{n=0}^{K}\phi_n(x,t)$ and ODM approximate solutions $\sum_{n=0}^{K}\varphi_n(x,t)$ for the the fractional Klein-Gordon Eq. (58) for some fractional orders $\alpha$, when $x = 1$, $a = 0.2$, $b = -0.4$ and $K = 4$. From the numerical results appeared in Fig. 8, we can deduce that our ODM algorithm improves the convergence and accuracy of the series solutions to Eq. (58) compared with ADM.

# 5 Conclusions

The ODM has been demonstrated as a powerful tool for solving many classes of nonlinear functional equations analytically compared with ADM. In this study, modified optimal decomposition algorithms for producing series solutions to broad classes of nonlinear fractional ODEs and PDEs have been introduced. There are some concluding remarks that should be mentioned here. Firstly, the proposed decomposition algorithms, which implement Taylor series linearization of the nonlinear fractional operators, have been optimally designed to produce analytical series solutions of the studied fractional functional equations. Secondly, the suggested ODM algorithms offer the best accuracy and increase the interval of convergence for the series solutions of the studied problems compared with ADM. Thirdly, it is believed that the presented ideas regarding the optimal decomposition of the series solutions can be further implemented for treating other nonlinear fractional functional equations.

# References

[1] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

[2] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc., New York, 1993.

[3] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, 1999.

[4] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Amsterdam, 1993.

[5] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, 2000.

[6] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston, 2012.

[7] R.L. Magin, Fractional Calculus in Bioengineering, Begell House Publishers, New York, 2006.

[8] J. Sabatier, O.P. Agrawal, J.A.T. Machado, Advances in Fractional Calculus, Springer, Dordrecht, 2007.

[9] M. Raberto, E. Scalas, F. Mainardi, Waiting-times and returns in high frequency financial data: An empirical study, Phys. A 314(2002) 749-755.

[10] S. Momani, Z. Odibat, Numerical approach to differential equations of fractional orders, J. Comput. Appl. Math. 207(1)(2007) 96-110.

[11] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston,1994.

[12] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135(1988) 501–544.

[13] F. Wu, L. L. Huang, Approximate solutions of fractional Riccati equations using the Adomian decomposition method, Abstract and Applied Analysis 2014(2014) https://doi.org/10.1155/2014/957590.

[14] S. Momani, N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, Appl. Math. Comput. 182(2)(2006) 1083-1092.

[15] H. Jafari, V.D. Gejji, Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method, Appl. Math. Comput. 180(2)(2006) 700-706.

[16] P. Guo, The Adomian decomposition method for a type of fractional differential equations, J. Appl. Math. Phys. 7(2019) 2459-2466.

[17] Y. Hua, Y. Luo, Z. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method, J. Comput. Appl. Math. 215(1)(2008) 220-229.

[18] J. S. Duan, T. Chaolu, R. Rach, L. Lu, The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations, Comput. Math. Appl. 66(5)(2013) 728-736.

[19] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Soliton Fract. 31(5)(2007) 1248–1255.

[20] N. Shawagfeh, The decomposition method for fractional differential equation, Appl. Math. Comput. 182(2)(2006) 1083-1092.

[21] Q. Yu, F. Liu, V. Anh, I. Turner, Solving linear and non-linear space–time fractional reaction–diffusion equations by the Adomian decomposition method, I. J. Numer. Meth. Engng. 74(1)(2008) 138–158.

[22] S.A. El-Wakil, A. Elhanbaly, M.A. Abdou, Adomian decomposition method for solving fractional nonlinear differential equations, Appl. Math. Comput. 182(1)(2006) 313-324.

[23] H. Jafari, V. D. Gejji, Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition, Appl. Math. Comput. 180(2)(2006) 488-497.

[24] J. Saelao, N. Yokchoo, The solution of Klein–Gordon equation by using modified Adomian decomposition method, Math. Comput. Simulat. 171(2020) 94-102.

[25] L. Song, W. Wang, A new improved Adomian decomposition method and its application to fractional differential equations, Appl. Math. Model. 37(3)(2013) 1590-1598.

[26] G.A. Birajdar, Numerical solution of time fractional Navier-Stokes equation by discrete Adomian decomposition method, Nonlinear Engineering 3(1)(2014) 21-26.

[27] S. Momani, Z. Odibat, Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method, Appl. Math. Comput. 177(2)(2006) 488–494.

[28] S. Momani, An explicit and numerical solutions of the fractional KdV equation, Math. Comput. Simulat. 70(2)(2005) 110-118.

[29] D.B. Dhaigude, G.A. Birajdar, V.R. Nikam, Adomain decomposition method for fractional Benjamin-Bona-Mahony-Burger's equations, I. J. Appl. Math. Mech. 8(12)( 2012) 42-51.

[30] Z. Odibat, S. Momani, Approximate solutions for boundary value problems of time-fractional wave equation, Appl. Math. Comput. 181(1)(2006) 767–774.

[31] J.S. Duan, R. Rach, D. Baleanu, A.M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, Commun. Frac. Calc. 3(2)(2012) 73–99.

[32] C. Li, Y. Wang, Numerical algorithm based on Adomian decomposition for fractional differential equations, Comput. Math. Appl. 57(10)(2009) 1672–1681.

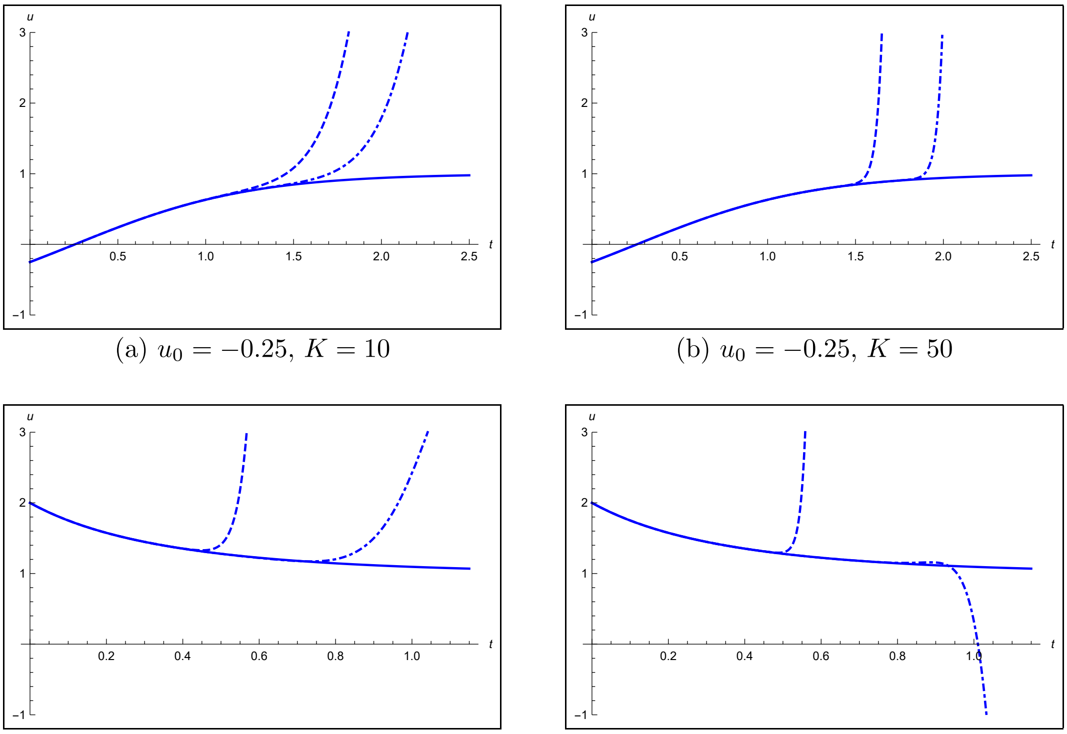
[33] Z. Odibat, An optimized decomposition method for nonlinear ordinary and partial differential equations, Phys. A 541(1)(2020) 123323.

[34] K. Abboui, Y. Cherruault, New ideas for proving convergence of decomposition methods, Comput. Math. Appl. 29(7)(1995) 103–108.

[35] Y. Cherruault, G. Adomian, K. Abbaoui, R. Rach, Further remarks on convergence of decomposition method, I. J. Bio-Medical Comput. 38(1)(1995) 89-93.

[36] R. A. Fisher. The wave of advance of advantageous genes, Annals of Eugenics 7(4)(1937) 355-369.

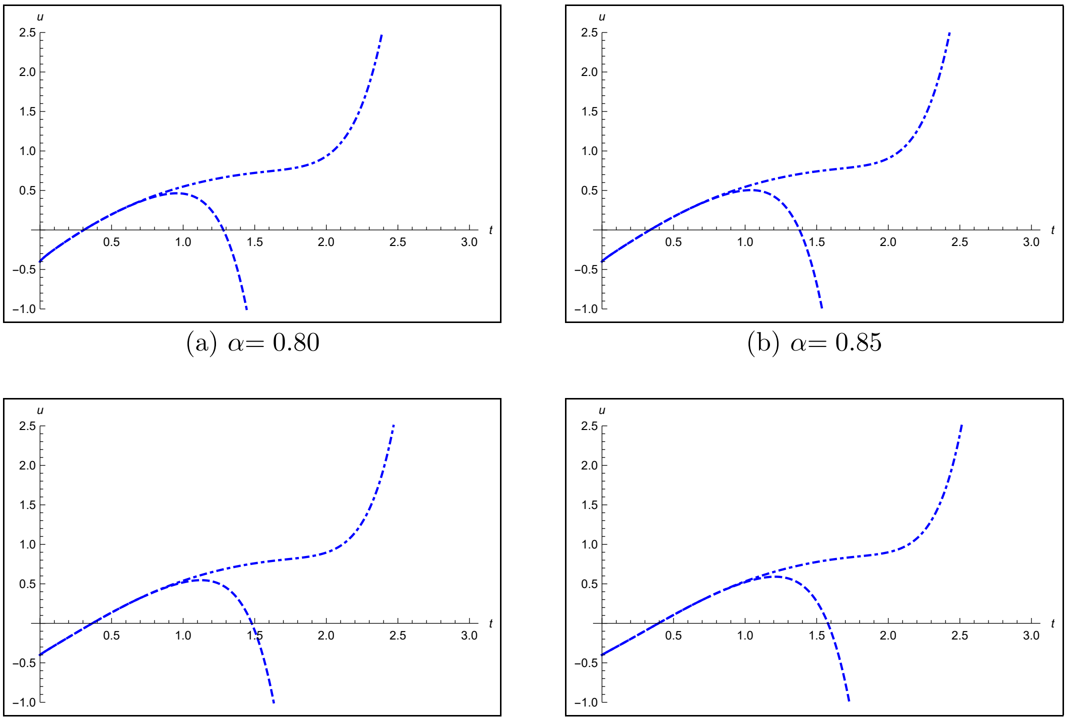
[37] B. H. Gilding, R. Kersner, Travelling Waves in Nonlinear Diffusion Convection Reaction. Birkhuser, Basel, (2004).




(a) $u_0 = -0.25,\ K = 10$ (b) $u_0 = -0.25,\ K = 50$

(c) $u_0 = 2,\ K = 20$ (d) $u_0 = 2,\ K = 40$

Figure 1: Graphs of approximate solutions and exact solution for Riccati Eq. (28), when $\alpha = 1$: exact solution (solid line); ADM approximate solution $\sum_{n=0}^{K} \phi_n(t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \psi_n(t)$ (dashed-dotted line).



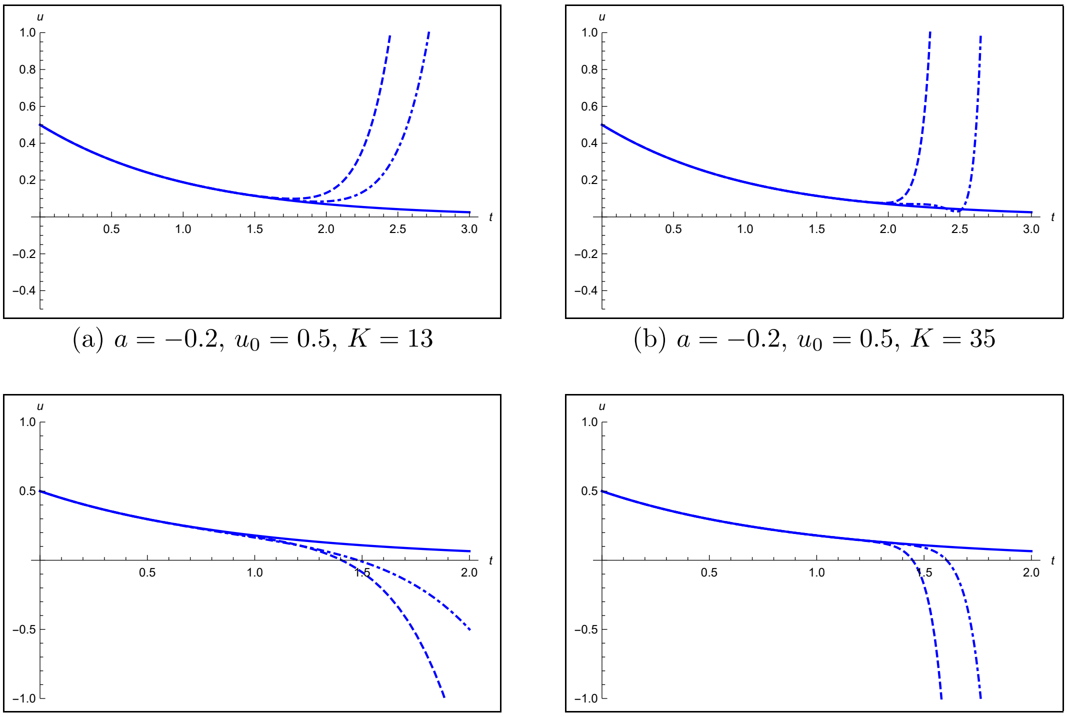

(a) $\alpha$= 0.80

(b) $\alpha$= 0.85

(c) $\alpha$= 0.90

(d) $\alpha$= 0.95

Figure 2: Graphs of approximate solutions for fractional Riccati Eq. (28), for some fractional orders $\alpha$, when $u_0 = -0.4$ and $K = 10$: ADM approximate solution $\sum_{n=0}^{K} \phi_n(t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \psi_n(t)$ (dashed-dotted line).



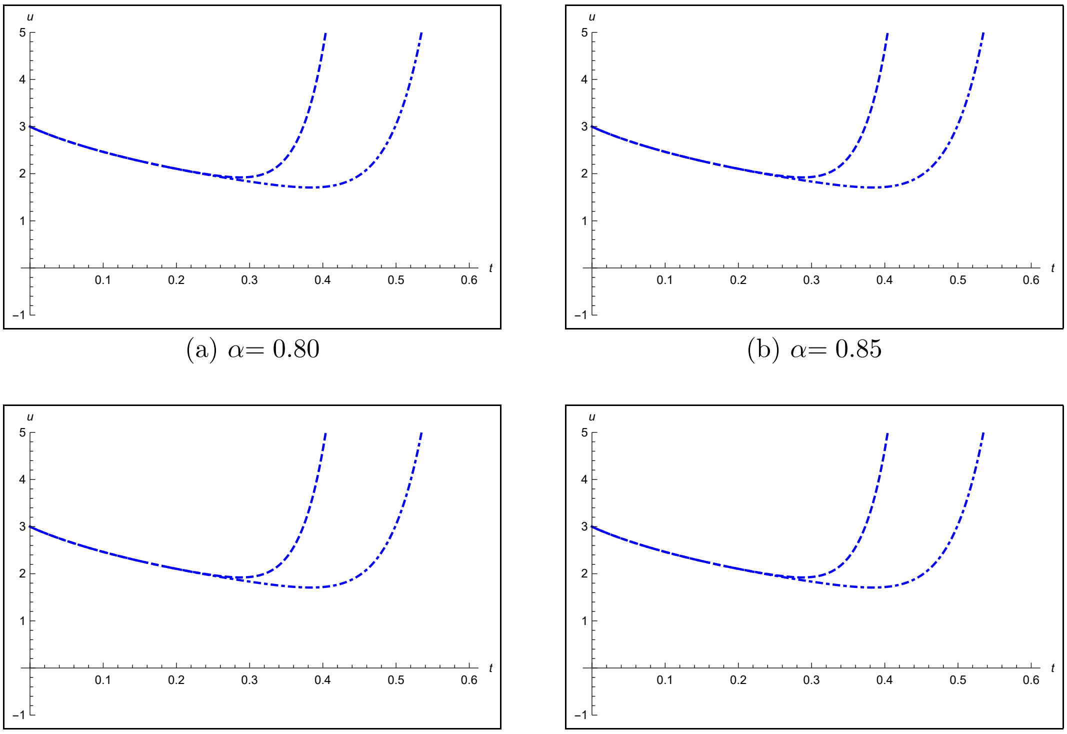

(a) $a = -0.2,\ u_0 = 0.5,\ K = 13$

(b) $a = -0.2,\ u_0 = 0.5,\ K = 35$

(c) $a = 0.25,\ u_0 = 0.5,\ K = 7$

(d) $a = 0.25,\ u_0 = 0.5,\ K = 25$

Figure 3: Graphs of approximate solutions and exact solution for differential Eq. (37), when $\alpha = 1$: exact solution (solid line); ADM approximate solution $\sum_{n=0}^{K} \phi_n(t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \psi_n(t)$ (dashed-dotted line).



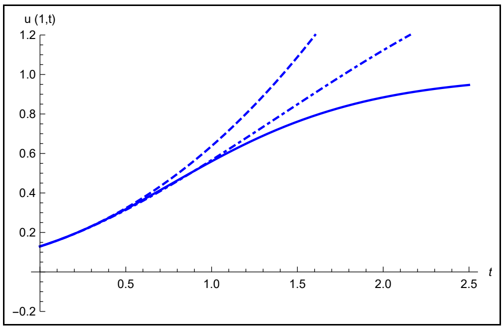

(a) $\alpha$= 0.80

(b) $\alpha$= 0.85

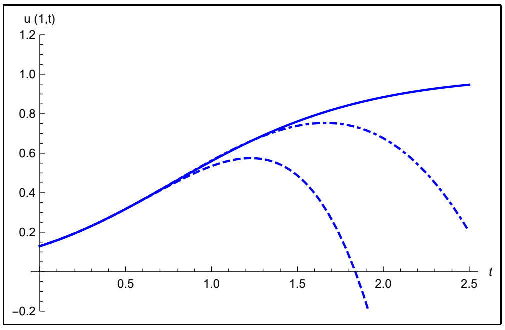
(c) $\alpha$= 0.90

(d) $\alpha$= 0.95

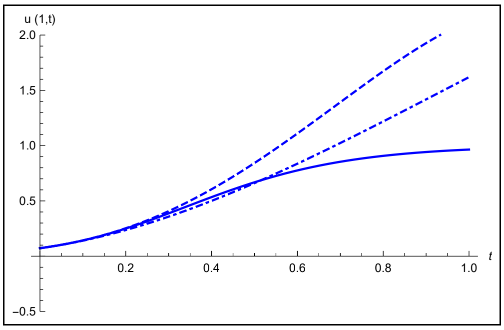
Figure 4: Graphs of approximate solutions for fractional differential Eq. (37), for some fractional orders $\alpha$, when $a = 0.1$, $u_0 = 3$ and $K = 12$: ADM approximate solution $\sum_{n=0}^{K} \phi_n(t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \psi_n(t)$ (dashed-dotted line).



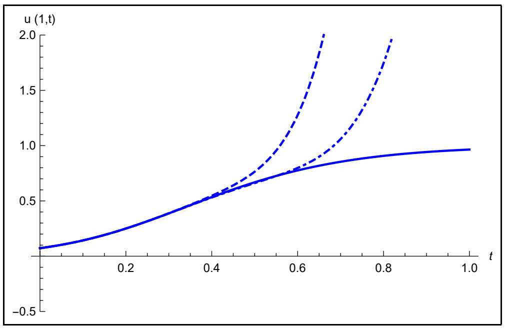

(a) $a = 2,\ K = 3$




(b) $a = 2,\ K = 5$

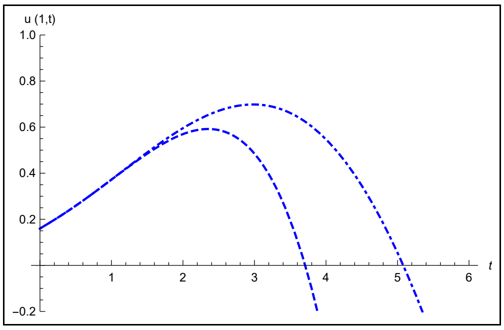



(c) $a = 6,\ K = 4$

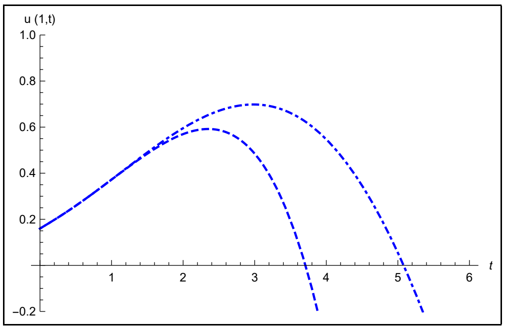



(d) $a = 6,\ K = 9$

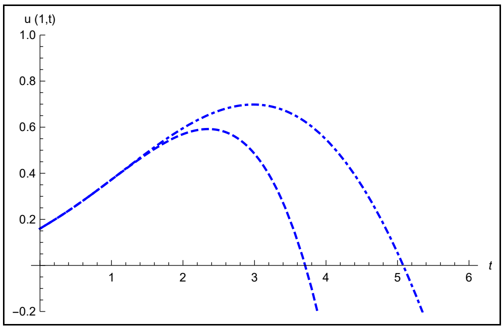
Figure 5: Graphs of approximate solutions and exact solution for Fisher Eq. (46), when $\alpha = 1$ and $x = 1$: exact solution (solid line); ADM approximate solution $\sum_{n=0}^{K} \phi_n(1,t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \varphi_n(1,t)$ (dashed-dotted line).



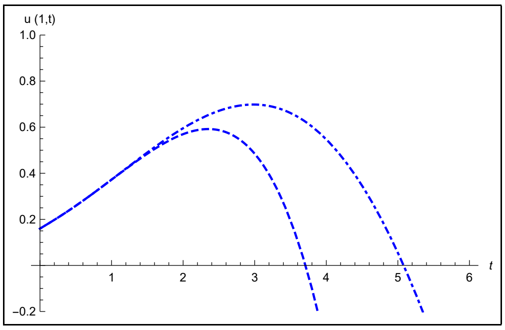

(a) $\alpha$= 0.80




(b) $\alpha$= 0.85

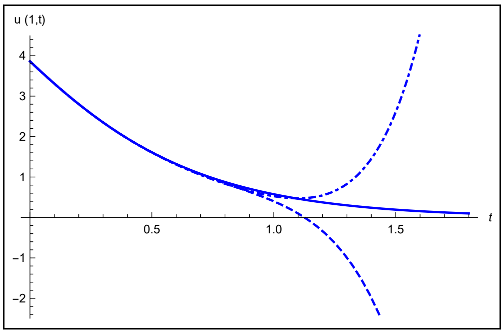



(c) $\alpha$= 0.90

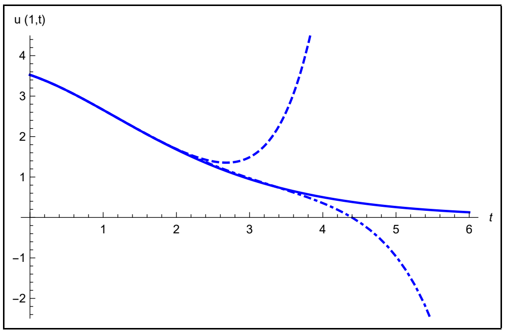



(d) $\alpha$= 0.95

Figure 6: Graphs of approximate solutions for fractional Fisher Eq. (46), for some fractional orders $\alpha$, when $a = 1$, $K = 5$ and $x = 1$: ADM approximate solution $\sum_{n=0}^{K} \phi_n(1,t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \varphi_n(1,t)$ (dashed-dotted line).

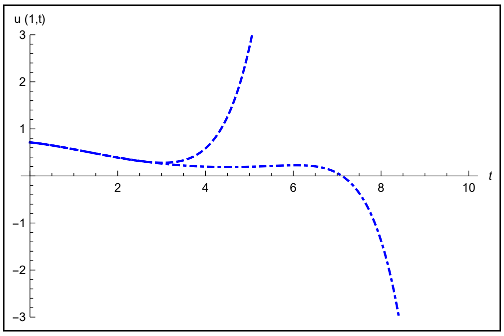



(a) $a = 2.5$, $b = -0.55$, $K = 3$

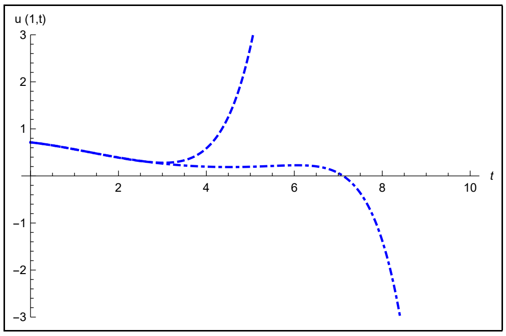



(b) $a = 0.25$, $b = -0.1$, $K = 5$

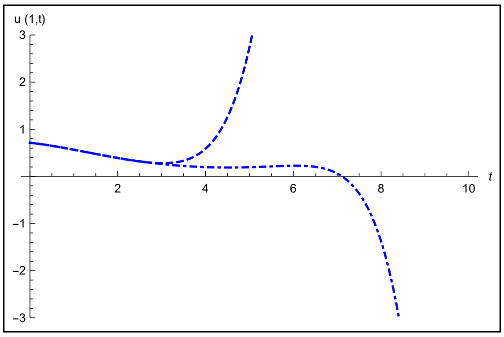
Figure 7: Graphs of approximate solutions and exact solution for Klein-Gordon Eq. (58), when $\alpha = 1$ and $x = 1$: exact solution (solid line); ADM approximate solution $\sum_{n=0}^{K} \phi_n(1,t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \varphi_n(1,t)$ (dashed-dotted line).



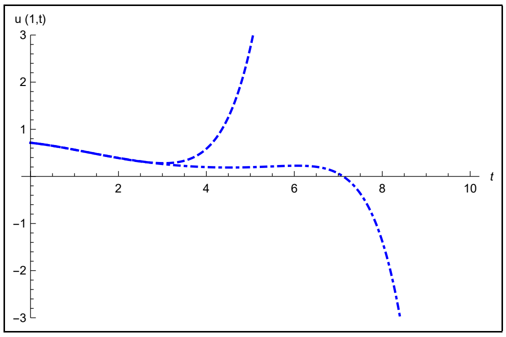

(a) $\alpha$= 1.80




(b) $\alpha$= 1.85




(c) $\alpha$= 1.90




(d) $\alpha$= 1.95

Figure 8: Graphs of approximate solutions for fractional Klein-Gordon Eq. (58), for some fractional orders $\alpha$, when $a = 0.2$, $b = -0.4$, $K = 4$ and $x = 1$: ADM approximate solution $\sum_{n=0}^{K} \phi_n(1,t)$ (dashed line); ODM approximate solution $\sum_{n=0}^{K} \varphi_n(1,t)$ (dashed-dotted line).