APPROXIMATE CONTROLLABILITY OF LINEAR PARABOLIC EQUATION WITH MEMORY

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Abstract. In this paper, we consider an optimal control problem governed by linear parabolic differential equations with memory. Under the assumption that the corresponding linear parabolic differential equation without memory term is approximately controllable, it is shown that the set of approximate controls is nonempty. The problem is first viewed as a constrained optimal control problem, and then it is approximated by an unconstrained problem with a suitable penalty function. The optimal pair of the constrained problem is obtained as the limit of the optimal pair sequence of the unconstrained problem. The result is proved by using the theory of strongly continuous semigroups and the Banach fixed point theorem. The approximation theorems, which guarantee the convergence of the numerical scheme to the optimal pair sequence, are also proved. Finally, we also present a numerical example to validate our main theoretical results.

Key words. Approximate controllability · Parabolic differential equations with memory · C_0- semigroup · Optimal control · Penalty function · Approximation theorems · Numerical result.

AMS subject classifications. 35B37 · 45K05 · 49J20 · 49J45 · 93B05 · 93C20

1. Introduction. Consider the following linear parabolic differential equation with memory and distributed control in abstract form

$$\frac{\partial y}{\partial t} + Ay(t) = \int_0^t B(t,s)y(s)ds + Gu(t), \quad t \in [0,T], \qquad y(0) = y_0 \in X, \tag{1.1}$$

where X denotes a real Hilbert space, y is a state variable, u represents a control variable, A is a self-adjoint, positive definite linear operator in X with dense domain $\mathcal{D}(A) \subset X$, $B(t,s)$ is also a linear and unbounded operator with $\mathcal{D}(A) \subset \mathcal{D}(B(t,s)) \subset X$ for $0 \le s \le t \le T$ and G is a bounded linear operator from the control space $L^2(0,T;U)$ to $L^2(0,T;X)$.

In applications that we have in mind, A represents a second-order linear self-adjoint elliptic partial differential operator defined on bounded domain Ω in $\mathbb{R}^d$ of the form

$$A = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + a_0(x)I \tag{1.2}$$

with homogeneous Dirichlet boundary condition, where the matrix $(a_{ij}(x))$ is symmetric and positive definite, $a_0 \ge 0$ on $\bar{\Omega}$. Here, $B(t,s)$ is a general second-order partial differential operator of the form

$$B(t,s) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(b_{i,j}(t,s;x)\frac{\partial}{\partial x_i}\right) + \sum_{j=1}^{d} b_j(t,s;x)\frac{\partial}{\partial x_j} + b_0(t,s;x)I, \tag{1.3}$$

and $G = I$. For the abstract form (1.1), we choose here $X = L^2(\Omega)$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\mathcal{D}(B) = H^2(\Omega)$ and throughout the article, we assume that the coefficients $b_{i,j}, b_j, b_0$, are smooth. Our subsequent analysis includes Gu as $u\chi_\omega$, where ω is a nonempty subdomain of Ω and χ_ω is a characteristic function which takes value 1 on ω and zero elsewhere.

Parabolic integro-differential equations of the type (1.1) occur in many applications such as heat conduction in materials with memory, compression of poroviscoelastic media, nuclear reactor dynamics, etc. (see, Cushman *et al.* [5], Dagan [6], Renardy *et al.* [28]).

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For control problems of the heat equation with memory, that is, when $A = -\Delta$ and $B(t,s) = -a(t-s)\Delta u(s)$ in (1.1), where $a(\cdot)$ is a completely monotone convolution kernel, Barbu *et al.* [1] have discussed approximate controllability using Carleman estimates, see also [20] and [35]. Later on, Pandolfi [25] has considered Dirichlet boundary controllability of heat equation with memory in one space dimension by employing Riesz systems. Subsequently, Fu *et al.* [7] have established controllability and observability results for a heat equation with hyperbolic memory kernel under general geometric conditions and using Carleman estimates. Further, Pandolfi [24] has employed a cosine operator approach to discussing the exact controllability results for the Dirichlet boundary control of the Gurtin-Pipkin model, which displays a hyperbolic behaviour. On second-order integro-differential equations, Kim [13, 14] has established reachability results using continuation arguments and multiplier techniques combined with compactness property. Wang *et al.* [33] have proved some sufficient conditions for the controllability of parabolic integro-differential systems in a Banach space. A result in the direction of approximate controllability of integro-differential equations (IDE) using Carleman estimates and continuation argument has been proved in [21] and using spectral analysis in [31] and [34]. However, Chaves-Silva *et al.* [2] established null controllability results for parabolic equations with memory terms by means of duality arguments and Carleman estimates. Loreti *et al.* [23] have analyzed reachability problems for a class of integro-differential equations using Hilbert uniqueness results. Kumar *et al.* [16] discussed controllability of mixed Volterra–Fredholm type integro-differential third-order dispersion equation by using the theory of semigroups and the Banach fixed point theorem. However, several negative results like lack of controllability of such systems are discussed in [9], [10] and [11].

Numerical solution by means of finite element methods has been investigated by several authors when u is a given function and $G = I$. In [32], Thomée *et al.* have considered the backward Euler method and obtained related error estimates for non-smooth data. Pani *et al.* [26] have used energy arguments and the duality technique to get error estimates for time-dependent parabolic integro-differential equations with smooth and non-smooth initial conditions. Lasiecka [17, 18] have considered optimal control problems for linear parabolic equations, which are approximated by a semidiscrete finite element method or Ritz-Galerkin scheme and then the convergence of optimal controls are derived. Moreover, Shen *et al.* [30] have developed the finite element and backward Euler scheme for space and time approximation of a constrained optimal control problem governed by a parabolic integro-differential equation. Further, in [29] Shen *et al.* have discussed mathematical formulation and optimality conditions for a quadratic optimal control problem for a quasi-linear integral differential equation and *a prior* error estimates are also established.

In the present article, an attempt has been made to discuss the approximate controllability of a distributed control problem for a general class of partial integro-differential equations of parabolic type (1.1), under the assumption that the corresponding parabolic equation without the memory term is approximately controllable. Firstly, the control problem is viewed as an optimal control problem, and using operator theoretic form, an optimal pair of the solution is derived, which, in turn, provides proof for the approximate controllability. The present proof is constructive in its approach and avoids using Carleman estimates and continuation of argument etc. Finally, some approximate theorems are established, and one numerical experiment using the finite element method are conducted to confirm our theoretical findings.

In order to motivate our main results, we first define the operator $\tilde{B}$ as

$$(\tilde{B}y)(t) = \int_0^t B(t,\tau)y(\tau)d\tau.$$

Since A generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X, then for a given $u(t) \in U$ and $y_0 \in X$, the mild solution for the system (1.1) is given by

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)\tilde{B}y(\tau)\,d\tau + \int_0^t S(t-\tau)Gu(\tau)\,d\tau \tag{1.4}$$

(see, Pazy [27]). This correspondence which assigns a unique $y \in L^2(0,T;X)$ to a given $u(t) \in U$, will be denoted by a solution operator, say W *i.e.* $Wu(t) = y(t)$.

The system (1.1) is said to be approximately controllable if for given functions y_0, $\hat{y} \in X$ and a $\delta > 0$, there exists a control $u(t) \in U$ such that the corresponding solution y of system (1.1) also satisfies $\|y(T) - \hat{y}\|_X \leq \delta$.

In view of (1.4), for such control u, we arrive at

$$\hat{y} = S(T)y_0 + \int_0^T S(T-\tau)\tilde{B}y(\tau)\, d\tau + \int_0^T S(T-\tau)Gu(\tau)d\tau, \tag{1.5}$$

where $\hat{y} = y(T)$.

Setting an operator L as

$$Lu = \int_0^T S(T-\tau)u(\tau)d\tau, \tag{1.6}$$

such that $\mathcal{D}(L) = \big\{u \in L^2(0,T;U) : Lu \text{ exists in } X\big\}$. Since $u \in L^2(0,T;U)$ or $u \in C([0,T];U)$ implies $Lu \in C(X)$ and hence, we have that $\mathcal{D}(L)$ is dense in $L^2(0,T;U)$. Moreover, L is a closed operator, see Lasiecka *et al.* [19] (chapter 0, below equation 0.32). Its adjoint L^*, defined by $(Lu, y)_X = (u, L^*y)_{L^2(0,T;U)}$ is the closed operator defined by

$$(L^*y)(t) = S(T-t)y,\ 0 \leq t \leq T,\ \ y \in \mathcal{D}(L^*),$$

where $\mathcal{D}(L^*) = \{y \in X : L^*y \in L^2(0,T;U)\}$. If G^* is the adjoint operator of the operator G, then it follows that

$$(G^*L^*y)(t) = G^*S(T-t)y,\ t \in [0,T] \text{ and } y \in X.$$

Thus, the equation (1.5) can be written equivalently as an operator equation

$$\hat{z} = L\tilde{B}y + LGu, \tag{1.7}$$

where $\hat{z} = \hat{y} - S(T)y_0$. Define for $\delta > 0$, the set $U_\delta \subset L^2(0,T;U)$ of admissible controls of (1.1) by

$$U_\delta = \Big\{u \in L^2(0,T;U) :\ \|L\tilde{B}y + LGu - \hat{z}\|_X \leq \delta\Big\}.$$

It is a closed, convex and bounded (possibly empty) subset of Y. Setting $Z = L^2(0,T;X)$ and $Y = L^2(0,T;U)$.

Definition 1.1. *Let y_0, $\hat{y} \in X$ and $T > 0$. We say the problem (1.1) is approximately controllable if for $\delta > 0$, there exists $u \in Y$ such that $U_\delta \neq \emptyset$.*

Now, our **main problem** is to prove $U_\delta \neq \emptyset$ and then to determine $u^*_\delta \in U_\delta$ such that

$$J(u^*_\delta) = \inf_{u \in U_\delta} J(u) \tag{1.8}$$

where $J(u) = \frac{1}{2}\|u\|^2_Y$.

Definition 1.2. *For a given $\delta > 0$, let $u^*_\delta \in U_\delta$ be a solution of the problem (1.8) with $y^*_\delta \in X$ as the corresponding mild solution of the system (1.1). Then the pair (u^*_δ, y^*_δ) is called optimal pair of the constrained optimal control problem (1.8).*

Our main thrust is to establish an existence of an optimal pair (u^*_δ, y^*_δ) of the constrained optimal control problem (1.8) and thereafter, present a numerical scheme for approximating the optimal pair. Under the assumption $B \equiv 0$, in section 2, we first show that the set U_δ of admissible controls is nonempty. Then the optimal pair (u^*_δ, y^*_δ) is obtained as a limit of the sequence of an optimal pair $(u^*_\epsilon, y^*_\epsilon)$, where u^*_ϵ minimizes the unconstrained functional $J_\epsilon(u)$ over the whole space Y defined by

$$J_\epsilon(u) = J(u) + \frac{1}{2\epsilon}\left\|Lu + L\tilde{B}Wu - \hat{z}\right\|^2_X, \tag{1.9}$$

where W is an operator which assigns to each control u_ϵ^* the solution y_ϵ^* of (1.1). We shall refer to $(u_\epsilon^*, y_\epsilon^*)$ as the optimal pair corresponding to the unconstrained problem.

The plan of this paper is as follows. In Section 2, it is shown that the set of admissible control U_δ is nonempty under the assumption that the corresponding system with $B \equiv 0$ is approximately controllable. The optimal pair (u_δ^*, y_δ^*) of the constrained problem (1.8) is obtained as a limit of the optimal pair sequence $\{(u_\epsilon^*, y_\epsilon^*)\}$, where u_ϵ^* minimizes the unconstrained functional $J_\epsilon(u)$ defined by (1.9). Approximation theorems that guarantee the convergence of the numerical scheme to the optimal pair are proved in Section 3. In Section 4, we discuss a technique for the computation of optimal pair $\{(u_n, y_n)\}$ through the finite-dimensional approximation scheme and demonstrate one numerical experiment to show the existence of the optimal control and the applicability of our results.

2. Existence of optimal control and convergence to the control problem. In this section, we first show that the set U_δ of admissible controls is nonempty. Here, we first make the following assumptions for the problem (1.1):

(A1) The set $\{S(t)\}_{t\geq 0}$ of C_0-semigroup of bounded linear operators on X, generated by $(-A)$ is uniformly bounded, that is, there exists $\beta > 0$ such that $\|S(t)\|_X \leq \beta$, for all $t \in [0, T]$.

(A2) The operator $B(t,\tau)$ is dominated by A together with certain derivatives with respect to t and τ, that is, $\|A^{-1}B(t,\tau)\varphi\| \leq \alpha\|\varphi\| \ \forall \ \varphi \in D(B(t,\tau)), \ 0 \leq \tau \leq t \leq T$.

(A3) The system (1.1) with $B \equiv 0$ is approximately controllable.

(A4) The operator $G : L^2(0,T;U) \to L^2(0,T;X)$ is a bounded linear operator.

The condition (A2) is not restrictive as it shows the dominance property of the main operator A, see Thomée and Zhang [32]. The assumption (A3) is on the approximate controllability of the corresponding linear parabolic problem.

The following lemma is related to the assumption $(A3)$.

Lemma 2.1. *The system (1.1) with $B \equiv 0$ is approximately controllable on $[0,T]$ if and only if one of the following statement holds:*

(i) $\overline{Range(LG)} = X$.

(ii) $Kernel(G^*L^*) = \{0\}$.

(iii) *For all $z \in X$, there holds for $\delta \in (0,1)$*

$$LGu_\delta = z - \delta\Big(\delta I + LGG^*L^*\Big)^{-1} z,$$

where $u_\delta := G^*L^*\Big(\delta I + LG\, G^*L^*\Big)^{-1} z.$

(iv) $\displaystyle\lim_{\delta\to 0^+} \delta\Big(\delta I + LG\, G^*L^*\Big)^{-1} z = 0.$

For a proof, we refer to Curtain *et al.* [3, 4] and Leiva *et al.* [22]. As a consequence, it is observed that

$$\lim_{\delta\to 0^+} LGu_\delta = z,$$

and the error $e_\delta z$ due to this approximation is given by

$$e_\delta z = \delta\Big(\delta I + LGG^*L^*\Big)^{-1} z,$$

which tends to zero as $\delta \to 0$.

For approximate controllability of the problem (1.1), we rewrite its controllability equation as

$$u_\delta := G^*L^*\Big(\delta I + LGG^*L^*\Big)^{-1}(\hat{z} - L\tilde{B}y), \tag{2.1}$$

where $\tilde{z} = y(T) - S(T)y_0$.

Now for a fixed $z \in Z$, consider the following linear parabolic integro-differential system which is indexed by z

$$\begin{aligned} \frac{\partial y_z}{\partial t} + Ay_z &= \int_0^t B(t,\tau)z(\tau)d\tau + Gu_z, \quad t \in [0,T], \\ y_z(0) &= y_0 \in X. \end{aligned} \tag{2.2}$$

The mild solution $y_z \in Z$ of the above system is given by

$$y_z(t) = S(t)y_0 + \int_0^t S(t-\tau)\tilde{B}z(\tau)\, d\tau + \int_0^t S(t-\tau)Gu_z(\tau)d\tau, \tag{2.3}$$

and hence,

$$\tilde{y}_z \equiv y_z(T) - S(T)y_0 = \int_0^T S(T-\tau)\tilde{B}z(\tau)\, d\tau + \int_0^T S(T-\tau)Gu_z(\tau)\, d\tau. \tag{2.4}$$

In operator theoretic form, (2.4) reduces to the operator equation

$$LGu_z = \hat{y}_z - L\tilde{B}z, \tag{2.5}$$

for each fixed z.

Now under the assumption $(A3)$, the system (1.1) with $B \equiv 0$ is approximately controllable, and hence, Lemma 2.1 implies that $(\delta I + G^*L^*LG)$ is boundedly invertible. For approximate controllability of (2.2), we observe that for a given state $\hat{z} = \hat{y} - S(T)y_0 \in X$, and for $\delta > 0$, a control $u_{\delta,z}$ solves

$$u_{\delta,z} = G^*L^*(\delta I + LG\, G^*L^*)^{-1}\left[\hat{z} - L\tilde{B}z\right]. \tag{2.6}$$

To keep the notation simple and wherever there is no confusion, we write $u_{\delta,z}$ simply by u_z.

Denote the operator $\mathcal{M}z := G^*L^*(\delta I + LGG^*L^*)^{-1}\left[\hat{z} - L\tilde{B}z\right]$ and consider for $\delta \in (0,1]$, the family of operators $R_\delta : Z \to Z$ which assigns a solution y_z of (1.1) (given by (2.3)), corresponding to $z \in Z$, that is,

$$R_\delta z(t) = S(t)y_0 + \int_0^t S(t-\tau)\left(\tilde{B}z(\tau) + LG\mathcal{M}z(\tau)\right)d\tau. \tag{2.7}$$

Now, define the operator $K : L^2(0,T;X) \to L^2(0,T;\mathcal{D}(A))$ by

$$(Ky)(t) = \int_0^t S(t-\tau)y(\tau)d\tau \tag{2.8}$$

which is linear and continuous. Rewriting equation (2.7) in operator form as

$$R_\delta z(t) = S(t)y_0 + K\tilde{B}z(t) + KG\mathcal{M}z(t). \tag{2.9}$$

First of all, we need to prove that for each fixed $\delta \in (0,1]$, the operator R_δ has fixed point, say, z_δ. The following lemmas deal with some properties of K and L.

LEMMA 2.2. *Let the assumptions $(A1)$ and $(A2)$ be satisfied and let the operators $L : L^2(0,T;U) \to X$ and $K : L^2(0,T;X) \to L^2(0,T;\mathcal{D}(A))$ be defined by (1.6) and (2.8), respectively. Then, the following estimates hold*

$$\|(K\tilde{B}y)(t)\|_X \le C\int_0^t \|y(s)\|_X ds,$$

and

$$\|L\tilde{B}y\|_X \le C\int_0^T \|y(s)\|_X ds.$$

Proof. From the definition of K and $\tilde{B}$, we rewrite using semigroup property as

$$\begin{aligned}(K\tilde{B}y)(t) &= \int_0^t S(t-\tau)\int_0^\tau B(\tau,s)y(s)dsd\tau \\ &= -\int_0^t \frac{d}{d\tau}S(t-\tau)\int_0^\tau A^{-1}B(\tau,s)y(s)dsd\tau.\end{aligned}$$

After integration by parts, we arrive at

$$\begin{aligned}(K\tilde{B}y)(t) = \int_0^t A^{-1}B(t,s)y(s)ds - \int_0^t S(t-\tau)\int_0^\tau A^{-1}B_\tau(\tau,s)y(s)dsd\tau \\ - \int_0^t S(t-\tau)A^{-1}B_\tau(\tau,\tau)y(\tau)d\tau.\end{aligned}$$

We note that

$$\begin{aligned}\|(K\tilde{B}y)(t)\|_X \le \int_0^t \|A^{-1}B(t,s)y(s)\|ds + \int_0^t \|S(t-\tau)\| \int_0^\tau \|A^{-1}B_\tau(\tau,s)y(s)\|dsd\tau \\ + \int_0^t \|S(t-\tau)\|\,\|A^{-1}B_\tau(\tau,\tau)y(\tau)\|d\tau,\end{aligned}$$

and hence,

$$\begin{aligned}\|(K\tilde{B}y)(t)\|_X &\le \alpha\int_0^t \|y(s)\|_X ds + \alpha\|A^{-1}\|(1+\beta)\int_0^t \|y(\tau)\|_X d\tau + \alpha\beta\int_0^t \|y(\tau)\|_X d\tau \\ &\le \alpha(1+\beta)(1+\|A^{-1}\|)\int_0^t \|y(\tau)\|_X d\tau.\end{aligned}$$

Thus, we now arrive at

$$\|(K\tilde{B}y)(t)\|_X \le C\int_0^t \|y(\tau)\|_X d\tau,$$

where C is a generic constant, which depends on α, β and $\|A^{-1}\|$. Similarly, a use of definition of L and $\tilde{B}$ yields

$$\|L\tilde{B}y\|_X \le C\int_0^T \|y(\tau)\|_X d\tau.$$

This completes the proof of the lemma. □

On the lines of Lemma 2.2, we have the following result.

Lemma 2.3. *Under the assumptions* (A1), (A2) *and* (A4), *the following estimate holds*

$$\left\|\Big(KLG\mathcal{M}y\Big)(t)\right\|_X \le C_1\left(\|\hat{z}\| + C\int_0^t \|y(\tau)\|_X d\tau\right),$$

where C_1 depends on $T, \beta, \|LG\|, \|G^*L^*\|$ and $\|(\delta I + LGG^*L^*)^{-1}\|$.

A variation of the Banach contraction mapping principle will help in the proof of the following theorem, which provides the approximate controllability of the system (1.1).

Theorem 2.4. *Under the assumption* $(A1)-(A4)$, *the operator R_δ^n is a contraction on the space Z for some positive integer n. Moreover, for any arbitrary $z_0 \in X$, the sequence of iterates $\{z_{\delta,k}\}$, defined by*

$$z_{\delta,k+1} = R_\delta^n z_{\delta,k}, \quad k = 0,1,2,\dots \tag{2.10}$$

with $z_{\delta,0} = y_0$ converges to y_δ^*, which is a mild solution of the system (1.1). Further, $u_{\delta,k} = \mathcal{M}z_{\delta,k}$ is such that $u_{\delta,k}$ converges to $u_\delta^* = \mathcal{M}y_\delta^*$, and the system (1.1) is approximately controllable.

Proof. Let $z_1, z_2 \in Z$. Then, a use of (2.9) yields

$$(R_\delta z_1 - R_\delta z_2)(t) = K\tilde{B}(z_1(t) - z_2(t)) + KG\mathcal{M}(z_1(t) - z_2(t)).$$

Apply Lemma 2.2 and 2.3 and obtain

$$\|(R_\delta z_1 - R_\delta z_2)(t)\|_X \le C \int_0^t \|z_1(\tau) - z_2(\tau)\|_X d\tau,$$

where C depends on $\beta, \alpha, T, \|LG\|, \|G^*L^*\|$ and $\|(\delta I + LGG^*L^*)^{-1}\|$. Hence,

$$\|R_\delta z_1 - R_\delta z_2\|_Z \le \frac{CT}{\sqrt{2}} \|z_1 - z_2\|_Z.$$

Proceeding inductively, we obtain a constant $\gamma_n = \frac{(2CT)^n}{\sqrt{2n}(3\cdot 5 \cdots 2n-1)}$, such that

$$\|R_\delta^n z_1 - R_\delta^n z_2\|_Z \le \gamma_n \|z_1 - z_2\|_Z.$$

Choose n large enough (independent of T and C) such that $\gamma_n < 1$, and hence, R_δ^n is a contraction. Therefore, by Banach contraction mapping theorem, R_δ^n has a unique fixed point, say, y_δ^*, which is the limit of the sequence defined by (2.10). This y_δ^* is also the unique fixed point of the operator R_δ, for fixed $\delta \in (0, 1]$.

In order to show $\mathcal{M}z_{\delta,k} \to \mathcal{M}y_\delta^*$, set $u_{\delta,k} = \mathcal{M}z_{\delta,k}$, where $z_{\delta,k}$ is the mild solution of the system (2.2) with control $u_{\delta,k}$. Then, we obtain

$$\left\|\left(\mathcal{M}z_{\delta,k} - \mathcal{M}y_\delta^*\right)\right\|_Z \le CT\|G^*L^*(\epsilon I + LGG^*L^*)^{-1}\| \, \|z_{\delta,k} - y_\delta^*\|_Z.$$

Since for each fixed $\delta \in (0, 1]$, the sequence $z_{\delta,k} \to y_\delta^*$ in Z. This implies that $\mathcal{M}z_{\delta,k} \to \mathcal{M}y_\delta^* = u_\delta^*$. As y_δ^* is the mild solution of the system (1.1) with control u_δ^*, as $z_{\delta,k} \to y_\delta^*$, it follows that $R_\delta z_{\delta,k} \to R_\delta y_\delta^* = y_\delta^*$. Using the definition of R_δ and with similar arguments as earlier, it follows that

$$R_\delta z_{\delta,k}(t) = S(t)y_0 + \int_0^t S(t-\tau)\tilde{B}z_{\delta,k}(\tau)\, d\tau + \int_0^t S(t-\tau)Gu_{\delta,k}(\tau)\, d\tau.$$

As $k \to \infty$, we obtain

$$y_\delta^*(t) = S(t)y_0 + \int_0^t S(t-\tau)\tilde{B}y_\delta^*(\tau)\, d\tau + \int_0^t S(t-\tau)Gu_\delta^*(\tau)\, d\tau,$$

and y_δ^* is the mild solution of the system (1.1), corresponding to control u_δ^* given by

$$u_\delta^* = \mathcal{M}y_\delta^* \tag{2.11}$$

$$= G^*L^*\left(\delta I + LGG^*L^*\right)^{-1}\left[\hat{z} - L\tilde{B}y_\delta^*\right]. \tag{2.12}$$

It remains to show that the problem (1.1) is approximately controllable. To this end, we observe that

$$\begin{aligned} LGu_\delta^* &= LGG^*L^*\left(\delta I + LGG^*L^*\right)^{-1}\left[\hat{z} - L\tilde{B}y_\delta^*\right] \\ &= \left((\delta I + LGG^*L^*) - \delta I\right)\left(\delta I + LGG^*L^*\right)^{-1}\left[\hat{z} - L\tilde{B}y_\delta^*\right] \\ &= [\hat{z} - L\tilde{B}y_\delta^*] - \delta\left(\delta I + LGG^*L^*\right)^{-1}\left[\hat{z} - L\tilde{B}y_\delta^*\right]. \end{aligned} \tag{2.13}$$

Since $\|\hat{z} - L\tilde{B}y_\delta^*\|$ is bounded, a use of Lemma 2.1 (iv) yields

$$\lim_{\delta \to 0^+} \left\| -\delta\left(\delta I + LGG^*L^*\right)^{-1}\left[\hat{z} - L\tilde{B}y_\delta^*\right]\right\| = 0,$$

and hence,

$$\lim_{\delta\to 0^+} \|LGu^*_\delta + L\tilde{B}y^*_\delta - \hat{z}\| = 0,$$

that is, for given any given $\delta_1 > 0$, there exists a $\delta_0 > 0$ such that for $0 < \delta \le \delta_0$

$$\|LGu^*_\delta + L\tilde{B}y^*_\delta - \hat{z}\| < \delta_1.$$

Hence, the system (1.1) is approximately controllable. This completes the rest of the proof. □

Remark 2.5. Note that $U_\delta \neq \emptyset$. Further, the error of the approximation in this case is given by

$$e_\delta z^*_\delta = \delta\left(\delta I + LGG^*L^*\right)^{-1}\left[\hat{z} - L\tilde{B}y^*_\delta\right].$$

Remark 2.6. Under assumption $(A1)$-$(A4)$, Theorem 2.4 implies that the system (1.1) is controllable without any inequality constraint on T.

Thus, we have $U_\delta \neq \emptyset$. The pair (u^*_δ, y^*_δ) so obtained need not be an optimal pair satisfying (1.8), and hence, the problem (1.8) remains unanswered.

We now change our strategy and examine the process of obtaining the optimal pair of the constrained problem through a sequence of optimal pairs of the unconstrained problems, as indicated in Section 1. For this purpose, we first define a sequence of functionals $\{J_\epsilon\}$ with $\epsilon > 0$ as

$$J_\epsilon(u) = \frac{1}{2}J(u) + \frac{1}{2\epsilon}P(u),\ u \in Y, \tag{2.14}$$

where penalty function $P(u)$ is of the form

$$P(u) = \left\|LGu + L\tilde{B}Wu - \hat{z}\right\|^2_X,\ u \in Y. \tag{2.15}$$

Now the problem under investigation is to seek $u^*_\epsilon \in U$ such that

$$J_\epsilon(u^*_\epsilon) = \inf_{u\in Y} J_\epsilon(u). \tag{2.16}$$

As in [8], roughly speaking, the approximate controllability can be viewed as the limit of a sequence of optimal control problems (2.16). We now make further assumption that

$(A5)$ The solution operator $W : U \to Z$ is completely continuous.

Remark 2.7. One of the sufficient condition for W to be completely continuous is that the semigroup $\{S(t)\}$ is compact.

Denote by $\mathcal{E}$ the operator $\mathcal{E}u = LGu + L\tilde{B}Wu$, where the operators L, $\tilde{B}$ and W are as defined before. Then, the functional J_ϵ defined through (2.14) can be written as

$$J_\epsilon(u) = \frac{1}{2}\|u\|^2_Y + \frac{1}{2\epsilon}\|\mathcal{E}u - \hat{z}\|^2_X. \tag{2.17}$$

Note that the operator $\mathcal{E}$ is a sum of continuous linear operator L and a completely continuous operator W, and hence, it is a weakly continuous operator.

THEOREM 2.8. *Under assumptions $(A1)$-$(A5)$, the unconstrained optimal control problem (2.16) has an optimal pair $(u^*_\epsilon, y^*_\epsilon)$ such that $u^*_\epsilon \in U$ minimizes $J_\epsilon(u)$ and y^*_ϵ solves (1.1) corresponding to the control u^*_ϵ.*

Proof. We first prove the weakly lower semicontinuity of the functional J_ϵ. Let $u^n_\epsilon \rightharpoonup u^*_\epsilon$ in Y, then, it follows that

$$\liminf_{n\to\infty} J_\epsilon(u^n_\epsilon \ge \liminf_{n\to\infty} \frac{1}{2}\|u^n_\epsilon\|^2_Y + \liminf_{n\to\infty}\frac{1}{2\epsilon}\|\mathcal{E}u^n_\epsilon - \hat{z}\|^2_X.$$

Observe that

$$\|\mathcal{E}u_\epsilon^n - \hat{z}\|_X^2 = \|LGu_\epsilon^n\|_X^2 + \|L\tilde{B}Wu_\epsilon^n - \hat{z}\|_X^2 + 2\left(Lu_\epsilon^n, L\tilde{B}Wu_\epsilon^n - \hat{z}\right)_X,$$

and hence,

$$\begin{aligned}\liminf_{n\to\infty} \|\mathcal{E}u_\epsilon^n - \hat{z}\|_X^2 \geq \liminf_{n\to\infty} \|LGu_\epsilon^n\|_X^2 + \liminf_{n\to\infty} \|L\tilde{B}Wu_\epsilon^n - \hat{z}\|_X^2 \\ +2\liminf_{n\to\infty}\left(LGu_\epsilon^n, L\tilde{B}Wu_\epsilon^n - \hat{z}\right)_X.\end{aligned}$$

From Lemma 2.2, we arrive at

$$\begin{aligned}\|L\tilde{B}\left(Wu_\epsilon^n - Wu_\epsilon^*\right)\|_X &\leq C\int_0^T \|(Wu_\epsilon^n - Wu_\epsilon^*)\|_X ds \\ &\leq CT^{1/2}\|Wu_\epsilon^n - Wu_\epsilon^*\|_Z.\end{aligned}$$

Since $u_\epsilon^n \rightharpoonup u_\epsilon^*$ in Y, and W is completely continuous, this implies that $Wu_\epsilon^n \to Wu_\epsilon^*$ in Z and hence, $L\tilde{B}Wu_\epsilon^n - \hat{z} \to L\tilde{B}Wu_\epsilon^* - \hat{z}$ in X. Using the fact that L is weakly continuous and W is completely continuous, we obtain $LGu_\epsilon^n \rightharpoonup LGu_\epsilon^*$, $L\tilde{B}Wu_\epsilon^n - \hat{z} \to L\tilde{B}Wu_\epsilon^* - \hat{z}$ and $\left(LGu_\epsilon^n, L\tilde{B}W\epsilon^n - \hat{z}\right) \to \left(LGu_\epsilon^*, L\tilde{B}Wu_\epsilon^* - \hat{z}\right)$ and along with the fact that the norm is weakly lower semicontinuous functional, we find that

$$\liminf_{n\to\infty} J_\epsilon(u_\epsilon^n) \geq \frac{1}{2}\|u_\epsilon^*\|_Y^2 + \frac{1}{2\epsilon}\|\mathcal{E}u_\epsilon^* - \hat{z}\|_X^2.$$

This proves the weakly lower semi-continuity of J_ϵ. Let $\{u_\epsilon^n\}$ be a minimizing sequence for the functional J_ϵ, that is, $\inf_{u\in Y} J_\epsilon(u) = \lim_{n\to\infty} J_\epsilon(u_\epsilon^n)$. Since J_ϵ is coercive, the sequence $\{u_\epsilon^n\}$ is bounded in Y. Then, there exists a subsequence which is also denoted by $\{u_\epsilon^n\}$ such that $u_\epsilon^n \rightharpoonup u_\epsilon^*$ weakly in Y. Since the functional (2.17) is weakly lower semicontinuous in Y, we arrive at

$$\inf_{u\in Y} J_\epsilon(u) = \lim_{n\to\infty} J_\epsilon(u_\epsilon^n) = \liminf_{n\to\infty} J_\epsilon(u_\epsilon^n) \geq J_\epsilon(u_\epsilon^*).$$

Therefore, we obtain

$$J_\epsilon(u_\epsilon^*) = \inf_{u\in Y} J_\epsilon(u).$$

As $y_\epsilon^n = Wu_\epsilon^n$ and $u_\epsilon^n \rightharpoonup u_\epsilon^*$, the complete continuity of W implies $y_\epsilon^n \to y_\epsilon^*$, where $y_\epsilon^* = Wu_\epsilon^*$. Thus, $(u_\epsilon^*, y_\epsilon^*)$ is the optimal pair for the unconstrained optimal control problem (2.16) and this completes the proof of the theorem. ☐

In our subsequent analysis, we need the following properties of the sequence of minimizers $\{u_\epsilon^*\}$.

Lemma 2.9. *Let $\epsilon > 0$ be arbitrary and let $u_\epsilon \in Y$ be a minimizer of $J_\epsilon(u)$ in Y, where $J_\epsilon(u)$ as defined by (2.14). For $\epsilon' < \epsilon$, the followings holds:*

(i) $J_\epsilon(u_\epsilon) \leq J_{\epsilon'}(u_{\epsilon'})$.
(ii) $P(u_\epsilon) \geq P(u_{\epsilon'})$.
(iii) $J(u_\epsilon) \leq J(u_{\epsilon'})$.
(iv) $J(u_\epsilon) \leq J_\epsilon(u_\epsilon) \leq J(u^*) + \frac{\delta_0^2}{2\epsilon}$.

Proof. As u_ϵ minimizes J_ϵ, it follows that

$$J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'}) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon'}P(u_{\epsilon'}) = J_{\epsilon'}(u_{\epsilon'}).$$

This proves (i). For (ii), let u_ϵ and $u_{\epsilon'}$ be the minimizers of J_ϵ and $J_{\epsilon'}$, respectively, then, we arrive at

$$J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'})$$

and

$$J_{\epsilon'}(u_{\epsilon'}) = J(u_{\epsilon'}) + \frac{1}{2\epsilon'}P(u_{\epsilon'}) \leq J(u_\epsilon) + \frac{1}{2\epsilon'}P(u_\epsilon).$$

On adding above inequalities, we get

$$P(u_\epsilon) \geq P(u_{\epsilon'}).$$

For (iii), note that

$$J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'}).$$

Hence, using (ii) it follows that

$$J(u_\epsilon) - J(u_{\epsilon'}) \leq \frac{1}{2\epsilon}\left(P(u_{\epsilon'}) - P(u_\epsilon)\right) \leq 0$$

and

$$J(u_\epsilon) \leq J(u_{\epsilon'}).$$

For (iv), again we observe that

$$J(u_\epsilon) \leq J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u^*) + \frac{1}{2\epsilon}P(u^*) \leq J(u^*) + \frac{\delta_0^2}{2\epsilon}.$$

This completes the rest of the proof. □

We are now in a position to state the main theorem of this article.

Theorem 2.10. *Assume that for a fixed* $\delta > 0$, $U_\delta \neq \emptyset$ *and assumptions* $(A1)$-$(A4)$ *hold. Let* $(u_\epsilon^*, y_\epsilon^*)$ *be an optimal pair of the unconstrained problem (2.16). As* $\epsilon \to 0$, *there exists a subsequence of* $(u_\epsilon^*, y_\epsilon^*)$ *converges to* (u_δ^*, y_δ^*), *where* (u_δ^*, y_δ^*) *is an optimal pair of the constrained optimal control problem (1.8). Furthermore, if* U_δ *is a singleton then the entire sequence* $(u_\epsilon^*, y_\epsilon^*)$ *converges to* (u_δ^*, y_δ^*).

Proof. For a fixed $\delta > 0$, $U_\delta \neq \emptyset$; the existence of the optimal pair $(u_\epsilon^*, y_\epsilon^*)$ to the unconstrained problem (2.16) follows from the Theorem 2.8. Let $u_{\epsilon'}^* \in U_\delta$. From Lemma 2.9, we have

$$J_\epsilon(u_\epsilon^*) \leq J_{\epsilon'}(u_{\epsilon'}^*) \text{ for } \epsilon' < \epsilon.$$

Thus, $\{J_\epsilon(u_\epsilon^*)\}$ is a monotone decreasing sequence which is bounded below and hence it converges. Similarly, $\{J(u_\epsilon^*)\}$ is also a convergent sequence. Now $\frac{1}{\epsilon}P(u_\epsilon^*)$, being the difference of two convergent sequence, also converges, which in turn, implies that $P(u_\epsilon^*) \to 0$ as $\epsilon \to 0$. Hence,

$$\lim_{\epsilon \to 0} \|\mathcal{E}u_\epsilon^* - \hat{z}\|_X = 0.$$

Since $\{u_\epsilon^*\}$ is a uniformly bounded sequence in Y, it has a subsequence, again denoted by $\{u_\epsilon^*\}$ such that $u_\epsilon^* \rightharpoonup u_\delta^*$ in Y. Weak continuity of $\mathcal{E}$ implies that $\mathcal{E}u_\epsilon^* \rightharpoonup \mathcal{E}u_\delta^*$. Hence $\mathcal{E}u_\delta^* = \hat{z}$ and $u_\delta^* \in U_\delta$. By the weak lower semicontinuity of the norm functional and Lemma 2.9, we arrive at

$$\|u_\delta^*\|_Y \leq \liminf_{\epsilon \to 0} \|u_\epsilon^*\|_Y \leq \limsup_{\epsilon \to 0} \|u_\epsilon^*\|_Y \leq \|u_\delta^*\|_Y,$$

and hence,

$$\lim_{\epsilon \to 0} \|u_\epsilon^*\|_Y = \|u_\delta^*\|_Y.$$

This along with the weak convergence of u_ϵ^* to u_δ^*, implies that

$$u_\epsilon^* \to u_\delta^* \text{ as } \epsilon \to 0.$$

Again from Lemma 2.9 and weak lower semicontinuity of the norm functional, we obtain the inequality

$$J(u_\delta^*) \leq \liminf_{\epsilon \to 0} J(u_\epsilon^*) \leq J(\tilde{u}),\ \tilde{u} \in U_\delta.$$

This, in turn, implies that

$$J(u_\delta^*) \le J(\tilde u) \;\forall\; \tilde u \in U_\delta.$$

Therefore, (u_δ^*, y_δ^*) is the optimal pair for the constrained optimal problem (1.8). It is also clear that if U_δ is a singleton then the entire sequence u_ϵ^* converges to u_δ^* in Y.

Next, we show the convergence of y_ϵ^* to y_δ^* in Z. From (1.4) and Lemma 2.2, we obtain

$$\begin{aligned}\|y_\epsilon^*(t)-y_\delta^*(t)\|_X &\le C\int_0^t \|y_\epsilon^*(\tau)-y_\delta^*(\tau)\|_X d\tau + \beta\int_0^t \|u_\epsilon^*(\tau)-u_\delta^*(\tau)\|_X d\tau\\ &\le C\int_0^t \|y_\epsilon^*(\tau)-y_\delta^*(\tau)\|_X d\tau + \beta T^{1/2}\|u_\epsilon^*-u_\delta^*\|_Y.\end{aligned}$$

Using Gronwall's lemma, we arrive that

$$\|y_\epsilon^*(t)-y_\delta^*(t)\|_X \le \beta T^{1/2}\|u_\epsilon^*-u_\delta^*\|_Y\, e^{CT},$$

and hence,

$$\|y_\epsilon^*-y_\delta^*\|_Z \le \beta T e^{CT}\|u_\epsilon^*-u_\delta^*\|_Y. \tag{2.18}$$

Since $u_\epsilon^* \to u_\delta^*$ in Y, from (2.18), we obtain $y_\epsilon^* \to y_\delta^*$ in Z as $\epsilon \to 0$. This completes the rest of the proof. □

3. Approximation theorems. In our analysis, we are interested in the computation of the optimal control pair for the unconstrained problem. We first begin by establishing some properties of the operator arising from the derivative of the functional J_ϵ, which is defined as follows:

$$J_\epsilon(u) = \frac{1}{2}\|u\|_Y^2 + \frac{1}{2\epsilon}\|\mathcal{E}u-\hat z\|_X^2, \tag{3.1}$$

where $\hat z$ is a fixed element in X. We first recall the unconstrained optimal control problem

$$J_\epsilon(u) = \inf_{v\in Y} J_\epsilon(v). \tag{3.2}$$

LEMMA 3.1. *The critical point of the functional J_ϵ is given by the solution of the operator equation*

$$u + \frac{1}{\epsilon}\mathcal{K}\left(\mathcal{E}u-\hat z\right) = 0 \tag{3.3}$$

where $\mathcal{K} = (LG+L\tilde BW)^*$, $\mathcal{E}u = (LG+L\tilde BW)u$ and $y = Wu$.

Proof. We note that

$$\begin{aligned}J_\epsilon(u+hv)-J_\epsilon(u) &= \frac{1}{2}\langle u+hv, u+hv\rangle\\ &\quad+\frac{1}{2\epsilon}\Big((LG+L\tilde BW)(u+hv)-\hat z, (LG+L\tilde BW)(u+hv)-\hat z\Big)\\ &\quad-\frac{1}{2}(u,u)-\frac{1}{2\epsilon}\Big((LG+L\tilde BW)(u)-\hat z, (LG+L\tilde BW)(u)-\hat z\Big)\\ &= h\,(u,v)+\frac{h^2}{2}(v,v)+\frac{h}{\epsilon}\Big(LGu+L\tilde BWu-\hat z, (LG+L\tilde BW)(v)\Big)\\ &\quad+\frac{h^2}{2\epsilon}\Big((LG+L\tilde BW)(v), (LG+L\tilde BW)(v)\Big).\end{aligned}$$

Then, $J_\epsilon^{'}(u)$ is given by

$$\begin{aligned}J_\epsilon^{'}(u)v &= \lim_{h\to 0}\frac{J_\epsilon(u+hv)-J_\epsilon(u)}{h}\\ &= (u,v)+\frac{1}{\epsilon}\Big(LGu+L\tilde BWu-\hat z, (LG+L\tilde BW)(v)\Big)\\ &= (u,v)+\frac{1}{\epsilon}\Big((LG+L\tilde BW)^*(LG+L\tilde BW)u-\hat z, v\Big),\end{aligned}$$

and hence,

$$J_{\epsilon}^{'}(u) = u + \frac{1}{\epsilon}(LG + L\tilde{B}W)^{*}(LG + L\tilde{B}W)u - \hat{z}).$$

If u is a critical point of J_ϵ, then it follows that

$$u + \frac{1}{\epsilon}\mathcal{K}(LGu + L\tilde{B}Wu - \hat{z}) = 0,$$

where $\mathcal{K} = (LG + L\tilde{B}W)^{*}$. This concludes the proof. $\square$

Note that, in the literature, the operator equation (3.3) is known as the Hammerstein equation (see, Joshi *et al.* [12]) Also, note that the operator $\mathcal{K}$ is bounded linear operator. We first assume that the critical point of J_ϵ is the unique minimizer of J_ϵ. Then the minimizing problem (3.2) is equivalent to the following solvability problem in the space Y:

$$u + \frac{1}{\epsilon}\mathcal{K}\mathcal{E}u = \hat{w}, \tag{3.4}$$

where $\hat{w} = \frac{1}{\epsilon}\mathcal{K}\hat{z}$. We now first begin approximating the main problem in the following way. Consider a family $\{X_m\}$ of finite dimensional subspaces of X such that

$$X_1 \subset X_2 \subset \ldots \subset X_m \ldots \subset X \text{ with } \overline{\bigcup_{m=1}^{\infty} X_m} = X.$$

Let $\{\phi_i\}_{i=1}^{\infty}$ be a basis for X. The approximating scheme for the space $Z = L^2(0,T;X)$ is then defined in a natural way by the family of subspaces $Z_m = L^2(0,T;X_m)$. Then the projection $P_m : X \to X_m$ is given by

$$P_m[y(t)] = \sum_{i=1}^{m} \alpha_i \phi_i, \ t \in [0,T],$$

where $X_m = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$. Similarly, let U_m be the finite dimensional spaces of U and $\{\psi_i\}_{i=1}^{\infty}$ be a basis for U such that $U_m = \text{span}\{\psi_1, \psi_2, \ldots, \psi_m\}$ and naturally the approximating scheme for the space $Y = L^2(0,T;U)$ is given by $Y_m = L^2(0,T;U_m)$ which induces a projection $\tilde{P}_m : Y \to Y_m$ given by

$$(\tilde{P}_m u)(t) = P_m G u(t).$$

The projections P_m and $\tilde{P}_m$ generate the approximating operators $\mathcal{K}_m$ and $\mathcal{E}_m$ defined by $\mathcal{K}_m = \tilde{P}_m\mathcal{K}$ and $\mathcal{E}_m u = P_m\mathcal{E}u$. Then, the approximated minimization problem is stated as: Find $u_m \in Y_m$ such that

$$J_{\epsilon,m}(u_m) = \inf_{u \in Y_m}\left[J_{\epsilon,m}(u) = \frac{1}{2}\|\tilde{P}_m u\|^2_{Y_m} + \frac{1}{2\epsilon}\|P_m\mathcal{E}\tilde{P}_m u - P_m\hat{z}\|^2_{X_m}\right]. \tag{3.5}$$

As in the case of problem (2.16), one can show that the problem (3.5) has a solution $u_m \in Y_m$, and hence, its critical point satisfying the operator equation in the approximating space Y_m as

$$u_m + \frac{1}{\epsilon}\mathcal{K}_m\left(\mathcal{E}_m u_m - P_m\hat{z}\right) = 0. \tag{3.6}$$

Let $u_m = \sum_{i=1}^{m}\beta_i\psi_i$ be the solution of (3.5) and $y_m = \sum_{i=1}^{m}\alpha_i\phi_i$ be the corresponding solution of the state equation. Then for each m, we shall refer to (u_m, y_m) as approximating optimal pair of (3.5) in (Y_m, Z_m).

Following theorem shows that the solution for the problem (3.6) is uniformly bounded in Y_m and the approximating pair (u_m^*, y_m^*) converges to (u^*, y^*), where (u^*, y^*) is an optimal pair of the constrained problem (1.8).

THEOREM 3.2. *Let $U_\delta \neq \emptyset$ and u_m^* be the solution to the problem (3.5). Then $\{u_m^*\}$ is uniformly bounded in Y_m. If in addition, J_ϵ possesses a unique minimizer in Y which is also the only critical point of J_ϵ, then (3.5) has an optimal pair (u_m^*, y_m^*) which converges to (u^*, y^*) in $Y \times Z$, where (u^*, y^*) is an optimal pair of the constrained problem (1.8).*

Proof. Existence of the optimal pair (u_m^*, y_m^*) to the optimal control problem (3.5) follows from Theorem 2.8. Let $u^* \in U_\delta$, then from the definition of U_δ, we have $\|\mathcal{E}u^* - \hat{z}\| \le \delta$. Define $u_m^* = \tilde{P}_m u^*$. Then

$$\begin{aligned}\frac{1}{2}\|u_m^*\|^2 \le J_{\epsilon,m}(u_m^*) &= \frac{1}{2}\|\tilde{P}_m u^*\|^2 + \frac{1}{2\epsilon}\|P_m\mathcal{E}u_m^* - P_m\hat{z}\|^2 \\ &\le \frac{1}{2}\|\tilde{P}_m u^*\|^2 + \frac{1}{2\epsilon}\|P_m\|^2\|\mathcal{E}u_m^* - \hat{z}\|^2 \\ &\le \frac{1}{2}\|\tilde{P}_m\|^2\ \|u^*\|^2 + \frac{1}{\epsilon}\|P_m\|^2\Big(\|\mathcal{E}u_m^* - \mathcal{E}u^*\|^2 + \|\mathcal{E}u^* - \hat{z}\|^2\Big).\end{aligned}$$

Since $u_m^* = \tilde{P}_m u^* \to u^*$ and $\mathcal{E}$ is weakly continuous, we have $\mathcal{E}u_m^* \rightharpoonup \mathcal{E}u^*$. Hence, both the term on right hand side is bounded. Therefore $\{u_m^*\}$ is uniformly bounded.

Since $\{u_m^*\}$ is uniformly bounded, it has a subsequence, still denoted by u_m^*, which converges weakly to u^* in Y. Then from the weak lower semicontinuity of the norm functional and Lemma 2.9, we arrive at

$$\|u^*\| \le \liminf_{m\to\infty}\|u_m^*\| \le \limsup_{m\to\infty}\|u_m^*\| \le \|u^*\|.$$

This implies

$$\lim_{m\to\infty}\|u_m^*\| = \|u^*\|.$$

Together with the fact that $u_m^* \rightharpoonup u^*$ in Y, we obtain

$$u_m^* \to u^* \ \text{ in } Y, \text{ as } m \to \infty.$$

As $y_m^* = Wu_m^*$ and $u_m^* \to u^*$, then continuity of the solution operator W implies that $y_m^* \to y^* = Wu^*$. This now completes the rest of the theorem. ☐

In the next step, we discretize in the direction of t. This leads to finite dimensional subspaces Z_m^k of each fixed Z_m which satisfy the following property

$$Z_m^1 \subset Z_m^2 \subset \ldots Z_m^k \subset \ldots \subset Z_m \ \text{ with } \overline{\bigcup_{k=1}^{\infty} Z_m^k} = Z_m.$$

Denoting the orthogonal projection Q_m^k from Z_m to Z_m^k which introduced the operators $\mathcal{K}_m^k = Q_m^k\mathcal{K}_m$ and $\mathcal{E}_m^k u = Q_m^k\mathcal{E}_m u$. For a fixed m, we approximate the minimization problem (3.5) by the following minimization problem in the finite dimensional subspace Y_m^k of Y_m.

Find $u_m^k \in Y_m^k$ such that

$$\Phi_\epsilon(u_m^k) = \inf_{u_m\in Y_m^k}\left[J_{\epsilon,m}^k(u_m) = \frac{1}{2}\|Q_m^k u_m\|^2_{Y_m^k} + \frac{1}{2\epsilon}\ \|Q_m^k\mathcal{E}_m u_m - \hat{z}_m\|^2_{X_m}\right]. \tag{3.7}$$

The unique minimizer of the problem (3.7) is given by the critical point of Φ_ϵ, which is equivalent to the following solvability problem in the space Y_m^k.

$$u_m^k + \frac{1}{\epsilon}\mathcal{K}_m^k\left(\mathcal{E}_m^k u_m^k - P_m\hat{z}\right) = 0. \tag{3.8}$$

Let $u_m^k = \sum_{i=1}^m \beta_i^k\psi_i$ be the solution of the minimization problem (3.7) and let $y_m^k = \sum_{i=1}^m \alpha_i^k\phi$, where $\alpha_i^k = \alpha_i(k\Delta t)$ and $\beta_i^k = \beta_i(k\Delta t), 1 \le k \le N, N = T/\Delta t$. For each m, we shall refer to (u_m^k, y_m^k) as approximating optimal pair of the problem (3.7) in (Y_m^k, Z_m^k).

On the lines of Theorem 3.2, we have the following theorem giving the convergence of the approximation optimal pair (u_m^k, y_m^k) as $k \to \infty$ with m fixed.

THEOREM 3.3. *Let the assumptions (A1) - (A5) be satisfied and $\{u_m^k\}$ be the solution of the problem (3.8). Then the approximating optimal pair (u_m^k, y_m^k) converges to (u_m^*, y_m^*) in (Y_m, Z_m).*

4. Application. Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. For fixed $T > 0$, let $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. Let A be a second order uniformly elliptic differential operator given by (1.2). Further, assume that the operator $B(t, s)$ is an unbounded partial differential operator of order $\beta \leq 2$ given by (1.3). Set $X = L^2(\Omega)$, $V = H_0^1(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $D(B) = H^2(\Omega)$. Then the weak formulation of the problem (1.1) is given by

$$(y_t, \phi) + A(y, \phi) = \int_0^t B(t, s; y(s), \phi)ds + (Gu, \phi) \ \ \forall \phi \in V, \ t \in [0, T] \tag{4.1}$$
$$y(0) = y_0,$$

where $A(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and $B(t, s; \cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ are the continuous bilinear forms corresponding the operators A and $B(t, s)$ given respectively by (1.2) and (1.3).

From Lumer–Phillips theorem (see, Pazy [27]), $(-A)$ generates a C_0–semigroup. For $y_0 \in D(A)$, the unique mild solution for the system (1.1) is given by

$$y(t) = S(t)y_0 + \int_0^t S(t - s)\tilde{B}y(s)ds + \int_0^t S(t - s)Gu(s)ds, \tag{4.2}$$

where $S(t) = e^{-tA}$ generates a C_0-semigroup.

For final time $t = T$, we obtain

$$y(T) = S(T)y_0 + L\tilde{B}y + LGu \tag{4.3}$$

where the operator $\tilde{B}$ and L are defined as before in Section 1. Since the linear operator A is self-adjoint and positive definite, one can show that A^{-1} exists and hence $\|A^{-1}B(t, s)\varphi\| \leq \alpha\|\varphi\|$ for $\varphi \in D(B)$ and $0 \leq s \leq t \leq T$, is satisfied.

Since all the hypotheses (A1-A4) are satisfied, an appeal to Theorem 2.4 ensures the approximate controllability of (4.1). Also, set $U = L^2(\Omega)$ and $Y = L^2(0, T, U)$, the solution operator $W : U \to Y$ is compact and an application to Theorem 2.8 and 2.10 shows the existence of optimal control.

Let $\{\mathcal{J}_h\}$ be a family of regular triangulation of Ω with $0 < h < 1$. For $K \in \mathcal{J}_h$, set $h_K = \text{diam}(K)$ and $h = \max(h_K)$. Let

$$X_h = \left\{v_h \in C^0(\bar{\Omega}) : v_h|_K \in \mathcal{P}_1(K), K \in \mathcal{J}_h, v_h = 0 \text{ on } \partial\Omega\right\},$$

where $\mathcal{P}_1(K)$ is the space of linear polynomials on K. Let U_h be the finite dimensional subspace of U consisting constant elements defined on the triangulation $\mathcal{J}_h$. Then, the semidiscrete Galerkin approximation of (4.1) is defined by

$$(y_{h,t}, \chi) + A(y_h, \chi) = \int_0^t B(t, s; y_h(s), \chi)ds + (Gu_h, \chi) \ \ \forall \chi \in X_h, \ t \in [0, T] \tag{4.4}$$
$$y_h(0) = y_{0h},$$

where, y_{0h} is the approximation of y_0 in X_h.

Let $P_h : X \to X_h$ is the L^2–projection and let $\{S_h(t)\}$ denote the finite element analogue of $S(t)$, defined by the semidiscrete equation (4.4) with $u = 0$ and $B = 0$. This operator on X_h may be defined as the semigroup generated by the discrete analogue $A_h : X_h \to X_h$ of A, where

$$(A_h v, \chi) = A(v, \chi) \ \ \forall \ v, \chi \in V_h.$$

Define the discrete analogue $B_h = B_h(t, s) : X_h \to X_h$ of $B = B(t, s)$ by

$$(B_h(t, s)v, \chi) = B(t, s; v, \chi) \ \ \forall \ v, \chi \in X_h, \ 0 \leq s \leq t \leq T.$$

Now we write the semidiscrete problem (4.4) in an abstract form

$$y_{h,t} + A_h y_h = \int_0^t B_h(t, s)y_h(s)ds + P_h G u_h \equiv \tilde{B}_h y_h + P_h G u_h, \ \text{ for } t \in [0, T], \tag{4.5}$$
$$y_h(0) = P_h y_0.$$

where $u_h(t) \in U_h$ and $y_h(t) \in X_h$. Using Duhamel's principle, the solution y_h of the semidiscrete problem (4.5) may be written as

$$y_h(t) = S_h(t)P_h y_0 + \int_0^t S_h(t-s)\tilde{B}_h y_h(s)ds + \int_0^t S_h(t-s)P_h G u_h(s)ds, \tag{4.6}$$

where the discrete counter part of $S(t) = e^{-tA}$ is defined by $S_h(t) = e^{-tA_h}$.

At time $t = T$, equation (4.6) becomes

$$y_h(T) = S_h(T)P_h y_0 + L_h \tilde{B}_h y_h + L_h P_h G u_h. \tag{4.7}$$

where L_h is defined by

$$L_h v_h = \int_0^T S_h(T-s)v_h(s)ds. \tag{4.8}$$

When m is replaced by h in equation (3.6), then the critical point u_h^* of the functional $J_{\epsilon,h}$ given by (3.5), is the solution of the following equation

$$u_h + \frac{1}{\epsilon}\mathcal{K}_h\left(\mathcal{E}_h u_h - P_h \hat{z}\right) = 0,$$

Here, P_h is the L^2-projection from the space X to X_h. Since all the conditions of Theorem 3.2 are satisfied, there exists an optimal pair (u_h^*, y_h^*) of (3.5) which converges to the optimal optimal pair of the constrained problem (1.8).

Full Discretization. In order to discretize in the direction of t, we partition the t-axis in a uniform partition (not necessary) by $0 = t_0 < t_1 \cdots < t_n < \cdots < t_N$ with $t_N = T$ and set $I_n = (t_{n-1}, t_n]$. Let W_N denotes the set of scalar functions on $[0,T]$ which reduces to polynomial of degree $q-1$ on each I_n. Let $Z_h^N = W_N \otimes X_h$ and $Y_h^N = W_N \otimes U_h$. In fact, Z_h^N consists of functions defined on $[0,T]$ whose restrictions to I_n is a polynomial of degree $\le q-1$ with its coefficients in X_h, that is, Z_h^N consists of piecewise constant on each I_n. Similarly, Y_h^N is defined for $q = 1$.

Let Δt be the step size in time, $t_n = n\Delta t,\ n = 1, 2, \cdots,\ N$, where $N = T/\Delta t$. Let $\phi^n = \phi(t_n)$. For $\phi \in C[0,T]$, set

$$\bar{\partial}_t \phi(t_n) = \frac{\phi(t_n) - \phi(t_{n-1})}{\Delta t}$$

Backward Euler Scheme: For $y_h^N \in Z_h^n$ and $u_h^N \in U_h^n$ with $y_h^N|_{I_n}$ denoted by $y_h^n \in X_h$ and $u_h^N|_{I_n} = u_h^n \in U_h$ for $n = 1, 2, \ldots, N$ and replacing the integral term by the left hand rectangular rule as

$$\int_0^{t_n} \varphi(s)ds \approx \Delta t \sum_{j=0}^{n-1} \varphi(t_j). \tag{4.9}$$

Then the backward Euler scheme is given by

$$\begin{aligned} \left(\bar{\partial}_t y_h^n, \chi\right) + A(y_h^n, \chi) &= \Delta t \sum_{j=0}^{n-1} B_h(t_n, t_j; y_h^j, \chi) + (G u_h^n, \chi), \quad \chi \in X_h, \\ y_h^0 &= y_{0h} \ \text{ in } \Omega, \end{aligned} \tag{4.10}$$

Let $y_h^n = \sum_{i=1}^{N_h} \alpha_i^n \varphi_i$ and $u_h^n = \sum_{i=1}^{M_h} \beta_i^n \psi_i$, where $\{\varphi_1, \varphi_2, \ldots, \varphi_{N_h}\}$ and $\{\psi_1, \psi_2, \ldots, \psi_{M_h}\}$ are basis of X_h and U_h, respectively. Note that the space Y_h^N can be identify as the space of matrices $\mathbb{M}$ of dimension $M_h \times (N+1)$. Therefore, the minimization problem (3.7) is equivalent to the following minimization problem in $\mathbb{M}$:

Find $\beta^* \in \mathbb{M}$ such that

$$\Phi_\epsilon(\beta^*) = \inf_{\beta \in \mathbb{M}} \left(\Phi_\epsilon(\beta) = \frac{1}{2}\|\beta\|_{\mathbb{M}}^2 + \frac{1}{2\epsilon}\|\mathcal{E}_m^k \beta - \hat{z}_m\|_{X_h} \right) \tag{4.11}$$

We shall use the following Matrix Optimization Algorithm (MOA) (for more details, see [15]) to get $\beta^* \in \mathbb{M}$ iteratively.
Step 1: Start with the initial guess $\beta^{(i)}(i=0)$, and ϵ. Set $\mathcal{G}^{(i)} = \Phi'_\epsilon(\beta^{(i)})$ and $\mathcal{D}^{(i)} = -\mathcal{G}^{(i)}$.
Step 2: Set $\beta^{(i+1)} = \beta^{(i)} + \alpha^{(i)}\mathcal{D}^{(i)}$, where $\alpha^{(i)}$ minimizes

$$\Phi_\epsilon(\beta^{(i+1)}) = \Phi_\epsilon(\beta^{(i)} + \alpha^{(i)}\mathcal{D}^{(i)})$$

Step 3: If $\|\beta^{(i+1)} - \beta^{(i)}\| \le \epsilon_1$, then set $\beta^{(i)} = \beta^{(i+1)}$ and go to Step (6) else go to Step (4).
Step 4: Compute $\mathcal{G}^{(i+1)} = \Phi'_\epsilon(\beta^{(i+1)})$ and set

$$\mathcal{D}^{(i+1)} = -\mathcal{G}^{(i+1)} + \zeta^{(i)}\mathcal{D}^{(i)}$$

where $\zeta^{(i)} = \frac{\langle \mathcal{G}^{(i+1)}, \mathcal{G}^{(i+1)} \rangle}{\langle \mathcal{G}^{(i)}, \mathcal{G}^{(i)} \rangle}$, $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{M}$.
Step 5: Set $i = i+1$ and go to Step (2).
Step 6: Set $\beta^{(i)} = \beta^{(i+1)}$ and $\epsilon = \epsilon + \delta,\ \delta > 0$.
Step 7: If $\|\beta^{(i+1)} - \beta^{(i)}\| \le \epsilon_2$, then set $\beta^* = \beta^{(i)}$ and stop. Else set $i = i+1$ and go to Step (2).

We note that corresponding to this minimizer β^*, $y_h^N \in Z_h^N$ with $y_h^N|_{I_n} = y_h^n \in X_h$ is computed through the backward Euler scheme (4.10). From the algorithm, it is clear that the inner loop finds the optimal control u_h^N for fixed values of N, h and ϵ and in the outer loop, we increment ϵ to find the optimal control u^* which is the optimal solution to the problem (1.8).

4.1. Numerical experiment. In this section, we present a numerical experiment to illustrate the computation of the minimizer u^* with the operator $G = I$, identity operator. We consider the following one-dimensional initial-boundary value problem

$$\begin{aligned} \frac{\partial y}{\partial t} &= \frac{\partial^2 y}{\partial x^2} + \int_0^t B(t,s)y(s)ds + u(t,x), \quad && \text{on } (0,T)\times(0,1) \\ y(0,x) &= y_0(x) && x \in (0,1) \\ y(t,0) &= 0 = y(1,t) && t \in [0,T] \end{aligned} \tag{4.1}$$

Set $T = 1,\ \Omega = (0,1) \subset \mathbb{R}^1$ with $B(t,s) = \exp(-\pi^2(t-s))I,\ y_0(x) = \sin(\pi x)$ and $\hat{y} = \exp(-\pi^2)\sin(\pi x)$. Note that for the above problem, $\hat{y} \in R(T, y_0)$, set of reachable states, since $y(t,x) = \exp(-\pi^2 t)\sin(\pi x)$ is an exact solution of the system (4.1) corresponding to the control function $u(t,x) = -t\exp(-\pi^2 t)\sin(\pi x)$ with $y(T,x) = \exp(-\pi^2)\sin(\pi x)$.

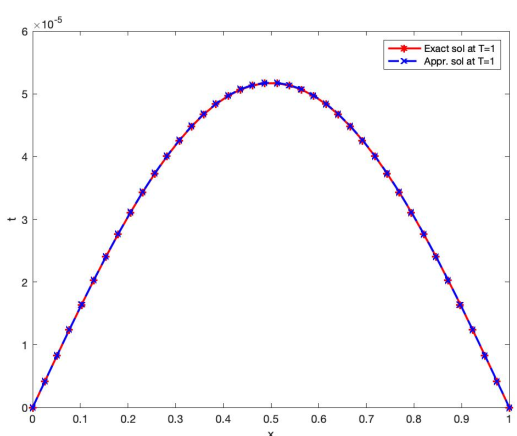
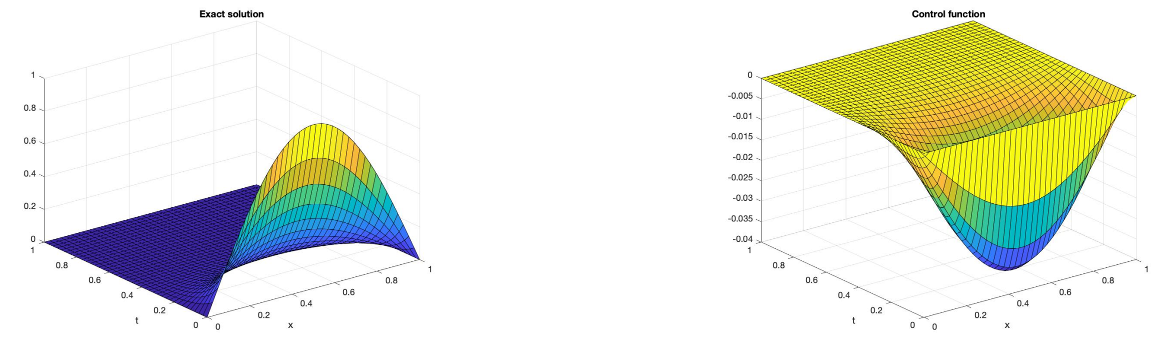


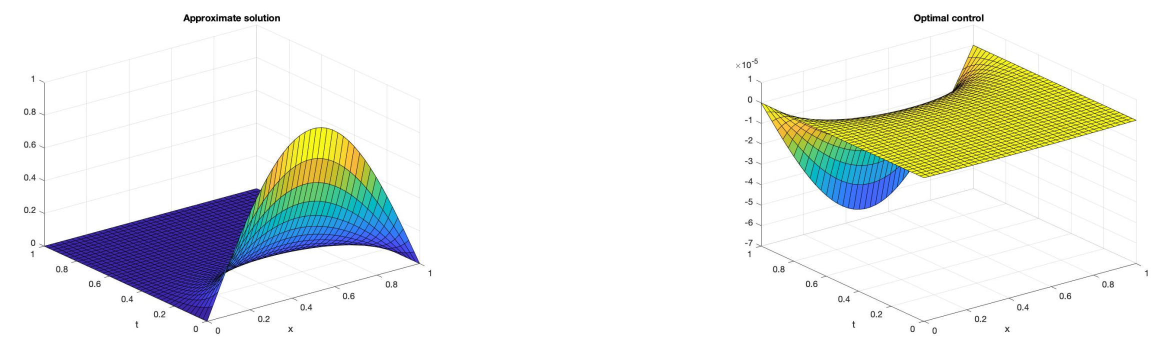
FIG. 1. *Comparison between $y(T)$ and $\hat{y}$.*

Here, we choose $\Delta t,\ h$ and $N = 1/\Delta t$. Using MOA algorithm, we compute $u_h^n, n = 1, 2, \ldots, N$ and then plot the graph of numerical results for $N = 40$. In Fig. 1, we plot the graph of the approximated state at time $T = 1$ and the given final state $\hat{y} = \exp(-\pi^2)\sin(\pi x)$ corresponding to the approximated optimal control u^*. Solution profile of the exact solution $y(t,x) = \exp(-\pi^2 t)\sin(\pi x)$ corresponding to the control function $u(t,x) = -t\exp(-\pi^2 t)\sin(\pi x)$ is shown in Fig. 2. Fig. 3 shows the solution profile of the optimal solution corresponding to the optimal control computed by using the MOA algorithm.



(a) (b)

FIG. 2. *Profile of exact solution with the control function* $u(t,x) = -t\exp(-\pi^2 t)\sin(\pi x)$.



(a) (b)

FIG. 3. *Profile of the approximate optimal solution with respect to the computed optimal control.*

5. Conclusion. In this work, the approximate controllability of the parabolic integro-differential equation is proved under a set of sufficient conditions. We note that such problems require a specific approach as the kernel $B(t,s)$ is unbounded. We develop this approach to obtain the optimal control u^*. In this approach, we have shown that the set of admissible controls U_δ is nonempty, and the approximate controllability is proved using the Banach fixed point theorem. We also proved approximation theorems, which guaranteed the convergence of the numerical scheme to the optimal pair sequence and presented a numerical example to validate our main theoretical results. Note that the system (1.1) is considered as a more general linear integro-differential equations, and the approximate controllability results are proved neither using multiplier techniques nor continuation arguments.

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REFERENCES

[1] V. Barbu and M. Iannelli. Controllability of the heat equation with memory. *Differential and Integral Equations*, 13(10-12):1393–1412, 2000.

[2] F. W. Chaves-Silva, X. Zhang, and E. Zuazua. Controllability of evolution equations with memory. *SIAM Journal on Control and Optimization*, 55(4):2437–2459, 2017.

[3] R. F. Curtain and A. J. Pitchard. *Infinite dimensional linear systems, Lecture Notes in Control and Information Sciences*, volume 8. Springer-Berlin, Germany, 1978.

[4] R. F. Curtain and H. J. Zwart. *An Introduction to Infinite Dimensional Linear Systems Theory*. Springer, 1995.

[5] J. H. Cushman and T. R. Ginn. Nonlocal dispersion in media with continuously evolving scales of heterogeneity. *Transport in porous media*, 13(1):123–138, 1993.

[6] G. Dagan. The significance of heterogeneity of evolving scales to transport in porous formations. *Water Resources Research*, 30(12):3327–3336, 1994.

[7] X. Fu, J. Yong, and X. Zhang. Controllability and observability of a heat equation with hyperbolic memory kernel. *Journal of Differential Equations*, 247(8):2395–2439, 2009.

[8] R. Glowinski, J. L. Lions, and J. He. *Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach (Encyclopedia of Mathematics and its Applications 117)*. Cambridge University Press, 2008.

[9] S. Guerrero and O. Y. Imanuvilov. Remarks on non controllability of the heat equation with memory. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(1):288–300, 2013.

[10] A. Halanay and L. Pandolfi. Lack of controllability of the heat equation with memory. *Systems & control letters*, 61(10):999–1002, 2012.

[11] S. Ivanov and L. Pandolfi. Heat equation with memory: Lack of controllability to rest. *Journal of Mathematical analysis and applications*, 355(1):1–11, 2009.

[12] M. C. Joshi and R. K. Bose. *Some topics in nonlinear functional analysis*. John Wiley & Sons, 1985.

[13] J. U. Kim. Control of a plate equation with large memory. *Differential and Integral Equations*, 5(2):261–279, 1992.

[14] J. U. Kim. Control of a second-order integro-differential equation. *SIAM journal on control and optimization*, 31(1):101–110, 1993.

[15] A. Kumar, M. C. Joshi, and A. K. Pani. On approximation theorems for controllability of non-linear parabolic problems. *IMA Journal of Mathematical Control and Information*, 24(1):115–136, 2007.

[16] M. Kumar and N. Sukavanam. Controllability of mixed Volterra–Fredholm type integrodifferential third order dispersion equations. *Journal of Applied Analysis*, 21(1):1–7, 2015.

[17] I. Lasiecka. Boundary control of parabolic systems: finite-element approximation. *Applied Mathematics and Optimization*, 6(1):31–62, 1980.

[18] I. Lasiecka. Ritz–Galerkin approximation of the time optimal boundary control problem for parabolic systems with dirichlet boundary conditions. *SIAM journal on control and optimization*, 22(3):477–500, 1984.

[19] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations: continuous and approximation theories. I, volume 74 of Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2000.

[20] R. Lavanya and K. Balachandran. Controllability results of linear parabolic integrodifferential equations. *Differential and Integral Equations*, 21(9-10):801–819, 2008.

[21] C.-G. Lefter and A. Lorenzi. Approximate controllability for an integro-differential control problem. *Applicable Analysis*, 91(8):1529–1549, 2012.

[22] H. Leiva, N. Merentes, and J. Sanchez. Approximate controllability of a semilinear heat equation. *International Journal of Partial Differential Equations*, 424309, 2013.

[23] P. Loreti and D. Sforza. Reachability problems for a class of integro-differential equations. *Journal of Differential Equations*, 248(7):1711–1755, 2010.

[24] L. Pandolfi. The controllability of the Gurtin-Pipkin equation: a cosine operator approach. *Applied Mathematics and Optimization*, 52(2):143–165, 2005.

[25] L. Pandolfi. Riesz systems and controllability of heat equations with memory. *Integral Equations and Operator Theory*, 64(3):429–453, 2009.

[26] A. K. Pani and R. K. Sinha. Error estimates for semidiscrete Galerkin approximation to a time dependent parabolic integro-differential equation with nonsmooth data. *Calcolo*, 37(4):181–205, 2000.

[27] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer, 1983.

[28] M. Renardy, W. J. Hrusa, and J. A. Nohel. *Mathematical problems in viscoelasticity*, volume 35. Longman Scientific & Technical, 1987.

[29] W. Shen, L. Ge, and D. Yang. Finite element methods for optimal control problems governed by linear quasi-parabolic integro-differential equations. *International Journal of Numerical Analysis & Modeling*, 10(3), 2013.

[30] W. Shen, L. Ge, D. Yang, and W. Liu. Sharp a posteriori error estimates for optimal control governed by parabolic integro-differential equations. *Journal of Scientific Computing*, 65(1):1–33, 2015.

[31] Q. Tao, H. Gao, B. Zhang, and Z. Yao. Approximate controllability of a parabolic integrodifferential equation. *Mathematical Methods in the Applied Sciences*, 37(15):2236–2244, 2014.

[32] V. Thomée and N. Y. Zhang. Error estimates for semidiscrete finite element methods for parabolic integro-differential equations. *Mathematics of Computation*, 53(187):121–139, 1989.

[33] J. Wang and W. Wei. Controllability of integrodifferential systems with nonlocal initial conditions in Banach spaces. *Journal of Mathematical Sciences*, 177(3):459–465, 2011.

[34] X. Zhou and H. Gao. Interior approximate and null controllability of the heat equation with memory. *Computers & Mathematics with Applications*, 67(3):602–613, 2014.

[35] Xiuxiang Zhou. Approximate and null controllability for the multidimensional Coleman–Gurtin model. *Mathematics of Control, Signals, and Systems*, 33(2):279–295, 2021.