Opinion Dynamic Games under One Step Ahead Optimal Control

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***Abstract*—This paper generalizes two recently proposed opinion dynamics models with control. The generalized model is made up of a standard model of agents interacting with each other, to which affine controls are added. The controls, influencing the opinions of agents, are exercised by entities called players, who specify targets, possibly conflicting, for agents. Three play procedures, sequential, parallel, and asynchronous are defined. Each player has knowledge of the current state of all agents, but *no other information about the other players*. We design the player controls using one step ahead optimization leading to the following novel results: easily computable controls for each player only dependent on its own information; conditions for convergence to the Nash equilibrium, and formulas for the latter.**

***Index Terms*—Opinion dynamics, De Groot model, Friedkin-Johnsen model, Hegselmann-Krause model, control, dynamic game, Jacobi iteration, parallel iteration, Gauss-Seidel iteration, sequential iteration, randomized Gauss-Seidel iteration, asynchronous iteration, one step ahead optimal control**

I. Introduction

Due to globalization and the ease of interpersonal interactions, such as online social networks and the Internet of Things (IoT), individual opinion constantly changes with the increase in the information received. This constant change affects not only the individual as a single point but the entire network around him. Understanding the dynamics involved in this kind of change is a fundamental process to analyzing society's interest in a subject, consequently opening up the possibility of carrying out control. The study of opinion dynamics and its models is widely used in several areas, including finance, group decision-making, and politics.

The literature on opinion dynamics models is extensive, starting with French [1], modeling the influence of interpersonal relationships of individuals on their opinions. In 1974, DeGroot [2] generalized this model to one in which each agent or individual has its own opinion, and is linked to other agents or nodes of a weighted graph $G = (V, E)$ that represents the social network connecting these agents. The weights model the extent of influence of each agent's neighbors. At each step, agent i interacts with neighboring agent j and updates its opinion based on a weighted average of its current opinion and the current neighbor's opinion with weight p_{ij}. In the last decade, several models have been discussed, such as Hegselmann-Krause (HK) model [3], Friedkin-Johnsen (FJ) model [4], Altafini model (antagonistic interactions) [5], DeGroot-Friedkin model [6], continuous opinions and discrete actions (CODA) models [7], informed agent models [8], and Markovian agent models [9]. Insightful presentations of the main opinion dynamics models and basic theory can be found in [10] and [11]. A comprehensive survey on modeling and analysis of dynamic social networks was carried out in [12], [13] and, more recently, [14] discusses recent trends and future challenges.

More recently, there has been interest in game-theoretic models of external control of opinion dynamics in social networks. These models introduce a network of agents, with opinions subject to one of the well known dynamic models, typically the de Groot model, and, in addition, a (smaller) number of entities, called players. Each player influences some or all of the agents, attempting to move some or all of the opinions to pre-specified target values. Veetaseveera et al. [15] introduce a model of opinion dynamics in which the opinion of agents in a social network is influenced by other agents, as usual, but also by two players (called marketers) who compete with each other. Varma et al. [16] in a related paper, describe a model in which opinion dynamics in a social network of two populations (called conformists and contrarians) with opposite beliefs (opinions) are influenced by an external entity called a marketer. Along these lines, Jiang et al. [17] introduce a game-theoretic model of opinion dynamics with control. Each agent (node) is associated with a (possibly empty) subset of players trying to influence the agent's opinion. The overall dynamics, considering the evolution of agent opinions under the influence of players, is assumed to be linear. Each player also has a payoff (objective function). The goal of control is to make the final opinion of all agents as close to the desired one as possible with minimum control costs.

This paper makes the following contributions: (i) it proposes a unified model of opinion dynamics with control that includes the models studied in [15]–[17]; (ii) it proposes the use of one step ahead optimal control (OSAOC) (recently introduced in [18]) with a quadratic performance index, showing that this approach provides a simple feedback control, that is tractable, analytically (for the de Groot and FJ models) as well as computationally (for the de Groot, FJ and HK models); (iii) it proposes a control in which each player uses only its own information to compute its optimal strategy, in contrast with [15]–[17] which all use the Riccati framework and require knowledge of the best response strategies of all adversaries; (iv) it highlights the importance of clearly defining the game playing procedure, showing the differences between the Jacobi and Gauss-Seidel procedures, studied in a general dynamic game context in [19], [20], and also introduces the more

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realistic randomized Gauss-Seidel procedure.

II. Opinion dynamics games and one step ahead optimal control

In order to make this paper self-contained, the definitions of opinion dynamics models, dynamic games and one step ahead optimal control are briefly recapitulated. We begin with a general model of opinion dynamics (between n agents) with control being exercised by p players who influence the opinions of selected agents.

A. General opinion dynamics model

A model of opinion dynamics consists of n agents, modeled as nodes of a directed graph, with the edges representing connections between pairs of agents. The weight of each edge incident on a node (agent) models the extent to which this node takes the opinions of neighbors into account when updating its own opinion. In order to encompass the most popular models, we denote the vector of agent opinions at instant k as $\mathbf{x}(k) \in \mathbb{R}^n$ and write the opinion updating dynamics as follows.

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k)) \tag{1}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a function defined in accordance with the model that we wish to describe (details will appear below). Generalizing [15], [17], we now define opinion dynamics with affine control $\mathbf{u} \in \mathbb{R}^p$ as follows.

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k)) + \mathbf{B}\mathbf{u}(k) \tag{2}$$

where $\mathbf{B} = (b_{ij}) \in \mathbb{R}^{n\times p}$ represents the existence and strength of the influence exerted by players on agents. Specifically, $b_{ij} > 0$ implies that player j who chooses control u_j influences the opinion of the ith player by the term $b_{ij}u_j$. In the sequel, the entries b_{ij} will all be assumed to belong to the interval $[0, 1]$, while the controls u_j can take positive or negative values.

B. Dynamic Games

Informally, a dynamic game consists of the following:

1) State Variables: These variables describe the current state of the game, which evolves as players make decisions. In this paper, the state at instant k is the vector $\mathbf{x}(k)$ of agent opinions.
2) Control Variables: These are variables that describe the decisions made by the players in the game. In this paper, the controls are the entries of the vector $\mathbf{u}$.
3) Outcome Variables: These are variables that describe the overall outcome of the game, which is typically determined by the state of the game after all players have made their decisions. In this paper, outcome variables are payoffs (performance indices) of each player.
4) Information structure: This details the information which each player has about agent states and, possibly, about the controls of the other players, at the instant it has to compute its next control.
5) Game playing procedure: This describes the order of play in which each player computes and applies its control.

Three procedures, with different information structures, are studied in this paper: parallel or Jacobi (J), in which all players have the same information about the previous states of all other agents and apply their controls simultaneously, causing all agent states to be updated; sequential or Gauss-Seidel (GS) in which an order is specified, and each player has access to the updated states of agents resulting from the preceding players in the specified order, when it computes its controls and updates all agent states; and finally, asynchronous or randomized Gauss-Seidel (RGS), in which the order is specified randomly, for each round of updates, from one iteration to the next.

The following notation is introduced to describe a general opinion dynamics game with n agents and p players. At instant k, the vector $\mathbf{x}(k) = (x_i(k)) \in \mathbb{R}^n$ is the vector of agent opinions, $\mathbf{u}(k) = (u_m(k)) \in \mathbb{R}^p$ is the vector of player controls, $f_m(\mathbf{x}(k), \mathbf{u}(k))$, describes the opinion dynamics of the mth agent, and $J_m(\mathbf{x}, u_m)$ is the payoff or performance index of the mth player. Player m is required to choose his controls from a feasible set U_m. If the players are labeled 1 through p, then, for Gauss-Seidel procedures, a permutation π of the integers 1 through p defines a play order, with the ith player in the order being the one who has label $\pi(i)$. A fixed play order can be defined for the entire game (standard Gauss-Seidel), or, a different play order π_k for the kth round of updates of the p players (randomized Gauss-Seidel).

More formal descriptions of dynamic games can be found in [18], [19].

C. One Step Ahead Optimal Control (OSAOC)

One step ahead optimal control (OSAOC), as defined in [18], can be described in the current dynamic game context as follows. Given the current value of the agent states, each player, in the order specified by the information structure and play procedure, computes its optimal control by optimizing the performance index only for the next step, as shown in (3). This control is then applied to the system, generating the updated state. This process continues with each player updating its state, until the end of the time horizon is reached. Since the updated state from each player is incorporated into the optimization of the subsequent states, this defines a state feedback scheme, unlike the traditional optimal control approach, in which the optimal controls are computed over the entire time horizon. In fact, the state feedback proposed in [15] is computed from an infinite horizon model, iteratively using the Riccati equation. Similarly, in [17], the infinite horizon approach with discounting and the resulting feedback control from the Riccati equation is used. The play procedure is not specified, but appears to be a Jacobi one (i.e., simultaneous update of all agent states by all players). In what follows, specific comparisons will be made, but we observe here that the proposed one step ahead optimal control approach is considerably simpler, both conceptually and computationally. In the standard control context (i.e., not in a game-theoretic setting), one step ahead optimal control was introduced under the name greedy control in [21].

D. Opinion dynamic game under OSAOC

Mathematically, the general description of an opinion dynamics game, with n agents and p players, for a fixed play order π, can be written as follows, for the mth player to update, at instant k:

$$\begin{aligned} \text{minimize} \quad & J_{\pi(m)}(\mathbf{x}(k+1), u_{\pi(m)}) \\ \text{s.t.} \quad & \mathbf{x}(k+1) = f(\mathbf{x}(k), u_{\pi(m)}(k)) \\ & u_{\pi(m)}(k) \in U_{\pi(m)}, \qquad \forall k \geq 0, \end{aligned} \tag{3}$$

When all p players have updated, in parallel, in the Jacobi case; or in some (random) sequential order in the (randomized) Gauss-Seidel case, one round of the game is said to have been completed and the iteration proceeds from the kth to the $(k+1)$th instant.

Finally, we observe that the Jacobi game playing procedure is not very realistic, since it assumes that all agents *simultaneously* update their states as a function of player inputs. Thus, in this paper, although we will derive results for both the Jacobi and Gauss-Seidel cases, we will emphasize the Gauss-Seidel procedure in the numerical examples and present the Jacobi case, only in comparison with earlier results in the literature, which all use only the Jacobi procedure.

III. The de Groot model with control (dGc)

We start with an analysis of the de Groot model with control. Let $\mathbf{A}$ be a given $n \times n$ row stochastic matrix. Suppose that the influence of p players on the opinion x_i of the ith agent ($i = 1, \ldots, n$) is given by an $n \times p$ matrix denoted $\mathbf{B}$, the columns of which are denoted $\mathbf{b}_i, i = 1, \ldots, p$ and the vector of agent opinions is denoted $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Each player's influence or control action is denoted by u_i and the vector of control actions by $\mathbf{u} \in \mathbb{R}^p$. Then the generalized de Groot opinion dynamics game involving n agents being influenced by p players is given by:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}, \mathbf{B} \in \mathbb{R}^{n \times p}$.

Equation (4) can also be written as:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \sum_{i=1}^{p} \mathbf{b}_i u_i \tag{5}$$

which makes it clearer that there are p players that compete to influence the agents' opinions. This formulation is the one used in the context of general dynamic games in [22].

We assume that the ith player has a set of targets or goals that he wishes each agent to attain, denoted by the vector $\mathbf{g}_i$. We denote the $n \times p$ matrix which has columns $\mathbf{g}_i$ as $\mathbf{G}$.

For the Jacobi game playing procedure, we assume that, at instant k, player i has access to all agent states at instant k. The one step ahead index for the ith player at instant k, denoted $J_i(k)$, is then defined in the standard way for quadratic indices, with γ_i being the control weight:

$$J_i(k) = (\mathbf{x}(k+1) - \mathbf{g}_i)^T (\mathbf{x}(k+1) - \mathbf{g}_i) + \gamma_i u_i^2 \tag{6}$$

Remark 1: In [17], the assumption is that each player defines the same target for every agent that he influences, i.e., if we denote this single target by $\bar{x}_i$, then $\mathbf{g}_i$ looks like $(0, \ldots, \bar{x}_i, \bar{x}_i, 0, 0, \bar{x}_i, \ldots)$, where the targets $\bar{x}_i$ are placed at the positions of the agents influenced by player i. In [15], a distinction is made between uniform broadcasting ($\mathbf{B}$ is a matrix of ones and all agents receive the same control and player i wishes to impose the target $\bar{x}_i$ on all agents, or the case of targeted broadcasting, in which $\mathbf{B}$ is the identity matrix and the control can be designed for each agent. Note that all the cases discussed in [15], [17] can be modeled using the proposed model (4), (6) with appropriate choices of $\mathbf{B}, \mathbf{g}_i$.

A. The dGc model under OSAOC: Jacobi procedure

In order to proceed with the computation of the one step ahead optimal control, we define the Jacobi procedure. In this procedure, we assume that the ith player optimizes its index J_i using the same state vector $\mathbf{x}(k)$, for all i. When all players have computed their optimal controls u_i, the control vector $\mathbf{u} = (u_1, \ldots, u_p)$ is applied to the right hand side of (4) and the next state computed, to be used in the next round of the Jacobi game.

The following notation is needed to state the main result.

$$\tilde{\mathbf{P}}_{\mathbf{b}_i} = \frac{\mathbf{b}_i \mathbf{b}_i^T}{\mathbf{b}_i^T \mathbf{b}_i + \gamma_i} \tag{7}$$

$$\mathbf{A}_{cl}^J = \left(\mathbf{I} - \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i} \right) \mathbf{A}. \tag{8}$$

The main result for the dGC model under the Jacobi procedure can now be stated.

Theorem 1: If $\mathbf{A}_{cl}^J$ has spectral radius strictly less than one, then the de Groot model dynamics (4) using OSAOC (3), under the Jacobi game playing procedure, is asymptotically stable and opinions converge to the Nash equilibrium point $\mathbf{x}^*$ defined as follows.

$$\mathbf{x}^* = \left(\mathbf{I} - \mathbf{A}_{cl}^J\right)^{-1} \left(\sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i} \mathbf{g}_i \right) \tag{9}$$

Proof: The partial derivatives of J_i are as follows.

$$\frac{\partial J_i}{\partial u_i} = 2\mathbf{b}_i^T (\mathbf{A}\mathbf{x} + \mathbf{b}_i u_i - \mathbf{g}_i) + 2\gamma_i u_i \tag{10}$$

Setting all partial derivatives to zero yields the one step ahead optimal control:

$$u_i^{os}(k) = \frac{\mathbf{b}_i^T (\mathbf{g}_i - \mathbf{A}\mathbf{x}(k))}{\mathbf{b}_i^T \mathbf{b}_i + \gamma_i} \tag{11}$$

Substituting $\mathbf{u}^{os} = (u_i^{os})$ into (4), yields the closed-loop dynamics:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}^{os}(k) \tag{12}$$

$$= \mathbf{A}\mathbf{x}(k) + \sum_{i=1}^{p} \frac{\mathbf{b}_i^T(\mathbf{g}_i - \mathbf{A}\mathbf{x}(k))}{\mathbf{b}_i^T\mathbf{b}_i + \gamma_i}\mathbf{b}_i \tag{13}$$

$$= \mathbf{A}\mathbf{x}(k) + \sum_{i=1}^{p} \frac{\mathbf{b}_i\mathbf{b}_i^T}{\mathbf{b}_i^T\mathbf{b}_i + \gamma_i}(\mathbf{g}_i - \mathbf{A}\mathbf{x}(k)) \tag{14}$$

$$= \mathbf{A}\mathbf{x}(k) - \sum_{i=1}^{p} \left(\tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{A}\mathbf{x}(k)\right) + \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{g}_i \tag{15}$$

$$= \left(\mathbf{I} - \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\right)\mathbf{A}\mathbf{x}(k) + \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{g}_i \tag{16}$$

The fixed point version of (16) immediately yields (9) and the assertion that the stable equilibrium is a Nash equilibrium follows from [19]. □

We now interpret theorem 1. If $\gamma_i = 0$, then $\tilde{\mathbf{P}}_{\mathbf{b}_i}$ is the orthogonal projector, denoted $\mathbf{P}_{\mathbf{b}_i}$ onto the ith control direction $\mathbf{b}_i$ of the ith player. The expression $\mathbf{g}_i - \mathbf{A}\mathbf{x}(k)$ represents the deviation of the next open-loop state ($\mathbf{A}\mathbf{x}(k)$) from the ith player's desired goal $\mathbf{g}_i$. Thus, if $\gamma_i = 0$, then the second term on the right hand side of (14) represents the sum of p projections of "residual error" $\mathbf{r}_i := \mathbf{g}_i - \mathbf{A}\mathbf{x}(k)$ onto the respective control direction $\mathbf{b}_i$. If γ_i is small, then the interpretation is approximately true and $\gamma_i > 0$ small means that the ith player can use control inputs with only a small penalty. If $\gamma_i = 0$, then player i can use impulsive control, which is not possible in real applications. We refer to $\tilde{\mathbf{P}}_{\mathbf{b}_i}$, with small γ, as an approximate projection operator. The equation (14) can be written as

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{r}_i \tag{17}$$

which, in words, means that one step ahead optimal control modifies the open-loop dynamics ($\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$) by adding, to the right hand side of the open-loop dynamics, the sum of approximate projections of the tracking errors $\mathbf{r}_i$ onto the respective control directions $\mathbf{b}_i$.

The steady-state residual error $\mathbf{r}_i^* = \mathbf{x}^* - \mathbf{g}_i$, under one step ahead optimal control applied by each player can be computed as follows:

$$\mathbf{r}_i^* = \left(\mathbf{I} - \left(\mathbf{I} - \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\right)\mathbf{A}\right)^{-1}\left(\sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{g}_i\right) - \mathbf{g}_i \tag{18}$$

The following lemma will be useful in interpreting the behavior of OSAOC, especially in the case of the Gauss-Seidel procedure.

Lemma 1: Assume that player i influences only agent i and sets a target only for this agent. For γ_i sufficiently small, starting from any state $\mathbf{x}(k)$, if only the ith player applies its control, then the ith component of the next state $\mathbf{x}(k+1)$, denoted $x_i(k+1)$, is approximately equal to the target g_i.

Proof: The assumptions are that $\mathbf{b}_i = b_i\mathbf{e}_i$, where $\mathbf{e}_i$ is the ith canonical basis vector (1 as the ith entry, zero otherwise), and $\mathbf{g}_i = g_i\mathbf{e}_i$. Substituting these values in (13), for any $\mathbf{x}(k)$, yields:

$$x_i(k+1) = \frac{\gamma_i}{b_i^2 + \gamma_i}\mathbf{a}_i^T\mathbf{x}(k) + \frac{b_i^2}{b_i^2 + \gamma_i}g_i \approx g_i \tag{19}$$

for γ_i sufficiently small, and b_i sufficiently large. □

Lemma 2: Assume averaging dynamics on a complete graph connecting n agents. Suppose that for $i = 1, \ldots, q \leq p$, player i influences only agent i (i.e., $b_i \neq 0$) and sets a target only for this agent. Then, under OSAOC, for γ_i sufficiently small and b_i sufficiently large, the opinions of the influenced agents ($i \leq q$) tend to their stipulated targets, while the opinions of uninfluenced agents ($i > q$) tend to a common consensus value.

Proof: In this case, $\mathbf{A} = \frac{1}{n}\mathbf{1}\mathbf{1}^T$, $\mathbf{b}_i = b_i\mathbf{e}_i$, $\mathbf{g}_i = g_i\mathbf{e}_i$ and, denoting the average as $\bar{x}(k)$, and $\eta_i = \frac{\gamma_i}{b_i^2+\gamma_i}$, (19) can be written as:

$$x_i(k+1) = \eta_i\bar{x}(k) + (1-\eta_i)g_i, \ i \leq q \tag{20}$$

$$x_i(k+1) = \bar{x}(k), \ i > q \tag{21}$$

Thus the closed-loop dynamics in matrix form is:

$$\mathbf{x}(k+1) = \mathbf{D}\mathbf{A}\mathbf{x}(k) + \mathbf{w} \tag{22}$$

where $\mathbf{D} = \mathrm{diag}(\eta_1, \ldots, \eta_q, 1, \ldots, 1)$, and $\mathbf{w} = ((1-\eta_1)g_1, \ldots, (1-\eta_q)g_q, 0, \ldots, 0)$. Using Cor. 3, it follows that $\rho(\mathbf{D}\mathbf{A}) < 1$, guaranteeing convergence to the closed-loop Nash equilibrium. The remaining assertions follow from (20), (21). □

B. The dGc model under OSAOC: Gauss-Seidel procedure

In the Gauss-Seidel procedure, first an update order is established. If the players are labeled 1 through p, then the update order is a permutation π of the positive integers 1 through p. The Gauss-Seidel update procedure stipulates that, at iteration k, following the given update order, the first player with label $\pi(1)$ updates the state $x(k)$ applying one step ahead optimal control - in other words, applies the control defined in (11). This state is passed onto the next player $\pi(2)$, who also uses (11) (with the state updated by the previous player), until player $\pi(p)$ is reached. Then, since all p players have updated, the next iteration $(k+1)$th is started. In order to express this mathematically, the following notation is introduced. State updates at iteration k are subscripted by (k) and superscripted by the label of the player updating the state. Observe that one update step of the Gauss-Seidel procedure for the ith player has the same form as (16). Thus, starting at state $\mathbf{x}(k)$ at iteration k, one round of Gauss-Seidel updates, leading to the next state $\mathbf{x}(k+1)$ can be written as follows:

$$\mathbf{x}_{(k)}^{\pi(1)} = (\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(1)}})\mathbf{A}\mathbf{x}(k) + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(1)}}\mathbf{g}_{\pi(1)} \tag{23}$$

$$\mathbf{x}_{(k)}^{\pi(2)} = (\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(2)}})\mathbf{A}\mathbf{x}_{(k)}^{\pi(1)} + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(2)}}\mathbf{g}_{\pi(2)} \tag{24}$$

$$\mathbf{x}_{(k)}^{\pi(3)} = (\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(3)}})\mathbf{A}\mathbf{x}_{(k)}^{\pi(2)} + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(3)}}\mathbf{g}_{\pi(3)} \tag{25}$$

$$\vdots \quad = \quad \vdots$$

$$\mathbf{x}_{(k)}^{\pi(p-1)} = (\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-1)}})\mathbf{A}\mathbf{x}_{(k)}^{\pi(p-2)} + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-1)}}\mathbf{g}_{\pi(p-1)} \tag{26}$$

$$\mathbf{x}_{(k)}^{\pi(p)} = (\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p)}})\mathbf{A}\mathbf{x}_{(k)}^{\pi(p-1)} + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p)}}\mathbf{g}_{\pi(p)} \tag{27}$$

$$\mathbf{x}(k+1) = \mathbf{x}_{(k)}^{\pi(p)} \tag{28}$$

We define

$$\mathbf{A}_{cl}^{GS} = \left(\prod_{i=1}^{p}[(\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(i)}})\mathbf{A}]\right) \tag{29}$$

$$\mathbf{d}^{GS} = \sum_{m=1}^{p-1}\left(\prod_{i=1}^{m}[(\mathbf{I} - \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-i+1)}})\mathbf{A}]\right)\tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p-m)}}\mathbf{g}_{\pi(p-m)} + \tilde{\mathbf{P}}_{\mathbf{b}_{\pi(p)}}\mathbf{g}_{\pi(p)} \tag{30}$$

Theorem 2: If $\mathbf{A}_{cl}^{GS}$ has spectral radius strictly less than one, then the de Groot dynamics (4) using OSAOC (3), under the Gauss-Seidel game playing procedure, is asymptotically stable and opinions converge to the Nash equilibrium point $\mathbf{x}^*$ defined as follows.

$$\mathbf{x}^* = \left(\mathbf{I} - \mathbf{A}_{cl}^{GS}\right)^{-1}\mathbf{d}^{GS} \tag{31}$$

Proof: From (23)-(28) it follows that the closed-loop dynamics is given by:

$$\mathbf{x}(k+1) = \mathbf{A}_{cl}^{GS}\mathbf{x}(k) + \mathbf{d}^{GS} \tag{32}$$

The fixed point version of (32) immediately yields (31) and the assertion that the stable equilibrium is a Nash equilibrium follows from [19]. □

Remark 2: Since matrix multiplication is not commutative, the spectral radius of the closed loop matrix depends on the permutation π, i.e., on the order of the Gauss-Seidel updates. For a fixed horizon, the proof of theorem 2 can be adapted for the randomized GS procedure, using permutations $\pi^{(k)}$ dependent on iteration k.

C. The dGc model under OSAOC: Randomized Gauss-Seidel procedure

We will use a simple game consisting of 2 agents and 2 players to illustrate the importance of permutation in the RGS procedure. Consider:

- Two agents with averaging dynamics: $\mathbf{A} = \frac{1}{2}\mathbf{1}\mathbf{1}^T$.
- Two players {1,2}: $\mathbf{b}_1 = \mathbf{e}_1$ and $\mathbf{b}_2 = \mathbf{e}_2$.
- Control cost $\gamma_i = 0.01, i = 1, 2$.
- Target values: $\mathbf{g}_1 = 0.8\mathbf{b}_1$, $\mathbf{g}_2 = 0.1\mathbf{b}_2$, i.e., $g_1 = 0.8, g_2 = 0.1$.
- Initial opinions: $\mathbf{x}_0 = (0.5, 0.5)$.

Without permutation (i.e., in the order (12)), player 2 evaluates its control after player 1 makes its move (modifying the state $x(k)$). By Lemma 1, for small γ_2, player 2, acting on this modified state, makes the second coordinate approximately equal to g_2. Figure 1 shows this: player 2 almost achieves its target, while player 1 does not.

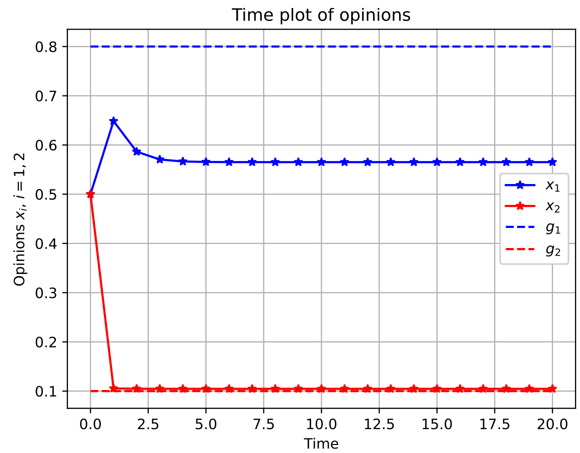

Figure 1. Opinion Dynamics for 2 agents and 2 players, using the sequential Gauss-Seidel procedure with fixed sequential update order 1,2.

Since adversarial players are unlikely to coordinate their actions in the real world, the use of the RGS procedure is more realistic. In fact, Figure 2 shows that, under RGS, agent opinions oscillate near the stipulated targets, with oscillations arising from the randomization of the order of play.

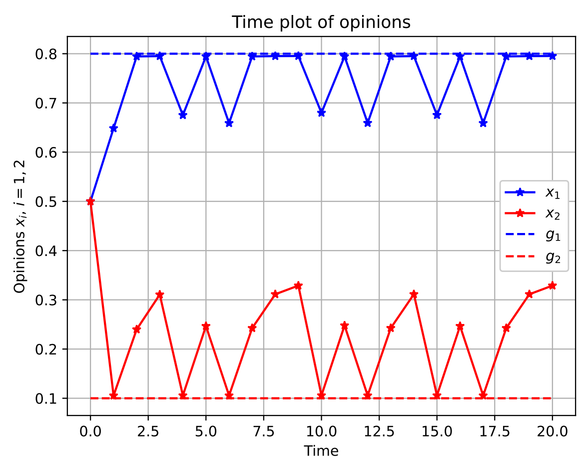

Figure 2. Opinions for 2 agents with RGS procedure. The randomized play order is (12)(21)(21)(12)(21)(12)(21)(21)(21)(12)(21)(12)(21) for the first thirteen rounds.

As shown in the inset of the phase plane plot in Figure 3, starting from the initial state $\mathbf{x}_0$, when round one in order (1,2) is complete, the second player reaches its target for agent 2. For round 2, in order (2,1) target 1 is attained, and so on. After a few rounds, opinions approach a limit cycle of high order, alternating between points on the two dotted target lines (but not attaining the intersection point of these lines), as shown in the main plot in Figure 3.

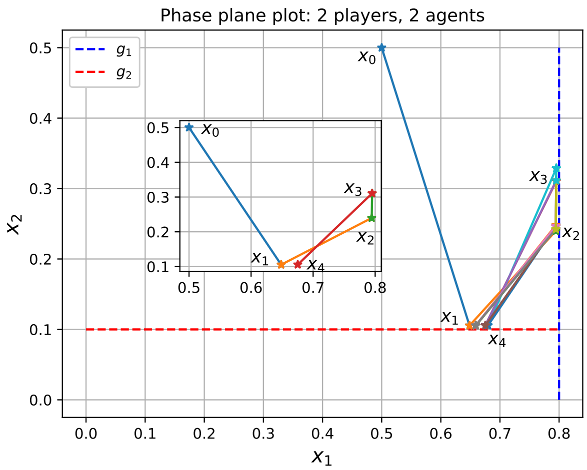

Figure 3. Phase plane plot for 20 instants, showing the approach to a high order limit cycle. The inset shows the initial portion of the randomized play order, i.e., (12)(21)(21)(12).

IV. The Friedkin-Johnsen model with control (FJc)

The Friedkin-Johnsen model with affine control (FJc) is a generalization of the de Groot model (dGc), written as follows.

$$\mathbf{x}(k+1) = (\mathbf{I} - \mathbf{\Theta})\mathbf{A}\mathbf{x}(k) + \mathbf{\Theta}\mathbf{x}_0 + \mathbf{B}\mathbf{u}(k) \qquad (33)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is row stochastic, $\mathbf{B} \in \mathbb{R}^{n \times p}$, $\mathbf{x}_0$ is the vector of initial opinions and Θ is the diagonal stubbornness matrix. In the absence of control, the FJ model uses a convex combination of the usual de Groot update rule and the initial opinion vector. In case $\theta_i = 1$, then the i-th agent will always hold the same opinion (i.e., *is completely stubborn*); if $\theta_i = 0$, the agent update rule is the same as the de Groot model; if $\theta_i \in (0, 1)$, the agent is said to be partially stubborn and the update occurs according to the convex combination.

We define

$$\mathbf{A}_{FJ} = (\mathbf{I} - \mathbf{\Theta})\mathbf{A} \qquad (34)$$

$$\mathbf{A}_{cl}^{FJ} = \left(\sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\right)\mathbf{A}^{FJ} \qquad (35)$$

$$\mathbf{d}^{FJ} = \left(\mathbf{I} - \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\right)\mathbf{\Theta}\mathbf{x}_0 + \sum_{i=1}^{p} \tilde{\mathbf{P}}_{\mathbf{b}_i}\mathbf{g}_i \qquad (36)$$

Theorem 3: If $\mathbf{A}_{cl}^{FJ}$ has spectral radius strictly less than one, then the FJc dynamics (33) using OSAOC (3), under the Jacobi game playing procedure, is asymptotically stable and opinions converge to the Nash equilibrium point $\mathbf{x}^*$ defined as follows.

$$\mathbf{x}^* = \left(\mathbf{I} - \mathbf{A}_{cl}^{FJ}\right)^{-1}\mathbf{d}^{FJ} \qquad (37)$$

The proof is entirely analogous to that of Theorem 1 and is omitted here. The theorem for FJc under OSAOC using the Gauss-Seidel procedure is also similar to Theorem 2 and is omitted here for brevity.

V. The Hegselmann-Krause model with control (HKc)

The well-known Hegselmann-Krause (HK) model [3] assumes that each agent has a certain level of confidence in its own opinion and is willing to change it to try to match the opinions of neighboring agents. In terms of the adjacency matrix $\mathbf{A}$ with entries a_{ij}, the HK model with affine control (HKc) can be written, for the ith agent, as follows.

$$x_i(k+1) = x_i(k) + \frac{1}{\vartheta_i}\sum_{j=1}^{N} a_{ij}\left[\Phi(x_i, x_j)(x_j - x_i)\right] + \mathbf{b}_i u_i, \qquad (38)$$

where $\Phi(x_i, x_j)$ is a threshold indicator function (a pulse) that dictates whether the opinion of agent j is close enough to that of agent i to be considered in the update, and $\vartheta_i = \sum \Phi(x_i, x_j)$, and $\mathbf{B}, \mathbf{u}$ are defined as in the previous section.

The presence of the function $\Phi(\cdot, \cdot)$ causes the model to become nonsmooth, so we define a smooth sigmoid-based function (that approximates Φ well and facilitates the use of numerical optimization to compute OSAOC) as follows.

$$\Phi(x_i, x_j) = \left(1 - \frac{1}{1 + \mathrm{e}^{\mu(d_{ij} + w_i)}}\right)\left(\frac{1}{1 + \mathrm{e}^{\mu(d_{ij} - w_i)}}\right), \qquad (39)$$

where $\mu \in \mathbb{R}^+$ is the slope of the sigmoid function, $d_{ij} \in \mathbb{R}$ is the difference between opinions of agents j and i at that instant $(x_j(k) - x_i(k))$, and $w_i \in \mathbb{R}^+$ is the confidence bound, i.e., $|d| > w_i \to \Phi = 0$ and $\Phi = 1$ otherwise, for sufficiently large μ. It is possible to consider asymmetry between positive and negative differences. To do so, just consider w_i^- for the lower limited confidence bound (first term) and w_i^+ for the upper one (second term), and if the confidence bound is the same for both, $w_i^- = w_i^+ = w_i$.

For the HKc model under OSAOC, analytical expressions for the closed-loop system and its equilibrium point are difficult to derive, but the OSAOC nonlinear program (3) is just as easy to implement as its versions for the de Groot and FJ models, leading to the numerical examples presented in subsequent sections.

VI. Numerical examples

This section gives some numerical examples of each of the three types of models under OSAOC, using the Jacobi and Gauss-Seidel procedures, respectively.

A. Comparison with the model of Jiang et al. [17]

Jiang et al. [17, sec.4.1] give an example of a 10 agent, 4 player game with the following data:

- Ten identical agents form a social network which is a complete graph on 10 nodes with averaging dynamics (i.e., $\mathbf{A} = \frac{1}{10}\mathbf{1}\mathbf{1}^T$)
- Players $1, 2, 3, 4$ influence agents $1, 4, 6, 9$, i.e., $\mathbf{b}_1 = \mathbf{e}_1, \mathbf{b}_2 = \mathbf{e}_4, \mathbf{b}_3 = \mathbf{e}_6$, $\mathbf{b}_4 = \mathbf{e}_9$.
- Control cost $\gamma = 0.01$ for all players.
- Target values are $(0.5, 0.7, 0.2, 0.3)$, i.e., $\mathbf{g}_1 = 0.5\mathbf{e}_1, \mathbf{g}_2 = 0.7\mathbf{e}_4, \mathbf{g}_3 = 0.2\mathbf{e}_6$, $\mathbf{g}_4 = 0.3\mathbf{e}_9$.

- Init. opinions: $(0.5, 0.7, 0.4, 0.4, 0.8, 0.7, 0.9, 0.6, 0.3, 0.5)$
- Final opinions: $(1.909, 0.425, 0.425, 5.967, 0.425, -4.028, 0.425, 0.425, -2.049, 0.425)$.

For the purposes of comparison, this example is revisited, using the proposed OSAOC method and the Figures 4, 5 should be compared with [17, Figs.1-3]. Note that, in the initialization in [17, sec.4.1], the opinions of agents 1 and 9 are already at the desired target values, so in this paper, the initial opinions of these agents were changed to $x_1^{\text{init}} = 0.1$ and $x_9^{\text{init}} = 0.9$

Desired targets	0.5	0.7	0.2	0.3
Equil.values [17]	1.91	5.97	-4.03	-2.05
Equil.values (this paper)	0.5	0.7	0.2	0.3
Equil. controls [17]	1.8	5.8	-4.2	-2.2
Equil. controls (this paper)	0.07	0.27	-0.22	-0.12

Table I
COMPARISON OF RESULTS FOR EXAMPLE IN [17, SEC.4.1]. THE TARGETS AND THE EQUILIBRIUM VALUES IN ROWS 1 THROUGH 3 OF TABLE I ARE FOR AGENTS $1, 4, 6, 9$ AND THE CONTROLS FOR PLAYERS $1, 2, 3, 4$.

From Table I, it is clear that the OSAOC method proposed in this paper is able to drive agent opinions very close to the desired targets (which follows from lemma 1), as opposed to the method proposed in [17]. Moreover, the control effort used is lower by an order of magnitude. In [17], agents without direct player influence (e.g., 2,3,5,7,8,10) reach a consensus value 0.425, and the same result is achieved using OSAOC.

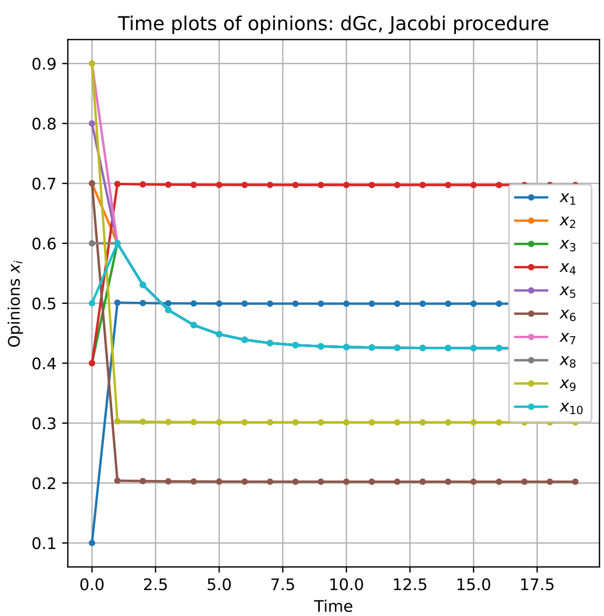

Figure 4. Opinion Dynamics for the example [17, sec.4.1] using OSAOC and Jacobi procedure.

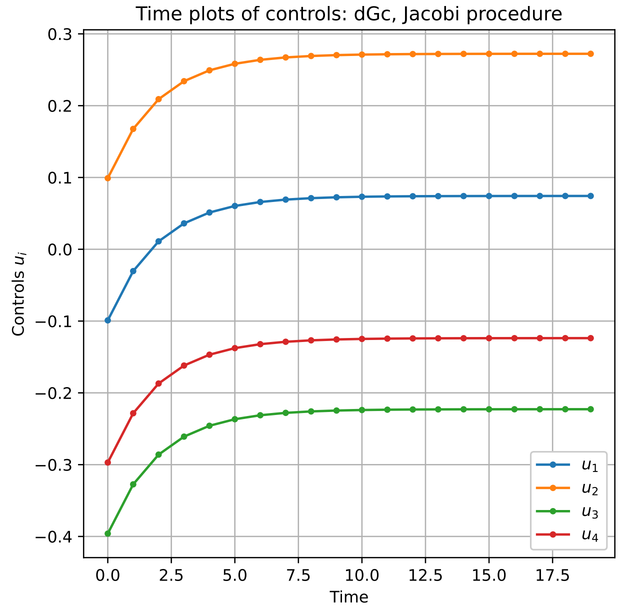

Figure 5. OSAO Controls of players for DeGroot model and Jacobi procedure.

B. FJc and HKc dynamics under OSAOC: Jiang et al.[17] example

For this section, the model used in subsection (VI-A) is used again for simulations with FJc and HKc. For FJc, the stubbornness matrix was chosen as $\Theta = \text{diag}(0.8, 0.2, 0.4, 0, 0.8, 0, 0.3, 0.4, 0.5, 0)$. Figure 6 shows the evolution of the agent opinions.

Unlike the DeGroot model, the FJ model contains stubborn agents whose opinions do not converge due to players' influence in the absence of control. However, OSAOC is able to drive opinions to their respective targets (see Table II).

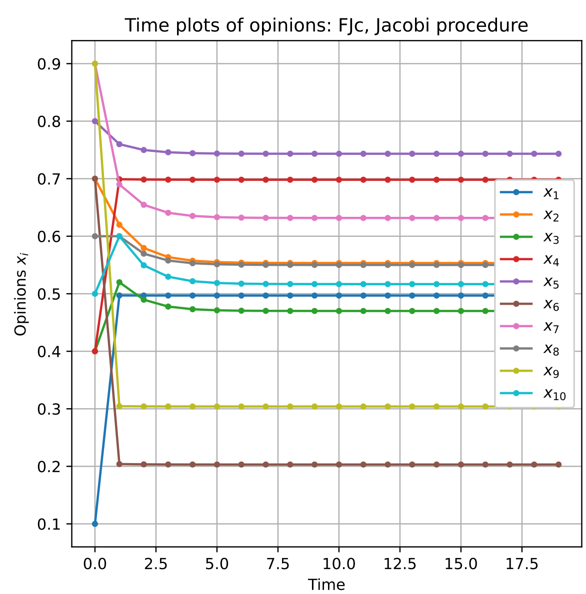

Figure 6. Opinion Dynamics for FJ model with OSAOC and Jacobi procedure.

For the HK model, the parameters of Φ were chosen as $\mu = 90$ and $w = 0.3$, leading to the evolution of opinions shown in Figure 7. The opinions take longer to converge to equilibrium with these values, and, once again, all stipulated targets are achieved. When the parameter w is changed to 0.4, the opinions dynamics have a short transient into five clusters of opinions (see Figure 8). Despite the differences in dynamics between this example (HKc) and the previous one (FJc), players in both cases drive the opinions to the desired targets (Table II), the only difference is in their control efforts.

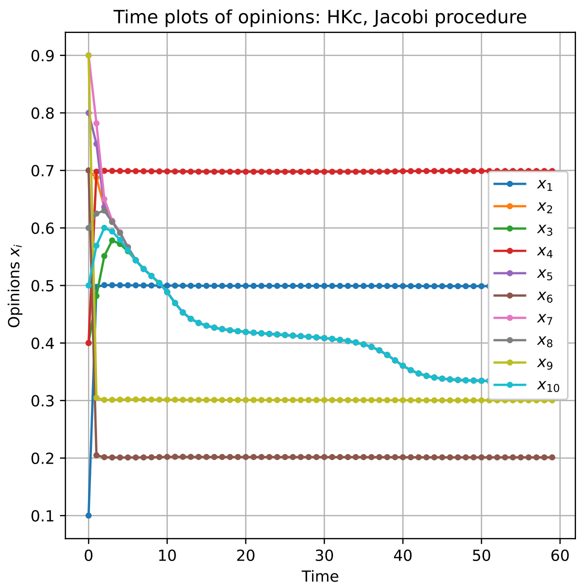

Figure 7. Opinion Dynamics for HK model with OSAOC, with $w = 0.3$, and Jacobi procedure

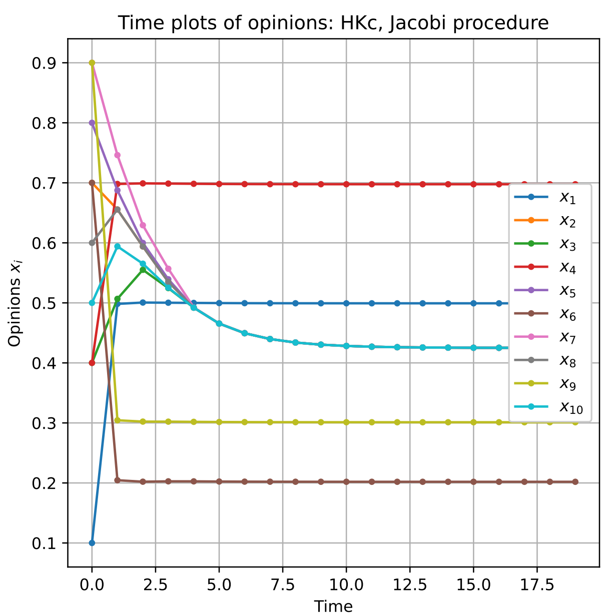

Figure 8. Opinion Dynamics for HK model with OSAOC, with $w = 0.4$, and Jacobi procedure

C. Comparison with the results of Veetaseveera et al. [15]

In [15], the example in Fig. 9 is introduced and studied in cases referred to as targeted advertising (when controls

Desired targets	0.5	0.7	0.2	0.3
Equil.values for FJc	0.50	0.69	0.20	0.30
Equil.values for HKc	0.49	0.69	0.20	0.30
Equil.values for HKc	0.49	0.69	0.20	0.30
Equil. controls for FJc	0.00	0.18	-0.31	-0.10
Equil. controls for HKc	0.12	0.10	-0.12	-0.03
Equil. controls for HKc	0.07	0.24	-0.19	-0.11

Table II
COMPARISON OF RESULTS FOR FJC AND HKC

and targets can be chosen for each agent in the network) and uniform broadcasting (in which all agents in the network receive the same control). In [15, p.257], it is stated that targeted advertising has an advantage if nodes with higher centrality are prioritized (the first four are, in order, for Fig. 9, nodes 1,5,9,2). They also state that if there are two players and both apply targeted advertising, with player 2 having control weight twice as large as player 1, then opinions converge to $(2.48, 0.96, 0.55, 0.42, 1.69, 0.02, -0.86, 0.2055, 1.19, -0.14)$. Parameters are specified in [15] as follows.
- The initial condition is $x_0 = (1, 2, -3, 0, 6, -5, 4, 3, -2, 4)$
- The targets are 2 for player 1 and -2 for player 2.
In our approach to this problem, we used the following parameters: - $\gamma_1 = 0.01, \gamma_2 = 2\gamma_1$ (as specified in [15]).
- Player 1 targets nodes 5 and 9, i.e., $\mathbf{b}_1 = \mathbf{e}_5 + \mathbf{e}_9$.
- Player 2 targets nodes 1 and 2, i.e., $\mathbf{b}_2 = \mathbf{e}_1 + \mathbf{e}_2$.
- The target vectors chosen as $\mathbf{g}_1 = 2\mathbf{b}_1$, $\mathbf{g}_2 = -2\mathbf{b}_2$, with these choices based on the centrality measures given above. With this choice of parameters, the results of applying the RGS procedure are shown in figure 10. Note that each player is able to drive the targeted agent opinions to mean values close to the desired target values (2 for agents 5 and 9 and -2 for agents 1 and 2. The untargeted agent opinions go to a consensus close to -2. This is to be contrasted with convergence to $(2.48, 0.96, 0.55, 0.42, 1.69, 0.02, -0.86, 0.2055, 1.19, -0.14)$ cited in [15].

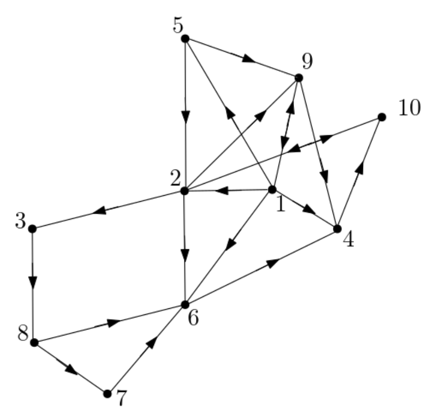

Figure 9. Directed graph of ten agents – redrawn based on [15, Fig. 1] for comparison.

For this example, the Jacobi procedure drives the agents 5 and 9 to a neighborhood of 1 and all other agents to a consensus of 0.

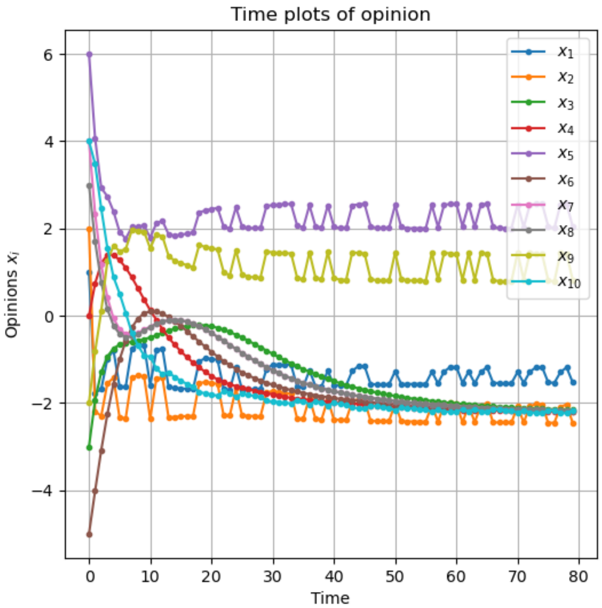

Figure 10. Opinion dynamics for targeted advertising on example in [15], using the randomized Gauss-Seidel procedure.

In addition, it is worth pointing out that the optimal control derived in [15] for each player depends on complete knowledge of the other player's control strategy, as well as the repeated solution of two algebraic Riccati equations, since each optimal control depends on these solutions. This is to be contrasted with each player's OSAOC strategy which depends only on the knowledge of its own parameters (targets $\mathbf{g}_i$, and influence coefficients $\mathbf{b}_i$) and does not need any information at all about the strategies of its adversaries. In addition, the OSAOC computation is based on matrix-vector products, which is much simpler and faster than solving Riccati equations repeatedly.

D. Inversion of polarization in a two cluster network

This section considers a two player, ten agent network with two clusters of five agents, shown in Figure 11. Agents 1 to 5 in cluster 1 have initial opinions positive and close to 1, while agents 6 to 10 have initial opinions negative and close to -1. Player 1 influences nodes 1 to 5 as well as node 7, while player 2 influences nodes 6 to 10, as well as node 3. The disputed nodes were chosen based on the out-degree centrality of the graph, with these nodes 3 and 7 being the largest using this centrality parameter. The objective of each player is to flip the opinion of the nodes it influences, thus player 1 has target -2 for nodes 1 through 5 and 7, while player 2 has target 2 for nodes 4, and 6 to 10. Results for the FJc using the RGS and GS procedures are presented below.

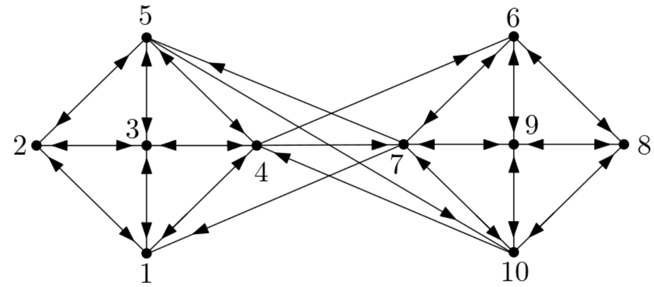

Figure 11. Directed graph of ten polarized agents in 2 clusters: agents $\{1, 2, 3, 4, 5\}$ have initial opinions close to 1 and agents $\{6, 7, 8, 9, 10\}$ have initial opinions close to -1. Two players target these clusters with the objective of flipping the polarization.

The parameter values described above are as follows.

- $\mathbf{x}_0 = (1, 0.7, 1, 0.9, 1, -1, -0.8, -1, -0.8, -1)$
- $\mathbf{b}_1 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_7$
- $\mathbf{b}_2 = \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8 + \mathbf{e}_9 + \mathbf{e}_{10} + \mathbf{e}_3$
- Goals: $\mathbf{g}_1 = -2\mathbf{b}_1$, $\mathbf{g}_2 = 2\mathbf{b}_2$
- Control cost $\gamma_i = 0.01, i = 1, 2$.

For the FJc model with RGS procedure, the stubbornness matrix used was $\Theta = \text{diag}(0.5, 0.3, 0, 0.7, 0, 0.1, 0, 0.3, 0, 0.6)$. Figure 12 shows the evolution of opinions over time. Note that both players are reasonably successful in flipping cluster opinions. As expected, agents 4 and 7 have the most significant oscillations in opinions since both players are disputing these agents.

For the FJc model with the Jacobi and Gauss-Seidel procedures, using the same stubbornness matrix, opinions evolve smoothly to the targeted inverted polarization (the Figures are not shown here for lack of space, but are available on the GitHub site (see sec. VII).

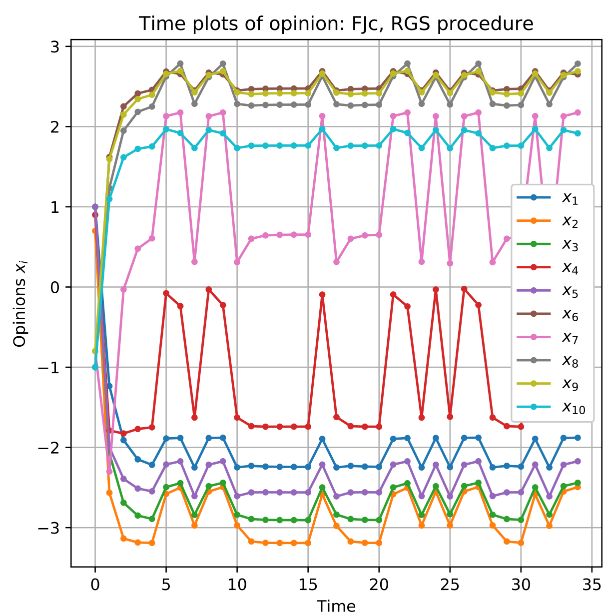

Figure 12. Opinion Dynamics for FJc model with OSA control and RGS procedure.

VII. Codes and GitHub

All codes are available on GitHub for download and experimentation by the interested reader. Extra comparisons not included here for lack of space can also be found on the GitHub site.

VIII. Concluding remarks

This paper has shown that a one step ahead optimal control approach with a sequential, parallel or asynchronous game playing procedure leads to easily computable and effective controls for players who influence agents' opinions connected by a weighted directed graph. To implement the proposed control, each player needs only global state information which is a standard assumption in all existing methods, but *no information on the policies of other players*. This result substantially improves existing results which use a Riccati equation framework and thus require much more computation and also require each player to have full knowledge of the controls used by its adversaries. Another novel feature introduced in this paper is the introduction of asynchronous game playing procedures which are more realistic than the synchronous parallel procedures, or sequential procedures following a fixed update order. Future work will investigate the use of delayed and noisy information, as well as the use of norms other than the 2-norm in the player performance indices.

IX. Background material on nonnegative matrices

Let $A, B \in \mathbb{R}^{n\times n}$. The following notation is used throughout the appendix. $B > 0$ (≥ 0) if al $b_{ij} > 0$ (≥ 0),$A > (\geq) B$, if $A - B > (\geq) 0$.

The following well known theorem [23] has a corollary which will also be useful.

Theorem 4: Let $A, B \in \mathbb{R}^{n\times n}$. Then

$$|A| \leq B \Rightarrow \rho(A) \leq \rho(|A|) \leq \rho(B).$$

Three corollaries of theorem 4 are as follows.

Corollary 1: If $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$.

Corollary 2: If $A \geq 0$ and $\tilde{A}$ is any principal submatrix of A, then $\rho(\tilde{A}) \leq \rho(A)$.

Corollary 3: If $0 \leq A < B$, then $\rho(A) < \rho(B)$.

Corollaries 1 and 2 are easily proved. We prove corollary 3.
Proof of corollary 3: There exists $\alpha > 1$, such that:

$$0 \leq A \leq \alpha A < B.$$

If $\rho(A) \neq 0$, the conclusion follows by corollary 1. If $\rho(A) = 0$, the conclusion follows by corollary 2. □

Acknowledgments

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References

[1] J. R. P. French, "A formal theory of social power," *Psychological Review*, vol. 63, no. 3, pp. 181–194, 1956.

[2] M. H. Degroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, pp. 118–121, 1974.

[3] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis and simulation," *Journal of Artificial Societies and Social Simulation*, vol. 5, pp. 1–33, Jul. 2002.

[4] N. E. Friedkin and E. C. Johnsen, "Social influence and opinions," *The Journal of Mathematical Sociology*, no. 3-4, pp. 193–206, 1990.

[5] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2013.

[6] P. Jia, A. Mirtabatabaei, N. E. Friedkin, and F. Bullo, "Opinion dynamics and the evolution of social power in influence networks," *SIAM Review*, vol. 57, pp. 367–397, 2015.

[7] A. Martins, "Discrete opinion dynamics with M choices," *The European Physical Journal B*, vol. 93, pp. 1–10, Jan. 2020.

[8] E. Ghezelbash, M. J. Yazdanpanah, M. Asadpour, and A. Yaghmaei, "Optimal selection of informed agents for influence opposition," *IEEE Transactions on Computational Social Systems*, vol. 8, no. 1, pp. 20–32, 2021.

[9] P. Bolzern, P. Colaneri, and G. De Nicolao, "Opinion dynamics in social networks: The effect of centralized interaction tuning on emerging behaviors," *IEEE Transactions on Computational Social Systems*, vol. 7, no. 2, pp. 362–372, 2020.

[10] M. O. Jackson, *Social and economic networks*. Princeton, NJ: Princeton University Press, 2008.

[11] F. Bullo, *Lectures on Network Systems*. Kindle Direct Publishing, 2022.

[12] A. V. Proskurnikov and R. Tempo, "A tutorial on modeling and analysis of dynamic social networks. part I," *Annual Reviews in Control*, vol. 43, pp. 65–79, 2017.

[13] A. V. Proskurnikov and R. Tempo, "A tutorial on modeling and analysis of dynamic social networks. part II," *Annual Reviews in Control*, vol. 45, pp. 166–190, 2018.

[14] H. Hassani, R. Razavi-Far, M. Saif, F. Chiclana, O. Krejcar, and E. Herrera-Viedma, "Classical dynamic consensus and opinion dynamics models: A survey of recent trends and methodologies," *Information Fusion*, vol. 88, pp. 22–40, 2022.

[15] J. Veetaseveera, V. S. Varma, and I.-C. Morarescu, "A dynamic game formulation for control of opinion dynamics over social networks," in *Network games, control and optimization*, vol. 1354, 2021, pp. 252–260.

[16] V. S. Varma, B. Adhikari, I.-C. Morarescu, and E. Panteley, "Optimal campaign strategy for social media marketing with a contrarian population," in *Network games, control and optimization*, vol. 1354, 2021, pp. 241–251.

[17] H. Jiang, V. V. Mazalov, H. Gao, and C. Wang, "Opinion dynamics control in a social network with a communication structure," *Dynamic Games and Applications*, vol. 13, pp. 412–434, Mar. 2023.

[18] E. Kaszkurewicz and A. Bhaya, *Business Dynamics Models: Optimization-Based One Step Ahead Optimal Control*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2022.

[19] T. Başar, "Relaxation techniques and asynchronous algorithms for on-line computation of non-cooperative equilibria," *Journal of Economic Dynamics and Control*, vol. 11, no. 4, pp. 531–549, 1987.

[20] A. Bhaya and E. Kaszkurewicz, "Dynamic advertising games in duopolies under one-step-ahead optimal control," *Dynamic Games and Applications*, Jan. 2023. DOI: 10.1007/s13235-023-00489-9.

[21] E. Lavretsky, "Greedy optimal control," in *Proceedings of the American Control Conference (ACC)*, vol. 6, Feb. 2000, pp. 3888–3892.

[22] J. Engwerda, *LQ Dynamic Optimization and Differential Games*. Chichester, England: John Wiley, 2005.

[23] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd. New York: Cambridge University Press, 2012.