The Riemann Zeta function using the Method of Stationary Phase Approximation of the Van der Pol Fourier Integral

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Abstract—The Van der Pol Fourier representation of the Riemann Zeta function in the critical strip is evaluated using the Method of Stationary Phase. The approximation allows us to propose new representation for the zeta function.

***Index Terms*— Riemann Zeta Function, Van der Pol Fourier Representation, Method of Stationary Phase, Non-trivial Zeros**

I. Introduction

Riemann's zeta function is an extension of the Euler sum for complex numbers $s = \sigma + it,\ \sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

Riemann was able to extend the function to the whole complex plane using analytic continuation [1]. With the extended definition of the zeta function, Riemann was able to address fundamental questions related to the distribution of prime numbers using infinitesimal calculus. Riemann's work is the foundation of analytical number theory. The non-trivial zeros of the zeta function are zeros with $0 \le \sigma \le 1$ (also called the critical strip). The claim (hypothesis), made by Riemann more than 160 years ago states that the non-trivial zeros of the zeta function lie on the critical line $\sigma = 1/2$.

Multiple unsuccessful attempts at proving Riemann's claim were made by some of the most accomplished mathematicians. The problem was so challenging that some of the same mathematicians who made fundamental contributions to understanding the zeta function were driven to the point where they believed that the Riemann claim is probably not true [2].

Numerical methods were developed to find any counter example to the claim [3]. All the trillions of zeros that have been computed so far to a convincing numerical precision are found to lie on the critical line, but we still cannot verify that the claim still holds for the zeros that are beyond modern day computers' capabilities. There is a belief that the difficulty of the problem is due to some hidden pattern in the zeta function itself and not because of the lack of analytical or computational tools that have yet to be invented.

The zeta function and Riemann hypothesis have no real-life applications except for a few examples such as the Miller prime numbers test algorithm [4] which is used for data encryption. This was before it was replaced with another efficient algorithm that does not even require the Riemann claim [5]. Contrary to popular belief, the zeta function and Riemann's claim have no applications in physics or engineering except for a few values of the zeta function used for evaluating some infinite sums found in some engineering and physics problems. The natural question that arises is: why should an engineer care about the Riemann claim or the zeta function?

Engineers bring a completely different perspective to problem solving. Van der Pol's work on the zeta function was a prime example of what engineers can bring to the table. Van der Pol is more known in academic engineering departments for his work on non-linear oscillators and not for contributions to analytical number theory. Van der Pol [6] showed that the Riemann's zeta function in the open critical strip $0 < \sigma < 1$ can be written as a Fourier transform:

$$\zeta(s) = -s \int_{-\infty}^{+\infty} e^{-\sigma\omega} \left(e^{\omega} - [e^{\omega}]\right) e^{-i\omega t} d\omega \quad (2)$$

where $[x]$ is the integer part of x (floor function). He proceeded to cutting the sawtooth function inside the integral on a circular disk. By using a motor to rotate the disk at a controlled speed and by shining light in the teeth of the function and measuring the light transmission using a photo sensor, Van der Pol was able to show actual physical measurements of the location of the first few zeros of the zeta function on the critical line. The measured values matched the known numerical values to within the precision allowed by the technology of his era.

Equation (2) appears to be a continuous transformation but (1) is a discrete sum. The apparent difference is misleading because the sawtooth function is not continuous at $\omega = \log(n)$ for all positive integers $n > 1$. An analytic integration of (2) for negative ω and using piece-wise integration between $\log(n)$ and $\log(n+1)$ for positive ω will show that (2) is the same as (1) to within the analytical continuation.

The present study applies two algorithms used commonly to approximate Fourier integrals to evaluate Van der Pol's representation of the zeta function: the first is the Fast Fourier Transform (FFT) and the second is to approximate (2) using the

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Method of Stationary Phase. We will show that while FFT has some limitations due to computational burden, the Method of Stationary Phase is tailored to work for large t which allows us to introduce a new representation of the zeta function in the critical strip.

II. Material and Methods

Analytical and computer simulation tools are used to compare the accuracy of FFT and Method of Stationary Phase in approximating the Van the der Pol Fourier integral. The Method of Stationary Phase is used introduce new representation of the zeta function in the critical strip.

III. Simplified Fourier Integral and FFT

From [7, page 14], we can write the zeta function that is valid for $\sigma > 0$ as

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2} \tag{3}$$

After making the substitution $x = e^\omega$, (3) can be written as

$$\zeta(s) = sI + \frac{1}{s-1} + \frac{1}{2} \tag{4}$$

where

$$I = \int_0^\infty e^{-\sigma\omega}\left([e^\omega] - e^\omega + \frac{1}{2}\right)e^{-i\omega t}d\omega \tag{5}$$

The difference between the positive sawtooth function used in (2) and the more symmetric sawtooth function in (5) allows the Fourier series expansion to be simplified for the upcoming Method of Stationary Phase. The main difference between the two forms is that the integral from (2) is simplified to reduce any numerical errors by computing the integral for negative ω analytically. Also, the simplified form is valid for all $\sigma > 0$ (Note: the critical strip included except for $\sigma = 0$).

There are many numerical methods [3] that can be used to evaluate the zeta function and there are efficient numerical methods that use the FFT method when estimating the location of the non-trivial zeros when combined with the different representations of the zeta function such as the Riemann-Siegel sums [3]. The FFT observations highlighted in this study only apply to estimating the Van der Pol integral.

We are going to focus on FFT with uniform sampling and not on non-uniform sampling algorithms such as NUFFT because, as stated earlier, the best non-uniform sampling strategy we can use to evaluate (5) is to perform piece-wise integration over ω between $\log(n)$ and $\log(n+1)$ for all positive integers $n > 1$. In that case, (5) will be reduced to evaluating the infinite sum as shown in (1) and the Fourier transform is no longer required.

There is no fundamental reason FFT cannot approximate any Fourier transform including the non-continuous decaying sawtooth function if we have enough computing power in terms of memory and computational speed. The precision and computational performance of the FTT approximation is defined by two numbers: ω_{max} that limits the integral upper limit, and $\Delta\omega$ that is the distance between the samples of the exponentially decaying sawtooth function. Truncating the integral to ω_{max} will result in the following upper bound error:

$$E_T \sim e^{-\sigma\omega_{max}}$$

A reasonable ω_{max} can be used to make the truncation error small. A reasonable $\Delta\omega$ must avoid aliasing by being able to capture the smallest tooth in the sawtooth function:

$$\Delta\omega \sim log(e^{\omega_{max}} + 1) - \omega_{max} \sim e^{-\omega_{max}}$$

This number is unreasonably small. The resolution Δt in the t domain where the zeros of the zeta function are evaluated is

$$\Delta t = \frac{2\pi}{\omega_{max}}$$

The maximum value t_{max} that can be computed using FFT is

$$t_{max} = \frac{\pi}{\Delta\omega}$$

Because FFT is discrete, finding the t value of where the zeta function crosses 0 will require some form of interpolation between the discrete FFT points, but it was proven in [7] that for large t, the non-trivial zeros of the zeta function form a dense set where the distance between consecutive zeros goes to 0. Because of that, a very large ω_{max} and a very small $\Delta\omega$ must be used to study asymptotic zeros which will be a severe computational limit on the FFT approximation in terms of memory usage and execution time. However, with modest computational resources, as we are going to show in the experimental section of this study, FFT can still be used to study the zeta function in the critical strip for small values of t where the error in locating the zeros is not very costly.

Van der Pol did not have to deal will all the FFT limitations because his electromechanical device was the equivalent of an analog computer. The device was limited by other factors such as mechanical tolerances, the precision of the motor rotational speed, and the consistency of the light source and the photo detector, but the device did not have to deal with the FFT approximation issues stated previously except for the truncation error due to a finite ω_{max} which is common to both approaches.

IV. The Method of Stationary Phase

A. Problem definition

We are going to apply the Method of Stationary Phase to approximate the integral shown in equation (5). The method was already used in [8] to show a simple derivation of the Riemann-Siegel sums which is the main method used for numerical analysis of non-trivial zeros. And to the best of our knowledge, the method was never used to approximate the Van der Pol integral directly.

We use the following Fourier serries representation of the sawtooth function:

$$[x] - x + \frac{1}{2} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{sin(2\pi nx)}{n}$$

After swapping the integral and the sum, equation (5) can be written as:

$$I = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty} e^{-\sigma\omega}\, sin(2\pi n e^{\omega}) e^{-i\omega t} d\,\omega \qquad (6)$$

We also use the following relation to complete the phase representation of the integral:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Equation (6) can be written as

$$I = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\big(I_1(n) - I_2(n)\big) \qquad (7)$$

where the integrals that need to be approximated are

$$I_1(n) = \int_0^{\infty} e^{-\sigma\omega}\, e^{i\varphi_n(\omega)} d\omega \qquad (8)$$

and

$$I_2(n) = \int_0^{\infty} e^{-\sigma\omega}\, e^{-i\psi_n(\omega)} d\omega \qquad (9)$$

with the associated phases

$$\varphi_n(\omega) = 2\pi n e^{\omega} - \omega t$$
$$\psi_n(\omega) = 2\pi n e^{\omega} + \omega t$$

The stationary phase is the value of ω where the derivative of the phase with respect to ω is zero. For positive ω where the integral is defined and for $t > 0$, $\varphi_n(\omega)$ has a single stationary point at

$$\omega_0(n) = \log\left(\frac{t}{2\pi n}\right) \qquad (10)$$

However, $\psi_n(\omega)$ does not have any stationary points. Because of that, the Method of Stationary Phase will be used to approximate $I_1(n)$ and the Method of Non-Stationary Phase will be used to evaluate $I_2(n)$.

The fact that both approximations must hold under the infinite sum (7) is a challenge that does not exist for the regular Method of Stationary Phase. Additionally, the standard method, as illustrated by equation (11), rely on an independent global phase gain parameter k that controls the approximation error.

$$\int_R g(x)\, e^{ikf(x)} dx \qquad (11)$$

But as shown in equations (8) and (9), the only parameter that can be made arbitrarily large is t. The lack of common phase gain is an additional complication that we must navigate.

B. Stationary phase integral

The Method of Stationary Phase works because the integral main contribution is at the stationary point. The stationary phase approximation of (11) when the phase has a single stationary point x_0 as shown in [10]:

$$\int_R g(x)\, e^{ikf(x)} dx = g(x_0) e^{ikf(x_0) + sign(f''(x_0))i\pi/4}\sqrt{\frac{2\pi}{k|f''(x_0)|}} + O(k^{-\frac{3}{2}})$$

We approximate φ_n around the stationary point $\omega_0(n)$ using:

$$\varphi_n(\omega) = \; t - t\,log\left(\tfrac{t}{2\pi n}\right) + \tfrac{t}{2}\left(\omega - log\left(\tfrac{t}{2\pi n}\right)\right)^2 + \cdots \quad (12)$$

After applying the Method of Stationary Phase using (12), we get:

$$I_1(n) \approx \left(\frac{t}{2\pi n}\right)^{-\sigma} e^{i\left(t - t\,log\left(\frac{t}{2\pi n}\right) + \frac{\pi}{4}\right)} \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}}$$

or equivalently

$$I_1(n) \approx e^{i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\sigma-it}\left(\frac{1}{n}\right)^{-\sigma-it}$$

The infinite sum over n can be replaced with a finite sum

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\big(I_1(n)\big) \approx \frac{1}{2\pi i} e^{i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\sigma-it}\sum_{n=1}^{\left[\frac{t}{2\pi}\right]}\left(\frac{1}{n}\right)^{1-\sigma-it}$$

where $[x]$ is the integer part of x (floor function). The infinite sum is truncated to $[t/(2\pi)]$ because the stationary points (10) are only valid for positive ω where the integral is defined.

Luckily, the parameter t in (12) is acting like the standard gain k shown in (11) because all the derivatives $\frac{d^m}{d\omega}\varphi_n(\omega)$ at the stationary points are equal to t for $m > 1$. We also showed that the infinite sum is replaced with a finite one. Because t is acting like k, the approximation error for the Method of Stationary Phase for each $I_1(n)$ should decay by $O\left(t^{-\frac{3}{2}}\right)$ for large t [10]. The error of the sum over n should be $O\left(t^{-\frac{1}{2}}\right)$ because we are simply multiplying $O\left(t^{-\frac{3}{2}}\right)$ by the number of elements in the finite sum $[t/(2\pi)]$. The stationary part can be written as:

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\big(I_1(n)\big) = \frac{1}{2\pi i} e^{i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\sigma-it}\sum_{n=1}^{\left[\frac{t}{2\pi}\right]}\left(\frac{1}{n}\right)^{1-\sigma-it} + O\left(t^{-\frac{1}{2}}\right)$$

C. Non-stationary part

The terms $I_2(n)$ are hard to estimate analytically and the only course of action is to estimate how much error is added if they are approximated to zero for all n.

The standard way to estimate errors for the Method of Non-Stationary Phase is to apply the Van der Corput's lemma [9, 10] or simply to integrate (9) by parts after dividing by the non-zero derivative $\dot{\psi}_n(\omega)$ of the phase $\psi_n(\omega)$:

$$I_2(n) = \left[\frac{e^{-\sigma\omega} e^{-i\psi_n(\omega)}}{-i\dot{\psi}_n(\omega)}\right]_0^{\infty} - \int_0^{\infty}\frac{d}{d\omega}\left(\frac{e^{-\sigma\omega}}{-i\dot{\psi}_n(\omega)}\right) e^{-i\psi_n(\omega)} d\omega$$

And since $\dot{\psi}_n(\omega) = 2\pi n e^{\omega} + t$, we can write

$$I_2(n) = -\frac{i}{2\pi n + t} - i\int_0^{\infty} \frac{d}{d\omega}\left(\frac{e^{-\sigma\omega}}{2\pi n e^{\omega} + t}\right) e^{-i\psi_n(\omega)} d\omega$$

After taking the absolute value, we can write

$$|I_2(n)| \le \frac{1}{2\pi n + t} + \int_0^{\infty} \left|\frac{d}{d\omega}\left(\frac{e^{-\sigma\omega}}{2\pi n e^{\omega} + t}\right)\right| d\omega$$

The next idea is based on Van der Corput's lemma.

Since $\frac{e^{-\sigma\omega}}{2\pi n e^{\omega}+t}$ is the product of two decreasing functions for $\omega \ge 0$, $\frac{e^{-\sigma\omega}}{2\pi n e^{\omega}+t}$ is a monotonic function. Therefore, $\frac{d}{d\omega}\left(\frac{e^{-\sigma\omega}}{2\pi n e^{\omega}+t}\right)$ has a fixed sign and we can swap the absolute value and the integral and write

$$|I_2(n)| \le \frac{2}{2\pi n + t}$$

Also, the error on the infinite sum has the following bound:

$$\left|\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\left(I_2(n)\right)\right| \le \frac{2}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{1}{n\left(n + \frac{t}{2\pi}\right)}$$

If we define

$$H(a) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+a}\right)$$

and use the limit

$$\lim_{a\to\infty}\left(H(a) - log(a)\right) = \gamma$$

where γ is Mascheroni's constant. We can write the following approximation for $a \gg 1$:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+a)} \approx \frac{\log(a)}{a}$$

Finally, for large t we get:

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\left(I_2(n)\right) = O\left(\frac{\log(t)}{t}\right)$$

This bound is much smaller than the stationary phase approximation error. The total error will be dominated by the stationary phase approximation $O\left(t^{-\frac{1}{2}}\right)$.

D. Approximation results

In this section, we are going to see how close the approximation is to actual numerical values of the zeta function in the critical strip. By combining the results of the last two sections, the approximation of the Van der Pol integral for large t is:

$$I = \frac{1}{2\pi i} e^{i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\sigma-it}\sum_{n=1}^{\left[\frac{t}{2\pi}\right]}\left(\frac{1}{n}\right)^{1-\sigma-it} + O\left(t^{-\frac{1}{2}}\right) \quad (13)$$

Relation (13) illustrates that we are still computing a similar discrete sum to (1) at $s = \sigma + it$. The work done so far allowed us to truncate the infinite sum and estimate the approximation error for $\sigma > 0$ and large t.

To compute zeta function in the half-closed critical strip $1 \ge \sigma > 0$ and as stated earlier, (13) is combined with the following relation:

$$\zeta(s) = sI + \frac{1}{s-1} + \frac{1}{2}$$

The error in computing $\zeta(s)$ will be the same as (13) because the Method of Stationary Phase can be applied again to approximate the multiplication sI. The MATLAB script shown in Appendix computes I in two ways. The first is using FFT and the second is using relation (13). Fig. 1 and Fig. 2 show the results for both methods for low and large t values.

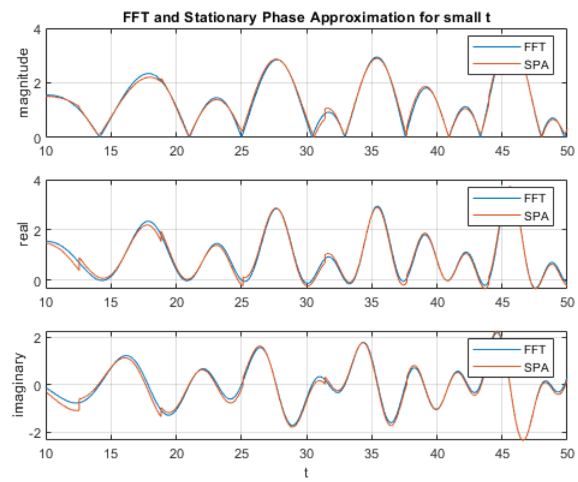

Fig. 1. FFT and Method of Stationary Phase Approximations (SPA) of the zeta function on the critical line ($\sigma = 0.5$) are shown for small t values. It is worth noting that the Method of Stationary Phase is close to the FFT given the fact the method should only be valid for large t. The zeros of the zeta function are the points where the magnitude = 0.

The unexpected observation from the data is that the Method of Stationary Phase gives reasonable results for small t values because the method is only valid for large t. We used the FFT method with ω_{max} = 500 and $\Delta\omega$ = 1.0e-6 to reduce the error for large t as shown in Fig. 2. The first zeros of the zeta function were compared against known values. The error for the first few zeros was 1.0e-5 using polynomial interpolation which is reasonable considering the computational constraints we were working with. But without very large ω_{max} and very small $\Delta\omega$, the FFT method will not be adequate for computing large zeros as stated previously.

Fig. 2. FFT and Method of the Method of Stationary Phase (SPA) approximations of the zeta function on the critical line ($\sigma = 0.5$) are shown for large t values. The difference between the two methods is very small when using FFT parameters ω_{max} = 500 and $\Delta\omega$ = 1.0e-6. The span of the t axis is the same as in Fig. 1. Notice that the number of zeros increase for large t as stated earlier in the study. The magnitude of the zeta function between zeros does not follow any pattern as observed by Van der Pol using his electromechanical device.

Because both methods are valid for $\sigma > 0$, Fig. 3 illustrates the Riemann hypothesis where zeta function does not have zeros (magnitude > 0) if $\sigma \neq 0.5$.

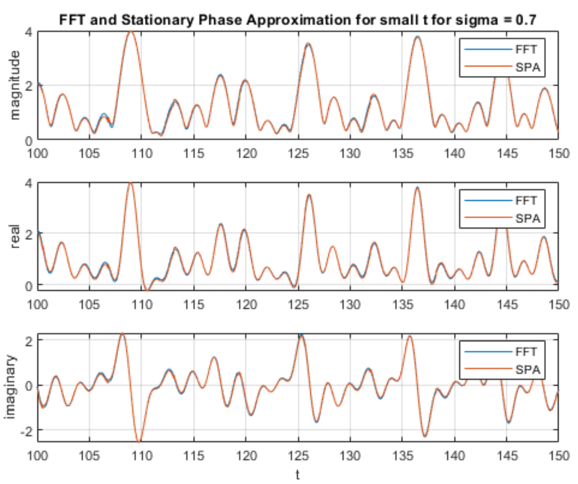

Fig. 3. FFT and Method of Stationary Phase (SPA) approximations of the zeta function away from the critical line. The zeta function magnitude no longer crosses 0, illustrating Riemann's hypothesis.

V. Further simplifications

In this section, we will try further simplifications of the representation of the zeta function using relation (13) in the critical script and on the critical line where Van der Pol performed his measurements.

Because of the Method of Stationary phase can be applied again to approximate the multiplication sI in [4] where s is replaced asymptotically by $s \approx it$ for large t, and if use the fact that $\frac{1}{s-1}$ in [4] decays faster that $O\left(t^{-\frac{1}{2}}\right)$, the zeta function in the critical strip $1 \geq \sigma > 0$ and for large t is:

$$\zeta(s) = \frac{1}{2} + e^{i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it}\sum_{n=1}^{\left[\frac{t}{2\pi}\right]}\left(\frac{1}{n}\right)^{1-\sigma-it} + O\left(t^{-\frac{1}{2}}\right) \quad (14)$$

On the critical line ($\sigma = 1/2$), the zeta function for large t is:

$$\zeta\left(\frac{1}{2} + it\right) = f(t) + O\left(t^{-\frac{1}{2}}\right) \quad (15)$$

with

$$f(t) = \frac{1}{2} + e^{i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-it}\sum_{n=1}^{\left[\frac{t}{2\pi}\right]}\left(\frac{1}{n}\right)^{\frac{1}{2}-it} \quad (16)$$

The function $f(t)$ would have been what Van der Pol would have measured if he could extend his electromechanical device to large t values because as seen in Fig. 4, the zeta function and $f(t)$ are undistinguishable for large t. Using relations (14) and (15) could be a better way to understand the zeta function in the critical strip and on the critical line because of their simplicity.

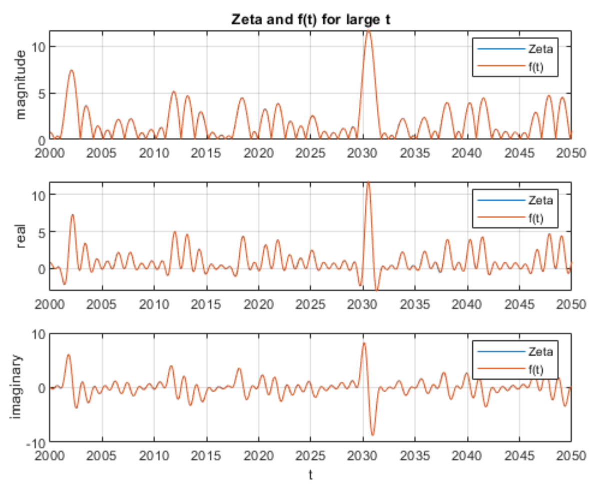

Fig. 4. Zeta function and $f(t)$ are undisguisable for large t.

However, using (16) to compute large zeros is problematic because the Riemann-Siegel sums [8] gets truncated at $\left[\sqrt{t/2\pi}\right]$ but the sum in $f(t)$ is truncated at a much larger value $[t/(2\pi)]$. This could be an expensive tradeoff for large t.

VI. Non-Trivial Zeros

Relation (14) may lack computational efficiency, but its simplicity will allow us to make new analytic predictions about the behavior of the non-trivial zeros for large t.

The complexity of (14) is dominated by the following single term:

$$P = \sum_{n=1}^{\left[\frac{t}{2\pi}\right]}\left(\frac{1}{n}\right)^{1-\sigma-it} \quad (17)$$

However, from (14) and for large t, P can be written as:

$$P = \left(\zeta(s) - \frac{1}{2}\right)e^{-i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}+\sigma+it}$$

At the zeros of the zeta function $\zeta(s) = 0$, we can write:

$$P = -\frac{1}{2}e^{-i\left(t+\frac{\pi}{4}\right)}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}+\sigma+it}$$

After using the absolute value, we get:

$$|P| = \frac{1}{2}\left(\frac{t}{2\pi}\right)^{-\frac{1}{2}+\sigma}$$

The new representation allowed us to simplify a very complicated relation (17) at the zeros of the zeta function. If the Riemann hypothesis is true, then all the non-trivial zeros of the zeta function are at $\sigma = 1/2$ where:

$$|P| = \frac{1}{2}$$

However, the challenge to prove that $\sigma = 1/2$ is the only valid solution remains.

VII. Conclusion

We used the Method of Stationary Phase to approximate the Van der Pol Fourier representation of the zeta function in the critical script. The approximation was used to introduce a new representation of the zeta function. Error analysis and computer simulations demonstrated the usefulness of proposed method in studying the zeta function. The advantage of the proposed method compared with other established computational methods is yet to be determined.

We tried to share proofs and computer scripts to make the ideas accessible not just to mathematicians but also to engineers. We believe that the new simplified representation of the zeta function could open new avenues for studying the zeta function in the critical strip.

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Appendix.

Matlab scripts used in the study

```
sigma = 0.5;
Max_Omega = 500;
delta_Omega = 1.0e-5;
Min_View_Value = 1000;
Max_View_Value = 1050;
jay = sqrt(-1);

%FFT method
Omega = 0:delta_Omega:Max_Omega;
L = numel(Omega);
F_Omega = exp(-sigma*Omega).*(-exp(Omega)+floor(exp(Omega))+1/2);
FFT_t = fft(F_Omega)*delta_Omega;
t = 2*pi*(0:fix(L/2))/Max_Omega;
Half_FFT_t = FFT_t(1:fix(L/2)+1);

%compute s and 1/(s-1)
s= sigma + jay*t;
pole = 1./(s-1);
%Zeta using FFT
Zeta_FFT = Half_FFT_t.*s + pole + 1/2;

%Selet viewing section
[ii,jj]=find(t>Min_View_Value & t < Max_View_Value);
t_FFT = t(jj);
Zeta_FFT = Zeta_FFT(jj);

%Method of Stationary Phase
t = Min_View_Value:pi/Max_Omega:Max_View_Value;
s = sigma + jay*t;
pole = 1./(s-1);
ratio = t/(2*pi);

Zeta_SPA= [];
%Select simplified or regular approximation
Use_Simplified_Verion = true;

if(~Use_Simplified_Verion)
  for i = 1 : length(t)
    n = 1 : floor(ratio(i));
    vect = ratio(i)./n;
    phase = t(i) - t(i)*log(vect) + pi/4;
    vect = 1./n;
    J = vect.^(1-sigma).*exp(jay*phase);
    approximation = -jay/(2*pi)*ratio(i)^(-1/2-sigma)*sum(J);
    %Zeta using Method of Stationary Phase
    Zeta_SPA(i) = approximation*s(i) + 1/2  + pole(i);
  end
else
  %Simplified Zeta function using Method of Stationary Phase
  for i = 1 : length(t)
    n = 1 : floor(ratio(i));
    phase = exp(jay*(t(i) + pi/4));
    sum_value = sum(n.^(sigma-1+jay*(t(i))));
    exp_value = 1/2-sigma-jay*t(i);
    %Simplified Zeta using  Method of Stationary Phase
    Zeta_SPA(i) = 1/2+sum_value*phase*(t(i)/(2*pi))^exp_value;
  end
end
```