Optimal Cost-Effectiveness Analysis of a Mathematical Model of Climate Change Induced by Excessive Emission of Carbon Dioxide

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Abstract

The current crisis of global climate change and its consequences which are manifested in form of different environmental disasters is attributed to excessive emission and accumulation of greenhouse gases in the atmosphere, key among which is carbon dioxide. Hence, remedies are needed to mitigate against this change in climate. A mathematical model on climate change incorporating good conservation policies, enlightenment programmes and direct air capture technology as mitigation measures is formulated and analysed using the concept of optimal control theory and cost-effectiveness analysis. The objective functional is set up to minimize both the excessive concentration of carbon dioxide in the atmosphere and the total cost of implementation of each mitigation measure, as the resources available to cater for the needs of the teeming human population are limited. By formulating a Hamiltonian function and using Pontryagin's Principle, the adjoint equations and characterisation of the optimal units were calculated. Using the optimality control system obtained, the numerical simulation was done in MATLAB using the Forward Backward Sweep algorithm of the Runge-Kutta Method. Seven different strategies of mitigation scenarios were simulated. From the results, each of these strategies has the potency to reduce the excessive concentration of carbon dioxide in the atmosphere. However, the best result was obtained using the strategy that combines all the three mitigation measures of good conservation policies, enlightenment programmes and direct air capture technology. Despite that this strategy (Strategy VII) appears the most desirable option to adopt, the cost of implementation of each strategy has to be considered since human resources are limited. Therefore, cost-effectiveness analysis techniques (Average Cost-Effectiveness Ratio and Incremental Cost-Effectiveness Ratio) were used to arrive at the most cost-friendly strategy. From the computations involving these two ratios, both indicated good conservation policies strategy as the cheapest option to adopt in reducing the excessive concentration of carbon dioxide in the atmosphere.

Keywords: Optimal Control, Cost-Effectiveness Analysis, Climate Change Model, Pontryagin's Principle, Adjoint Equations, Carbon Dioxide Emission, Average Cost-Effectiveness Ratio, Incremental Cost-Effectiveness Ratio.

1 Introduction

The crisis of global climate change and its consequences which are manifested in form of different environmental disasters is attributed to excessive emission and accumulation of greenhouse gases in the atmosphere, key among which is carbon dioxide. There are many sources of emission of greenhouse gases into the atmosphere. One of such is the combustion of fossil fuels which releases carbon dioxide in large amounts into the atmosphere [1]. The combustion of fossil fuels to generate electricity has contributed to emission of gases such as carbon dioxide, nitrous oxide and methane which are negative influencers of the current issue of global warming and climate change [2]. The European 2020 strategy had its focus on decreasing greenhouse gases emissions to about 20%, raising renewable energy share consumption to about 20% and improving energy efficiency to about 20% in a bid to mitigate against climate change [3]. Between 2005 and 2014, greenhouse gases emission rose to about 65% in the atmosphere [1]. There is a projection of increase in the emitted greenhouse gases emission from anthropogenic sources to a value range of about $25 - 90\%$ by 2030 [1]. Current narrative of the negative impact of climate change is evident in the change in patterns of climate variables such as ice, wind, snow, humidity, rainfall, precipitation, temperature and many others [4]. Energy consumption contributes to about 75% of the current greenhouse gases emissions globally [5]. Dating back to the industrial revolution era, there has been about 35% and 148% rise in the concentration of carbon dioxide and methane respectively [2]. Warmer temperatures are being experienced as a result of uncontrolled deforestation activities perpetrated by humans in the quest to build homes, industries and attain some level of civilization. Global warming is majorly caused by excessive emission of the greenhouse gases and the depletion of the ozone layer [2]. Due to the fact that many natural processes such as rainfall, precipitation, snowfall, increase in sea level, hailstorms are influenced by a number of factors, it is difficult to precisely predict the consequences of global warming. Consequences of global warming could result in flooding, drought, crop failure, famine, shrinking glaciers, melting of ice, heat waves, violent rainfall, hailstorms and thunderstorms, rising sea levels, earthquakes, disease outbreak (E.g dengue fever and malaria), migration of wild and aquatic lives from their natural habitats, bush fires and extinction of some species of plants and animals. Global warming can also affect human health due to outbreak of diseases such as dengue fever, malaria, West Nile Virus, Lyme disease which thrive under warmer temperatures [2]. If the current trends in global warming are not properly checked, there could be grave consequences such as climate change, extreme weather events, increase in sea levels, negative natural, environmental and social effects [2]. The issue of excessive carbon dioxide emission and accumulation is a problem common to all countries of the world, the most affected (with the largest emission) been China [6]. Sources of excessive emission of carbon dioxide into the atmosphere include combustion of coal and oil, deforestation, manufacture of cement [2]. In order to mitigate against the increasing patterns of carbon dioxide emissions, it is vitally important for climate experts and policymakers to comprehend the various variables that influence and affect carbon dioxide emission [7]. Increase in energy consumption, growth in trade, industrial expansion, increase in energy consumption and people's income are some of the indices that are used to measure globalization through economic treaties signed by different countries [7]. Adoption and practice of good conservation policies can reduce the excessive emission of the greenhouse gases (most importantly carbon dioxide) in the atmosphere to a considerable concentration levels. Good conservation policies include the use of renewable energy, afforestation, reforestation, recycling, discouragement of unregulated deforestation and forest degradation. Good conservation policies such as planned felling, thinning, reforestation and fire prevention enhances photosynthetic biomass to attain their optimal carbon capturing potential [4]. The replacement of fossil fuel-based energy production with renewable and purer energy sources as viable options can go a long way to reduce the excessive emission of carbon dioxide into the atmosphere. Air pollution and the health risks that come with it can be reduced by substituting a great portion of fossil fuel with renewable domestic resources. Renewable, cleaner and alternative sources of energy like hydro, solar, wind, biomass and geothermal can go a long way to reduce the current emission quagmire experienced globally [2]. Advantages of using cleaner and renewable sources of energy include: they have low or no pollution ability, ecologically friendly and have no danger tendencies towards the balance of the ecosystem. In removing the excessive accumulated carbon dioxide in the atmosphere, sustainable low-temperature adsorbent technology with high efficiency capacity that will meet the energy demand and cost of maintaining direct air capture (DAC) equipment can also be considered [1]. Direct air capture (DAC) and carbon capture and storage (CCS) can remove gaseous carbon dioxide from the atmosphere on a large scale [1]. Also, there is high level of ignorance of people as to the current rise in factors that contribute to climate change and the dangers that come with it that can upset the balance in the ecosystem. Hence, there is need to put in place enlightenment programmes to educate people on the dangers of climate change and possible measures that can be undertaken to reduce activities that contribute to the emission and accumulation of these gases.

Optimal control theory is one of the most commonly applied techniques in mathematical biology to put in control and prevention measures to pressing problems that require optimal results. [8] worked on the mathematical analysis of a model for Chlamydia and Gonorrhea co-dynamics using optimal control theory. They considered treatment as control by adopting four control functions. Their findings were that implementing any of the controls led to a reduction in the total number of individuals co-infected with Chlamydia and gonorrhea. The highest co-infected cases was averted by a strategy that combined implementing female Chlamydia trachomatis treatment and male gonorrhea treatment. [9] made a research on the backward bifurcation and optimal control in a co-infection model for SARS-CoV-2 and ZIKV. They put in SARS-CoV-2, Zika and co-infection prevention strategies as time dependent control variables in the optimal model formulation. Their results showed that the prevention of either SARS-CoV-2 or Zika significantly lowered the burden of co-infection within the given population. [10] studied a mathematical model and optimal control of HIV and COVID-19 co-infection. Using time dependent controls of COVID-19 and HIV prevention interventions and COVID-19 treatment in the optimal model, they found out that HIV prevention measures greatly decreased the burden of co-infection with COVID-19 and treatment of COVID-19 reduced co-infection with HIV/AIDS. The COVID-19 only prevention strategy as well as the HIV only prevention control each averted about 10 500 new cases of co-infection of these two infections. [11] worked on the mathematical model and optimal control of Newcastle disease by considering three control measures. Their results revealed that the combination of effective revaccination programmes and optimal efficacy of the vaccines significantly decreased the population of infectious productive birds while simultaneously enhancing the productivity of the birds. [12] worked on the mathematical modelling, analysis and optimal control of corruption dynamics by incorporating media campaign and punishment as control measures against corruption. His findings revealed that the integrated control measures would greatly reduce the number of corrupt individuals. [13] modelled the dynamics of Campylobacteriosis using non-standard finite difference approach with optimal control by employing some time dependent control measures. Their findings revealed that the best result in combating this infection was obtained by combining the control measures (this combination has the potency of reducing the interactions causing the infection to an appreciable minimum). [14] worked on the fractional modelling and optimal control of COVID-19 transmission in Portugal by considering preventive measures and vaccination as time dependent variables in the optimal model formulation. The cost effectiveness analysis done revealed preventive measures were more effective compared to the vaccination strategy. [15] did a research on optimal control strategies applied to reduce the unemployed population by considering the effect of policy action towards job creation and provision of skilled manpower as control strategies. Their results showed that the unemployed population was maximized as a result of implementation of government policies while the cost of policy making was minimized at the same time. [16] developed a mathematical model of COVID-19 in Ethiopia using optimal control analysis. They introduced time dependent control variables representing prevention, medication and awareness creation into the original model. Their findings revealed that the strategy that combined all the control measures gave the highest efficiency within a short period of time. [17] worked on optimal control analysis of a mathematical model for unemployment by employing provision of employment by government and creation of vacancies as control strategies. Their findings revealed that successful implementation of government policies in creating new vacancies and control programmes greatly reduced the population of unemployed individuals. [18] worked on optimal control model for criminal gang population in a limited-resource setting by incorporating control functions representing crime prevention strategy for the susceptible population and case finding control for the criminal gang population. From their results, employment programmes and job creation, reducing out of school children could lead to a freer and better society. From the cost-effectiveness analysis, the least costly and most effective strategy to dopt in combating crime was arresting and sentencing of criminals for corrective measures. [19] worked on optimal control of an HIV immunology model by considering two different treatment strategies as control measures. His results showed that treatment could reduce the infection but not remove it. [20] worked on optimal control of the coronavirus pandemic with both pharmaceutical and non-pharmaceutical interventions. Their findings revealed that the COVID-19 infection can be greatly reduced by strict and strong compliance with the two strategies considered. [21] worked on optimal HIV treatment by maximising immune response. [22] worked on optimal control analysis of pneumonia and meningitis co-infection by considering four time-dependent controls. Using cost-effectiveness analysis technique, he revealed that prevention only strategy has the highest impact in bringing down the spread of pneumonia and meningitis within a short period compared to the remaining four strategies. Human resources are scarce and limited. Inasmuch as we would like to adopt the best strategy got by combining a number of interventions, the cost of achieving it may not be feasible due to cost constraint. Hence, the need to do a cost analysis on different formulated strategies that would be effective and cheapest to implement.

Cost-effectiveness analysis has been employed in a number of mathematical models to determine the cost of implementing strategies devised as intervention in optimal control problems such as disease outbreak, finance, economics, engineering, environmental disasters, social, moral and psychological issues and many others. [23] worked on a mathematical modelling of anthrax and listeriosis co-dynamics with optimal control. His findings form using cost-effectiveness analysis technique showed that treatment of both infectious humans and vectors has the highest impact compared to other strategies. [24] worked on optimal control strategies and cost-effectiveness analysis of a malaria model. They considered three mitigation measures. From their analysis of the cost of implementation of the different strategies considered, the most cost effective option was a combination of spray of insecticides and treatment of infected individuals. [25] formulated and analysed an optimal control model of maize streak virus pathogen interaction with pest invasion in maize plant. Using cost-effectiveness analysis technique, they revealed that the combination of prevention and quarantine represented the best and most effective choice with reference to costs and health benefits. [26] in their work on the cost-effectiveness analysis on mathematical modelling of HIV/AIDS with optimal control strategy revealed that a combination of prevention and screening gave the most most cost effective strategy that could reduce the spread of the infection. [27] worked on modelling and optimal control of pneumonia disease with cost-effective strategies by incorporating three control measures of education, treatment and screening. Using cost-effectiveness analysis technique, the most cost effective strategy in militating against pneumonia infection was the combination of prevention and treatment. [28] worked on a mathematical model of hepatitis B disease using optimal control theory and cost-effectiveness analysis technique. They considered two time-dependent controls involving prevention and vaccination of newborns. From their results, prevention was adopted as the most cost effective strategy. [29] worked on mathematical model for co-infection of pneumonia and typhoid fever disease with optimal control From his findings, prevention of typhoid fever and treatment of pneumonia was the best bet in militating against the co-infection of individuals with these two diseases. [30] formulated and analysed a mathematical model on anthrax using optimal control theory and cost-effectiveness analysis technique. From their findings, vaccination of animals and prevention by humans are the best options in combating the evolution and progression of anthrax disease. An optimal control problem (OCP) is designed with the sole aim of finding a piecewise continuous control say, $v(t)$, and a state variable say $w(t)$ that minimizes or maximizes a defined objective functional subject to some mathematical constraint (s) [31]. [31] formulated a mathematical model on climate change by incorporating coastal greenbelt and desulfurization as mitigation strategies. Their findings revealed that global warming, climate change and greenhouse gas concentrations can be reduced by the combination of the two mitigation strategies considered. [4] formulated an optimal control model made up of three dependent variables : live biomass, $B(t)$, burned area, $I(t)$ and intrinsic growth, $r(t)$ to manage forest plantations. The characterization of their model was done using Pontryagin's Principle while the numerical simulation was done using the Runge-Kutta fourth-order method. From their findings, decreasing felling rate and extending the rotation age of the forest plantation can increase carbon capture. [32] used optimal control theory to obtain effective and useful distribution of investments in reforestation and encouragement of certain technology to attain a carbon dioxide emission schedule for 2020 in the Legal Brazilian Amazon (BA). This was to be achieved by increasing the forest area which has been greatly depleted by cattle ranching to sequester carbon dioxide from the atmosphere. Their mathematical model comprised of a coupled system of nonlinear ODEs of three compartments: Amount of carbon dioxide released into the atmosphere annually, regional Gross Domestic Product and the forest area, all being functions of time. Their simulation results showed that a forest area of about $3.7 \times 10^6\ km^2$ was needed for carbon dioxide emission target of 3.76×10^8 tonnes in 2020, which could require about $4.5 \times 10^5\ km^2$ reforestation out of this total land area.

Carbon dioxide is very crucial to the existence of lives in the ecosystem. Organisms with chlorophyll make food for the sustenance of the ecosystem using it and in turn, help in maintaining a natural balance in the biosphere. However, the sequesters (photosynthetic biomass) are greatly depleted due to a number of factors resulting from the activities of the increasing global population. Hence, remedies are needed to mitigate against climate change. In this current research work, an optimal control mathematical model on climate change incorporating good conservation policies, enlightenment programmes and direct air capture technology as auxiliary mitigation measures is formulated and analysed using the concept of optimal control theory and cost-effectiveness analysis. This optimal climate model is an extension of the model presented in [33]. The objective functional is set up to minimize the excessive concentration of carbon dioxide in the atmosphere as well as the total cost of implementation of each mitigation measure, as the resources available to cater for the needs of the teeming human population are limited.

2 Model Formulation and Analysis

In this section, we formulate the optimal control climate model to minimize both the excessive concentration of carbon dioxide in the atmosphere and the cost of implementation of the mitigation measures by extending the model in [33] using optimal control theory. By assumption, excessive concentration of carbon dioxide can be checked by the following measures: good conservation policies, enlightenment programmes and direct air capture technology. To obtain the optimal mitigation measures for reducing the excessive concentration of carbon dioxide and the cost of implementing these measures, time variant control variables $u_1(t)$, $u_2(t)$, $u_3(t)$ are added to the original model given in [33]. By incorporating these control variables into the original climate model given in [33], the optimal control climate model is given by the equations:

$$\frac{dC}{dt} = \beta C \left(1 - \frac{C}{C_m}\right) - d_1 CP - \left[(u_1 + d_2) + (u_2 + d_3) + (u_3 + d_4) + \mu_0\right] C. \tag{1}$$

$$\frac{dP}{dt} = \omega P \left(1 - \frac{P}{N}\right) + \phi CP + (u_1 + \tau)PR - (\mu_1 + \mu_2)P. \tag{2}$$

$$\frac{dR}{dt} = a_1 C - a_0 R. \tag{3}$$

$$\frac{dE}{dt} = b_1 C - b_0 E. \tag{4}$$

$$\frac{dT}{dt} = m_1 C - m_0 T. \tag{5}$$

Where,

u_1 =control unit representing good conservation policies;

u_2 =control unit representing enlightenment programmes;

u_3 =control unit representing direct air capture technology.

$u_j \in [0, 1]$, for $j = 1, 2, 3$.

Since the aim of taking the direction of optimal control theory is to mitigate against climate change by minimizing excessive concentration of CO_2 in the atmosphere and the corresponding costs of implementation of the employed mitigation measures, the objective function to achieve this aim is formulated as:

$$M = \min_{U} \int_0^{t_f} M_1 dt, \tag{6}$$

where

$$M_1 = \eta C + \frac{1}{2}(\kappa_1 u_1^2 + \kappa_2 u_2^2 + \kappa_3 u_3^2). \tag{7}$$

η is a balancing constant factor or coefficient of the excessive concentration of carbon dioxide and κ_1, κ_2 and κ_3 are positive weight constants. When $u_1 = 0, u_2 = 0, u_3 = 0$, the optimal model given by equations (1) to (5) reduces to the classical climate model presented in [33]. The terms $\frac{1}{2}\kappa_1 u_1^2$, $\frac{1}{2}\kappa_2 u_2^2$ and $\frac{1}{2}\kappa_3 u_3^2$ are the costs of implementing the mitigation measures associated with the control variables u_1, u_2 and u_3 respectively. Since we are interested in reducing both excessive concentration of CO_2 and the associated costs, we look for optimal control set $U^* = (u_1^*, u_2^*, u_3^*)$ such that $M(u_1^*, u_2^*, u_3^*) = \min M(u_1, u_2, u_3 | u_j \in U, j = 1, 2, 3$. Here, u_j is Lebesgue measurable on $0 \leq u_j \leq 1$ for $0 \leq t \leq t_f$, $j = 1, 2, 3$.

Table 1: Model Variables and Parameters Adopted from [33]

Symbols	Descriptions
t	Time
$C(t)$	Excessive concentration of carbon dioxide in the atmosphere
$P(t)$	Photosynthetic biomass density
$R(t)$	Good conservation policies density
$E(t)$	Enlightenment programmes density
$T(t)$	Direct air capture technology density
β	Intrinsic rate of accumulation of carbon dioxide in the atmosphere
C_m	Maximum tolerated concentration of carbon dioxide beyond which the model becomes meaningless
d_1	Rate of decrease in carbon dioxide concentration due to interaction between carbon dioxide and the photosynthetic biomass
d_2	Rate of decrease in carbon dioxide concentration due to implementation of good conservation policies
d_3	Rate of decrease in carbon dioxide concentration due to implementation of enlightenment programmes
d_4	Rate of decrease in carbon dioxide concentration due to implementation of direct air capture technology
μ_0	Natural rate of depletion in concentration of carbon dioxide
ω	Intrinsic rate of growth of the photosynthetic biomass
N	Carrying capacity for the photosynthetic biomass
ϕ	Rate of increase in the photosynthetic biomass due to the interaction between the biomass and carbon dioxide (photosynthetic rate)
τ	Rate of increase in photosynthetic biomass due to interaction between good conservation policies and the photosynthetic biomass
μ_1	Rate of decrease in photosynthetic biomass due to natural phenomena
μ_2	Rate of decrease in photosynthetic biomass due to human activities
a_1	Rate of success of good conservation policies
a_0	Rate of negligence or evasion of good conservation policies
b_1	Rate of success of enlightenment programmes
b_0	Rate of Ignorance, negligence and evasion of the enlightenment programmes
m_1	Rate of success of direct air capture technology
m_0	Rate of decline in the implementation of direct air capture technology

2.1 The Hamiltonian and Optimality Systems

In order to establish the necessary conditions for the optimization of the objective function, M, the Pontryagin Principle as used in [34] is employed. To achieve this, we formulate the Hamiltonian function, H, for the optimal climate model given by equations (1) to (5) as:

$$H = M_1 + \nu_1 \frac{dC}{dt} + \nu_2 \frac{dP}{dt} + \nu_3 \frac{dR}{dt} + \nu_4 \frac{dE}{dt} + \nu_5 \frac{dT}{dt}. \tag{8}$$

Substituting equation (7) and equations (1) to (5) into equation (8):

$$\begin{aligned} H =& \eta C + \frac{1}{2}(\kappa_1 u_1^2 + \kappa_2 u_2^2 + \kappa_3 u_3^2) + \nu_1 \Bigg[\beta C \left(1 - \frac{C}{C_m}\right) - d_1 CP - ((u_1 + d_2) + (u_2 + d_3) \\ &+ (u_3 + d_4) + \mu_0)C \Bigg] + \nu_2 \Bigg[\omega P \left(1 - \frac{P}{N}\right) + \phi CP + (u_1 + \tau)PR - (\mu_1 + \mu_2)P \Bigg] \\ &+ \nu_3 \big(a_1 C - a_0 R\big) + \nu_4 \big(b_1 C - b_0 E\big) + \nu_5 \big(m_1 C - m_0 T\big). \end{aligned} \tag{9}$$

In the formulated Hamiltonian function, $\nu_1, \nu_2, \nu_3, \nu_4$ and ν_5 are the adjoint variables associated with the model variables C, P, R, E and T respectively. To determine them (adjoint variables) and the control set,the following theorem is proposed.

Theorem 1 *For optimal controls units u_1^*, u_2^*, u_3^* that minimize an objective function, M, over U, there are associated adjoint variables $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$ with transversality conditions that $\nu_i(t_f) = 0, i = 1, 2, 3, 4, 5$ such that*

$$\begin{aligned} \frac{d\nu_1(t)}{dt} &= -\eta - \nu_1 \Bigg[\beta \left(1 - \frac{2C}{C_m}\right) - d_1 P - ((u_1 + d_2) + (u_2 + d_3) + (u_3 + d_4) + \mu_0) \Bigg] \\ &\quad - \phi P \nu_2 - a_1 \nu_3 - b_1 \nu_4 - m_1 \nu_5 && (10) \\ \frac{d\nu_2(t)}{dt} &= d_1 C \nu_1 - \nu_2 \Bigg[\omega \left(1 - \frac{2P}{N}\right) + \phi C + (u_1 + \tau)R - (\mu_1 + \mu_2) \Bigg] && (11) \\ \frac{d\nu_3(t)}{dt} &= -(u_1 + \tau)P\nu_2 + a_0 \nu_3; && (12) \\ \frac{d\nu_4(t)}{dt} &= b_0 \nu_4; && (13) \\ \frac{d\nu_5(t)}{dt} &= m_0 \nu_5. && (14) \end{aligned}$$

Furthermore, the control set $U^* = (u_1^*, u_2^*, u_3^*)$ is characterized by

$$u_1^* = \max\{0, \min(1, \theta_1)\}, \theta_1 = \frac{C\nu_1 - PR\nu_2}{\kappa_1} \tag{15}$$

$$u_2^* = \max\{0, \min(1, \theta_2)\}, \theta_2 = \frac{C\nu_1}{\kappa_2} \tag{16}$$

$$u_3^* = \max\{0, \min(1, \theta_3)\}, \theta_3 = \frac{C\nu_1}{\kappa_3} \tag{17}$$

Proof 1 *The Hamiltonian function from equation (9) is given as*

$$
\begin{aligned}
H =& \eta C + \frac{1}{2}(\kappa_1 u_1^2 + \kappa_2 u_2^2 + \kappa_3 u_3^2) + \nu_1 \left[\beta C \left(1 - \frac{C}{C_m} \right) - d_1 CP - ((u_1 + d_2) + (u_2 + d_3) \right. \\
& \left. + (u_3 + d_4) + \mu_0) C \right] + \nu_2 \left[\omega P \left(1 - \frac{P}{N} \right) + \phi CP + (u_1 + \tau) PR - (\mu_1 + \mu_2) P \right] \\
& + \nu_3 \big(a_1 C - a_0 R \big) + \nu_4 \big(b_1 C - b_0 E \big) + \nu_5 \big(m_1 C - m_0 T \big).
\end{aligned}
$$

Obtaining the partial derivatives of H with respect to the variables $C, P, R, E, T, u_1, u_2, u_3$:

$$
\begin{aligned}
\frac{\partial H}{\partial C} &= \eta + \nu_1 \left[\beta \left(1 - \frac{2C}{C_m} \right) - d_1 P - ((u_1 + d_2) + (u_2 + d_3) + (u_3 + d_4) + \mu_0) \right] + \phi P \nu_2 \\
& \quad + a_1 \nu_3 + b_1 \nu_4 + m_1 \nu_5; \qquad (18) \\
\frac{\partial H}{\partial P} &= -d_1 C \nu_1 + \nu_2 \left[\omega \left(1 - \frac{2P}{N} \right) + \phi C + (u_1 + \tau) R - (\mu_1 + \mu_2) \right]; \qquad (19) \\
\frac{\partial H}{\partial R} &= (u_1 + \tau) P \nu_2 - a_0 \nu_3; \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial E} &= -b_0 \nu_4; \qquad (21) \\
\frac{\partial H}{\partial T} &= -m_0 \nu_5; \qquad (22) \\
\frac{\partial H}{\partial u_1} &= \kappa_1 u_1 - C \nu_1 + PR \nu_2; \qquad (23) \\
\frac{\partial H}{\partial u_2} &= \kappa_2 u_2 - C \nu_1; \qquad (24) \\
\frac{\partial H}{\partial u_3} &= \kappa_3 u_3 - C \nu_1. \qquad (25)
\end{aligned}
$$

The objective function, M, can be shown to represent indeed a minimization problem if the second partial derivatives of the Hamiltonian function, H with respect to the control variables are positive. Thus,

$$
\frac{\partial^2 H}{\partial u_1^2} = \kappa_1 > 0; \frac{\partial^2 H}{\partial u_2^2} = \kappa_2 > 0; \frac{\partial^2 H}{\partial u_3^2} = \kappa_3 > 0;
$$

Hence, the objective function, M, represents a minimization problem.

By the application of the Pontryagin Principle [28], the associated adjoint equations for the optimal climate model are:

$$
\begin{aligned}
\frac{d\nu_1(t)}{dt} &= -\frac{\partial H}{\partial C} = -\eta - \nu_1 \left[\beta \left(1 - \frac{2C}{C_m} \right) - d_1 P - ((u_1 + d_2) + (u_2 + d_3) + (u_3 + d_4) + \mu_0) \right] \\
& \quad - \phi P \nu_2 - a_1 \nu_3 - b_1 \nu_4 - m_1 \nu_5 \qquad (26) \\
\frac{d\nu_2(t)}{dt} &= -\frac{\partial H}{\partial P} = d_1 C \nu_1 - \nu_2 \left[\omega \left(1 - \frac{2P}{N} \right) + \phi C + (u_1 + \tau) R - (\mu_1 + \mu_2) \right] \qquad (27) \\
\frac{d\nu_3(t)}{dt} &= -\frac{\partial H}{\partial R} = -(u_1 + \tau) P \nu_2 + a_0 \nu_3; \qquad (28) \\
\frac{d\nu_4(t)}{dt} &= -\frac{\partial H}{\partial E} = b_0 \nu_4; \qquad (29) \\
\frac{d\nu_5(t)}{dt} &= -\frac{\partial H}{\partial T} = m_0 \nu_5. \qquad (30)
\end{aligned}
$$

Using $\frac{\partial H}{\partial u_j} = 0$, the optimal controls units, u_j^*, for $j = 1, 2, 3$ are obtained as follows:

$$\frac{\partial H}{\partial u_1} = \kappa_1 u_1 - C\nu_1 + PR\nu_2 = 0 \implies u_1 = \frac{C\nu_1 - PR\nu_2}{\kappa_1}$$
$$\therefore u_1^* = \frac{C\nu_1 - PR\nu_2}{\kappa_1}. \tag{31}$$
$$\frac{\partial H}{\partial u_2} = \kappa_2 u_2 - C\nu_1 = 0 \implies u_2 = \frac{C\nu_1}{\kappa_2}$$
$$\therefore u_2^* = \frac{C\nu_1}{\kappa_2} \tag{32}$$
$$\frac{\partial H}{\partial u_3} = \kappa_3 u_3 - C\nu_1 = 0 \implies u_3 = \frac{C\nu_1}{\kappa_3}$$
$$\therefore u_3^* = \frac{C\nu_1}{\kappa_3}. \tag{33}$$

with transversality conditions that $\nu_1(t_f) = 0$, $\nu_2(t_f) = 0$, $\nu_3(t_f) = 0$, $\nu_4(t_f) = 0$, $\nu_5(t_f) = 0$; that is $\nu_i(t_f) = 0$, $i = 1, 2, 3, 4, 5$.

The characterization of the control units is done within the interior of the control set u_j for $0 \leq u_j \leq 1$; $j = 1, 2, 3$. Suppose $u_j = \theta_j \ \forall j = 1, 2, 3$, then:

$$u_1^* = \begin{cases} 0, \theta_1 \leq 0 \\ \theta_1, 0 < \theta_1 < 1 \\ 1, \theta_1 \geq 1 \end{cases} \tag{34}$$

$$u_2^* = \begin{cases} 0, \theta_2 \leq 0 \\ \theta_2, 0 < \theta_2 < 1 \\ 1, \theta_2 \geq 1 \end{cases} \tag{35}$$

$$u_3^* = \begin{cases} 0, \theta_3 \leq 0 \\ \theta_3, 0 < \theta_3 < 1 \\ 1, \theta_3 \geq 1 \end{cases} \tag{36}$$

These results could be rewritten in compact forms as:

$$u_1^* = \max\{0, \min(1, \theta_1)\} = \max\left\{0, \min\left(1, \frac{C\nu_1 - PR\nu_2}{\kappa_1}\right)\right\} \tag{37}$$
$$u_2^* = \max\{0, \min(1, \theta_2)\} = \max\left\{0, \min\left(1, \frac{C\nu_1}{\kappa_2}\right)\right\} \tag{38}$$
$$u_3^* = \max\{0, \min(1, \theta_3)\} = \max\left\{0, \min\left(1, \frac{C\nu_1}{\kappa_3}\right)\right\} \tag{39}$$

From the optimal control system (otherwise called the state system) in equations (1) to (5) and the adjoint variable system got as given in equations (26) to (30) , the optimality system is formed by incorporating the characterized control set, initial and transversal conditions as follows:

$$\frac{dC}{dt} = \beta C\left(1 - \frac{C}{C_m}\right) - d_1 CP - ((u_1 + d_2) + (u_2 + d_3) + (u_3 + d_4) + \mu_0)C. \tag{40}$$

$$\frac{dP}{dt} = \omega P\left(1 - \frac{P}{N}\right) + \phi CP + (u_1 + \tau)PR - (\mu_1 + \mu_2)P. \tag{41}$$

$$\frac{dR}{dt} = a_1 C - a_0 R. \tag{42}$$

$$\frac{dE}{dt} = b_1 C - b_0 E. \tag{43}$$

$$\frac{dT}{dt} = m_1 C - m_0 T \tag{44}$$

$$\frac{d\nu_1}{dt} = -\eta - \nu_1\left[\beta\left(1 - \frac{2C}{C_m}\right) - d_1 P - ((u_1 + d_2) + (u_2 + d_3) + (u_3 + d_4) + \mu_0)\right]$$
$$- \phi P\nu_2 - a_1\nu_3 - b_1\nu_4 - m_1\nu_5 \tag{45}$$

$$\frac{d\nu_2}{dt} = d_1 C\nu_1 - \nu_2\left[\omega\left(1 - \frac{2P}{N}\right) + \phi C + (u_1 + \tau)R - (\mu_1 + \mu_2)\right] \tag{46}$$

$$\frac{d\nu_3}{dt} = -(u_1 + \tau)P\nu_2 + a_0\nu_3; \tag{47}$$

$$\frac{d\nu_4}{dt} = b_0\nu_4; \tag{48}$$

$$\frac{d\nu_5}{dt} = m_0\nu_5 \tag{49}$$

$$u_1^* = \max\{0, \min(1, \theta_1)\} = \max\left\{0, \min\left(1, \frac{C\nu_1 - PR\nu_2}{\kappa_1}\right)\right\} \tag{50}$$

$$u_2^* = \max\{0, \min(1, \theta_2)\} = \max\left\{0, \min\left(1, \frac{C\nu_1}{\kappa_2}\right)\right\} \tag{51}$$

$$u_3^* = \max\{0, \min(1, \theta_3)\} = \max\left\{0, \min\left(1, \frac{C\nu_1}{\kappa_3}\right)\right\} \tag{52}$$

where $\nu_i(t_f) = 0$ for $i = 1, 2, 3, 4, 5$; $C(0) = C_0$, $P(0) = P_0$, $R(0) = R_0$, $E(0) = E_0$, $T(0) = T_0$.

3 Numerical Simulation

The parameter values as also used for the classical model without optimal control in [33] were used for the simulation of the optimal control climate model. The numerical simulation of the optimality control system

Table 2: Values of Model Parameters Adopted from [33]

Parameters	Values	Units	References
β	1 and 6	$\mu mol\ per\ mol\ m^{-2}$	[35, 36]
C_m	25	m^{-2}	Fixed
d_1	0.05	m^{-2}	[37]
d_2	0.013	m^{-2}	Fixed
d_3	0.01	m^{-2}	Fixed
d_4	0.5	m^{-2}	[4]
μ_0	0.016	$year^{-1}$	Fixed
ω	0.8 and 1.0	$year^{-1}$	[36–38]
N	80	$kg\ m^{-2}$	[38]
ϕ	0.1	m^{-2}	[37]
τ	0.03	m^{-2}	[37]
μ_1	0.02	$year^{-1}$	[36]
μ_2	0.04	$year^{-1}$	[38]
a_1	0.008	m^{-2}	Assumed
a_0	0.001	m^{-2}	Assumed
b_1	0.0078	m^{-2}	Assumed
b_0	0.0019	m^{-2}	Assumed
m_1	0.0068	m^{-2}	Assumed
m_0	0.0012	m^{-2}	Assumed

given by equations (40) to (52) is done in MATLAB using the Runge-Kutta method (adopting the forward backward sweep algorithm). The balancing factor is fixed as $\eta = 3$. The total cost functions $\frac{1}{2}\kappa_1 {u_1}^2$,

$\frac{1}{2}\kappa_2 {u_2}^2$, $\frac{1}{2}\kappa_3 {u_3}^2$ for each mitigation measure considered are computed over a period of $t_f = 10$ years. By assuming that the cost of implementation of enlightenment programmes on the dangers of excessive emission and accumulation of carbon dioxide is least expensive while the cost of installing direct air capture technology is the most expensive, the weight functions are set as $\kappa_1 = 35$, $\kappa_2 = 20$ and $\kappa_3 = 65$, that is $\kappa_2 < \kappa_1 < \kappa_3$. The initial conditions for the model solution are set as: $C(0) = 1$, $P(0) = 4$, $R(0) = 2$, $E(0) = 5$ and $T(0) = 7$.

4 Discussion of Results

4.1 *Strategy I: Good Conservation Policies* $(u_1 \neq 0)$

In Figure 1, the simulated results for the model without control and the optimality control system given by equations (40) to (52) are presented. Here, the strategy considered for the reduction of excessive concentration of carbon dioxide in the atmosphere is good conservation policies. The model without control produced a maximum excessive concentration of 19.679. With the strategy involving good conservation policies, the maximum excessive concentration obtained is 9.3773. This is equivalent to 52.35% reduction in the maximum excessive concentration of carbon dioxide in the atmosphere. Furthermore, the total accumulated excessive concentration of carbon dioxide for a period of 10 years without control is 14 292. But with strategy I, the total accumulated excessive concentration became 3 202.1 for the same period. This is equivalent to 11 089.9 total excessive concentration of carbon dioxide averted (representing 77.60% total excessive concentration of carbon dioxide averted in 10 years).

4.2 *Strategy II: Enlightenment Programmes* $(u_2 \neq 0)$

In Figure 2, excessive concentration of carbon dioxide results against time are presented for the model without control and the model with control. Here, the strategy of mitigation considered is enlightenment programmes. Simulated result involving enlightenment programmes gave the maximum value of the excessive concentration of carbon dioxide as 15.497 compared to the value of 19.679 obtained for the model without any control. This represents about 21.25% reduction in the value of the maximum excessive concentration of carbon dioxide. The total accumulated excessive concentration of carbon dioxide obtained with strategy II for a period of 10 years was 11 267 compared to 14 292 obtained for same period for the model without control. By calculation, this gives 3 025 total averted excessive concentration of carbon dioxide in the atmosphere (21.17% reduction in the total accumulated excessive concentration of carbon dioxide in the atmosphere for 10 years).

Table 3: Maximum and Minimum Excessive Concentration Values of Carbon Dioxide Without and With Control

Figure	M_1	M_2	M_3	M_4
1	19.679	9.3773	1.3832×10^{-11}	52.35
2	19.679	15.497	1	21.25
3	19.679	13.476	1	31.52
4	19.679	5.9603	5.8265×10^{-13}	69.71
5	19.679	4.6106	1.7845×10^{-13}	76.57
6	19.679	9.3579	0.32017	52.45
7	19.679	2.5522	2.5955×10^{-15}	87.03

M_1: Maximum Excessive Concentration of Carbon Dioxide Without Control

M_2: Maximum Excessive Concentration of Carbon Dioxide With Control

M_3: Minimum Excessive Concentration of Carbon Dioxide With Control

M_4: Percentage Change in Maximum Excessive Concentration of Carbon Dioxide

Table 4: Ascending Order Arrangement of Mitigation Strategies According to Total Excessive Concentration Averted

Strategy	W_1	W_2	W_3	W_4 (US Dollars)	ACER
II : $(u_2 \neq 0)$	14 292	11 267	3 025	20 000	6.6116
III : $(u_3 \neq 0)$	14 292	9 920.6	4 371.4	65 000	14.8694
VI : $(u_2 \neq 0, u_3 \neq 0)$	14 292	7 334.3	6 957.7	85 000	12.2167
I : $(u_1 \neq 0)$	14 292	3 202.1	11 089.9	35 000	3.1560
IV : $(u_1 \neq 0, u_2 \neq 0)$	14 292	2 192.1	12 099.9	55 000	4.5455
V : $(u_1 \neq 0, u_3 \neq 0)$	14 292	1 768	12 524	100 000	7.9847
VII : $(u_1 \neq 0, u_2 \neq 0, u_3 \neq 0)$	14 292	1 050	13 242	120 000	9.0621

W_1: Total Excessive Concentration of Carbon Dioxide Without Control for a Period of 10 Years

W_2: Total Excessive Concentration of Carbon Dioxide With Control for a Period of 10 Years

W_3: Total Excessive Concentration of Carbon Dioxide Averted for a Period of 10 Years

W_4: Cost of Implementation of Strategies for a Period of 10 Years

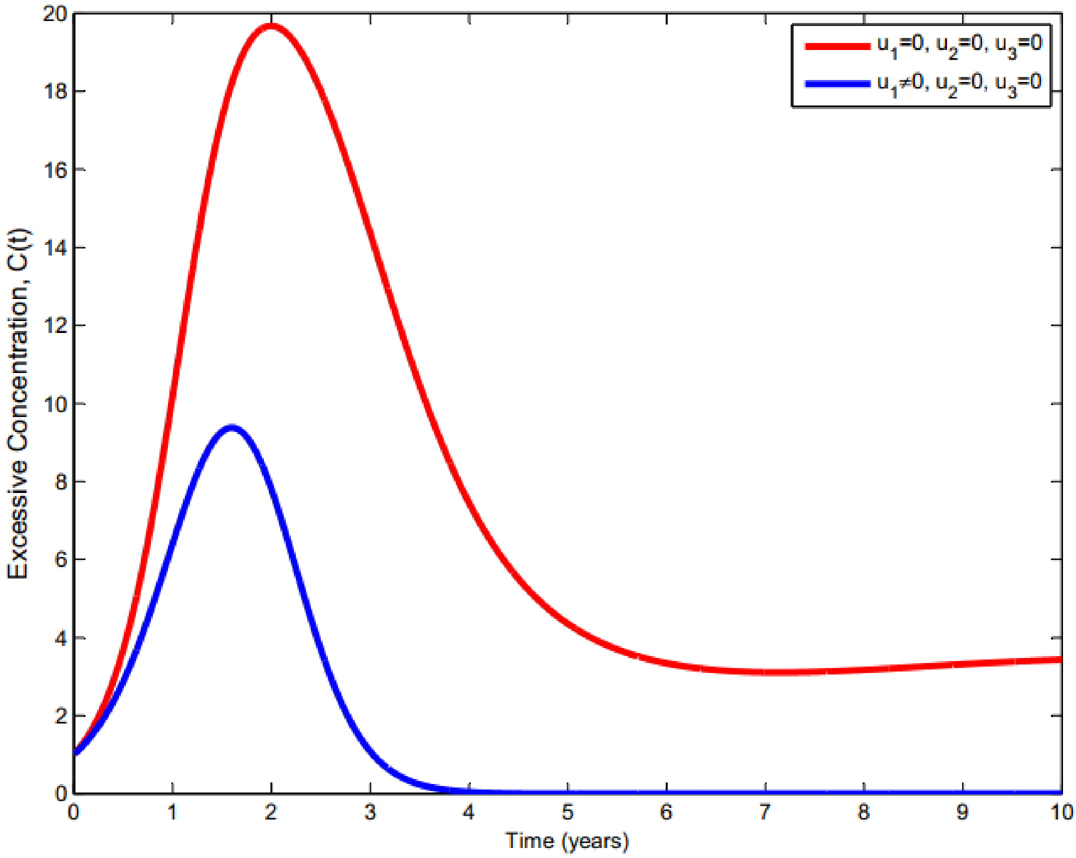

Figure 1: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy I implementation

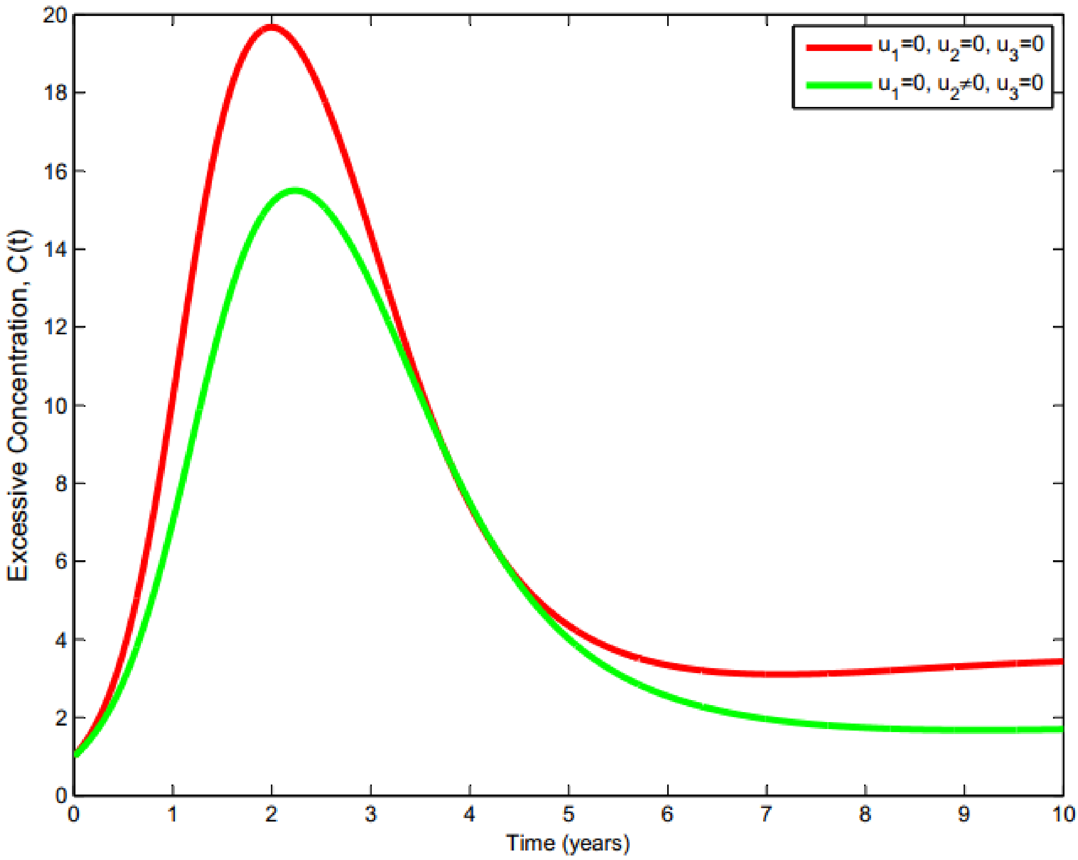

Figure 2: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy II implementation

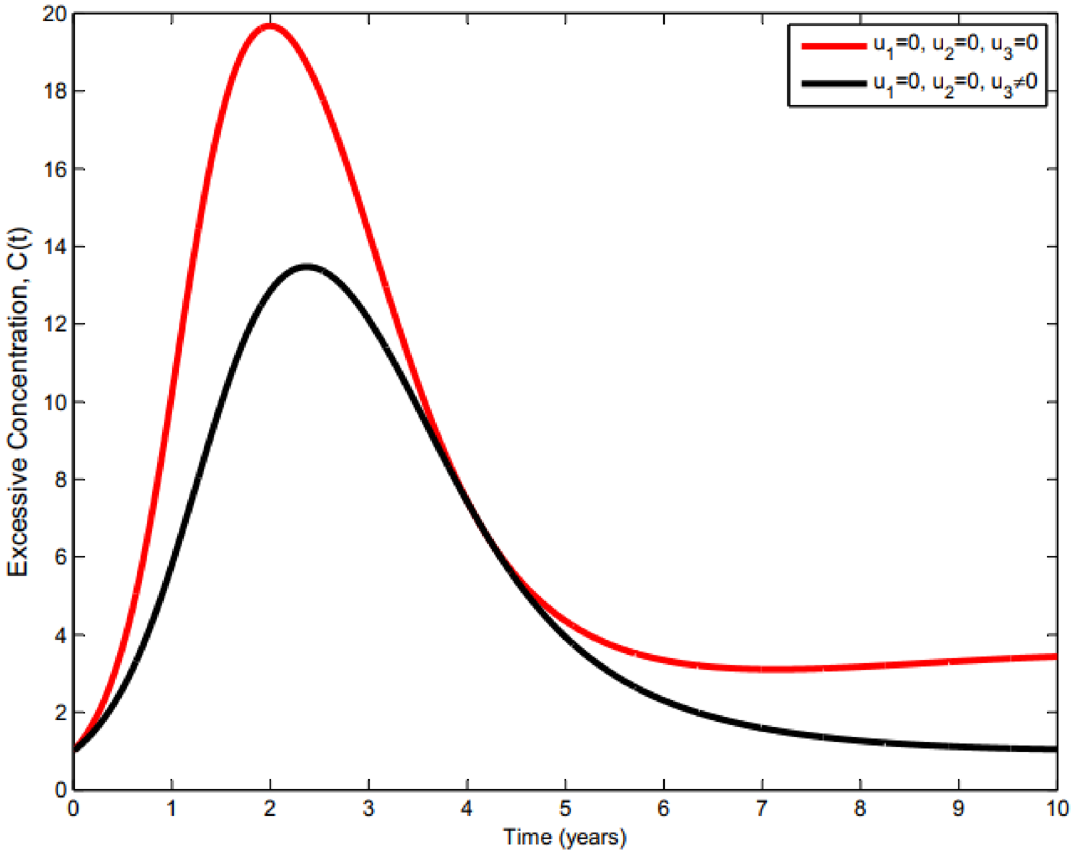

Figure 3: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy III implementation

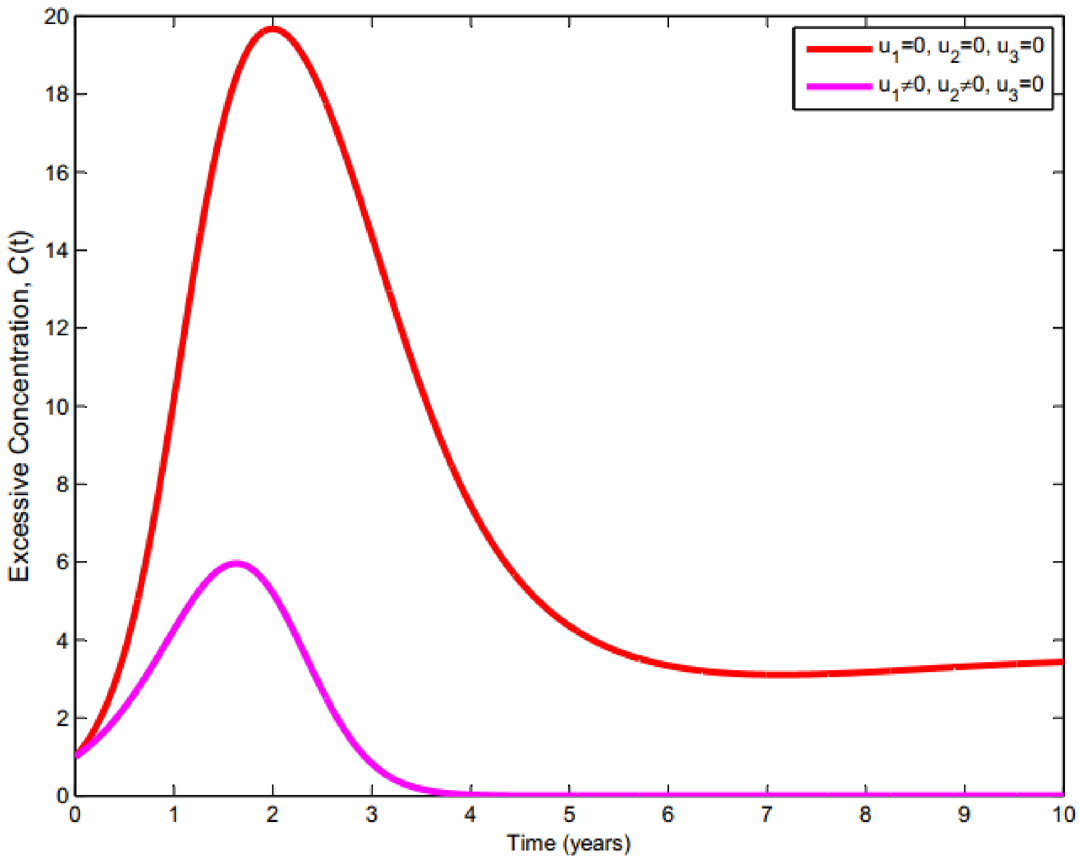

Figure 4: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy IV implementation

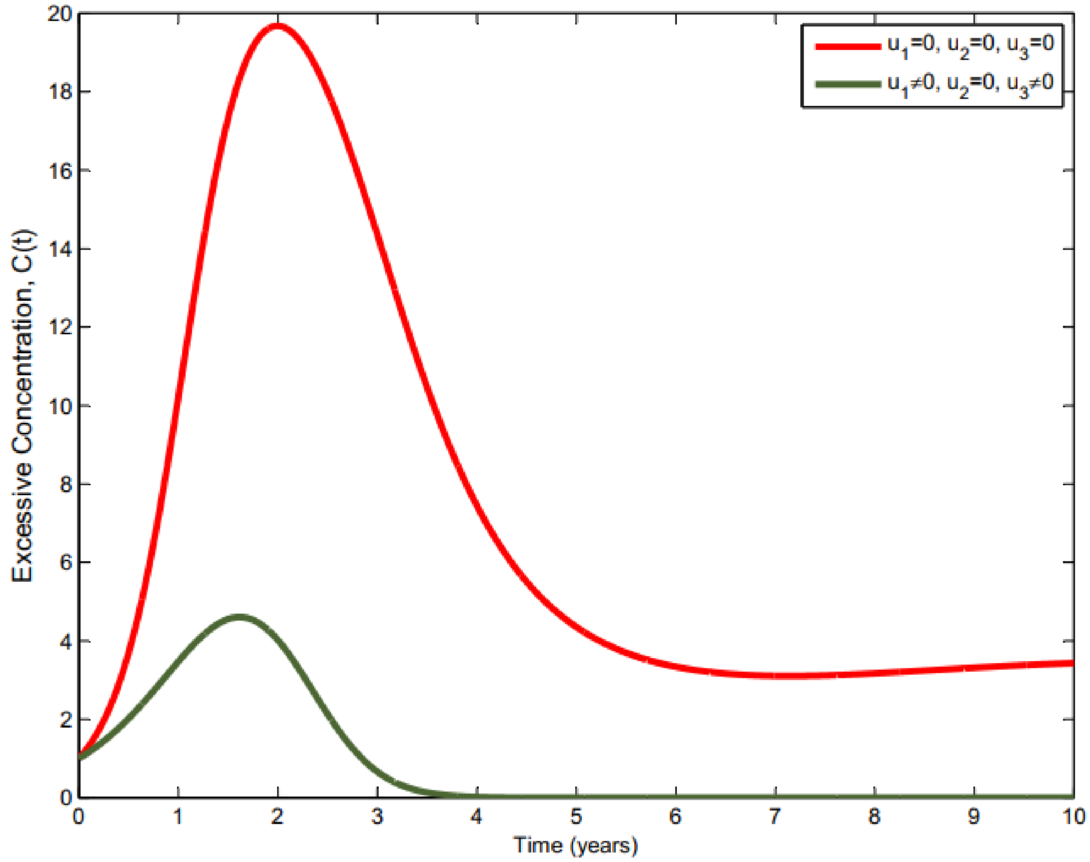

Figure 5: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy V implementation

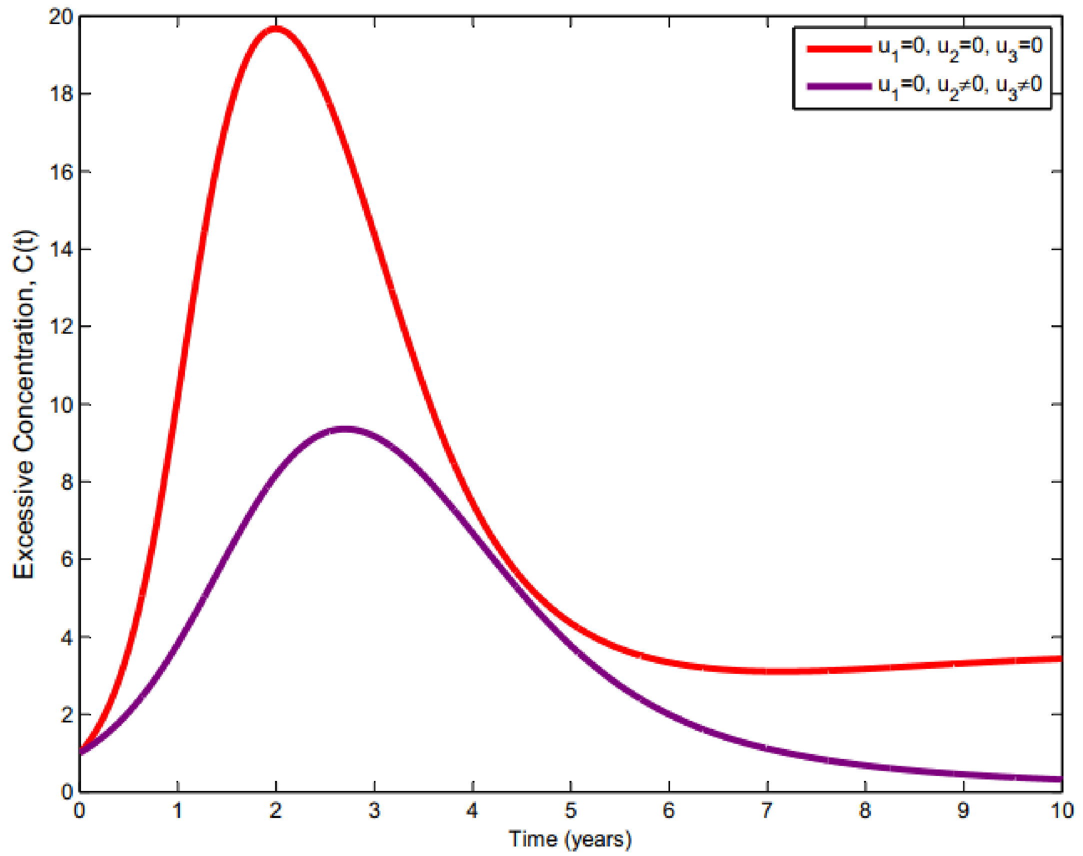

Figure 6: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy VI implementation

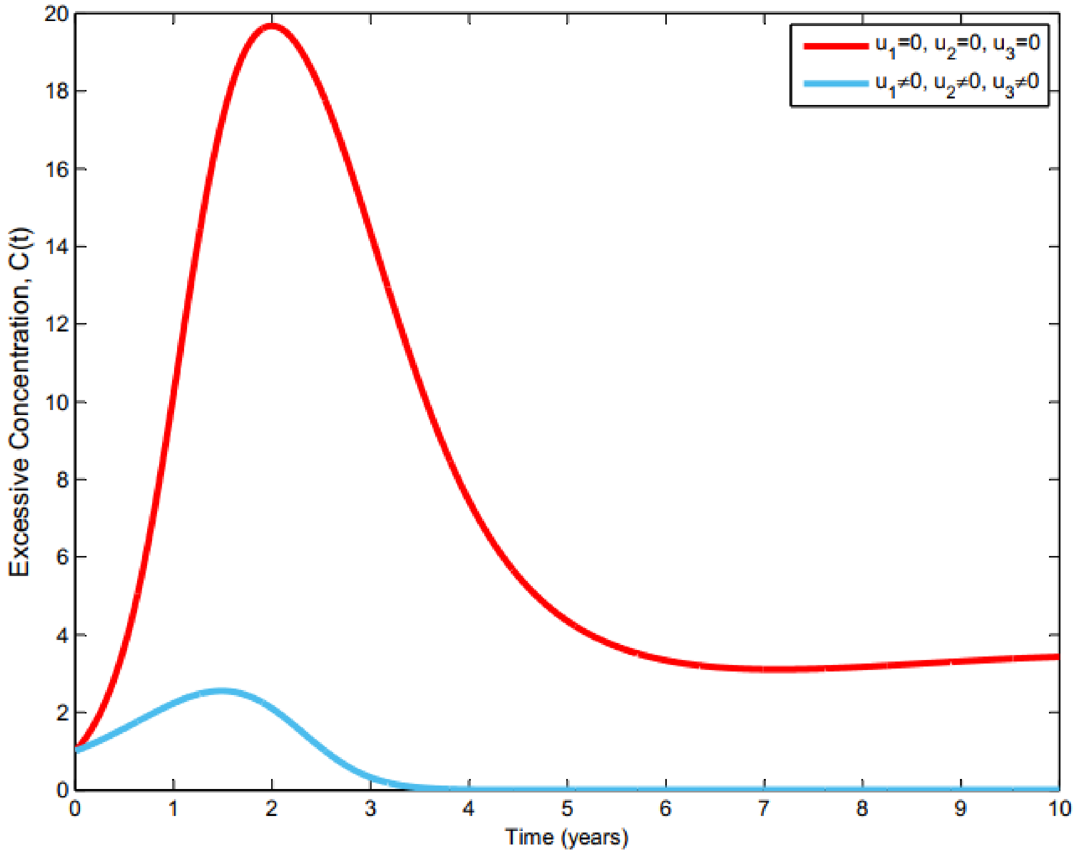

Figure 7: Excessive concentration of carbon dioxide in the atmosphere without optimal control and with Strategy VII implementation

4.3 *Strategy III : Direct Air Capture Technology* $(u_3 \neq 0)$

Excessive concentration of carbon dioxide without and with control are presented in Figure 3. Strategy III employed here involves the use of direct air capture technology in reducing the excessive concentration of carbon dioxide in the atmosphere. The simulated value for the maximum excessive concentration of carbon dioxide is 13.476 involving this strategy of direct air capture technology. A reduction in the maximum excessive concentration of about 31.52% is recorded, considering that the maximum excessive concentration without control is 19.679. Also, a total accumulated excessive concentration value of $9\ 920.6$ was obtained compared to $14\ 292$ for the model without control. This is equivalent to a total averted excessive concentration value of $4\ 371.4$, depicting about 30.59% reduction in the total accumulated excessive concentration of carbon dioxide present in the atmosphere in 10 years.

4.4 *Strategy IV : Good Conservation Policies and Enlightenment Programmes* $(u_1 \neq 0, u_2 \neq 0)$

In Figure 4, the excessive concentration of carbon dioxide without control and with control are presented. Here, the strategy considered involves a combination of good conservation policies and enlightenment programmes as mitigation measures in the reduction of the excessive emission of carbon dioxide from the atmosphere. The effect of this strategy from the simulated results produced a maximum excessive concentration of carbon dioxide to a value of 5.9603 compared to a value of 19.679 without control. This is equivalent to about 69.71% reduction in the maximum excessive concentration of carbon dioxide. Additionally, a total accumulated concentration value of $2\ 192.1$ was obtained as against $14\ 292$ without control. This represents a total accumulated excessive concentration of carbon dioxide averted in 10 years to be $12\ 099.9$ (84.66% total accumulated excessive concentration of carbon dioxide averted).

4.5 *Strategy V : Good Conservation Policies and Direct Air Capture Technology* $(u_1 \neq 0, u_3 \neq 0)$

Excessive concentration of carbon dioxide with the effect of the strategy comprising good conservation policies and direct air capture technology as mitigation measures as well as the excessive concentration without control are presented in Figure 5. With this strategy, the maximum excessive concentration of carbon dioxide obtained is 4.6106 as against 19.679 for the model without control. This represents about 76.57% decrease in the maximum excessive concentration of carbon dioxide. Also, a total accumulated excessive concentration of $1\ 768$ was obtained with the help of this strategy for a period of 10 years compared to a value of $14\ 292$ without control for the same period. Thus, this gives a total accumulated excessive concentration of carbon dioxide averted value of $12\ 524$ (an 87.63% equivalence of total accumulated excessive concentration of carbon dioxide averted in 10 years).

4.6 *Strategy VI : Enlightenment Programmes and Direct Capture Technology* $(u_2 \neq 0, u_3 \neq 0)$

In Figure 6, excessive concentration of carbon dioxide without control and with enlightenment programmes and direct air capture technology as controls are presented. A maximum excessive concentration value of 9.3579 was obtained unlike the value of 19.679 for the model without control, representing 52.45% reduction in the maximum excessive concentration of carbon dioxide present in the atmosphere. For a period of 10 years, this strategy produced a total accumulated excessive concentration value of $7\ 334.3$ compared to a value of $14\ 292$ obtained for the model without control. In this case, a total accumulated excessive concentration value of $6\ 957.7$ was averted (representing a total accumulated excessive concentration of carbon dioxide averted to be 48.68%).

4.7 *Strategy VII : Good Conservation Policies, Enlightenment Programmes and Direct Air Capture Technology* $(u_1 \neq 0, u_2 \neq 0, u_3 \neq 0)$

In Figure 7, the simulated results for the excessive concentration of carbon dioxide without control and with control are depicted. The optimal controls considered are a combination of good conservation policies, enlightenment programmes and direct air capture technology. Using the optimality control system given by equations (40) to (52), the simulated results revealed that a maximum excessive concentration of 2.5522 was obtained using this strategy as opposed to the value of 19.679 obtained for the model without control. This represents 87.03% decrease in the maximum excessive concentration of carbon dioxide in the atmosphere. A total accumulated excessive concentration of carbon dioxide of about $1\ 050$ was recorded for a period of 10 years involving this strategy as against $14\ 292$ for the model without control. Thus, the total accumulated excessive concentration of carbon dioxide averted for a period of 10 years is $13\ 242$ (representing 92.65% total accumulated excessive concentration of carbon dioxide averted in 10 years).

5 Cost-Effectiveness Analysis

The determination of the most cost-effective mitigation strategy is done by using the average cost-effectiveness ratio (ACER) and the incremental cost-effectiveness ratio (ICER) approaches as used by [39], [28]. ACER involves weighing one mitigation measure against the corresponding implementation cost of that particular measure. Using the analogy of the definition of ACER for infection model control given by [39], [28], ACER with respect to reduction of excessive concentration of carbon dioxide could be defined as well. Hence, ACER is the ratio of the total cost of implementation of a mitigation measure to the total accumulated excessive concentration of carbon dioxide averted over a specified period of time. Mathematically, if W_A is the total cost of a mitigation strategy A and M_A is the total accumulated excessive concentration of carbon dioxide averted, then

$$ACER = \frac{Total\ cost\ of\ a\ mitigation\ strategy}{Total\ accumulated\ excessive\ concentration\ of\ carbon\ dioxide\ averted} = \frac{W_A}{M_A}. \tag{53}$$

ICER on the other hand considers a relative comparison of two mitigation strategies and the cost involvement in implementing those strategies. It could be defined in this particular case to be the ratio of the difference between the costs of two selected mitigation strategies and the difference between their respective total accumulated excessive concentration of carbon dioxide averted. The excessive concentration of carbon dioxide that is averted is equivalent to the difference between the total accumulated excessive concentration of carbon dioxide without control and the total accumulated excessive concentration of carbon dioxide with control over the period under consideration (10 years in this case). If strategies A and B say cost W_A and W_B and averted M_A and M_B total accumulated excessive concentration of carbon dioxide respectively, then the $ICER$ is defined mathematically thus:

$$\begin{aligned} ICER &= \frac{Difference\ between\ the\ costs\ of\ the\ two\ mitigation\ strategies}{Difference\ between\ their\ total\ accumulated\ excessive\ concentration\ averted} \\ &= \frac{W_B - W_A}{M_B - M_A}. \end{aligned} \tag{54}$$

The cost-effectiveness analysis ($ACER$ and $ICER$) for the current research work is done using the total accumulated excessive concentration of carbon dioxide averted and the total cost of implementation of the various strategies presented in Table 4 as similarly calculated for epidemiological model [28, 39, 40]. In other words, the total cost of each mitigation strategy is approximated using the formula:

$$Total\ Cost\ of\ Each\ Strategy\ = \frac{1}{2}\int_0^{t_f} \left(\kappa_1 {u_1}^2 + \kappa_2 {u_2}^2 + \kappa_3 {u_3}^2\right) dt$$

The total cost of each mitigation measure of good conservation policies, enlightenment programmes and direct air capture technology is calculated using their respective cost functions of $\frac{1}{2}\kappa_1 {u_1}^2$, $\frac{1}{2}\kappa_2 {u_2}^2$ and $\frac{1}{2}\kappa_3 {u_3}^2$. The total cost of implementing each strategy obtained from the simulation is rounded to the nearest thousand US Dollars while $ACER$ and $ICER$ values have been rounded to 4 decimal places (as represented in Table 4). Based on the ACER values obtained (as presented in Table 4), the least costly of the strategies is strategy I (good conservation policies) with ACER value of 3.1560 and the most costly is strategy III (direct air capture technology) with an ACER value of 14.8694. Hence, due to limited and scarce resources, strategy I is the cheapest option to consider followed by strategies IV, II, V, VII, VI, III in that order.

Using the arrangement of strategies in Table 4, the incremental cost-effectiveness ratios (ICERs) for the different strategies considered are calculated thus:

Computation of the ICER for strategy II (enlightenment programmes $(u_2 \neq 0)$) and strategy III (direct air capture technology $(u_3 \neq 0)$) are :

$$ICER(II) = \frac{20000}{3025} = 6.6116$$
$$ICER(III) = \frac{65000 - 20000}{4371.4 - 3025} = 33.4225.$$

From these calculations of ICER, it is observed that $ICER(III) = 33.4225$ is higher than $ICER(II) = 6.6116$. This implies that strategy III is more costly compared to strategy II. Based on this result, strategy III is eliminated in subsequent computations of ICER. The ICERs values for strategies II and III are presented in Table 5. Next, we compare Strategies II and VI (Table 6).

$$ICER(II) = \frac{20000}{3025} = 6.6116$$
$$ICER(VI) = \frac{85000 - 20000}{6957.7 - 3025} = 16.5281.$$

The computation of ICERs for Strategy II (enlightenment programmes $(u_2 \neq 0)$) and Strategy VI (enlightenment programmes and direct air capture technology $(u_2 \neq 0, u_3 \neq 0)$) revealed that ICER(VI)=16.5281 is higher in value compared to ICER (II)=6.6116. Hence, it is inferred that Strategy VI is more costly than Strategy II. Therefore, Strategy VI is eliminated from the subsequent calculations. Next, we compute and compare the ICER values for Strategies II and I (Table 7).

$$ICER(II) = \frac{20000}{3025} = 6.6116$$
$$ICER(I) = \frac{35000 - 20000}{11089.9 - 3025} = 1.8599.$$

From these ICER values computed for Strategies II enlightenment programmes $(u_2 \neq 0)$) and Strategy I (good conservation policies $(u_1 \neq 0)$), ICER (II)=6.6116 is higher in value compared to ICER (I)=1.8599, implying that Strategy II is more costly to implement than Strategy I. Hence, Strategy II is dropped in subsequent computations and Strategy I selected as the new base for comparison. Next, we compute and compare ICER values for Strategies I and IV (Table 8).

$$ICER(I) = \frac{35000}{11089.9} = 3.1560$$
$$ICER(IV) = \frac{55000 - 35000}{12099.9 - 11089.9} = 19.8020.$$

The evaluation of ICERs for Strategy I (good conservation policies ($u_1 \neq 0$)) and Strategy IV (good conservation policies and enlightenment programmes ($u_1 \neq 0, u_2 \neq 0$)) revealed that ICER (IV)=19.8020 is greater than ICER (I)=3.1560. This implies that Strategy IV is more costly to implement compared to Strategy I. Therefore, Strategy IV is removed in the next calculations. Next, we evaluate and compare ICER values for Strategies I and V (Table 9).

$$ICER(I) = \frac{35000}{11089.9} = 3.1560$$
$$ICER(V) = \frac{100000 - 35000}{12524 - 11089.9} = 45.3246.$$

The computation of ICERs for Strategy I (good conservation policies ($u_1 \neq 0$)) and Strategy V (good conservation policies and direct air capture technology ($u_1 \neq 0, u_3 \neq 0$)) showed that Strategy V is more costly to implement that Strategy I. This is because ICER (V)=45.3246 is higher than ICER (I)=3.1560. Hence, Strategy V is dropped and Strategy I is still used in subsequent calculations. Next, we calculate and compare ICERs for Strategies I and VII (Table 10).

$$ICER(I) = \frac{35000}{11089.9} = 3.1560$$
$$ICER(VII) = \frac{120000 - 35000}{13242 - 11089.9} = 39.4963$$

The computation of ICERs for Strategy I (good conservation policies ($u_1 \neq 0$)) and Strategy VII (good conservation policies, enlightenment programmes and direct air capture technology ($u_1 \neq 0, u_2 \neq 0 u_3 \neq 0$)) revealed that Strategy VII is more costly to implement that Strategy I. This is because ICER (VII)=39.4963 is far higher than ICER (I)=3.1560. From all the results of ICERs obtained for the seven different strategies, we can infer that Strategy I (good conservation policies) can be adopted as the most cost-effective strategy in providing mitigation to excessive emission and accumulation of carbon dioxide in the atmosphere.

Table 5: ICER Computations for Strategies II and III

Strategy	Total Concentration Averted	Total Cost	$ACER$	$ICER$
II	3 025	20 000	6.6116	6.6116
III	4 371.4	65 000	14.8694	33.4225

Table 6: ICER Computations for Strategies II and VI

Strategy	Total Concentration Averted	Total Cost	$ACER$	$ICER$
II	3 025	20 000	6.6116	6.6116
VI	6 957.7	85 000	12.2167	16.5281

Table 7: ICER Computations for Strategies II and I

Strategy	Total Concentration Averted	Total Cost	$ACER$	$ICER$
II	3 025	20 000	6.6116	6.6116
I	11 089.9	35 000	3,1560	1.8599

Table 8: ICER Computations for Strategies I and IV

Strategy	Total Concentration Averted	Total Cost	$ACER$	$ICER$
I	11 089.9	35 000	3.1560	3.1560
IV	12 099.9	55 000	4.5455	19.8020

Table 9: ICER Computations for Strategies I and V

Strategy	Total Concentration Averted	Total Cost	$ACER$	$ICER$
I	11 089.9	35 000	3.1560	3.1560
V	12 524	100 000	7.9847	45.3246

Table 10: ICER Computations for Strategies I and VII

Strategy	Total Concentration Averted	Total Cost	$ACER$	$ICER$
I	11 089.9	35 000	3.1560	3.1560
VII	13 242	120 000	9.0621	39.4963

6 Conclusions

A mathematical model of climate change due to excessive emission and accumulation of carbon dioxide is analysed using optimal control theory and cost-effectiveness analysis techniques. Since the aim of the research is to reduce the excessive concentration of carbon dioxide in the atmosphere by employing some mitigation measures, a minimization objective function is set up comprising terms made up of the excessive concentration of carbon dioxide and the costs of the mitigation measures to be implemented. From the Hamiltonian function formulated, the adjoint equations and characterizations were obtained using the Pontryagin's Principle. Seven different mitigation strategies were considered and simulated for the model. These strategies included: good conservation policies only, enlightenment programmes only, direct air capture technology only, good conservation policies and enlightenment programmes, good conservation policies and direct air capture technology, enlightenment programmes and direct air capture technology, good conservation policies, enlightenment programmes and direct air capture technology. From the simulated results (presented as Tables 3,4 and Figures 1-7), despite the fact that implementation of each of the seven strategies can reduce the excessive concentration of carbon dioxide in th atmosphere, the best result was obtained using strategy VII (combines all the mitigation measures considered) , leading to a maximum excessive concentration value of 2.5522 (as against 19.679 without control) and a total accumulated excessive concentration of carbon dioxide to a value of $13\ 242$ out of a possible $14\ 292$ (representing 92.65% total excessive concentration of carbon dioxide that can be averted). Even though strategy VII appears the most desirable strategy to adopt, the constraint of cost is another variable or factor to be considered in selecting a strategy. Human resources are scarce and there are many needs to be satisfied using the meagre available resources. Hence, the need to consider cost-effectiveness analysis techniques (in this case, Average Cost-Effectiveness Ratio, ACER and Incremental Cost-Effectiveness Ratio, ICER). From these analyses, the least ACER value (3.1560) and the least ICER value (3.1560) coincided for Strategy I. Hence, the most cost-effective strategy (cheapest strategy) to implement is Strategy I.

Data Availability Statement

No underlying data was collected or produced in this study. The data used in the numerical simulations and other calculations in this study were got from research materials whose sources are appropriately cited as seen in Table (2) while others were fixed or assumed.

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Conflict of Interest

The authors declare no conflict of interest.

Authorship Contributions Statement

Peter Achimugwu performed all the calculations, simulations and partly wrote the original draft of the manuscript.

He also edited and worked on the comments from the co-authors. **Mathew Kinyanjui** helped with the design of the ideas presented in the manuscript, reviewed, commented, supervised and edited the work. **David Malonza** also contributed to the draft of the ideas presented in the manuscript, reviewed, commented and supervised the work. The authors have read, agreed and consented to the continuation with the publication process for this manuscript.

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