Supporting Information for "Passive source reverse time migration based on the spectral element method"

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Introduction

In this supporting information, we show how to get the weak-form solutions for wavefield decomposition. If you are familiar with the spectral element method, please go to section 3 directly. For detailed deviations, please refer to Fichtner (2010).

Text S1. Notes for understanding the weak solutions

1. Weak Solutions for the elastic wave-equation

The strong displacement-stress variant of the equations of motion:

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) - \nabla\cdot\sigma(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) \quad , \tag{1}$$

$$\sigma(\mathbf{x},t) = \mathbf{C}(\mathbf{x}) : \nabla\mathbf{u}(\mathbf{x},t) \quad , \tag{2}$$

subject to the boundary and initial conditions

$$\mathbf{n}\cdot\sigma|_{\mathbf{x}\in\partial G} = 0 \quad , \quad \mathbf{u}|_{t=0} = \dot{\mathbf{u}}|_{t=0} = 0 \quad . \tag{3}$$

For the moment we disregard dissipation, i.e., the time dependence of the elastic tensor $\mathbf{C}$. Multiply Eq.(1) by an arbitrary, differentiable, time-independent test function $\mathbf{w}$ and integrating over G gives

$$\int_G \rho\mathbf{w}\cdot\ddot{\mathbf{u}}d^3\mathbf{x} - \int_G \mathbf{w}\cdot(\nabla\cdot\sigma)d^3\mathbf{x} = \int_G \mathbf{w}\cdot\mathbf{f}d^3\mathbf{x} \quad . \tag{4}$$

Invoking the identity

$$\mathbf{w}\cdot(\nabla\cdot\sigma) = \nabla\cdot(\mathbf{w}\cdot\sigma) - \nabla\mathbf{w} : \sigma \quad . \tag{5}$$

Poof:

$$\nabla\cdot(\mathbf{w}\cdot\sigma) = \partial_i(w_j\sigma_{ij}) = (\partial_i w_j)\sigma_{ij} + w_j(\partial_i\sigma_{ij}) = \nabla\mathbf{w} : \sigma + \mathbf{w}\cdot(\nabla\cdot\sigma) \quad .$$

Together with Gauss's theorem, yields,

$$\int_G \rho\mathbf{w}\cdot\ddot{\mathbf{u}}d^3\mathbf{x} - \int_{\partial G} \mathbf{w}\cdot\sigma\cdot\mathbf{n}d^2\mathbf{x} + \int_G \nabla\mathbf{w} : \sigma d^3\mathbf{x} = \int_G \mathbf{w}\cdot\mathbf{f}d^3\mathbf{x} \quad . \tag{6}$$

Upon inserting the free surface boundary condition, Eq. (6) condenses to

$$\int_G \rho\mathbf{w}\cdot\ddot{\mathbf{u}}d^3\mathbf{x} + \int_G \nabla\mathbf{w} : \sigma d^3\mathbf{x} = \int_G \mathbf{w}\cdot\mathbf{f}d^3\mathbf{x} \quad . \tag{7}$$

Finding a weak solution to the equations of motion means finding a displacement field $\mathbf{u}$ that satisfies the integral relation Eq. (7) and

$$\int_G \mathbf{w} \cdot \sigma d^3\mathbf{x} = \int_G \mathbf{w} \cdot (\mathbf{C} : \nabla \mathbf{u}) d^3\mathbf{x} \quad . \tag{8}$$

for any test function $\mathbf{w}$ and subject to the initial conditions.

2. Discretisation of the Equations of Motion

By using the Galerkin method, we approximate the p-component u_p of the displacement field $\mathbf{u}$ by a superposition of basis functions

$$\psi_{ijk}(\mathbf{x}) = \psi_{ijk}(x_1, x_2, x_3) \quad , \tag{9}$$

weighted by expansion coefficients u_p^{ijk}:

$$u_p(\mathbf{x}, t) \approx \bar{u}_p(\mathbf{x}, t) = \sum_{i,j,k=1}^{N+1} u_p^{ijk}(t) \psi_{ijk}(\mathbf{x}) \quad . \tag{10}$$

The corresponding approximation of the stress tensor components σ_{pq} is

$$\sigma_{pq}(\mathbf{x}, t) \approx \bar{\sigma}_{pq}(\mathbf{x}, t) = \sum_{i,j,k=1}^{N+1} \sigma_{pq}^{ijk}(t) \psi_{ijk}(\mathbf{x}) \quad . \tag{11}$$

To find a weak solution in the Galerkin sense, we replace the exact weak formulation forms from Eqs. (7) and (8) by the requirement that approximations $\bar{\mathbf{u}}$ and $\bar{\sigma}$ satisfy

$$\int_G \rho \mathbf{w} \cdot \ddot{\mathbf{u}} d^3\mathbf{x} + \int_G \nabla \mathbf{w} : \bar{\sigma} d^3\mathbf{x} = \int_G \mathbf{w} \cdot \mathbf{f} d^3\mathbf{x} \quad ,$$
$$\int_G \mathbf{w} \cdot \bar{\sigma} d^3\mathbf{x} = \int_G \mathbf{w} \cdot (\mathbf{C} : \nabla \bar{\mathbf{u}}) d^3\mathbf{x} \quad ,$$

for any test function, $w_{ijk}^p = \psi_{ijk} \mathbf{e}_p$ in the form of

$$\int_{G_e} \rho \psi_{ijk} \mathbf{e}_p \cdot \ddot{\mathbf{u}} d^3\mathbf{x} + \int_{G_e} \nabla(\psi_{ijk} \mathbf{e}_p) : \bar{\sigma} d^3\mathbf{x} = \int_{G_e} \psi_{ijk} \mathbf{e}_p \cdot \mathbf{f} d^3\mathbf{x} \tag{12}$$

$$\int_{G_e} \psi_{ijk} \mathbf{e}_p \cdot \bar{\sigma} d^3\mathbf{x} = \int_{G_e} \psi_{ijk} \mathbf{e}_p \cdot (\mathbf{C} : \nabla \bar{\mathbf{u}}) d^3\mathbf{x} \quad . \tag{13}$$

Here Eqs. (12) and (13) already assume that u_p and σ_{pq} are considered inside an element $G_e \subset \mathbb{R}^3$,where they can be represented by $(N+1)^3$ basis functions.

For the first term on the left-hand side of Eq. (12), we find

$$\begin{aligned}\mathbb{F}_{qrs}(\rho\ddot{u}_p) &:= \int_{G_e} \rho\psi_{qrs}\mathbf{e}_p \cdot \ddot{\mathbf{u}} d^3\mathbf{x} = \sum_{i,j,k=1}^{N+1} \int_{G_e} \rho\psi_{qrs}\ddot{u}_p^{ijk}\psi_{ijk} d^3\mathbf{x} \\ &= \sum_{i,j,k=1}^{N+1} \int_{\Lambda} \rho[\mathbf{x}(\xi)]\psi_{qrs}[\mathbf{x}(\xi)]\ddot{u}_p^{ijk}(t)\psi_{ijk}[\mathbf{x}(\xi)]J(\xi)d^3\xi \quad .\end{aligned} \tag{14}$$

For the basis function $\psi_{ijk}[\mathbf{x}(\xi)]$ we chose the product of three Lagrange polynomial collocated at the GLL points:

$$\psi_{ijk}[\mathbf{x}(\xi)] = l_i(\xi_1)l_j(\xi_2)l_k(\xi_3) \quad , \tag{15}$$

then we have

$$\mathbb{F}_{qrs}(\rho\ddot{u}_p) = \sum_{i,j,k=1}^{N+1} \int_{\Lambda} \rho(\xi)l_q(\xi_1)l_r(\xi_2)l_s(\xi_3)\ddot{u}_p^{ijk}(t)l_i(\xi_1)l_j(\xi_2)l_k(\xi_3)J(\xi^{qrs})d^3\xi \quad . \tag{16}$$

Apply the GLL quadrature rule to Eq. (16) yields the following simple expression:

$$\begin{aligned}\mathbb{F}_{qrs}(\rho\ddot{u}_p) &= \sum_{i,j,k=1}^{N+1} \sum_{f,g,h=1}^{N+1} w_f(\xi_1)w_g(\xi_2)w_h(\xi_3) \\ \cdot\rho(\xi^{fgh})\ddot{u}_p^{ijk}(t)J(\xi^{gh})&l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)l_i^f(\xi_1)l_j^g(\xi_2)l_k^h(\xi_3) \\ &= \rho(\xi^{qrs})w_q(\xi_1)w_r(\xi_2)w_s(\xi_3)\ddot{u}_p^{qrs}(t)J(\xi^{qrs}) \quad ,\end{aligned} \tag{17}$$

here, J represents the Jacobin matrix. For the second term on the left-hand side of Eq. (12), we need to know that, according to the proof for Eq. (5), $\nabla\mathbf{w} : \sigma$ actually represents a double inner product for two matrices, which is in the form of $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij}B_{ij}$.

Therefore, $\nabla \mathbf{w} : \sigma = \sum_{i,j=1}^{3} \partial_j w_i \sigma_{i,j}$ then we have

$$
\begin{aligned}
\mathbb{F}_{qrs}(\nabla : \sigma)_p &:= \int_{G_e} \nabla(\psi_{qrs}\mathbf{e}_p) : \bar{\sigma} d^3\mathbf{x} \\
&= \int_{G_e} \sum_{m,n=1}^{3} [\frac{\partial_n(\psi_{qrs}e_p)}{dx_m}] : \bar{\sigma} d^3\mathbf{x} \\
&= \int_{G_e} \sum_{m,n=1}^{3} [\frac{\partial(\psi_{qrs}e_p^n)}{dx_m}] \bar{\sigma}_{mn} d^3\mathbf{x} \\
&= \int_{G_e} \sum_{m,n=1}^{3} \frac{\partial(\psi_{qrs})}{dx_m} \delta_n^p \bar{\sigma}_{mn} d^3\mathbf{x} \\
&= \int_{G_e} \sum_{m=1}^{3} \frac{\partial(\psi_{qrs})}{dx_m} \bar{\sigma}_{mp} d^3\mathbf{x} \\
&= \int_{\Lambda} \sum_{m,n=1}^{3} \frac{\partial(\psi_{qrs})}{d\xi_n} \frac{d\xi_n}{dx_m}(\xi) \bar{\sigma}_{mp}(\xi) J(\xi) d^3\xi \\
&= \int_{\Lambda} \sum_{m,n=1}^{3} \frac{\partial[l_q(\xi_1)l_s(\xi_2)]}{d\xi_n} \frac{d\xi_n}{dx_m} \bar{\sigma}_{mp}[x(\xi)] J[x(\xi)] d^3\xi \\
&= \int_{\Lambda} \sum_{m}^{3} \dot{l}_q(\xi_1) l_r(\xi_2) l_s(\xi_3) \frac{d\xi_1}{dx_m} \bar{\sigma}_{mp}(\xi) J(\xi) d^3\xi \\
&+ \int_{\Lambda} \sum_{m}^{3} l_q(\xi_1) \dot{l}_r(\xi_2) l_s(\xi_3) \frac{d\xi_2}{dx_m} \bar{\sigma}_{mp}(\xi) J(\xi) d^3\xi \\
&+ \int_{\Lambda} \sum_{m}^{3} l_q(\xi_1) l_r(\xi_2) \dot{l}_s(\xi_3) \frac{d\xi_3}{dx_m} \bar{\sigma}_{mp}(\xi) J(\xi) d^3\xi \quad .
\end{aligned}
\tag{18}
$$

Now if we bring the GLL quadrature rule to Eq. (19), we have

$$
\begin{aligned}
\mathbb{F}_{qrs}(\nabla:\sigma)_p = &\sum_{i,j,k=1}^{N+1}\sum_{m=1}^{3} w_q w_r w_s \dot{l}_q^i(\xi_1) l_r^j(\xi_2) l_s^k(\xi_3)\frac{d\xi_1}{dx_m}(\xi^{ijk})\bar{\sigma}_{mp}(\xi^{ijk})J(\xi^{ijk})d^3\xi \\
&+\sum_{i,j,k=1}^{N+1}\sum_{m=1}^{3} w_q w_r w_s l_q^i(\xi_1) \dot{l}_r^r(\xi_2) l_s^k(\xi_3)\frac{d\xi_2}{dx_m}(\xi^{ijk})\bar{\sigma}_{mp}(\xi^{ijk})J(\xi^{ijk})d^3\xi \\
&+\sum_{i,j,k=1}^{N+1}\sum_{m=1}^{3} w_q w_r w_s l_q(\xi_1) l_r(\xi_2) \dot{l}_s(\xi_3)\frac{d\xi_3}{dx_m}(\xi^{ijk})\bar{\sigma}_{mp}(\xi^{ijk})J(\xi^{ijk})d^3\xi \\
&= \sum_{m=1}^{3}\sum_{i=1}^{N+1} w_q w_r w_s \dot{l}_q^i(\xi_1)\frac{d\xi_1}{dx_m}(\xi^{irs})\bar{\sigma}_{mp}(\xi^{irs})J(\xi^{irs})d^3\xi \\
&+\sum_{m=1}^{3}\sum_{i=1}^{N+1} w_q w_r w_s \dot{l}_r^i(\xi_2)\frac{d\xi_2}{dx_m}(\xi^{qis})\bar{\sigma}_{mp}(\xi^{qis})J(\xi^{qis})d^3\xi \\
&+\sum_{m=1}^{3}\sum_{i=1}^{N+1} w_q w_r w_s \dot{l}_s^i(\xi_3)\frac{d\xi_3}{dx_m}(\xi^{qri})\bar{\sigma}_{mp}(\xi^{qri})J(\xi^{qri})d^3\xi \quad .
\end{aligned}
\tag{19}
$$

Repeating the above procedure for the source term in Eq. (12)gives

$$
\mathbb{F}_{qrs}(\mathbf{f}_p) := w_q w_r w_s f_p(\xi^{qrs})J(\xi^{qrs}) \tag{20}
$$

It remains to consider the approximate weak form of the constitutive relation as specified by Eq. (13). For the left-hand term:

$$
\begin{aligned}
\mathbb{F}_{qrs}(\sigma_{mn}) &:= \int_{G_e}\psi_{qrs}(\mathbf{e}_m\cdot\bar{\sigma})_n d^3\mathbf{x} \\
&= \int_{G_e}\psi_{qrs}\sum_{i,j,k=1}^{N+1}\sigma_{mn}^{ijk}(t)\psi_{ijk}(\mathbf{x})d^3\mathbf{x} \\
&= \int_{\Lambda}\psi_{qrs}\sum_{i,j,k=1}^{N+1}\sigma_{mn}^{ijk}(t)\psi_{ijk}(\xi)J(\xi)d^3\xi \\
&= \int_{\Lambda}\sum_{i,j,k=1}^{N+1}\sigma_{mn}^{ijk}(t) l_q(\xi_1)l_r(\xi_2)l_s(\xi_3)l_i(\xi_1)l_j(\xi_2)l_k(\xi_3)J(\xi)d^3\xi \\
&= w_p(\xi_1)w_r(\xi_2)w_s(\xi_3)\sigma_{mn}^{qrs}(t)J(\xi^{qrs}) \quad ,
\end{aligned}
\tag{21}
$$

while the right-hand term could be simplified by

$$
\begin{aligned}
\mathbb{F}_{qrs}(\mathbf{C}:\nabla\mathbf{u})_{mn} &:= \int_{G_e}[\psi_{qrs}\mathbf{e}_m\cdot(\mathbf{C}:\nabla\bar{\mathbf{u}})]_n d^3\mathbf{x} \\
&= \int_{G_e}\psi_{qrs}\sum_{a,b=1}^{3}C_{mnab}(\nabla\bar{\mathbf{u}})_{ab}d^3\mathbf{x} \\
&= \int_{G_e}\sum_{a,b=1}^{3}\psi_{qrs}C_{mnab}\sum_{i,j,k=1}^{N+1}\frac{\partial\psi_{ijk}}{dx_a}u_b^{ijk}d^3\mathbf{x} \\
&= \int_{\Lambda}\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}\psi_{qrs}C_{mnab}\frac{\partial(l_i(\xi_1)l_j(\xi_2)l_k(\xi_3))}{d\xi_1}\frac{d\xi_1}{dx_a}u_b^{ijk}Jd^3\xi \\
&+ \int_{\Lambda}\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}\psi_{qrs}C_{mnab}\frac{\partial(l_i(\xi_1)l_j(\xi_2)l_k(\xi_3))}{d\xi_2}\frac{d\xi_2}{dx_a}u_b^{ijk}Jd^3\xi \\
&+ \int_{\Lambda}\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}\psi_{qrs}C_{mnab}\frac{\partial(l_i(\xi_1)l_j(\xi_2)l_k(\xi_3))}{d\xi_3}\frac{d\xi_3}{dx_a}u_b^{ijk}Jd^3\xi \\
&= \int_{\Lambda}\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}[l_q(\xi_1)l_s(\xi_2)]C_{mnab}\dot{l}_i(\xi_1)l_j(\xi_2)l_k(\xi_3)\frac{d\xi_1}{dx_a}[\xi]u_b^{ijk}J(\xi)d^3\xi \\
&+ \int_{\Lambda}\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}[l_q(\xi_1)l_r(\xi_2)l_s(\xi_3)]C_{mnab}l_i(\xi_1)\dot{l}_j(\xi_2)l_k(\xi_3)\frac{d\xi_2}{dx_a}[\xi]u_b^{ijk}J(\xi)d^3\xi \\
&+ \int_{\Lambda}\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}[l_q(\xi_1)l_r(\xi_2)l_s(\xi_3)]C_{mnab}l_i(\xi_1)l_j(\xi_2)\dot{l}_k(\xi_3)\frac{d\xi_2}{dx_a}[\xi]u_b^{ijk}J(\xi)d^3\xi \\
&= \sum_{f,g,h=1}^{N+1}w_f w_g w_h\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}[l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)]C_{mnab}\dot{l}_i^f(\xi_1)l_j^g(\xi_2)l_j^h(\xi_3)\frac{d\xi_1}{dx_a}(\xi^{fgh})u_b^{ijk}J(\xi^{fgh}) \\
&+ \sum_{f,g,h=1}^{N+1}w_f w_g w_h\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}[l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)]C_{mnab}l_i^f(\xi_1)\dot{l}_j^g(\xi_2)l_j^h(\xi_3)\frac{d\xi_1}{dx_a}(\xi^{fgh})u_b^{ijk}J(\xi^{fgh}) \\
& \sum_{f,g,h=1}^{N+1}w_f w_g w_h\sum_{a,b=1}^{3}\sum_{i,j,k=1}^{N+1}[l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)]C_{mnab}l_i^f(\xi_1)l_j^g(\xi_2)\dot{l}_j^h(\xi_3)\frac{d\xi_1}{dx_a}(\xi^{fgh})u_b^{ijk}J(\xi^{fgh}) \\
&= w_q w_r w_s\sum_{a,b=1}^{3}\sum_{i=1}^{N+1}C_{mnab}\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_a}(\xi^{qrs})u_b^{iqs}J(\xi^{qrs}) \\
&+ w_q w_r w_s\sum_{a,b=1}^{3}\sum_{i=1}^{N+1}C_{mnab}\dot{l}_i^r(\xi_2)\frac{d\xi_2}{dx_a}(\xi^{qrs})u_b^{qik}J(\xi^{qrs}) \\
&+ w_q w_r w_s\sum_{a,b=1}^{3}\sum_{i=1}^{N+1}C_{mnab}\dot{l}_i^s(\xi_2)\frac{d\xi_2}{dx_a}(\xi^{qrs})u_b^{qri}J(\xi^{qrs}) \quad .
\end{aligned}
\tag{22}
$$

Let's see if the above equation is right or not (with the 2D case for simple) with the Voigt notation for the tensor index as:

$$\begin{array}{ccccccc} ij & = & 11 & 22 & 33 & 23,32 & 13,31 & 12,21 \\ \Downarrow & = & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \alpha & = & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{23}$$

$$C_{ijkl} \Rightarrow C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \tag{24}$$

For the isotropic case, it only has 2 independent elements:

$$C_{\alpha\beta} = \begin{bmatrix} K+4\mu/3 & K-2\mu/3 & K-2\mu/3 & 0 & 0 & 0 \\ K-2\mu/3 & K+4\mu/3 & K-2\mu/3 & 0 & 0 & 0 \\ K-2\mu/3 & K-2\mu/3 & K+4\mu/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{25}$$

$$
\begin{aligned}
\sigma_{11}^{qrs} = c_{1111} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_1^{qi}] \\
+c_{1112} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_2^{qi}] \\
+c_{1121} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_1^{qi}] \\
+c_{1122} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_2^{qi}] \\
\sigma_{12}^{qrs} = c_{1211} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_1^{qi}] \\
+c_{1212} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_2^{qi}] \\
+c_{1221} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_1^{qi}] \\
+c_{1222} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_2^{qi}] \\
\sigma_{21}^{qrs} = c_{2111} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_1^{qi}] \\
+c_{2112} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_2^{qi}] \\
+c_{2121} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_1^{qi}] \\
+c_{2122} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_2^{qi}] \\
\sigma_{22}^{qrs} = c_{2211} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_1^{qi}] \\
+c_{2212} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_1}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_1}u_2^{qi}] \\
+c_{2221} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_1^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_1^{qi}] \\
+c_{2222} \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1)\frac{d\xi_1}{dx_2}u_2^{is} + \dot{l}_i^1(\xi_2)\frac{d\xi_2}{dx_2}u_2^{qi}]
\end{aligned}
\tag{26}
$$

Here for the isotropic, $K = \lambda + 2/3\mu$, so that $c_{1111} = \lambda + 2\mu$,$c_{1122} = \lambda$, $c_{1112} = c_{1121} = \mu$, therefore, Eq. (26) could be simplified by

$$
\begin{aligned}
\sigma_{11}^{qrs} &= (\lambda + 2\mu) \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_1} u_1^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_1} u_1^{qi}] \\
&+ \lambda \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_2} u_2^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_2} u_2^{qi}] \\
\sigma_{12}^{qrs} &= \mu \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_1} u_2^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_1} u_2^{qi}] \\
&+ \mu \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_2} u_1^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_2} u_1^{qi}] \\
\sigma_{21}^{qrs} &= \mu \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_1} u_2^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_1} u_2^{qi}] \\
&+ \mu \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_2} u_1^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_2} u_1^{qi}] \\
\sigma_{22}^{qrs} &= \lambda \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_1} u_1^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_1} u_1^{qi}] \\
&+ (\lambda + 2\mu) \sum_{i=1}^{N+1} [\dot{l}_i^q(\xi_1) \frac{d\xi_1}{dx_2} u_2^{is} + \dot{l}_i^1(\xi_2) \frac{d\xi_2}{dx_2} u_2^{qi}]
\end{aligned}
\tag{27}
$$

3. Isotropic weak solutions for the PS decoupling

Given the equation for separating amplitude-preserved vector S wave fields as

$$
\mathbf{u^s} = -\nabla \times (v_s^2 \nabla \times \mathbf{u}) \quad . \tag{28}
$$

To get a weak solution, we use the test function as

$$
\int_{G_e} \mathbf{w} \cdot \mathbf{u^s} d^3\mathbf{x} = -\int_{G_e} \mathbf{w} \cdot \nabla \times (v_s^2 \nabla \times \mathbf{u}) d^3\mathbf{x} \quad . \tag{29}
$$

Since $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$, and note $\Phi := v_s^2 \nabla \times \mathbf{u}$ therefore,

$$
\begin{aligned}
\int_{G_e} \mathbf{w} \cdot \mathbf{u^s} d^3\mathbf{x} &= -\int_{G_e} \mathbf{w} \cdot (\nabla \times \mathbf{\Phi}) d^3\mathbf{x} \\
&= \int_{G_e} \nabla \times \mathbf{w} \cdot \mathbf{\Phi} d^3\mathbf{x} \quad .
\end{aligned}
\tag{30}
$$

For any test function, $w_{ijk}^p = \psi_{ijk}\mathbf{e}_p$, the right hand of Eq. (1.30) has the form of

$$
\begin{aligned}
\mathbb{F}_{qrs}(\Phi)_p &:= \int_{G_e} \nabla \times \psi_{qrs}\mathbf{e}_p \cdot \bar{\mathbf{\Phi}} d^3\mathbf{x} \\
&= \int_{G_e} \sum_{m,n,p=1}^{3} \epsilon_{mnp} \frac{\partial \psi_{qrs} e_p}{\partial x_n} \Phi_m d^3\mathbf{x} \\
&= \int_{G_e} \sum_{m,n,p=1}^{3} \epsilon_{mnp} \dot{l}_q(\xi_1) l_r(\xi_2) l_s(\xi_3) \frac{\partial \xi_1}{\partial x_n}(\xi) \Phi_m e_p J(\xi) d^3\xi \\
&+ \int_{G_e} \sum_{m,n,p=1}^{3} \epsilon_{mnp} l_q(\xi_1) \dot{l}_r(\xi_2) l_s(\xi_3) \frac{\partial \xi_2}{\partial x_n}(\xi) \Phi_m e_p J(\xi) d^3\xi \\
&+ \int_{G_e} \sum_{m,n,p=1}^{3} \epsilon_{mnp} l_q(\xi_1) l_r(\xi_2) \dot{l}_s(\xi_3) \frac{\partial \xi_3}{\partial x_n}(\xi) \Phi_m e_p J(\xi) d^3\xi \\
&= \sum_{m,n,p=1}^{3} \epsilon_{mnp} \sum_{i=1}^{N+1} w_i w_r w_s \dot{l}_q^i(\xi_1) \frac{\partial \xi_1}{\partial x_n}(\xi^{irs}) \Phi_m e_p J(\xi^{irs}) d^3\xi \\
&+ \sum_{m,n,p=1}^{3} \epsilon_{mnp} \sum_{i=1}^{N+1} w_q w_i w_s \dot{l}_q^i(\xi_2) \frac{\partial \xi_2}{\partial x_n}(\xi^{qis}) \Phi_m e_p J(\xi^{qis}) d^3\xi \\
&+ \sum_{m,n,p=1}^{3} \epsilon_{mnp} \sum_{i=1}^{N+1} w_q w_r w_i \dot{l}_q^i(\xi_3) \frac{\partial \xi_3}{\partial x_n}(\xi^{qri}) \Phi_m e_p J(\xi^{qri}) d^3\xi \quad .
\end{aligned}
\tag{31}
$$

Now let's solve $\Phi := v_s^2 \nabla \times \mathbf{u}$ by neglecting the v_s^2 term, similarly, for any test function,

$$
\begin{aligned}
\mathbb{F}_{qrs}[\nabla\times\mathbf{u}]_m &:= \int_{G_e}\mathbf{w}\cdot\nabla\times\mathbf{u}d^3\mathbf{x} \\
&= \int_{G_e}[\psi_{qrs}\mathbf{e}]_m\cdot[\nabla\times\mathbf{u}]_m d^3\mathbf{x} \\
&= \int_{G_e}[\psi_{qrs}\mathbf{e}]_m\cdot[\sum_{m,n,p=1}^{3}\epsilon_{mnp}\frac{\partial u_p}{\partial x_n}e_m]d^3\mathbf{x} \\
&= \int_{G_e}\sum_{m,n,p=1}^{3}\psi_{qrs}\epsilon_{mnp}\frac{\partial u_p}{\partial x_n}e_m d^3\mathbf{x} \\
&= \int_{G_e}\sum_{m,n,p=1}^{3}\psi_{qrs}\epsilon_{mnp}[\frac{\partial u_p}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_n}+\frac{\partial u_p}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_n}+\frac{\partial u_p}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_n}]e_m J(\xi)d^3\xi \\
&= \int_{G_e}\sum_{m,n,p=1}^{3}\psi_{qrs}\epsilon_{mnp}e_m J(\xi)\sum_{i,j,k=1}^{N+1}\dot{l}_i(\xi_1)l_j(\xi_2)l_k(\xi_3)\frac{\partial \xi_1}{\partial x_n}(\xi)d^3\xi \\
&+ \int_{G_e}\sum_{m,n,p=1}^{3}\psi_{qrs}\epsilon_{mnp}e_m J(\xi)\sum_{i,j,k=1}^{N+1}l_i(\xi_1)\dot{l}_j(\xi_2)l_k(\xi_3)\frac{\partial \xi_2}{\partial x_n}(\xi)d^3\xi \\
&+ \int_{G_e}\sum_{m,n,p=1}^{3}\psi_{qrs}\epsilon_{mnp}e_m J(\xi)\sum_{i,j,k=1}^{N+1}l_i(\xi_1)l_j(\xi_2)\dot{l}_k(\xi_3)\frac{\partial \xi_3}{\partial x_n}(\xi)d^3\xi \\
&= \sum_{f,g,h=1}^{N+1}w_f w_g w_h l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)\sum_{m,n,p=1}^{3}\epsilon_{mnp}e_m J(\xi^{fgh})\sum_{i,j,k=1}^{N+1}\dot{l}_i^f(\xi_1)l_j^g(\xi_2)l_k^h(\xi_3)\frac{\partial \xi_1}{\partial x_n}(\xi) \\
&+ \sum_{f,g,h=1}^{N+1}w_f w_g w_h l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)\sum_{m,n,p=1}^{3}\epsilon_{mnp}e_m J(\xi^{fgh})\sum_{i,j,k=1}^{N+1}l_i^f(\xi_1)\dot{l}_j^g(\xi_2)l_k^h(\xi_3)\frac{\partial \xi_2}{\partial x_n}(\xi) \\
&+ \sum_{f,g,h=1}^{N+1}w_f w_g w_h l_q^f(\xi_1)l_r^g(\xi_2)l_s^h(\xi_3)\sum_{m,n,p=1}^{3}\epsilon_{mnp}e_m J(\xi^{fgh})\sum_{i,j,k=1}^{N+1}l_i^f(\xi_1)l_j^g(\xi_2)\dot{l}_k^h(\xi_3)\frac{\partial \xi_3}{\partial x_n}(\xi) \\
&= w_q w_r w_s\sum_{m,n,p=1}^{3}\epsilon_{mnp}e_m J(\xi^{qrs})\sum_{i=1}^{N+1}\dot{l}_i^q(\xi_1)\frac{\partial \xi_1}{\partial x_n}(\xi_{qrs}) \\
&+ w_q w_r w_s\sum_{m,n,p=1}^{3}\epsilon_{mnp}e_m J(\xi^{qrs})\sum_{i=1}^{N+1}\dot{l}_i^r(\xi_2)\frac{\partial \xi_2}{\partial x_n}(\xi_{qrs}) \\
&+ w_q w_r w_s\sum_{m,n,p=1}^{3}\epsilon_{mnp}e_m J(\xi^{qrs})\sum_{i=1}^{N+1}\dot{l}_i^s(\xi_3)\frac{\partial \xi_3}{\partial x_n}(\xi_{qrs}) \quad .
\end{aligned}
\tag{32}
$$

If given the equation for separating amplitude-preserved vector P wave fields as

$$
\mathbf{u}^{\mathbf{P}} = \nabla(v_p^2\nabla\cdot\mathbf{u}) \quad , \tag{33}
$$

then the weak solution could be

$$\int_{G_e} \mathbf{w} \cdot \mathbf{u^p} d^3\mathbf{x} = \int_{G_e} \mathbf{w} \cdot \nabla(v_p^2 \nabla \cdot \mathbf{u}) d^3\mathbf{x} \quad . \tag{34}$$

According to the Wiki ($https : //en.m.wikipedia.org/wiki/Vector_calculus_identities$), the integral by parts of the vector dot product is in the form of

$$\iiint \alpha \nabla \cdot A d\mathbf{V} = \oint_{\partial V} \alpha A \cdot d\mathbf{S} - \iiint A \cdot \nabla \alpha d\mathbf{V} \quad . \tag{35}$$

According to Eq. (3), we get the integral by paths of Eq. (34) in the form of

$$\int_{G_e} \mathbf{w} \cdot \mathbf{u^p} d^3\mathbf{x} = \int_{\partial G} \alpha \mathbf{w} \cdot \mathbf{n} d^2\mathbf{x} - \int_{G_e} \alpha \nabla \cdot \mathbf{w} \quad . \tag{36}$$

If we take the boundary condition into the first term of the last equation, we have

$$\int_{G_e} \mathbf{w} \cdot \mathbf{u^p} d^3\mathbf{x} = - \int_{G_e} \alpha \nabla \cdot \mathbf{w} \quad , \tag{37}$$

where we note $\alpha = v_p^2 \nabla \cdot u$. Now let's try to get the weak form solution,

$$
\begin{aligned}
\mathbb{F}_{qrs}[\nabla \cdot \mathbf{w}]_p := \int_{G_e} [\nabla \cdot (\psi_{qrs}\mathbf{e_p})] \alpha^{qrs} d^3\mathbf{x} \\
= \int_{G_e} \frac{\partial \psi^{qrs}}{\partial x_p} \alpha d^3\mathbf{x} \\
= \int_{G_e} \dot{l}_q l_r l_s \frac{\partial \xi_1}{\partial x_p} \alpha(\xi) J(\xi) d^3\xi \\
+ \int_{G_e} l_q \dot{l}_r l_s \frac{\partial \xi_2}{\partial x_p} \alpha(\xi) J(\xi) d^3\xi \\
+ \int_{G_e} l_q l_r \dot{l}_s \frac{\partial \xi_3}{\partial x_p} \alpha(\xi) J(\xi) d^3\xi \\
= \sum_{ijk}^{N+1} w_i w_j w_k \dot{l}_q^i l_r^j l_s^k \frac{\partial \xi_1}{\partial x_p}(\xi^{ijk}) \alpha(\xi^{ijk}) J(\xi^{ijk}) \\
+ \sum_{ijk}^{N+1} w_i w_j w_k l_q^i \dot{l}_r^j l_s^k \frac{\partial \xi_2}{\partial x_p}(\xi^{ijk}) \alpha(\xi^{ijk}) J(\xi^{ijk}) \\
+ \sum_{ijk}^{N+1} w_i w_j w_k l_q^i \dot{l}_r^j l_s^k \frac{\partial \xi_3}{\partial x_p}(\xi^{ijk}) \alpha(\xi^{ijk}) J(\xi^{ijk}) \\
= \sum_{i=1}^{N+1} w_i w_r w_s \dot{l}_q^i \frac{\partial \xi_1}{\partial x_p}(\xi^{irs}) \alpha(\xi^{irs}) J(\xi^{irs}) \\
+ \sum_{i=1}^{N+1} w_i w_r w_s \dot{l}_r^i \frac{\partial \xi_2}{\partial x_p}(\xi^{qis}) \alpha(\xi^{qis}) J(\xi^{irs}) \\
+ \sum_{i=1}^{N+1} w_i w_r w_s \dot{l}_s^i \frac{\partial \xi_3}{\partial x_p}(\xi^{qri}) \alpha(\xi^{qri}) J(\xi^{irs}) \quad .\mathring{a}
\end{aligned}
\tag{38}
$$

It is not necessary to get this weak form of this term $\alpha = v_p^2 \nabla \cdot u$, since it is fairly easy.

References

Fichtner, A. (2010). *Full seismic waveform modelling and inversion.* Springer Science & Business Media.