Maximum principle for optimal control of fully coupled mean-field forward-backward stochastic differential equations with Teugels martingales under partial observation

G. Saranya*, P. Muthukumar*[1] and Mokhtar Hafayed**

*Department of Mathematics,

The Gandhigram Rural Institute (Deemed to be University),

Gandhigram - 624 302, Tamil Nadu, India.

**Laboratory of Mathematical Analysis, Probability and Optimizations,

PO Box 145, University of Biskra, BISKRA, 7000, Algeria

Abstract: The necessary conditions for the optimal control of partially observed, fully coupled forward-backward mean-field stochastic differential equations driven by Teugels martingales are discussed in this paper. In this context, we make the assumption that the forward diffusion coefficient and the martingale coefficient are independent of the control variable, and the control domain may not necessarily be convex. For this class of optimal control problems, we derive the stochastic maximum principle based on the classical method of spike variations and the filtering techniques. The adjoint processes that are related to the variational equations are determined by the solutions of proposed forward-backward stochastic differential equations in finite-dimensional spaces. Further, the Hamiltonian function is used to obtain the maximum principle for the optimality of the given control system. Our results are then applied to the mean-field type problem of linear quadratic stochastic optimization.

Keywords: fully coupled forward-backward stochastic differential equations; partially observed system; stochastic maximum principle; Teugels martingales.

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1. Introduction

Stochastic optimal control problems (SOCP) have attained significant attention from researchers and have a lot of applications in various fields, including finance, engineering, economics, and operations research. Several methods are used to solve SOCP. One such important method is the Pontryagin maximum principle. The paper [1] extensively examined the variational principle of optimality for

[1]Corresponding Author. Email: pmuthukumargri@gmail.com, Phone: 91-451-2452371, Fax: 91-451-2454466
saranyaganesangri@gmail.com (G. Saranya) hafa.mokh@yahoo.com (Mokhtar Hafayed)

stochastic systems with mixed initial and terminal conditions, specifically addressing the general case of a non-convex control domain. The consideration of mixed initial and terminal conditions and the cost functional in [1] indicates that the problem studied by the author is more broad and inclusive.

Forward-Backward Stochastic Differential Equations (FBSDE) has lot of applications in finance. One such problem is risk minimizing portfolio problem. While investigating SOCPs, Pardoux and Peng [2] introduced general nonlinear backward stochastic differential equations (BSDEs). A forward-backward stochastic differential equation (FBSDE) is constructed by coupling a BSDE with a forward stochastic differential equation (SDE). For the forward-backward stochastic jump-diffusion differential systems with observation noises, the maximum principle is obtained [3]. The authors in [4] conducted research into the fully coupled forward-backward stochastic control system using convex variation methods and duality techniques. On the other hand, in [5], the authors established the maximum principle for FBSDEs with non-convex control domain. The paper [6] discussed the optimal control for fully coupled forward-backward doubly SDEs. As an application, they also addressed the linear quadratic (LQ) problem.

Kac [7] and McKean [8] initially introduced the mean-field SDE (MF-SDE) for investigating physical systems with a large number of interacting particles. Solutions of MF-SDEs typically occur as a limit in law of an increasing number of identically distributed interacting processes. Applications of the mean-field models to economics and finance were further developed [9]. Since then, the optimal control problem (OCP) of MF- SDE has significantly improved. Mean-field FBSDEs (MF-FBSDE), which are used to describe how a large population of interacting agents or particles behaves and play an important role in statistics and financial engineering. The necessary conditions for the OCPs of MF-BSDEs with delay and noisy memory in the infinite horizon, as well as for the McKean Vlasov FBSDEs, were established in [10] and [11]. The optimal transport problem was studied and the necessary conditions for the optimality of the extended mean field control systems have been developed [12]. To gain a more comprehensive understanding of FBSDEs and MF-FBSDEs, readers are encouraged to consult references [13–15].

It is common knowledge that the SOCPs related to Teugels martingales can accurately explain the randomness in the environment. Subsequently, several studies have examined BSDEs and SDEs that are influenced by the combined impact of Brownian motion and Teugels martingales. The authors in [16] investigated the necessary and sufficient conditions for SDEs and discovered the maximum principle for Lévy process-related backward stochastic systems. For additional insights into Lévy processes and Teugels martingales, readers are encouraged to refer [17–20] for further relevant findings and real-world applications.

Recently authors in [21, 22] discussed the maximum principle for the stochastic systems with full information. However, in many practical situations, such as the well-known stochastic recursive optimal problems [23], [24] and the risk-sensitive optimal portfolio problems [25], the states of the system cannot be observed directly, and the controllers have to make a decision partially according to their observable information. The objective of the optimal control problem with partial observation, is to derive a suitable optimal control to the model, where the controller has less information than the complete information filtration. In particular, a partially observed optimal control problem can be used to build an economic model in which there are information gaps among economic agents. Thus, studying partially-observed optimal control problems (PO-OCP) is natural and necessary. The necessary conditions for the optimality of the PO-OCPs of McKean-Vlasov type systems were discussed in [26] and the LQ nonzero-sum stochastic differential game with Markov jump was studied [27]. FBSDEs with jumps and regime switching under partial observation was studied by the authors in [28].

While investigating the PO-OCPs of SDEs, many authors derived the necessary conditions for optimality under the assumption of convex control domain. Readers can refer to [29–31]. But in most of the cases, the control domain is need not be convex. Thus the aim of this paper is to deepen the investigation of optimal control of partially observed, fully coupled FBSDE of mean-field type driven by Teugels martingales under non-convex control domain. Compared our work with [1], a novel combination of Teugels martingales and mean-field theory, we newly discussed the necessary conditions for partially observed MF-FBSDEs with Teugels martingales.

The following are the major contributions in this paper:

- Teugels martingales provide a more realistic representation of certain real-world phenomena, such as financial asset prices, where the sudden jumps or discontinuous movements will occur. Thus, the partially observed and fully coupled FBSDE of mean-field type driven by Teugels martingales serves as a novel system.
- Incorporating the mean field terms in the coupled system makes the system more complex.
- The maximum principle for optimal control of the proposed system is extended to accommodate a non-convex control domain, allowing for more comprehensive applicability.
- The mean-field LQ stochastic optimization problem provides insight into the proposed theoretical results.

The organization of this paper is as follows:

Section 2 of this paper is dedicated to formulating the fully coupled MF-FBSDEs with partial observation, driven by Teugels martingales. In Section 3, we derive the variational inequality (V.I.) using spike variational techniques and variational equations for our main result. Furthermore, we derive the maximum principle for the proposed control system governed by fully coupled MF-FBSDE. In Section 4, the LQ stochastic OCP is showcased as an application.

2. Model Formulation

In this article, $\mathbf{R}^n$ is used to denote the n-dimensional Euclidean space. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. The transpose of a matrix is denoted by $\top$ in the superscripts. $\bar{K}$ is always used to represent a positive constant and ς denotes time.

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_\varsigma), \mathcal{P})$ be a complete filtered probability space with a normal filtration $\mathfrak{F}_\varsigma$ which is generated by three mutually independent stochastic processes: A d-dimensional Brownian motion $\{\mathcal{B}(\varsigma)\}_{\varsigma \geq 0}$, an $\mathfrak{r}$-dimensional Brownian motion $\{\mathcal{W}(\varsigma)\}_{\varsigma \geq 0}$ and a 1-dimensional Lévy process $\{\tilde{\mathcal{L}}(\varsigma)\}_{\varsigma \geq 0}$, where the Lévy measure ν of $\{\tilde{\mathcal{L}}(\varsigma)\}_{\varsigma \geq 0}$ satisfies $\int_{\mathbf{R}}(1 \wedge x^2)\nu(dx) < \infty$, and for some $\varepsilon > 0$ and $\lambda > 0, \int_{(-\varepsilon,\varepsilon)} exp(\lambda|x|)\nu(dx) < \infty$. For some $\lambda > 0$, we denote $\tilde{\mathcal{L}}^1(\varsigma) = \tilde{\mathcal{L}}(\varsigma)$ and $\tilde{\mathcal{L}}^i(\varsigma) = \sum_{0<\zeta\leq\varsigma} (\triangle L(\zeta))^i$ for $(i \geq 2)$, where $\triangle\tilde{\mathcal{L}}(\zeta) = \tilde{\mathcal{L}}(\zeta) - \tilde{\mathcal{L}}(\zeta_-), S^i(\varsigma) = \tilde{\mathcal{L}}^i(\varsigma) - \mathtt{E}[\tilde{\mathcal{L}}^i(t)]$ is the compensated power jump process of order i. Denote $H^i(\varsigma) = c_{i,i}S^i(\varsigma) + c_{i,i-1}S^{i-1}(\varsigma) + \cdots + c_{i,1}S^1(\varsigma)$, and $\{H^i(\varsigma)\}_{i=1}^{\infty}$ is called Teugels martingales associated with the Lévy process $\{\tilde{\mathcal{L}}(\varsigma)\}_{\varsigma \geq 0}$. Let $T_f > 0$ be a fixed time horizon. $L^2(\Omega, \mathfrak{F}^{\mathcal{B}}_{T_f}; \mathbf{R}^n)$ denotes the space of all $\mathbf{R}^n$-valued $\mathfrak{F}^{\mathcal{B}}_{T_f}$-measurable random variables ξ such that $\mathtt{E}[\ |\xi|^2\ ] < \infty, L^2_{\mathfrak{F}^{\mathcal{B}}}([0, T_f]; \mathbf{R}^n)$ denotes the space of all $\mathbf{R}^n$-valued ${\mathfrak{F}_\varsigma}^{\mathcal{B}}$-adapted processes Ψ_ς such that $\mathtt{E}[\int_0^{T_f} |\Psi_\varsigma|^2 dt] < \infty$.

Let $\mathbb{U}$ be a non empty, non-convex subset of $\mathbf{R}^k$. An admissible control variable is denoted as $v : [0, T_f] \times \Omega \longrightarrow \mathbb{U}$ and meets the following criteria: it is $\mathfrak{F}^{\mathcal{W}}_\varsigma$-adapted, and its absolute moments are bounded. That is $\sup_{0\leq\varsigma\leq T_f} \mathtt{E}|v(\varsigma)|^i < \infty$, for all positive integers i. Let $\mathcal{U}_{ad}$ represents the collection of all admissible control variables.

Let $v(\cdot) \in \mathcal{U}_{ad}$ be given. Now, consider the following control system governed by fully-coupled FBSDE of mean-field type:

$$dp_v(\varsigma) = h(\varsigma, p_v(\varsigma), \mathtt{E}[p_v(\varsigma)], q_v(\varsigma), \mathtt{E}[q_v(\varsigma)], r_v(\varsigma), \mathtt{E}[r_v(\varsigma)], s(\varsigma), \mathtt{E}[s(\varsigma)], v(\varsigma))d\varsigma$$

$$+\sum_{j=1}^{d}\sigma^{j}(\varsigma,p_v(\varsigma),\mathtt{E}[p_v(\varsigma)],q_v(\varsigma),\mathtt{E}[q_v(\varsigma)],r_v(\varsigma),\mathtt{E}[r_v(\varsigma)],s(\varsigma),\mathtt{E}[s(\varsigma)])d\mathcal{B}^{j}(\varsigma)$$
$$+\sum_{j=1}^{\infty}f^{j}(\varsigma,p_v(\varsigma_-),\mathtt{E}[p_v(\varsigma_-)],q_v(\varsigma_-),\mathtt{E}[q_v(\varsigma_-)],r_v(\varsigma),\mathtt{E}[r_v(\varsigma)],s(\varsigma),\mathtt{E}[s(\varsigma)])dH^{j}(\varsigma),$$
$$-dq_v(\varsigma)=k(\varsigma,p_v(\varsigma),\mathtt{E}[p_v(\varsigma)],q_v(\varsigma),\mathtt{E}[q_v(\varsigma)],r_v(\varsigma),\mathtt{E}[r_v(\varsigma)],s(\varsigma),\mathtt{E}[s(\varsigma)],v(\varsigma))d\varsigma-\sum_{j=1}^{d}(r_v)^{j}(\varsigma)d\mathcal{B}^{j}(\varsigma)$$
$$-\sum_{j=1}^{\infty}s^{j}(\varsigma)dH^{j}(\varsigma),$$
$$p_v(0)=p_0,$$
$$q_v(T_f)=l(p_v(T_f),\mathtt{E}[p_v(T_f)]). \quad (1)$$

Here, the state processes $(p_v(\varsigma),\mathtt{E}[p_v(\varsigma)],q_v(\varsigma),\mathtt{E}[q_v(\varsigma)],r_v(\varsigma),\mathtt{E}[r_v(\varsigma)])\equiv(p_v(\varsigma,\omega),\mathtt{E}[p_v(\varsigma,\omega)],q_v(\varsigma,\omega),$ $\mathtt{E}[q_v(\varsigma,\omega)],r_v(\varsigma,\omega),\mathtt{E}[r_v(\varsigma,\omega)])\in\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^{m\times d}\times\mathbf{R}^{m\times d},0\le\varsigma\le T_f,\omega\in\Omega,p_0\in\mathbf{R}^n$ is deterministic and

$h:[0,T_f]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^{m\times d}\times\mathbf{R}^{m\times d}\times l^2(\mathbf{R}^m)\times l^2(\mathbf{R}^m)\times\mathbb{U}\longrightarrow\mathbf{R}^n,$

$\sigma^j:[0,T_f]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^{m\times d}\times\mathbf{R}^{m\times d}\times l^2(\mathbf{R}^m)\times l^2(\mathbf{R}^m)\longrightarrow\mathbf{R}^n,$

$f:[0,T_f]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^{m\times d}\times\mathbf{R}^{m\times d}\times l^2(\mathbf{R}^m)\times l^2(\mathbf{R}^m)\longrightarrow l^2(\mathbf{R}^n),$

$k:[0,T_f]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^{m\times d}\times\mathbf{R}^{m\times d}\times l^2(\mathbf{R}^m)\times l^2(\mathbf{R}^m)\times\mathbb{U}\longrightarrow\mathbf{R}^m,$

$l:\mathbf{R}^n\times\mathbf{R}^n\longrightarrow\mathbf{R}^m.$

In our assumptions, the state processes $(p_v(\varsigma),\mathtt{E}[p_v(\varsigma)],q_v(\varsigma),\mathtt{E}[q_v(\varsigma)],r_v(\varsigma),\mathtt{E}[r_v(\varsigma)])$ are not directly observable. However, the controllers are equipped with the capability to observe a correlated noisy process $\mathcal{W}(\cdot)$ that is associated with the state process. This relationship is characterized by

$$d\mathcal{W}(\varsigma)=g(\varsigma,p_v(\varsigma),\mathtt{E}[p_v(\varsigma)],q_v(\varsigma),\mathtt{E}[q_v(\varsigma)],v(\varsigma))d\varsigma+d\tilde{\mathcal{B}}(\varsigma),$$
$$\mathcal{W}(0)=0. \quad (2)$$

Here $\tilde{\mathcal{B}}(\cdot)$ represents an $\mathfrak{r}$-dimensional stochastic process that depends on $v(\cdot)$ and

$g:[0,T_f]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbb{U}\longrightarrow\mathbf{R}^{\mathfrak{r}}.$

To guarantee the existence and uniqueness of the solutions for the aforementioned fully coupled MF-FBSDE (1), we impose the subsequent assumptions: **(A1)** and the $\mathcal{G}$-monotonic conditions **(A2)** as employed in [32].

For this let us take an $m\times n$ matrix $\mathcal{G}$ of full rank and for any $v(\cdot)\in\mathcal{U}_{ad}$, we employ the notations

$\Gamma_v := (p_v, \mathtt{E}[p_v], q_v, \mathtt{E}[q_v], r_v, \mathtt{E}[r_v])$ and $\mathcal{A}(\varsigma, \Gamma_v, s, \mathtt{E}[s], v) := \begin{pmatrix} -\mathcal{G}^\top k \\ \mathcal{G}h \\ \mathcal{G}\sigma \end{pmatrix} (\varsigma, \Gamma_v, s, \mathtt{E}[s], v).$

Here, $\mathcal{G}\sigma \equiv (\mathcal{G}\sigma_1, \mathcal{G}\sigma_2, \cdots, \mathcal{G}\sigma_d)$. We consider the following assumption to be valid for our investigation.

(A1) Lipschitz and linear growth conditions for nonlinear functions in system:

(i) $\forall \Gamma_v, v(\cdot) \in \mathcal{U}_{ad}$ and $(s, \mathtt{E}[s])$, $\mathcal{A}(\cdot, \Gamma_v, v) \in L^2_{\mathfrak{F}^W}([0, T_f]; \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \times \mathbf{R}^{m\times d} \times l^2(\mathbf{R}^n) \times l^2(\mathbf{R}^n))$,

(ii) h, k are differentiable in Γ_v and its derivatives are continuous. It has uniformly bounded, uniformly Lipschitz partial derivatives and are bounded by $\bar{K}(1 + |p| + |\mathtt{E}[p]| + |q| + |\mathtt{E}[q]| + |r| + |\mathtt{E}[r]| + |v|)$,

(iii) g is differentiable in $(p, \mathtt{E}[p], q, \mathtt{E}[q])$, continuous in v and uniformly bounded. Its derivatives are continuous and all uniformly bounded,

(iv) For each $j = 1, 2, 3, \cdots, d$, σ^j is continuously differentiable in Γ_v, its partial derivatives are uniformly bounded. Additionally, σ^j is bounded by $\bar{K}(1+|p|+|\mathtt{E}[p]|+|q|+|\mathtt{E}[q]|+|r|+|\mathtt{E}[r]|)$,

(v) For each $j = 1, 2, \ldots \infty$, f^j is uniformly Lipschitz with respect to $(\Gamma_v, s, \mathtt{E}[s])$,

(vi) l is continuously differentiable in $(p, \mathtt{E}[p])$, $l_p, l_{\mathtt{E}[p]}$ are uniformly bounded and l is bounded by $\bar{K}(1 + |p| + |\mathtt{E}[p]|)$,

(vii) For each $(p, \mathtt{E}[p]) \in \mathbf{R}^n \times \mathbf{R}^n, l(p, \mathtt{E}[p]) \in L^2(\Omega, \mathfrak{F}^{\mathcal{B}}_{T_f}; \mathbf{R}^m)$.

(A2) G-Monotonic conditions:

$$\Big\langle \mathcal{A}(\varsigma, \Gamma_v, s, \mathtt{E}[s], v) - \mathcal{A}(\varsigma, \bar{\Gamma}_v, \bar{s}, \mathtt{E}[\bar{s}], v), \Gamma_v - \bar{\Gamma}_v \Big\rangle + \sum_{j=1}^{\infty} \Big\langle \mathcal{G}f^j(\varsigma, \Gamma_v, s, \mathtt{E}[s]) - \mathcal{G}f^j(\varsigma, \bar{\Gamma}_v, \bar{s}, \mathtt{E}[\bar{s}]), \bar{s}^j \Big\rangle$$
$$\leq -\beta_1\Big(|\mathcal{G}\hat{p}_v|^2 + |\mathcal{G}\mathtt{E}\hat{p}_v|^2\Big) - \beta_2\Big(|\mathcal{G}\hat{q}_v|^2 + |\mathcal{G}\mathtt{E}\hat{q}_v|^2 + |\mathcal{G}\hat{r}_v|^2 + |\mathcal{G}\mathtt{E}\hat{r}_v|^2 + \sum_{j=1}^{\infty} |\mathcal{G}^\top \bar{s}^j|^2\Big),$$

$$\Big\langle l(p_v, \mathtt{E}[p_v]) - l(\bar{p}_v, \mathtt{E}[\bar{p}_v]), \mathcal{G}((p_v, \mathtt{E}[p_v]) - (\bar{p}_v, \mathtt{E}[\bar{p}_v])) \Big\rangle \geq \mu_1\Big(|\mathcal{G}\hat{p}_v|^2 + |\mathcal{G}\mathtt{E}[\hat{p}_v]|^2\Big),$$

where $\Gamma_v = (p_v, q_v, r_v)$, $\bar{\Gamma}_v = (\bar{p}_v, \bar{q}_v, \bar{r}_v)$, $\hat{p}_v = p_v - \bar{p}_v, \hat{q}_v = q_v - \bar{q}_v$, $\hat{r}_v = r_v - \bar{r}_v$, and the given non-negative constants β_1, β_2 and μ_1 with $\beta_1+\beta_2 > 0, \beta_2+\mu_1 > 0$. Moreover we have $\mu_1 > 0, \beta_1 > 0 (\beta_2 > 0)$ when $m > n (m < n)$.

Remark 2.1. *For any $v(\cdot) \in \mathcal{U}_{ad}$, under the assumptions **(A1)** and **(A2)**, we can establish that the MF-FBSDE* (1) *possesses a unique solution* $\Gamma_v(\cdot) \equiv (p_v, \mathtt{E}[p_v], q_v, \mathtt{E}[q_v], r_v, \mathtt{E}[r_v])$. *This result is derived from Theorem 2.6 in* [32], *and the solution is commonly referred to as the corresponding trajectory.*

In order to find the optimal control for our partially observed system (1) with (2), we define the following change of measure. That is $d\mathcal{P}^v := \mathcal{Y}_v(\varsigma) d\mathcal{P}$, where

$$\begin{aligned}\mathcal{Y}_v(\varsigma) :=& exp\bigg\{ \int_0^\varsigma \langle g(\zeta, p_v(\zeta), \mathtt{E}[p_v(\zeta)], q_v(\zeta), \mathtt{E}[q_v(\zeta)], v(\zeta)), d\mathcal{W}(\zeta)\rangle \\ &- \frac{1}{2}\int_0^\varsigma |g(\zeta, p_v(\zeta), \mathtt{E}[p_v(\zeta)], q_v(\zeta), \mathtt{E}[q_v(\zeta)], v(\zeta))|^2 d\zeta \bigg\}.\end{aligned} \tag{3}$$

It is evident that $\mathcal{Y}_v(\cdot)$ is the unique solution of the following equation (4), which is adapted to $\mathfrak{F}_\varsigma^{\mathcal{W}}$.

$$\begin{aligned}&d\mathcal{Y}_v(\varsigma) = \mathcal{Y}_v(\varsigma)\langle g(\zeta, p_v(\zeta), \mathtt{E}[p_v(\zeta)], q_v(\zeta), \mathtt{E}[q_v(\zeta)], v(\zeta)), d\mathcal{W}(\zeta)\rangle, \\ &\mathcal{Y}_v(0) = 1.\end{aligned} \tag{4}$$

By employing Itô's formula, one can demonstrate that $\sup\limits_{0 \le \varsigma \le T_f} \mathtt{E}|\mathcal{Y}_v(\varsigma)|^i < \infty$, for all positive integers i. As a result of Girsanov's theorem and **(A1)**, $\mathcal{P}^v$ is a new probability measure and $\mathbf{R}^{d+\mathfrak{r}}$-valued standard Brownian motion $(\mathcal{B}(\cdot), \tilde{\mathcal{B}}(\cdot))$ which is defined on the new probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\varsigma\}, \mathcal{P}^v)$.

Remark 2.2. *The equation* (3) *is known as Doléans-Dade exponential. It allows us to transform a stochastic integral involving a local martingale into a true martingale. This transformation simplifies the analysis of stochastic processes and provides a powerful tool for solving various problems in stochastic finance, option pricing, and other areas involving stochastic modeling.*

The cost functional for the proposed System (1) is given by the following equation:

$$\begin{aligned}\mathcal{J}^*(v(\cdot)) := \mathtt{E}^v\bigg\{ &\int_0^{T_f} \varphi(\varsigma, p_v(\varsigma), \mathtt{E}[p_v(\varsigma)], q_v(\varsigma), \mathtt{E}[q_v(\varsigma)], r_v(\varsigma), \mathtt{E}[r_v(\varsigma)], s_v(\varsigma), \mathtt{E}[s_v(\varsigma)], v(\varsigma)) d\varsigma \\ &+ \phi(p_v(T_f), \mathtt{E}[p_v(T_f)]) + \gamma(q_v(0), \mathtt{E}[q_v(0)]) \bigg\}.\end{aligned} \tag{5}$$

In this context, $\mathtt{E}^v$ signifies the expectation on the probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_\varsigma, \mathcal{P}^v)$ and

$\varphi : [0, T_f] \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \times \mathbf{R}^{m\times d} \times l^2(\mathbf{R}^m) \times l^2(\mathbf{R}^m) \times \mathbb{U} \longrightarrow \mathbf{R}$,

$\phi : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$,

$\gamma : \mathbf{R}^m \times \mathbf{R}^m \longrightarrow \mathbf{R}$.

We need the following hypothesis.

(A3) Lipschitz and linear growth conditions for nonlinear functions in cost functional:

(i) φ is continuous in v, continuously differentiable in $(p, \mathtt{E}[p], q, \mathtt{E}[q], r, \mathtt{E}[r], s, \mathtt{E}[s])$, its partial derivatives are continuous in $(p, \mathtt{E}[p], q, \mathtt{E}[q], r, \mathtt{E}[r], s, \mathtt{E}[s], v)$ and bounded by $\bar{K}(1+|p|+|\mathtt{E}[p]|+|q| + |\mathtt{E}[q]| + |r| + |\mathtt{E}[r]| + |s| + |\mathtt{E}[s]| + |v|)$,

(ii) ϕ is continuously differentiable and ϕ_p is bounded by $\bar{K}(1 + |p|)$ and $\phi_{\mathtt{E}[p]}$ is bounded by $\bar{K}(1 + |\mathtt{E}[p]|)$,

(iii) γ is continuously differentiable, γ_q is bounded by $\bar{K}(1 + |q|)$ and $\gamma_{E[q]}$ is bounded by $\bar{K}(1 + |\mathtt{E}[q]|)$.

The primary objective of the proposed PO-OCP is to minimize (5), while considering $v(\cdot) \in \mathcal{U}_{ad}$ as the admissible control, subject to the constraints given by (1) and (2).
In other words, our goal is to find $u(\cdot) \in \mathcal{U}_{ad}$ that satisfies

$$\mathcal{J}^*(u(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}} \mathcal{J}^*(v(\cdot)). \tag{6}$$

Based on [33], the cost functional (5) is modified as follows:

$$\begin{aligned}\mathcal{J}^*(v(\cdot)) := \mathtt{E}^v\Big\{ \int_0^{T_f} &\mathcal{Y}_v(\varsigma)\varphi(\varsigma, p_v(\varsigma), \mathtt{E}[p_v(\varsigma)], q_v(\varsigma), \mathtt{E}[q_v(\varsigma)], r_v(\varsigma), \mathtt{E}[r_v(\varsigma)], s_v(\varsigma), \mathtt{E}[s_v(\varsigma)], v(\varsigma))d\varsigma \\ &+ \mathcal{Y}_v(T_f)\phi(p_v(T_f), \mathtt{E}[p_v(T_f)]) + \gamma(q_v(0), \mathtt{E}[q_v(0)])\Big\}.\end{aligned} \tag{7}$$

Therefore, the original problem (6) can be reformulated as minimizing (7) with respect to $v(\cdot) \in \mathcal{U}_{ad}$ while satisfying the constraints (1) and (4).

3. Partially Observed Stochastic Maximum Principle

3.1. **Spike Variation and ε-order Estimations.** Suppose that $u(\cdot) \in \mathcal{U}_{ad}$ is an optimal control, $\Gamma(\cdot) \equiv (p(\cdot), \mathtt{E}[p(\cdot)], q(\cdot), \mathtt{E}[q(\cdot)], r(\cdot), \mathtt{E}[r(\cdot)])$ represents the optimal trajectory of (1). Similarly, let $\mathcal{Y}(\cdot)$ denote the solution of (4). Here, we use the spike variation which is used in [33] as follows:

$$u^\varepsilon(\varsigma) := \begin{cases} v, \text{ if } \delta \le \varsigma \le \delta + \varepsilon, \\ u(\varsigma), \text{ otherwise.} \end{cases} \tag{8}$$

Assuming that ε is a positive value which is sufficiently small, $0 \le \delta \le T_f$ is fixed, and $v \in \mathbb{U}$ is an arbitrary random variable that is measurable with respect to $\mathfrak{F}_\delta^{\mathcal{W}}$ such that $\sup_{\omega\in\Omega} |v(\omega)| < +\infty$. Clearly, $u^\varepsilon(\cdot)$ is admissible.

Let $\Gamma^\varepsilon(\cdot) \equiv (p^\varepsilon(\cdot), \mathtt{E}[p^\varepsilon(\cdot)], q^\varepsilon(\cdot), \mathtt{E}[q^\varepsilon(\cdot)], r^\varepsilon(\cdot), \mathtt{E}[r^\varepsilon(\cdot)])$ denote the perturbed trajectory of the proposed system described by equation (1). Furthermore, let $\mathcal{Y}^\varepsilon(\cdot)$ represent the solution of equation (4) which corresponds to $u^\varepsilon(\cdot)$.

We use the following simplified notations:

$$\kappa(u^\varepsilon(\varsigma)) := \kappa(\varsigma, p(\varsigma), \mathtt{E}[p(\varsigma)], q(\varsigma), \mathtt{E}[q(\varsigma)], r(\varsigma), \mathtt{E}[r(\varsigma)], s(\varsigma), \mathtt{E}[s(\varsigma)], u^\varepsilon(\varsigma)),$$

$$\kappa(u(\varsigma)) := \kappa(\varsigma, p(\varsigma), \mathtt{E}[p(\varsigma)], q(\varsigma), \mathtt{E}[q(\varsigma)], r(\varsigma), \mathtt{E}[r(\varsigma)], s(\varsigma), \mathtt{E}[s(\varsigma)], u(\varsigma)),$$

$$\tilde{\kappa}(\varsigma) := \tilde{\kappa}(\varsigma, p(\varsigma), \mathtt{E}[p(\varsigma)], q(\varsigma), \mathtt{E}[q(\varsigma)], r(\varsigma), \mathtt{E}[r(\varsigma)], s(\varsigma), \mathtt{E}[s(\varsigma)]),$$

where $\kappa = h, k, g, \varphi$, $\tilde{\kappa} = \sigma, f$ and similarly we can use the simplified notations for their partial derivatives in the optimal trajectory $(p, \mathtt{E}[p], q, \mathtt{E}[q], r, \mathtt{E}[r])$. Here after we use the notations $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ for $\mathtt{E}[p], \mathtt{E}[q], \mathtt{E}[r], \mathtt{E}[s]$ respectively. Now consider the variational equations (a linear FBSDE) corresponding to (1) as follows:

$$\begin{aligned}
dp^1(\varsigma) = \Big[& h_p(u(\varsigma))p^1(\varsigma) + h_{\tilde{p}}(u(\varsigma))\mathtt{E}[p^1(\varsigma)] + h_q(u(\varsigma))q^1(\varsigma) + h_{\tilde{q}}(u(\varsigma))\mathtt{E}[q^1(\varsigma)] + h_r(u(\varsigma))r^1(\varsigma) \\
& + h_{\tilde{r}}(u(\varsigma))\mathtt{E}[r^1(\varsigma)] + h_s(u(\varsigma))s^1(\varsigma) + h_{\tilde{s}}(u(\varsigma))\mathtt{E}[s^1(\varsigma)] + h(u^\varepsilon(\varsigma)) - h(u(\varsigma)) \Big] d\varsigma \\
& + \sum_{j=1}^{d} \Big[\sigma_p^j(\varsigma)p^1(\varsigma) + \sigma_{\tilde{p}}^j(\varsigma)\mathtt{E}[p^1(\varsigma)] + \sigma_q^j(\varsigma)q^1(\varsigma) + \sigma_{\tilde{q}}^j(\varsigma)\mathtt{E}[q^1(\varsigma)] + \sigma_r^j(\varsigma)r^1(\varsigma) \\
& + \sigma_{\tilde{r}}^j(\varsigma)\mathtt{E}[r^1(\varsigma)] + \sigma_s^j(\varsigma)s^1(\varsigma) + \sigma_{\tilde{s}}^j(\varsigma)\mathtt{E}[s^1(\varsigma)] \Big] d\mathcal{B}^j(\varsigma) \\
& + \sum_{j=1}^{\infty} \Big[f_p^j(\varsigma)p^1(\varsigma) + f_{\tilde{p}}^j(\varsigma)\mathtt{E}[p^1(\varsigma)] + f_q^j(\varsigma)q^1(\varsigma) + f_{\tilde{q}}^j(\varsigma)\mathtt{E}[q^1(\varsigma)] + f_r^j(\varsigma)r^1(\varsigma) \\
& + f_{\tilde{r}}^j(\varsigma)\mathtt{E}[r^1(\varsigma)] + f_s^j(\varsigma)s^1(\varsigma) + f_{\tilde{s}}^j(\varsigma)\mathtt{E}[s^1(\varsigma)] \Big] dH^j(\varsigma) \\
-dq^1(\varsigma) = \Big[& k_p(u(\varsigma))p^1(\varsigma) + k_{\tilde{p}}(u(\varsigma))\mathtt{E}[p^1(\varsigma)] + k_q(u(\varsigma))q^1(\varsigma) + k_{\tilde{q}}(u(\varsigma))\mathtt{E}[q^1(\varsigma)] + k_r(u(\varsigma))r^1(\varsigma) \\
& + k_{\tilde{r}}(u(\varsigma))\mathtt{E}[r^1(\varsigma)] + k_s(u(\varsigma))s^1(\varsigma) + k_{\tilde{s}}(u(\varsigma))\mathtt{E}[s^1(\varsigma)] + k(u^\varepsilon(\varsigma)) - k(u(\varsigma)) \Big] d\varsigma - \sum_{j=1}^{d} (r^1)^j(\varsigma)
\end{aligned}$$

$$\times d\mathcal{B}^j(\varsigma) - \sum_{j=1}^{\infty}(s^1)^j(\varsigma)dH^j(\varsigma),$$

$$p^1(0) = 0,$$

$$q^1(T_f) = l_p(p(T_f), \mathtt{E}[p(T_f)])p^1(\varsigma) + l_{\tilde{p}}(p(T_f), \mathtt{E}[p(T_f)])\mathtt{E}[p^1(\varsigma)], \tag{9}$$

and corresponding to (4) which is a linear SDE:

$$\begin{aligned} d\mathcal{Y}^1(\varsigma) &= \langle \mathcal{Y}^1(\varsigma)g(u(\varsigma)) + \mathcal{Y}(\varsigma)g_p(u(\varsigma))p^1(\varsigma) + \mathcal{Y}(\varsigma)g_{\tilde{p}}(u(\varsigma))\mathtt{E}[p^1(\varsigma)] + \mathcal{Y}(\varsigma)g_q(u(\varsigma))q^1(\varsigma) \\ &\quad + \mathcal{Y}(\varsigma)g_{\tilde{q}}(u(\varsigma))\mathtt{E}[q^1(\varsigma)] + \mathcal{Y}(\varsigma)(g(u^\varepsilon(\varsigma) - g(u(\varsigma))), d\mathcal{W}(\varsigma)\rangle, \\ \mathcal{Y}^1(0) &= 0. \end{aligned} \tag{10}$$

Clearly by the assumptions **(A1)** and **(A2)**, (9) and(10) admit unique adapted solutions $(p^1(\cdot), \mathtt{E}[p^1(\cdot)], q^1(\cdot), \mathtt{E}[q^1(\cdot)], r^1(\cdot), \mathtt{E}[r^1(\cdot)])$ and $\mathcal{Y}^1(\cdot)$ respectively.

3.2. **Variational Inequality.** The main objective of this subsection is to calculate some estimations for order of ε for $\mathcal{Y}^1(\cdot)$ and $\mathcal{Y}^\varepsilon(\cdot) - (\mathcal{Y}(\cdot) + \mathcal{Y}^1(\cdot))$. Then we derive V.I. using these estimations.

Lemma 3.1. *Assuming that **(A1)** and **(A2)** are satisfied, we can obtain the following result:*

$$(i) \sup_{0\le\varsigma\le T_f} \mathtt{E}|\mathcal{Y}^1(\varsigma)|^2 \le \bar{K}\varepsilon, \ \ (ii) \sup_{0\le\varsigma\le T_f} \mathtt{E}|\mathcal{Y}^1(\varsigma)|^4 \le \bar{K}\varepsilon^2, \tag{11}$$

$$(iii) \sup_{0\le\varsigma\le T_f} \mathtt{E}|\mathcal{Y}^\varepsilon(\varsigma) - \mathcal{Y}(\varsigma) - \mathcal{Y}^1(\varsigma)|^2 \le \bar{K}_\varepsilon\varepsilon^2. \tag{12}$$

Proof. First we prove the inequality (ii) in (11), then obviously the inequality (i) in (11) holds. From (10) and using the inequalities of Holder and Davis-Burkholder-Gundy, we obtain

$$\begin{aligned} \mathcal{Y}^1(\varsigma) &= \int_0^\varsigma \langle \mathcal{Y}^1(\zeta)g(u(\zeta)) + \mathcal{Y}(\zeta)g_p(u(\zeta))p^1(\zeta) + \mathcal{Y}(\zeta)g_{\tilde{p}}(u(\zeta))\mathtt{E}[p^1(\zeta)] + \mathcal{Y}(\zeta)g_q(u(\zeta))q^1(\zeta) \\ &\quad + \mathcal{Y}(\zeta)g_{\tilde{q}}(u(\zeta))\mathtt{E}[q^1(\zeta)] + \mathcal{Y}(\zeta)(g(u^\varepsilon(\zeta) - g(u(\zeta))), d\mathcal{W}(\zeta)\rangle \\ \mathtt{E}|\mathcal{Y}^1(t)|^4 &\le \bar{K}\Bigg(\mathtt{E}\Big|\int_0^t \mathcal{Y}^1(\zeta)\langle g(u(\zeta)), d\mathcal{W}(\zeta)\rangle\Big|^4 + \mathtt{E}\Big|\int_0^t \mathcal{Y}(\zeta)\langle g_p(u(\zeta))p^1(\zeta), d\mathcal{W}(\zeta)\rangle\Big|^4 \\ &\quad + \mathtt{E}\Big|\int_0^\varsigma \mathcal{Y}(\zeta)\langle g_{\tilde{p}}(u(\zeta))\mathtt{E}[p^1(\zeta)], d\mathcal{W}(\zeta)\rangle\Big|^4 + \mathtt{E}\Big|\int_0^\varsigma \mathcal{Y}(\zeta)\langle g_q(u(\zeta))q^1(\zeta), d\mathcal{W}(\zeta)\rangle\Big|^4 \\ &\quad + \mathtt{E}\Big|\int_0^\varsigma \mathcal{Y}(\zeta)\langle g_{\tilde{q}}(u(\zeta))\mathtt{E}[q^1(\zeta)], d\mathcal{W}(\zeta)\rangle\Big|^4 + \mathtt{E}\Big|\int_0^\varsigma \mathcal{Y}(\zeta)\langle g(u^\varepsilon(\zeta) - g(u(\zeta))), d\mathcal{W}(\zeta)\rangle\Big|^4 \\ &\le \bar{K}\Bigg(\int_0^\varsigma \mathtt{E}|\mathcal{Y}^1(\zeta)|^4 d\zeta + \mathtt{E}\Big[\int_0^\varsigma |\mathcal{Y}(\zeta)p^1(\zeta)|^2 d\zeta\Big]^2 + \mathtt{E}\Big[\int_0^\varsigma |\mathcal{Y}(\zeta)\mathtt{E}[p^1(\zeta)]|^2 d\zeta\Big]^2 \end{aligned}$$

$$
\begin{aligned}
&+\mathrm{E}\bigg[\int_0^{\varsigma}|\mathcal{Y}(\zeta)q^1(\zeta)|^2d\zeta\bigg]^2+\mathrm{E}\bigg[\int_0^{\varsigma}|\mathcal{Y}(\zeta)\mathrm{E}[q^1(\zeta)]|^2d\zeta\bigg]^2+\varepsilon\int_{\delta}^{\delta+\varepsilon}\mathrm{E}|\mathcal{Y}(\zeta)(g(u^{\varepsilon}\zeta)\\
&-g(u(\zeta))|^4d\zeta\bigg)\\
\le\ &\bar{K}\bigg(\int_0^{\varsigma}\mathrm{E}|\mathcal{Y}^1(\zeta)|^4d\zeta+\mathrm{E}\bigg[\int_0^{\varsigma}|\mathcal{Y}(\zeta)|^4d\zeta\cdot\int_0^{\varsigma}|p^1(\zeta)|^4d\zeta\bigg]+\mathrm{E}\bigg[\int_0^{\varsigma}|\mathcal{Y}(\zeta)|^4d\zeta\\
&\cdot\int_0^{\varsigma}|\mathrm{E}[p^1(\zeta)]|^4d\zeta\bigg]+\mathrm{E}\bigg[\int_0^{\varsigma}|\mathcal{Y}(\zeta)|^4d\zeta\cdot\int_0^{\varsigma}|q^1(\zeta)|^4d\zeta\bigg]+\mathrm{E}\bigg[\int_0^{\varsigma}|\mathcal{Y}(\zeta)|^4d\zeta\\
&\cdot\int_0^{\varsigma}|\mathrm{E}[q^1(\zeta)]|^4d\zeta\bigg]\bigg)+\bar{K}\varepsilon^2\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathcal{Y}(\zeta)|^4\\
\le\ &\bar{K}\bigg(\int_0^{\varsigma}\mathrm{E}|\mathcal{Y}^1(\zeta)|^4d\zeta+\bigg[T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathcal{Y}(\zeta)|^8\bigg]^{\frac{1}{2}}\cdot\bigg[\mathrm{E}\int_0^{\varsigma}|p^1(\zeta)|^8d\zeta\bigg]^{\frac{1}{2}}\\
&+\bigg[T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathcal{Y}(\zeta)|^8\bigg]^{\frac{1}{2}}\cdot\bigg[\mathrm{E}\int_0^{\varsigma}|\mathrm{E}[p^1(\zeta)]|^8d\zeta\bigg]^{\frac{1}{2}}+\bigg[T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathcal{Y}(\zeta)|^8\bigg]^{\frac{1}{2}}\cdot\bigg[\mathrm{E}\int_0^{\varsigma}|q^1(\zeta)|^8d\zeta\bigg]^{\frac{1}{2}}\\
&+\bigg[T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathcal{Y}(\zeta)|^8\bigg]^{\frac{1}{2}}\cdot\bigg[\mathrm{E}\int_0^{\varsigma}|\mathrm{E}[q^1(\zeta)]|^8d\zeta\bigg]^{\frac{1}{2}}\bigg)+\bar{K}\varepsilon^2\\
\le\ &\bar{K}\bigg(\int_0^{\varsigma}\mathrm{E}|\mathcal{Y}^1(\zeta)|^4d\zeta+\sqrt{T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|p^1(\zeta)|^8}+\sqrt{T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathrm{E}[p^1(\zeta)]^8|}\\
&+\sqrt{T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|q^1(\zeta)|^8}+\sqrt{T_f\sup_{0\le\varsigma\le T_f}\mathrm{E}|\mathrm{E}[q^1(\zeta)]^8|}\bigg)+\bar{K}\varepsilon^2.
\end{aligned}
$$

By Lemma 2.5 of [33] and the Gronwall inequality we get inequality (ii) in (11). We now prove the inequality (iii) in (12). At first, we have to estimate $\mathcal{Y}^{\varepsilon}(\varsigma)-\mathcal{Y}(\varsigma)-\mathcal{Y}^1(\varsigma)$. For this, let us consider

$$
\begin{aligned}
&\int_0^{\varsigma}\mathcal{Y}^1(\zeta)\langle g(u(\zeta)),d\mathcal{W}(\zeta)\rangle+\int_0^{\varsigma}\mathcal{Y}(\zeta)\langle g(\zeta,p(\zeta)+p^1(\zeta),\mathrm{E}[p(\zeta)]+\mathrm{E}[p^1(\zeta)],q(\zeta)+q^1(\zeta),\mathrm{E}[q(\zeta)]\\
&\quad+\mathrm{E}[q^1(\zeta)],u^{\varepsilon}(\zeta)),d\mathcal{W}(\zeta)\rangle\\
&=\int_0^{\varsigma}\mathcal{Y}^1(\zeta)\langle g(u(\zeta)),d\mathcal{W}(\zeta)\rangle+\int_0^{\varsigma}\mathcal{Y}(\zeta)\langle g(u^{\varepsilon}(\zeta)),d\mathcal{W}(\zeta)\rangle+\int_0^{\varsigma}\mathcal{Y}(\zeta)\bigg\langle\int_0^1 g_p(\zeta,p(\zeta)+\lambda p^1(\zeta),\\
&\quad\mathrm{E}[p(\zeta)]+\lambda\mathrm{E}[p^1(\zeta)],q(\zeta)+\lambda q^1(\zeta),\mathrm{E}[q(\zeta)]+\lambda\mathrm{E}[q^1(\zeta)],u^{\varepsilon}(\zeta))p^1(\zeta)d\lambda,d\mathcal{W}(\zeta)\bigg\rangle+\int_0^{\varsigma}\mathcal{Y}(\zeta)\\
&\quad\bigg\langle\int_0^1 g_{\tilde{p}}(\zeta,p(\zeta)+\lambda p^1(\zeta),\mathrm{E}[p(\zeta)]+\lambda\mathrm{E}[p^1(\zeta)],q(\zeta)+\lambda q^1(\zeta),\mathrm{E}[q(\zeta)]+\lambda\mathrm{E}[q^1(\zeta)],u^{\varepsilon}(\zeta))\\
&\quad\mathrm{E}[p^1(\zeta)]d\lambda,d\mathcal{W}(\zeta)\bigg\rangle+\int_0^{\varsigma}\mathcal{Y}(\zeta)\bigg\langle\int_0^1 g_q(\zeta,p(\zeta)+\lambda p^1(\zeta),\mathrm{E}[p(\zeta)]+\lambda\mathrm{E}[p^1(\zeta)],q(\zeta)+\lambda q^1(\zeta),\\
&\quad\mathrm{E}[q(\zeta)]+\lambda\mathrm{E}[q^1(\zeta)],u^{\varepsilon}(\zeta))q^1(\zeta)d\lambda,d\mathcal{W}(\zeta)\bigg\rangle+\int_0^{\varsigma}\mathcal{Y}(\zeta)\bigg\langle\int_0^1 g_{\tilde{q}}(\zeta,p(\zeta)+\lambda p^1(\zeta),\mathrm{E}[p(\zeta)]\\
&\quad+\lambda\mathrm{E}[p^1(\zeta)],q(\zeta)+\lambda q^1(\zeta),\mathrm{E}[q(\zeta)]+\lambda\mathrm{E}[q^1(\zeta)],u^{\varepsilon}(\zeta))\mathrm{E}[q^1(\zeta)]d\lambda,d\mathcal{W}(\zeta)\bigg\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\varsigma} \langle \mathcal{Y}^1(\zeta) g(u(\zeta)) + \mathcal{Y}(\zeta) g_p(u(\zeta)) p^1(\zeta) + \mathcal{Y}(\zeta) g_{\tilde{p}}(u(\zeta)) \mathtt{E}[p^1(\zeta)] + \mathcal{Y}(\zeta) g_q(u(\zeta)) q^1(\zeta) \\
&\quad + \mathcal{Y}(\zeta) g_{\tilde{q}}(u(\zeta)) \mathtt{E}[q^1(\zeta)] + \mathcal{Y}(\zeta)(g(u^{\varepsilon}(\zeta)) - g(u(\zeta))), d\mathcal{W}(\zeta)\rangle + \int_0^{\varsigma} \mathcal{Y}(\zeta) \langle g(u(\zeta)), d\mathcal{W}(\zeta)\rangle \\
&\quad + \int_0^{\varsigma} \mathcal{Y}(\zeta) \langle A_2^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle \\
&= \mathcal{Y}(\varsigma) - 1 + \mathcal{Y}^1(\varsigma) + \int_0^{\varsigma} \mathcal{Y}(\zeta) \langle A_2^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
A_2^{\varepsilon}(\zeta) &:= \int_0^1 [g_p(\zeta, p(\zeta) + \lambda p^1(\zeta), \mathtt{E}[p(\zeta)] + \lambda \mathtt{E}[p^1(\zeta)], q(\zeta) + \lambda q^1(\zeta), \mathtt{E}[q(\zeta)] + \lambda \mathtt{E}[q^1(\zeta)],\ u^{\varepsilon}(\zeta)) \\
&\quad - g_p(u(\zeta))] d\lambda p^1(\zeta) + \int_0^1 [g_{\tilde{p}}(\zeta, p(\zeta) + \lambda p^1(\zeta), \mathtt{E}[p(\zeta)] + \lambda \mathtt{E}[p^1(\zeta)], q(\zeta) + \lambda q^1(\zeta), \mathtt{E}[q(\zeta)] \\
&\quad + \lambda \mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta)) - g_p(u(\zeta))] d\lambda \mathtt{E}[p^1(\zeta)] + \int_0^1 [g_q(\zeta, p(\zeta) + \lambda p^1(\zeta), \mathtt{E}[p(\zeta)] + \lambda \mathtt{E}[p^1(\zeta)], q(\zeta) \\
&\quad + \lambda q^1(\zeta), \mathtt{E}[q(\zeta)] + \lambda \mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta)) - g_q(u(\zeta))] d\lambda q^1(\zeta) + \int_0^1 [g_{\tilde{q}}(\zeta, p(\zeta) + \lambda p^1(\zeta), \mathtt{E}[p(\zeta)] \\
&\quad + \lambda \mathtt{E}[p^1(\zeta)], q(\zeta) + \lambda q^1(\zeta), \mathtt{E}[q(\zeta)] + \lambda \mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta)) - g_p(u(\zeta))] d\lambda \mathtt{E}[q^1(\zeta)].
\end{aligned}
$$

By Lemma 2.7 of [32],

$$
\sup_{0 \le \varsigma \le T_f} \mathtt{E} \left(\int_0^{\varsigma} \mathcal{Y}(\zeta) A_2^{\varepsilon}(\zeta) d\mathcal{W}(\zeta) \right)^2 \le \bar{K}_{\varepsilon} \varepsilon^2. \tag{14}
$$

Now

$$
\mathcal{Y}^{\varepsilon}(\varsigma) = 1 + \int_0^{\varsigma} \mathcal{Y}^{\varepsilon}(\zeta) \langle g(\zeta, p^{\varepsilon}(\zeta), \mathtt{E}[p^{\varepsilon}(\zeta)], q^{\varepsilon}(\zeta), \mathtt{E}[q^{\varepsilon}(\zeta)], u^{\varepsilon}(\zeta)), d\mathcal{W}(\zeta)\rangle. \tag{15}
$$

Using (13) and (15) we can get

$$
\begin{aligned}
&\mathcal{Y}^{\varepsilon}(\varsigma) - \mathcal{Y}(\varsigma) - \mathcal{Y}^1(\varsigma) \\
&= \int_0^{\varsigma} \mathcal{Y}^{\varepsilon}(\zeta) \langle g(\zeta), p^{\varepsilon}(\zeta), \mathtt{E}[p^{\varepsilon}(\zeta)], q^{\varepsilon}(\zeta), \mathtt{E}[q^{\varepsilon}(\zeta)], u^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle - \int_0^{\varsigma} \mathcal{Y}^1(\zeta) \langle g(u(\zeta)), d\mathcal{W}(\zeta)\rangle \\
&\quad - \int_0^{\varsigma} \mathcal{Y}(\zeta) \langle g(\zeta, p(\zeta) + p^1(\zeta), \mathtt{E}[p(\zeta)] + \mathtt{E}[p^1(\zeta)], q(\zeta) + q^1(\zeta), \mathtt{E}[q(s)] \\
&\quad + \mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta)), d\mathcal{W}(\zeta)\rangle + \int_0^{\varsigma} \mathcal{Y}(\zeta) \langle A^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle \\
&= \int_0^{\varsigma} (\mathcal{Y}^{\varepsilon}(\zeta) - \mathcal{Y}(\zeta) - \mathcal{Y}^1(\zeta)) \langle g(s), p^{\varepsilon}(\zeta), \mathtt{E}[p^{\varepsilon}(\zeta)], q^{\varepsilon}(\zeta), \mathtt{E}[q^{\varepsilon}(\zeta)], u^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle \\
&\quad + \int_0^{\varsigma} (\mathcal{Y}(\zeta) + \mathcal{Y}^1(\zeta)) \langle B_2^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle + \int_0^{\varsigma} \mathcal{Y}^1(\zeta) B_3^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle + \int_0^{\varsigma} \mathcal{Y}^1(\zeta) \langle g(u^{\varepsilon}(\zeta)) \\
&\quad - g(u(\zeta)), d\mathcal{W}(\zeta)\rangle + \int_0^{\varsigma} \mathcal{Y}(\zeta) \langle A_2^{\varepsilon}(\zeta), d\mathcal{W}(\zeta)\rangle,
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
B_2^{\varepsilon} := & \int_0^{\varsigma} g_p(p(\zeta)+p^1(\zeta)+\lambda(p^{\varepsilon}(\zeta)-p(\zeta)-p^1(\zeta)), \mathtt{E}[p(\zeta)]+\mathtt{E}[p^1(\zeta)]+\lambda(\mathtt{E}[p^{\varepsilon}(\zeta)]-\mathtt{E}[p(\zeta)] \\
& -\mathtt{E}[p^1(\zeta)]), q(\zeta)+q^1(\zeta)+\lambda(q^{\varepsilon}(\zeta)-q(\zeta)-q^1(\zeta)), \mathtt{E}[q(\zeta)]+\mathtt{E}[q^1(\zeta)]+\lambda(\mathtt{E}[q^{\varepsilon}(\zeta)] \\
& -\mathtt{E}[q(\zeta)]-\mathtt{E}[q^1(\zeta)]), u^{\varepsilon}(\zeta))d\lambda(p^{\varepsilon}(\zeta)-p(\zeta)-p^1(\zeta))+\int_0^{\varsigma} h_{\tilde{p}}(p(\zeta)+p^1(\zeta)+\lambda(p^{\varepsilon}(\zeta) \\
& -p(\zeta)-p^1(\zeta)), \mathtt{E}[p(\zeta)]+\mathtt{E}[p^1(\zeta)]+\lambda(\mathtt{E}[p^{\varepsilon}(\zeta)]-\mathtt{E}[p(\zeta)]-\mathtt{E}[p^1(\zeta)]), q(\zeta)+q^1(\zeta) \\
& +\lambda(q^{\varepsilon}(\zeta)-q(\zeta)-q^1(\zeta)), \mathtt{E}[q(\zeta)]+\mathtt{E}[q^1(\zeta)]+\lambda(\mathtt{E}[q^{\varepsilon}(\zeta)]-\mathtt{E}[q(\zeta)]-\mathtt{E}[q^1(\zeta)]), u^{\varepsilon}(\zeta)) \\
& \times d\lambda(\mathtt{E}[p^{\varepsilon}(\zeta)]-\mathtt{E}[p(\zeta)]-\mathtt{E}[p^1(\zeta)])+\int_0^{\varsigma} g_q(p(\zeta)+p^1(\zeta)+\lambda(p^{\varepsilon}(\zeta)-p(\zeta) \\
& -p^1(\zeta)), \mathtt{E}[p(\zeta)]+\mathtt{E}[p^1(\zeta)]+\lambda(\mathtt{E}[p^{\varepsilon}(\zeta)]-\mathtt{E}[p(\zeta)]-\mathtt{E}[p^1(\zeta)]), q(\zeta)+q^1(\zeta)+\lambda(q^{\varepsilon}(\zeta) \\
& -q(\zeta)-q^1(\zeta)), \mathtt{E}[q(\zeta)]+\mathtt{E}[q^1(\zeta)]+\lambda(\mathtt{E}[q^{\varepsilon}(\zeta)]-\mathtt{E}[q(\zeta)]-\mathtt{E}[q^1(\zeta)]), u^{\varepsilon}(\zeta))d\lambda(q^{\varepsilon}(\zeta) \\
& -q(\zeta)-q^1(\zeta))+\int_0^{\varsigma} g_{\tilde{q}}(p(\zeta)+p^1(\zeta)+\lambda(p^{\varepsilon}(\zeta)-p(\zeta)-p^1(\zeta)), \mathtt{E}[p(\zeta)]+\mathtt{E}[p^1(\zeta)] \\
& +\lambda(\mathtt{E}[p^{\varepsilon}(\zeta)]-\mathtt{E}[p(\zeta)]-\mathtt{E}[p^1(\zeta)]), q(\zeta)+q^1(\zeta)+\lambda(q^{\varepsilon}(\zeta)-q(\zeta)-q^1(\zeta)), \mathtt{E}[q(\zeta)] \\
& +\mathtt{E}[q^1(\zeta)]+\lambda(\mathtt{E}[q^{\varepsilon}(\zeta)]-\mathtt{E}[q(\zeta)]-\mathtt{E}[q^1(\zeta)]), u^{\varepsilon}(\zeta))d\lambda(\mathtt{E}[q^{\varepsilon}(\zeta)]-\mathtt{E}[q(\zeta)]-\mathtt{E}[q^1(\zeta)]) \\
B_3^{\varepsilon} := & \int_0^1 g_p(p(\zeta)+\lambda p^1(\zeta), \mathtt{E}[p(\zeta)]+\lambda\mathtt{E}[p^1(\zeta)], q(\zeta)+\lambda q^1(\zeta), \mathtt{E}[q(\zeta)]+\lambda\mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta)) \\
& \times d\lambda p^1(\zeta)+\int_0^1 g_{\tilde{p}}(p(\zeta)+\lambda p^1(\zeta), \mathtt{E}[p(\zeta)]+\lambda\mathtt{E}[p^1(\zeta)], q(\zeta)+\lambda q^1(\zeta), \mathtt{E}[q(\zeta)] \\
& +\lambda\mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta))d\lambda\mathtt{E}[p^1(\zeta)]+\int_0^1 g_q(p(\zeta)+\lambda p^1(\zeta), \mathtt{E}[p(\zeta)]+\lambda\mathtt{E}[p^1(\zeta)], q(\zeta) \\
& +\lambda q^1(\zeta), \mathtt{E}[q(\zeta)]+\lambda\mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta))d\lambda q^1(\zeta)+\int_0^1 g_{\tilde{q}}(p(\zeta)+\lambda p^1(\zeta), \mathtt{E}[p(\zeta)] \\
& +\lambda\mathtt{E}[p^1(\zeta)], q(\zeta)+\lambda q^1(\zeta), \mathtt{E}[q(\zeta)]+\lambda\mathtt{E}[q^1(\zeta)], u^{\varepsilon}(\zeta))d\lambda\mathtt{E}[q^1(\zeta)]
\end{aligned}
$$

By Lemma 2.5 of [33], it becomes apparent that:

$$
\sup_{0\le\varsigma\le T_f} \mathtt{E}\left(\int_0^{\varsigma} \mathcal{Y}(\zeta)B_2^{\varepsilon}(\zeta)d\mathcal{W}(\zeta)\right)^2 \le \bar{K}_{\varepsilon}\varepsilon^2. \tag{17}
$$

By (14),(16) and (17), we have

$$
\mathtt{E}|\mathcal{Y}^{\varepsilon}(\varsigma)-\mathcal{Y}(\varsigma)-\mathcal{Y}^1(\varsigma)|^2
$$

$$
\begin{aligned}
&\leq \Big\{ \bar{K}\int_0^{\varsigma} \mathtt{E}|\mathcal{Y}^{\varepsilon}(\zeta)-\mathcal{Y}(\zeta)-\mathcal{Y}^1(\zeta)|^2 d\zeta + \mathtt{E}\int_0^{\varsigma} |\mathcal{Y}^1(\zeta)B_2^{\varepsilon}(\zeta)|^2 d\zeta + \mathtt{E}\int_0^{\varsigma} |\mathcal{Y}^1(\zeta)B_3^{\varepsilon}(\zeta)|^2 d\zeta \\
&\quad + \mathtt{E}\int_0^{\varsigma} |\mathcal{Y}^1(\zeta)(g(u^{\varepsilon}(\zeta))-g(u(\zeta)))|^2 d\zeta + \sup_{0\leq\varsigma\leq T_f} \mathtt{E}\left(\int_0^{\varsigma} \mathcal{Y}(\zeta)A_2^{\varepsilon}(\zeta)d\mathcal{W}(\zeta)\right)^2 \\
&\quad + \sup_{0\leq\varsigma\leq T_f} \mathtt{E}\left(\int_0^{\varsigma} \mathcal{Y}(\zeta)B_2^{\varepsilon}(\zeta)d\mathcal{W}(\zeta)\right)^2 \Big\} \\
&\leq \bar{K}\int_0^{\varsigma} \mathtt{E}|\mathcal{Y}^{\varepsilon}(\zeta)-\mathcal{Y}(\zeta)-\mathcal{Y}^1(\zeta)|^2 d\zeta + \bar{K}_{\varepsilon}\varepsilon^2.
\end{aligned}
$$

By the Gronwall inequality, we get

$$
\sup_{0\leq\varsigma\leq T_f} \mathtt{E}|\mathcal{Y}^{\varepsilon}(\varsigma)-\mathcal{Y}(\varsigma)-\mathcal{Y}^1(\varsigma)|^2 \leq \bar{K}_{\varepsilon}\varepsilon^2.
$$

Hence the inequality (iii) in (12) holds. □

To prove the partially-observed stochastic maximum principle, we need the following V.I.

Lemma 3.2. *Under the assumptions **(A1)**∼**(A3)** and considering $u(\cdot)$ as an optimal control for the proposed PO-OCP* (6)*, we can deduce the following:*

$$
\begin{aligned}
\mathtt{E}^u \Big\{ &\int_0^{\varsigma} [\eta(\varsigma)\varphi(u(\varsigma)) + \varphi_p^{\top}(u(\varsigma))p^1(\varsigma) + \varphi_{\tilde{p}}^{\top}\mathtt{E}[p^1(\varsigma)], \varphi_q^{\top}(u(\varsigma))q^1(\varsigma) + \varphi_{\tilde{q}}^{\top}\mathtt{E}[q^1(\varsigma)] \\
&+ \varphi_r^{\top}(u(\varsigma))r^1(\varsigma) + \varphi_{\tilde{r}}^{\top}\mathtt{E}[r^1(\varsigma)] + \varphi_r^{\top}(u(\varsigma))r^1(\varsigma) + \varphi_{\tilde{r}}^{\top}\mathtt{E}[r^1(\varsigma)] + \varphi(u^{\varepsilon}(\varsigma)) - \varphi(u(\varsigma))]d\varsigma \\
&+ \eta(T_f)\phi(p(T_f),\mathtt{E}[p(T_f)]) + \phi_p^{\top}(p(T_f),\mathtt{E}[p(T_f)])p^1(T_f) + \phi_{\tilde{p}}^{\top}(p(T_f),\mathtt{E}[p(T_f)])\mathtt{E}[p^1(T_f)] \\
&+ \gamma_q(q(0),\mathtt{E}[q(0)])q^1(0) + \gamma_{\tilde{q}}(q(0),\mathtt{E}[q(0)])\mathtt{E}[q^1(0)] \Big\} \geq o(\varepsilon).
\end{aligned}
\tag{18}
$$

Here, $\eta(\cdot)$ represents the solution to the subsequent SDE:

$$
\begin{aligned}
d\eta(\varsigma) &= \langle g_p(u(\varsigma))p^1(\varsigma) + g_{\tilde{p}}(u(\varsigma))\mathtt{E}[p^1(\varsigma)] + g_q(u(\varsigma))q^1(\varsigma) + g_{\tilde{q}}(u(\varsigma))\mathtt{E}[q^1(\varsigma)] + g(u^{\varepsilon}(\varsigma)) \\
&\qquad - g(u(\varsigma)), d\tilde{\mathcal{B}}(\varsigma)\rangle, \\
\eta(0) &= 0,
\end{aligned}
\tag{19}
$$

and $o(\varepsilon)$ denotes the order of ε in the Taylor's expansion such that $\frac{o(\varepsilon)}{\varepsilon} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Proof. Given that $u(\varsigma)$ is optimal for $0 \leq \varsigma \leq T_f$, it follows that:

$0 \leq \mathcal{J}^*(u^{\varepsilon}(\varsigma)) - \mathcal{J}^*(u(\varsigma)$

$$= \mathtt{E}\int_0^{\varsigma}\bigg[\mathcal{Y}^{\varepsilon}(\varsigma)\varphi(\varsigma,p^{\varepsilon}(\varsigma),\mathtt{E}[p^{\varepsilon}(\varsigma)],q^{\varepsilon}(\varsigma),\mathtt{E}[q^{\varepsilon}(\varsigma)],r^{\varepsilon}(\varsigma),\mathtt{E}[r^{\varepsilon}(\varsigma)],s^{\varepsilon}(\varsigma),\mathtt{E}[s^{\varepsilon}(\varsigma)],u^{\varepsilon}(\varsigma))-\mathcal{Y}(\varsigma)\varphi(u(\varsigma))\bigg]d\varsigma$$
$$+\mathtt{E}\bigg[\mathcal{Y}^{\varepsilon}(T_f)\phi(p^{\varepsilon}(T_f),\mathtt{E}[p^{\varepsilon}(T_f)])-\mathcal{Y}(T_f)\phi(p(T_f),\mathtt{E}[p(T_f)])\bigg]+\mathtt{E}\bigg[\gamma(q^{\varepsilon}(0),\mathtt{E}[q^{\varepsilon}(0)])-\gamma(q(0),\mathtt{E}[q(0)])\bigg].$$
$$=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3, \tag{20}$$

where

$$\mathcal{J}_1=\mathtt{E}\int_0^{\varsigma}\bigg[\mathcal{Y}^{\varepsilon}(\varsigma)\varphi(\varsigma,p^{\varepsilon}(\varsigma),\mathtt{E}[p^{\varepsilon}(\varsigma)],q^{\varepsilon}(\varsigma),\mathtt{E}[q^{\varepsilon}(\varsigma)],r^{\varepsilon}(\varsigma),\mathtt{E}[r^{\varepsilon}(\varsigma)],s^{\varepsilon}(\varsigma),\mathtt{E}[s^{\varepsilon}(\varsigma)],u^{\varepsilon}(\varsigma))-\mathcal{Y}(\varsigma)\varphi(u(\varsigma))\bigg]d\varsigma,$$
$$\mathcal{J}_2=\mathtt{E}\bigg[\mathcal{Y}^{\varepsilon}(T_f)\phi(p^{\varepsilon}(T_f),\mathtt{E}[p^{\varepsilon}(T_f)])-\mathcal{Y}(T_f)\phi(p(T_f),\mathtt{E}[p(T_f)])\bigg],$$
$$\mathcal{J}_3=\mathtt{E}\bigg[\gamma(q^{\varepsilon}(0),\mathtt{E}[q^{\varepsilon}(0)])-\gamma(q(0),\mathtt{E}[q(0)])\bigg].$$

First we are going to compute J_3. For this purpose, it is evident that:

$$\mathtt{E}\bigg[\gamma(q^{\varepsilon}(0),\mathtt{E}[q^{\varepsilon}(0)])-\gamma(q(0)+q^1(0),\mathtt{E}[q(0)+q^1(0)])\bigg]=o(\varepsilon).$$

Hence,

$$\mathcal{J}_1=\mathtt{E}\bigg[\gamma(q^{\varepsilon}(0),\mathtt{E}[q^{\varepsilon}(0)])-\gamma(q(0),\mathtt{E}[q(0)])\bigg]=\mathtt{E}\bigg[\gamma_q^{\top}(q(0),\mathtt{E}[q(0)])q^1(0)+\gamma_{\tilde{q}}^{\top}(q(0),\mathtt{E}[q(0)])\mathtt{E}[q^1(0)]\bigg]$$
$$+o(\varepsilon).$$

Next we are going to evaluate the term $\mathcal{J}_2$. That is

$$\mathcal{J}_2=\mathtt{E}\bigg[\mathcal{Y}^{\varepsilon}(T_f')\phi(p^{\varepsilon}(T_f),\mathtt{E}[p^{\varepsilon}(T_f)])-\mathcal{Y}(T_f)\phi(p(T_f),\mathtt{E}[p(T_f)])\bigg]$$
$$=\mathtt{E}\bigg[\mathcal{Y}^1(T_f)\phi(p(T_f),\mathtt{E}[p(T_f)])\bigg]+\mathtt{E}\bigg[\mathcal{Y}(T_f)\phi_p^{\top}(p(T_f),\mathtt{E}[p(T_f)])p^1(T_f)+\phi_{\tilde{p}}(p(T_f),\mathtt{E}[p(T_f)])\mathtt{E}[p^1(T_f)]\bigg]$$
$$+\mathtt{E}\bigg[(\mathcal{Y}(T_f)+\mathcal{Y}^1(T_f))\phi_p^{\top}(p(T_f)+p^1(T_f),\mathtt{E}[p(T_f)+p^1(T_f)])(p^{\varepsilon}(T_f)-p(T_f)-p^1(T_f))+\phi_{\tilde{p}}^{\top}(p(T_f)$$
$$+p^1(T_f),\mathtt{E}[p(T_f)+p^1(T_f)])(\mathtt{E}[p^{\varepsilon}(T_f)]-\mathtt{E}[p(T_f)]-\mathtt{E}[p^1(T_f)])\bigg]+\mathtt{E}\bigg[\mathcal{Y}^1(T_f)[\phi_p^{\top}(p(T_f),\mathtt{E}[p(T_f)])p^1(T_f)$$
$$+\phi_{\tilde{p}}^{\top}(p(T_f),\mathtt{E}[p(T_f)])\mathtt{E}[p^1(T_f)]]\bigg]+\mathtt{E}\bigg[(\mathcal{Y}^{\varepsilon}(T_f)-\mathcal{Y}(T_f)-\mathcal{Y}^1(T_f))\phi(p^{\varepsilon}(T_f),\mathtt{E}[p^{\varepsilon}(T_f)])\bigg]+o(\varepsilon).$$

Finally,

$$\mathcal{J}_3=\mathtt{E}\int_0^{T_f}[\mathcal{Y}^{\varepsilon}(\varsigma)\varphi(\varsigma,p^{\varepsilon}(\varsigma),\mathtt{E}[p^{\varepsilon}(\varsigma)],q^{\varepsilon}(\varsigma),\mathtt{E}[q^{\varepsilon}(\varsigma)],r^{\varepsilon}(\varsigma),\mathtt{E}[r^{\varepsilon}(\varsigma)],s^{\varepsilon}(\varsigma),\mathtt{E}[s^{\varepsilon}(\varsigma)],u^{\varepsilon}(\varsigma))-\mathcal{Y}(\varsigma)\varphi(u(\varsigma))]d\varsigma$$

$$
\begin{aligned}
&= \mathtt{E}\int_0^{T_f} \mathcal{Y}^1(\varsigma)\varphi(u(\varsigma))d\varsigma + \mathtt{E}\int_0^{T_f} (\mathcal{Y}^\varepsilon(\varsigma) - \mathcal{Y}(\varsigma) - \mathcal{Y}^1(\varsigma))[\varphi(\varsigma, p(\varsigma) + p^1(\varsigma), \mathtt{E}[p(\varsigma) + p^1(\varsigma)], q(\varsigma) \\
&\quad + q^1(\varsigma), \mathtt{E}[q(\varsigma) + q^1(\varsigma)], r(\varsigma) + r^1(\varsigma), \mathtt{E}[r(\varsigma) + r^1(\varsigma)], s(\varsigma) + s^1(\varsigma), \mathtt{E}[s(\varsigma) + s^1(\varsigma)], u^\varepsilon(\varsigma))]d\varsigma \\
&\quad + \mathtt{E}\int_0^{T_f} (\mathcal{Y}(\varsigma) + \mathcal{Y}^1(\varsigma))[\varphi(\varsigma, p(\varsigma) + p^1(\varsigma), \mathtt{E}[p(\varsigma) + p^1(\varsigma)], q(\varsigma) + q^1(\varsigma), \mathtt{E}[q(\varsigma) + q^1(\varsigma)], r(\varsigma) \\
&\quad + r^1(\varsigma), \mathtt{E}[r(\varsigma) + r^1(\varsigma)], s(\varsigma) + s^1(\varsigma), \mathtt{E}[s(\varsigma) + s^1(\varsigma)], u^\varepsilon(\varsigma)) - \varphi(u^\varepsilon(\varsigma))]d\varsigma - \mathtt{E}\int_0^{T_f} (\mathcal{Y}(\varsigma) \\
&\quad + \mathcal{Y}^1(\varsigma))[\varphi(\varsigma, p(\varsigma) + p^1(\varsigma), \mathtt{E}[p(\varsigma) + p^1(\varsigma)], q(\varsigma) + q^1(\varsigma), \mathtt{E}[q(\varsigma) + q^1(\varsigma)], r(\varsigma) + r^1(\varsigma), \mathtt{E}[r(\varsigma) \\
&\quad + r^1(\varsigma)], s(\varsigma) + s^1(\varsigma), \mathtt{E}[s(\varsigma) + s^1(\varsigma)], u(\varsigma)) - \varphi(u(\varsigma))]d\varsigma + \mathtt{E}\int_0^{T_f} (\mathcal{Y}(\varsigma) + \mathcal{Y}^1(\varsigma))[\varphi(u^\varepsilon(\varsigma)) - \varphi(u(\varsigma))] \\
&\quad + \mathtt{E}\int_0^{T_f} (\mathcal{Y}(\varsigma) + \mathcal{Y}^1(\varsigma))[\varphi(\varsigma, p(\varsigma) + p^1(\varsigma), \mathtt{E}[p(\varsigma) + p^1(\varsigma)], q(\varsigma) + q^1(\varsigma), \mathtt{E}[q(\varsigma) + q^1(\varsigma)], r(\varsigma) \\
&\quad + r^1(\varsigma), \mathtt{E}[r(\varsigma) + r^1(\varsigma)], s(\varsigma) + s^1(\varsigma), \mathtt{E}[s(\varsigma) + s^1(\varsigma)], u(\varsigma)) - \varphi(u(\varsigma))]d\varsigma \\
&= \mathtt{E}\int_0^{T_f} \Big[\mathcal{Y}^1(\varsigma)\varphi(u(\varsigma)) + \mathcal{Y}(\varsigma)\varphi_p^\top(u(\varsigma))p^1(\varsigma) + \mathcal{Y}(\varsigma)\varphi_{\tilde{p}}^\top(u(\varsigma))\mathtt{E}[p^1(\varsigma)] + \mathcal{Y}(\varsigma)\varphi_q^\top(u(\varsigma))q^1(\varsigma) \\
&\qquad + \mathcal{Y}(\varsigma)\varphi_{\tilde{q}}^\top(u(\varsigma))\mathtt{E}[q^1(\varsigma)] + \mathcal{Y}(\varsigma)\varphi_r^\top(u(\varsigma))r^1(\varsigma) + \mathcal{Y}(\varsigma)\varphi_{\tilde{r}}^\top(u(\varsigma))E[r^1(\varsigma)] + \mathcal{Y}(\varsigma)\varphi_s^\top(u(\varsigma))s^1(\varsigma) \\
&\qquad + \mathcal{Y}(\varsigma)\varphi_{\tilde{s}}^\top(u(\varsigma))\mathtt{E}[s^1(\varsigma)]\Big]d\varsigma + o(\varepsilon).
\end{aligned}
$$

Using $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ in (20), we obtain

$$
\begin{aligned}
&\mathtt{E}^u\Big\{\int_0^\varsigma [\mathcal{Y}^1(\varsigma)\varphi(u(\varsigma)) + \mathcal{Y}(\varsigma)\varphi_p^\top(u(\varsigma))p^1(\varsigma) + \mathcal{Y}(\varsigma)\varphi_{\tilde{p}}^\top\mathtt{E}[p^1(\varsigma)] + \mathcal{Y}(\varsigma)\varphi_q^\top(u(\varsigma))q^1(\varsigma) + \mathcal{Y}(\varsigma)\varphi_{\tilde{q}}^\top\mathtt{E}[q^1(\varsigma)] \\
&\quad + \mathcal{Y}(\varsigma)\varphi_r^\top(u(\varsigma))r^1(\varsigma) + \mathcal{Y}(\varsigma)\varphi_{\tilde{r}}^\top\mathtt{E}[r^1(\varsigma)] + \mathcal{Y}(\varsigma)\varphi_s^\top(u(\varsigma))s^1(\varsigma) + \mathcal{Y}(\varsigma)\varphi_{\tilde{s}}^\top\mathtt{E}[s^1(\varsigma)] + \mathcal{Y}(\varsigma)(\varphi(u^\varepsilon(\varsigma)) \\
&\quad - \varphi(u(\varsigma)))]d\varsigma + \mathcal{Y}^1(T_f)\phi(p(T_f), \mathtt{E}[p(T_f)]) + \mathcal{Y}(T_f)\phi_p^\top(p(T_f), \mathtt{E}[p(T_f)])p^1(T_f) + \mathcal{Y}(T_f)\phi_{\tilde{p}}^\top(p(T_f), \mathtt{E}[p(T_f)]) \\
&\quad \times \mathtt{E}[p^1(T_f)] + \gamma_q(q(0), \mathtt{E}[q(0)])q^1(0) + \gamma_{\tilde{q}}(q(0), \mathtt{E}[q(0)])\mathtt{E}[q^1(0)]\Big\} \geq o(\varepsilon). \qquad (21)
\end{aligned}
$$

Applying Ito's formula in (10), we can get

$$
\begin{aligned}
\mathcal{Y}^1(\varsigma) = \mathcal{Y}(\varsigma)\int_0^\varsigma &\langle g_p(u(\zeta))p^1(\zeta) + g_{\tilde{p}}(u(\zeta))E[p^1(\zeta)] + g_q(u(\zeta))q^1(\zeta) + g_{\tilde{q}}(u(\zeta))\mathtt{E}[q^1(\zeta)] + g(u^\varepsilon(\zeta)) \\
&- g(u(\zeta)), d\tilde{\mathcal{B}}(\zeta)\rangle.
\end{aligned}
$$

By (4), we can get

$$
d\mathcal{Y}^{-1}(\varsigma) = -\mathcal{Y}^{-1}(\varsigma)\langle g(u(\varsigma)), d\mathcal{W}(\varsigma)\rangle + \mathcal{Y}^{-1}(\varsigma)|g(u(\varsigma))|^2 d\varsigma.
$$

By applying Ito's formula to the expression $\eta(\cdot) = \mathcal{Y}^{-1}(\cdot)\mathcal{Y}^1(\cdot)$, we obtain equation (19). Consequently, based on equation (21), we can conclude that V.I. (18) holds. Hence, the proof is now complete. $\square$

3.3. **Partially observed Stochastic Mean-Field Maximum Principle.** The goal of this subsection is to derive the partially-observed stochastic maximum principle, To achieve this, we start by eliminating the variational processes $(p^1(\cdot), \mathtt{E}[p^1(\cdot)], q^1(\cdot), \mathtt{E}[q^1(\cdot)], r^1(\cdot), \mathtt{E}[r^1(\cdot)])$ and the auxiliary process $\eta(\cdot)$ from V.I. (18). Subsequently, we utilize Ito's formula to derive the necessary conditions for optimality. In order to handle the process $\eta(\cdot) \in \mathbf{R}$, consider the auxiliary BSDE as follows:

$$
\begin{aligned}
-d\tilde{P}(\varsigma) &= \varphi(u(\varsigma))d\varsigma - \langle \tilde{Q}(\varsigma), d\tilde{\mathcal{B}}(\varsigma)\rangle,\\
\tilde{P}(T_f) &= \Upsilon(p(T_f), \mathtt{E}[p(T_f)]).
\end{aligned}
\tag{22}
$$

Based on the assumptions **(A1)** and **(A3)**, it can be readily verified that (22) possesses a unique solution denoted by $(\tilde{P}(\cdot), \tilde{Q}(\cdot))$.

The following system represents the adjoint equations for (1):

$$
\begin{aligned}
d\Phi(\varsigma) = \Bigg[& k_q^\top(u(\varsigma))\Phi(\varsigma) + k_{\tilde{q}}^\top(u(\varsigma))\mathtt{E}[\Phi(\varsigma)] - h_q^\top(u(\varsigma))\Psi(\varsigma) - h_{\tilde{q}}^\top(u(\varsigma))\mathtt{E}[\Psi(\varsigma)]\\
& - \sum_{j=1}^{d}\left(\sigma_q^{j\,\top}(\varsigma)a^j(\varsigma) + \sigma_{\tilde{q}}^{j\,\top}(\varsigma)\mathtt{E}[a^j(\varsigma)]\right) - \sum_{j=1}^{\infty}\left(f_q^j(\varsigma)b^j(\varsigma) + f_{\tilde{q}}^j(\varsigma)\mathtt{E}[b^j(\varsigma)]\right)\\
& - g_q^\top(u(\varsigma))\tilde{Q}(\varsigma) - g_{\tilde{q}}^\top(u(\varsigma))\mathtt{E}[\tilde{Q}(\varsigma)] - \varphi_q((u(\varsigma))) - \varphi_{\tilde{q}}(u(\varsigma))\Bigg]d\varsigma\\
& + \sum_{j=1}^{d}\Bigg[k_{r^j}^\top(u(\varsigma))\Phi(\varsigma) + k_{\tilde{r}^j}^\top(u(\varsigma))\mathtt{E}[\Phi(\varsigma)] - h_{r^j}^\top(u(\varsigma))\Psi(\varsigma) - h_{\tilde{r}^j}^\top(u(\varsigma))\mathtt{E}[\Psi(\varsigma)]\\
& + \sum_{i=1}^{d}\left(\sigma_{r^j}^{i\,\top}(\varsigma)a^i(\varsigma) + \sigma_{\tilde{r}^j}^{i\,\top}(\varsigma)\mathtt{E}[a^i(\varsigma)]\right) + \sum_{i=1}^{\infty}\left(f_{r^j}^i(\varsigma)b^i(\varsigma) + f_{\tilde{r}^j}^i(\varsigma)\mathtt{E}[b^i(\varsigma)]\right)\\
& - \varphi_{r^j}(u(\varsigma)) - \varphi_{\tilde{r}^j}(u(\varsigma))\Bigg]d\mathcal{B}^j(\varsigma) + \sum_{j=1}^{d}\Bigg[k_{s^j}^\top(u(\varsigma))\Phi(\varsigma) + k_{\tilde{s}^j}^\top(u(\varsigma))\mathtt{E}[\Phi(\varsigma)] - h_{s^j}^\top(u(\varsigma))\Psi(\varsigma)\\
& - h_{\tilde{s}^j}^\top(u(\varsigma))\mathtt{E}[\Psi(\varsigma)] + \sum_{i=1}^{d}\left(\sigma_{s^j}^{i\,\top}(\varsigma)a^i(\varsigma) + \sigma_{\tilde{s}^j}^{i\,\top}(\varsigma)\mathtt{E}[a^i(\varsigma)]\right)\\
& + \sum_{i=1}^{\infty}\left(f_{s^j}^i(\varsigma)b^i(\varsigma) + f_{\tilde{s}^j}^i(\varsigma)\mathtt{E}[b^i(\varsigma)]\right) - \varphi_{s^j}(u(\varsigma)) - \varphi_{\tilde{s}^j}(u(\varsigma))\Bigg]dH^j(\varsigma),\\
-d\Psi(\varsigma) = \Bigg[& -k_p^\top(u(\varsigma))\Phi(\varsigma) - k_{\tilde{p}}^\top(u(\varsigma))\mathtt{E}[\Phi(\varsigma)] + h_p^\top(u(\varsigma))\Psi(\varsigma) + h_{\tilde{p}}^\top(u(\varsigma))\mathtt{E}[\Psi(\varsigma)]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=1}^{d}\Big(\sigma_p^{j\top}(\varsigma)a^j(\varsigma) + \sigma_{\tilde p}^{j\top}(\varsigma)\mathtt{E}[a^j(\varsigma)]\Big) - \sum_{j=1}^{\infty}\Big(f_p^j(\varsigma)b^j(\varsigma) + f_{\tilde p}^j(\varsigma)\mathtt{E}[b^j(\varsigma)]\Big)\\
&+ g_p^\top(u(\varsigma))\tilde Q(\varsigma) + g_{\tilde p}^\top(u(\varsigma))\mathtt{E}[\tilde Q(\varsigma)] - \varphi_p((u(\varsigma))) - \varphi_{\tilde p}(u(\varsigma))\Big]d\varsigma - \sum_{j=1}^{d} a^j(\varsigma)d\mathcal{B}^j(\varsigma)\\
&- \sum_{j=1}^{\infty} b^j(\varsigma)dH^j(\varsigma),\\
\Phi(0) &= -\Big[\gamma_q(q(0),\mathtt{E}[q(0)]) + \gamma_{\tilde q}(q(0),\mathtt{E}[q(0)])\Big],\\
\Psi(T_f) &= -\Big[l_p^\top(p(T_f),\mathtt{E}[p(T_f)])\Phi(T_f) + l_{\tilde p}^\top(p(T_f),\mathtt{E}[p(T_f)])\mathtt{E}[\Phi(T_f)] + \phi_p(p(T_f),\mathtt{E}[p(T_f)])\\
&+ \phi_{\tilde p}^\top(p(T_f),\mathtt{E}[p(T_f)])\Big]. \qquad (23)
\end{aligned}
$$

Likewise, under the assumptions **(A1)**$\sim$**(A3)**, it can be verified that (23) has a unique solution given by $(\Phi(\cdot),\Psi(\cdot),a(\cdot),b(\cdot))$.

From [33], let us define the Hamiltonian function using the system (1) with cost functional (7) $\mathcal{H}:[0,T_f]\times\mathbf{R}^n\times\times\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}^m\times\mathbf{R}^{m\times d}\times\mathbf{R}^{m\times d}\times l^2(\mathbf{R}^m)\times l^2(\mathbf{R}^m)\times U\times\mathbf{R}^m\times\mathbf{R}^n\times\mathbf{R}^{n\times d}\times l^2(\mathbf{R}^m)\times\mathbf{R}^r\longrightarrow\mathbf{R}$ as

$$
\begin{aligned}
&\mathcal{H}(\varsigma,p,\tilde p,q,\tilde q,r,\tilde r,s,\tilde s,v,\Phi(\cdot),\Psi(\cdot),k(\cdot),a(\cdot),\tilde Q) := \langle\Psi, h(\varsigma,p,\tilde p,q,\tilde q,r,\tilde r,s,\tilde s,v)\rangle\\
&- \langle\Phi, k(\varsigma,p,\tilde p,q,\tilde q,r,\tilde r,s,\tilde s,v)\rangle + \langle\sigma(\varsigma,p,\tilde p,q,\tilde q,r,\tilde r,s,\tilde s),a(\varsigma)\rangle + \sum_{j=1}^{\infty} b^j(\varsigma)f^j(\varsigma,p,\tilde p,q,\tilde q,r,\tilde r,s,\tilde s)\\
&+ \langle\tilde Q, g(\varsigma,p,\tilde p,q,\tilde q,v)\rangle + \varphi(\varsigma,p,\tilde p,q,\tilde q,v). \qquad (24)
\end{aligned}
$$

Then, (23) can be reformulated as a stochastic Hamiltonian system of the following type:

$$
\begin{aligned}
d\Phi(\varsigma) &= -\Big[\mathcal{H}_q(u(\varsigma)) + \mathtt{E}[\mathcal{H}_{\tilde q}(u(\varsigma))]\Big]d\varsigma + \sum_{j=1}^{d}\Big[\mathcal{H}_{r^j}^j(u(\varsigma)) + \mathtt{E}[\mathcal{H}_{\tilde r^j}^j(u(\varsigma))]\Big]d\mathcal{B}^j(\varsigma)\\
&\quad - \sum_{j=1}^{\infty}\Big[\mathcal{H}_{s^j}^j(u(\varsigma)) + \mathtt{E}[\mathcal{H}_{\tilde s^j}^j(u(\varsigma))]\Big]dH^j(\varsigma),\\
-d\Psi(\varsigma) &= \Big[\mathcal{H}_p(u(\varsigma)) + \mathtt{E}[\mathcal{H}_{\tilde p}(u(\varsigma))]\Big]d\varsigma - \sum_{j=1}^{d} a^j(\varsigma)d\mathcal{B}^j(\varsigma) - \sum_{j=1}^{\infty} b^j(\varsigma)dH^j(\varsigma),\\
\Phi(0) &= -\Big[\gamma_q(q(0),\mathtt{E}[q(0)]) + \mathtt{E}[\gamma_{\tilde q}(q(0),\mathtt{E}[q(0)])]\Big],
\end{aligned}
$$

$$\Psi(T_f) = -\Big[l_p^\top(p(T_f), \mathtt{E}[p(T_f)])\Phi(T_f) + \mathtt{E}[l_{\tilde{p}}^\top(p(T_f), \mathtt{E}[p(T_f)])\Phi(T_f)] + \phi_p(p(T_f), \mathtt{E}[p(T_f)]) \\ + \mathtt{E}[\phi_{\tilde{p}}^\top(p(T_f), \mathtt{E}[p(T_f)])]\Big]. \tag{25}$$

Here the partial derivative of $\mathcal{H}$ with respect to p is denoted as
$\mathcal{H}_p(u(\varsigma)) := \mathcal{H}_p(\varsigma, p(\varsigma), \mathtt{E}[p(\varsigma)], q(\varsigma), \mathtt{E}[q(\varsigma)], r(\varsigma), \mathtt{E}[r(\varsigma)], s(\varsigma), \mathtt{E}[s(\varsigma)], u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))$ and similarly we can use the simplified notations for other partial derivatives.

The subsequent theorem presents the key result of this manuscript.

Theorem 3.3 (Partially-Observed Mean Field Stochastic Maximum Principle)**.** *We assume **(A1)**$\sim$ **(A3)** hold and considering $u(\cdot)$ as an optimal control for our PO-OCP described by* (6)*, let $(p(\cdot), \mathtt{E}[p(\cdot)], q(\cdot), \mathtt{E}[q(\cdot)], r(\cdot), \mathtt{E}[r(\cdot)])$ represents the corresponding optimal path, and $\mathcal{Y}(\cdot)$ be the solution of* (4)*. Moreover, let $(\tilde{P}(\cdot), \tilde{Q}(\cdot))$denote the solution of* (22) *and $(\Phi(\cdot), \Psi(\cdot), a(\cdot), b(\cdot))$ be the solution of adjoint equations*(23)*.*

Then we obtain the following:

$$\mathtt{E}^u\Big[\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), v, \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma)) \\ - \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))|\mathfrak{F}_\varsigma^{\mathcal{W}}\Big] \geq 0 \tag{26}$$

$$\forall v \in \mathbb{U},\ a.e.\ \varsigma \in [0, T_f], a.s.,$$

where $\mathcal{H}$ is the Hamiltonian function given in (24)*.*

Proof. By employing Ito's formula to $\langle p^1(\cdot), \Psi(\cdot)\rangle + \langle q^1(\cdot), \Phi(\cdot)\rangle + \eta(\cdot)\tilde{P}(\cdot)$ and using the auxiliary BSDE (22) and adjoint equations (23), as well as the variational equations (9) and V.I. (18), we get

$$\begin{aligned}
&\mathtt{E}^u\bigg[\int_0^{T_f}\bigg(\eta(\cdot)\varphi(u(\varsigma)) + \varphi_p^\top(u(\varsigma))p^1(\varsigma) + \varphi_{\tilde{p}}^\top(u(\varsigma))\mathtt{E}[p^1(\varsigma)] + \varphi_q^\top(u(\varsigma))q^1(\varsigma) + \varphi_{\tilde{q}}^\top(u(\varsigma))\mathtt{E}[q^1(\varsigma)] \\
&\quad + \varphi_r^\top(u(\varsigma))r^1(\varsigma) + \varphi_{\tilde{r}}^\top(u(\varsigma))\mathtt{E}[r^1(\varsigma)] + \varphi_s^\top(u(\varsigma))s^1(\varsigma) + \varphi_{\tilde{s}}^\top(u(\varsigma))\mathtt{E}[s^1(\varsigma)] + \varphi(u^\varepsilon(\varsigma)) - \varphi(u(\varsigma))\bigg)d\varsigma \\
&\quad + \eta(T_f)\phi(p(T_f), \mathtt{E}[p(T_f)]) + \phi_p^\top(p(T_f), \mathtt{E}[p(T_f)])p^1(T_f) + \phi_{\tilde{p}}^\top(p(T_f), \mathtt{E}[p(T_f)])\mathtt{E}[p^1(T_f)] \\
&\quad + \gamma_q(q(0), \mathtt{E}[q(0)])q^1(0) + \gamma_{\tilde{q}}(q(0), \mathtt{E}[q(0)])\mathtt{E}[q^1(0)]\bigg] \\
&= \mathtt{E}^u\bigg[\int_0^{T_f}\bigg(\langle\Psi(\varsigma), h(u^\varepsilon(\varsigma)) - h(u(\varsigma))\rangle - \langle\Phi(\varsigma), k(u^\varepsilon(\varsigma)) - k(u(\varsigma))\rangle + \langle\tilde{Q}, g(u^\varepsilon(\varsigma)) - g(u(\varsigma))\rangle \\
&\quad + \varphi(u^\varepsilon(\varsigma)) - \varphi(u(\varsigma))\bigg)d\varsigma\bigg]
\end{aligned}$$

$$= \mathtt{E}^u\Big[\int_0^{T_f}\Big(\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), u^{\varepsilon}(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma)$$
$$- \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma)s(\varsigma), \tilde{s}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma)\Big)d\varsigma\Big] \geq o(\varepsilon)$$

Based on the definition of $u^{\varepsilon}(\cdot)$, we have

$$\mathtt{E}^u\Big[\int_\delta^{\delta+\varepsilon}\Big(\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma)v, \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{\Psi}(\varsigma))$$
$$- \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma)\Big)d\varsigma\Big] \geq o(\varepsilon)$$

Dividing the above equation by ε, we get

$$\lim_{\varepsilon \longrightarrow 0}\frac{1}{\varepsilon}\ \mathtt{E}^u\int_\delta^{\delta+\varepsilon}\Big(\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), v, \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))$$
$$- \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))\Big)d\varsigma \geq 0.$$

Hence, we have

$$\mathtt{E}^u\Big[\mathcal{H}(p(\delta), \tilde{p}(\delta), q(\delta), \tilde{q}(\delta), r(\delta), \tilde{r}(\delta), s(\delta), \tilde{s}(\delta), v, \Phi(\delta), \Psi(\delta), a(\delta), b(\delta), \tilde{Q}(\delta))$$
$$- \mathcal{H}(\delta, p(\delta), \tilde{p}(\delta), q(\delta), \tilde{q}(\delta), r(\delta), \tilde{r}(\delta), s(\delta), \tilde{s}(\delta), u(\delta), \Phi(\delta), \Psi(\delta), a(\delta), b(\delta), \tilde{Q}(\delta))\Big] \geq 0$$
$$a.e.\ \delta \in [0, T_f].$$

Consider a deterministic element $c \in \mathbb{U}$ and an arbitrary element $F \in \mathfrak{F}_\varsigma^{\mathcal{W}}$. We define $w^*(\varsigma) = c\mathbf{I}_F + u(\varsigma)\mathbf{I}_{\Omega - F}$, where $\mathbf{I}$ denotes the indicator function.

It is evident that, $w^*(\cdot)$ is also an admissible control. As $0 \leq \delta \leq \varsigma_f$, for any bounded $\mathbb{U}$-valued, ${\mathfrak{F}_\varsigma}^{\mathcal{W}}$-measurable random variable v such that $\sup_{\omega\in\Omega}|v(\omega)| < \infty$, we obtain the following:

$$\mathtt{E}^u\Big[\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), v, \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))$$
$$- \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))\Big]d\varsigma \geq 0,\ \ a.e.\ \varsigma \in [0, T_f].$$

By utilizing the aforementioned inequality for $w^*(\cdot)$, one can get

$$\mathtt{E}^u\Big[\mathbf{I}(\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), c, \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))$$

$$- \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma), s(\varsigma), \tilde{s}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma)))\Big] d\varsigma \geq 0$$

$$\forall F \in \mathfrak{F}_\varsigma^{\mathcal{W}}, a.e.\ \varsigma \in [0, T_f].$$

This leads to the conclusion that

$$\mathtt{E}^u\Big[\mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}(\varsigma), r(\varsigma), \tilde{r}(\varsigma), v, \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))$$
$$- \mathcal{H}(\varsigma, p(\varsigma), \tilde{p}(\varsigma), q(\varsigma), \tilde{q}, r(\varsigma), \tilde{r}(\varsigma), u(\varsigma), \Phi(\varsigma), \Psi(\varsigma), a(\varsigma), b(\varsigma), \tilde{Q}(\varsigma))|\mathfrak{F}_\varsigma^{\mathcal{W}}\Big] \geq 0,$$
$$a.e.\ \varsigma \in [0, T_f], a.s.$$

Thus (26) holds. □

4. Application in Linear Quadratic Mean Field Optimal Control Problem

In this section, we are going to discuss an LQ example to illustrate our theoretical results in the previous section. Let us consider the following control system governed by MF-FBSDE with $m = n = 1$.

$$\begin{aligned}
dp_v(\varsigma) &= [\mathcal{A}_1(\varsigma)p_v(\varsigma) + \mathcal{A}_2(\varsigma)\mathtt{E}[p_v(\varsigma)] + \mathcal{A}_3(\varsigma)q_v(\varsigma) + \mathcal{A}_4(\varsigma)\mathtt{E}[q_v(\varsigma)] + \mathcal{A}_5(\varsigma)v(\varsigma)]d\varsigma + \sum_{j=1}^{d} \mathcal{C}^j(\varsigma)d\mathcal{B}^j(\varsigma) \\
&\quad + \sum_{j=1}^{\infty} D^j(\varsigma)dH^j(\varsigma), \\
-dq_v(\varsigma) &= [B_1(\varsigma)p_v(\varsigma) + B_2(\varsigma)\mathtt{E}[p_v(\varsigma)] + B_3(\varsigma)q_v(\varsigma) + B_4(\varsigma)\mathtt{E}[q_v(\varsigma)] + B_5(\varsigma)v(\varsigma)]d\varsigma - \sum_{j=1}^{d} (r_v^j)(\varsigma)d\mathcal{B}^j(\varsigma) \\
&\quad - \sum_{j=1}^{\infty} s^j(\varsigma)dH^j(\varsigma),\ \varsigma \in [0, T_f], \\
p_v(0) &= p_0, \\
q_v(T_f) &= C_1 p_v(T_f) + C_2\mathtt{E}[p_v(T_f)],
\end{aligned} \tag{27}$$

and observation

$$\begin{aligned}
d\mathcal{W}(\varsigma) &= G(\varsigma)d\varsigma + d\tilde{\mathcal{B}}(\varsigma),\ \varsigma \in [0, T_f], \\
\mathcal{W}(0) &= 0.
\end{aligned} \tag{28}$$

Here, the functions $\mathcal{A}_i(\cdot), B_i(\cdot)$ for $i = 1, 2, \cdots, 5$, $\mathcal{C}^j(\cdot)$ for $j = 1, 2, \ldots d$ and $D^j(\cdot)$ for $j = 1, 2, \ldots$ are all bounded, deterministic and satisfying the assumptions **(A1)** and **(A2)** in Section 2.

The cost functional is

$$\mathcal{J}^*(v(\cdot)) = \frac{1}{2}\mathtt{E}^v\Bigg[\int_0^{T_f}[S_1(\varsigma)(p_v(\varsigma))^2 + S_2(\varsigma)(\mathtt{E}[p_v(\varsigma)])^2 + S_3(\varsigma)(q_v(\varsigma))^2 + S_4(\varsigma)(\mathtt{E}[q_v(\varsigma)])^2$$
$$+ S_5(\varsigma)u^2(\varsigma)]d\varsigma + M_1(p_v(T_f))^2 + M_2(\mathtt{E}[p_v(T_f)])^2 + N_1(q_v(0))^2 + N_2(\mathtt{E}[q_v(0)])^2\Bigg], \quad (29)$$

where the functions $S_i(\cdot) \geq 0,\ \forall\ i = 1, 2, 3, 4, S_5(\cdot) > 0$ and the constants $M_1, M_2, N_1, N_2 \geq 0$.

The function $S_5^{-1}(\cdot)$ is also bounded. The two-dimensional standard Brownian motion $(\mathcal{B}(\cdot), \mathcal{W}(\cdot))$ is defined on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\varsigma\}, P)$. By (28), it is clear that $(\mathcal{B}(\cdot), \tilde{\mathcal{B}}(\cdot))$ constitutes a standard Brownian motion which is two dimensional, defined on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_\varsigma\}, \mathcal{P}^v)$, a new probability space, where $\mathcal{P}^v$ represents a new probability measure.

It is simple to prove that the condition **(A2)** in Section 2 holds for any given $v(\cdot)$. Hence, the FBSDE (27) admits a unique solution $(p_v(\cdot), \mathtt{E}[p_v(\cdot)], q_v(\cdot), \mathtt{E}[q_v(\cdot)], r_v(\cdot), \mathtt{E}[r_v(\cdot)])$. The Hamiltonian function is given by

$$\mathcal{H}(\varsigma, p, \bar{p}, q, \bar{q}, r, \bar{r}, v, \Phi, \Psi, a, b, \tilde{Q}) = \Psi(\mathcal{A}_1(\varsigma)p + \mathcal{A}_2(\varsigma)\bar{p} + \mathcal{A}_3(\varsigma)q + \mathcal{A}_4(\varsigma)\bar{q} + \mathcal{A}_5(\varsigma)v) - \Phi(B_1(\varsigma)p + B_2(\varsigma)\bar{p}$$
$$+ B_3(\varsigma)q + B_4(\varsigma)\bar{q} + B_5(\varsigma)v) + \sum_{j=1}^{d} a^j(\varsigma)C^j(\varsigma) + \sum_{j=1}^{\infty} b^j(\varsigma)D^j(\varsigma)$$
$$+ \tilde{Q}\bar{F}(\varsigma) + \frac{1}{2}[S_1(\varsigma)p^2 + S_2(\varsigma)(\bar{p})^2 + S_3(\varsigma)q^2 + S_4(\varsigma)(\bar{q})^2 + S_5(\varsigma)v^2]. \quad (30)$$

By Theorem 2.2, if $u(\cdot)$ is optimal, then

$$u(\varsigma) = -S_5^{-1}(\varsigma)\Big(\mathcal{A}_5(\varsigma)\mathtt{E}^u[\Psi(\varsigma)|\mathfrak{F}_\varsigma] - B_5(\varsigma)\mathtt{E}^u[\Phi(\varsigma)|\mathfrak{F}_\varsigma]\Big), \quad (31)$$

where $(\Phi(\cdot), \Psi(\cdot))$ represents the solution to the FBSDE specified as follows :

$$d\Phi(\varsigma) = \Big(B_3(\varsigma)\Phi(\varsigma) + B_4(\varsigma)\mathtt{E}[\Phi(\varsigma)] - \mathcal{A}_3(\varsigma)\Psi(\varsigma) - \mathcal{A}_4(\varsigma)\mathtt{E}[\Psi(\varsigma)] - S_3(\varsigma)q_u(\varsigma) - S_4(\varsigma)\mathtt{E}[q_u(\varsigma)]\Big)d\varsigma,$$
$$-d\Psi(\varsigma) = \Big(-B_1(\varsigma)\Phi(\varsigma) - B_2(\varsigma)\mathtt{E}[\Phi(\varsigma)] + \mathcal{A}_1(\varsigma)\Psi(\varsigma) + \mathcal{A}_2(\varsigma)\mathtt{E}[\Psi(\varsigma)] - S_1(\varsigma)p_u(\varsigma) - S_2(\varsigma)\mathtt{E}[p_u(\varsigma)]\Big)d\varsigma$$
$$- \sum_{j=1}^{d} a^j(\varsigma)d\mathcal{B}^j(\varsigma) - \sum_{j=1}^{\infty} b^j(\varsigma)dH^j(\varsigma),$$

$$\Phi(0) = -N_1(q_v(0)) - N_2\mathtt{E}[q_v(0)],$$

$$\Psi(T_f) = -C_1\Phi(T_f) - C_2\mathtt{E}[\Phi(T_f)] + M_1(p_u(T_f)) + M_2(\mathtt{E}[p_u(T_f)]). \tag{32}$$

Similarly, one can verify that the condition **(A2)** is satisfied, in which case the FBSDE (32) admits a unique solution $(\Phi(\cdot), \Psi(\cdot), a(\cdot), b(\cdot))$.

Moreover, one can demonstrate that the admissible control (31), which satisfies the necessary optimality conditions, is indeed optimal. The expectations $\mathtt{E}^u$ and $\mathtt{E}^v$ are equivalent. Then for any admissible control $v(\cdot)$, the following holds:

$$\begin{aligned}
\mathcal{J}^*(v(\cdot)) - \mathcal{J}^*(u(\cdot)) = \frac{1}{2}\mathtt{E}^u\Bigg[\int_0^{T_f} & S_1(\varsigma)(p_v(\varsigma) - p_u(\varsigma))^2 + 2S_1(\varsigma)p_u(\varsigma)(p_v(\varsigma) - p_u(\varsigma)) \\
& + S_2(\varsigma)(\mathtt{E}[p_v(\varsigma)] - \mathtt{E}[p_u(\varsigma)])^2 + 2S_2(\varsigma)\mathtt{E}[p_v(\varsigma)](\mathtt{E}[p_v(\varsigma)] - \mathtt{E}[p_u(\varsigma)]) \\
& + S_3(\varsigma)(q_v(\varsigma) - q_u(\varsigma))^2 + 2S_3(\varsigma)q_u(\varsigma)(q_v(\varsigma) - q_u(\varsigma)) \\
& + S_4(\varsigma)(\mathtt{E}[q_v(\varsigma)] - \mathtt{E}[q_u(\varsigma)])^2 + 2S_4(\varsigma)\mathtt{E}[q_v(\varsigma)](\mathtt{E}[q_v(\varsigma)] - \mathtt{E}[q_u(\varsigma)]) \\
& + S_5(\varsigma)(v(\varsigma) - u(\varsigma))^2 + 2S_5(\varsigma)u(\varsigma)(v(\varsigma) - u(\varsigma)) + M_1(p_v(T_f) - p_u(T_f))^2 \\
& + 2M_1p_u(T_f)(p_v(T_f) - p_u(T_f)) + M_2(\mathtt{E}[p_v(T_f)] - \mathtt{E}[p_u(T_f)])^2 \\
& + 2M_2\mathtt{E}[p_u(T_f)](\mathtt{E}[p_v(T_f)] - \mathtt{E}[p_u(T_f)]) + N_1(q_v(0) - q_u(0))^2 \\
& + 2N_1q_u(0)(q_v(0) - q_u(0)) + N_2(\mathtt{E}[q_v(0)] - \mathtt{E}[q_u(0)])^2 \\
& + 2N_2\mathtt{E}[q_u(0)](\mathtt{E}[q_v(0)] - \mathtt{E}[q_u(0)])\Bigg].
\end{aligned} \tag{33}$$

Applying Ito's formula to $(p_v(\varsigma) - p_u(\varsigma))\Psi(\varsigma) + (q_v(\varsigma) - q_u(\varsigma))\Phi(\varsigma)$, we have

$$\begin{aligned}
&\mathtt{E}^u\Big[(M_1p_u(T_f) + M_2\mathtt{E}[p_u(T_f)])(p_v(T_f) - p_u(T_f)) + (N_1q_u(0) + N_2\mathtt{E}[q_u(0)])(q_v(0) - q_u(0))\Big] \\
&\quad = \mathtt{E}^u\Bigg[\int_0^{T_f}\Big((p_v(\varsigma) - p_u(\varsigma))S_1(\varsigma)p_u(\varsigma) + (p_v(\varsigma) - p_u(\varsigma))S_2(\varsigma)\mathtt{E}[p_u(\varsigma)] + \Psi(\varsigma)\mathcal{A}_5(\varsigma)(v(\varsigma) - u(\varsigma)) \\
&\qquad + (q_v(\varsigma) - q_u(\varsigma))S_3(\varsigma)q_u(\varsigma) + (q_v(\varsigma) - q_u(\varsigma))S_4(\varsigma)\mathtt{E}[q_u(\varsigma)] - \Phi(\varsigma)B_5(\varsigma)(v(\varsigma) - u(\varsigma))d\varsigma\Big)\Bigg].
\end{aligned} \tag{34}$$

As $S_i(\varsigma) \geq 0 \ \forall i = 1,2,3,4, \ S_5(\varsigma) > 0 \ \forall t \in [0, T_f]$ and $M_1, M_2, N_1, N_2 \geq 0$, observing that $(\Phi(\cdot), \Psi(\cdot))$ are not completely observable, we obtain

$$\begin{aligned}
\mathcal{J}^*(v(\cdot)) - \mathcal{J}^*(u(\cdot)) \geq \ & \mathtt{E}^u\Big[\int_0^{T_f} \Big(S_1(\varsigma)p_u(\varsigma)(p_v(\varsigma) - p_u(\varsigma)) + S_2(\varsigma)\mathtt{E}[p_u(\varsigma)]\mathtt{E}[p_v(\varsigma) - p_u(\varsigma)] \\
& + S_3(\varsigma)q_u(\varsigma)(q_v(\varsigma) - q_u(\varsigma)) + S_4(\varsigma)\mathtt{E}[q_u(\varsigma)]\mathtt{E}[q_v(\varsigma) - q_u(\varsigma)] \\
& + S_5(\varsigma)u(\varsigma)(v(\varsigma) - u(\varsigma))\Big)dt + M_1 p_u(T_f)(p_v(T_f) - p_u(T_f)) \\
& + M_2\mathtt{E}[p_u(T_f)]\mathtt{E}[p_v(T_f) - p_u(T_f)] + N_1 q_u(0)(q_v(0) - q_u(0)) \\
& + N_2\mathtt{E}[q_u(0)]\mathtt{E}[q_v(0) - q_u(0)]\Big] \\
= \ & \mathtt{E}^u\Big[\int_0^{T_f} \Big(S_5(\varsigma)u(\varsigma)(v(\varsigma) - u(\varsigma)) + (\mathcal{A}_5(\varsigma)\Psi(\varsigma) - B_5(\varsigma)\Phi(\varsigma))(v(\varsigma) - u(\varsigma))\Big)dt\Big] \\
= \ & \mathtt{E}^u\Big[\int_0^{T_f} \Big(S_5(\varsigma)\Big(- S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\mathtt{E}^u[\Psi(\varsigma)|\mathfrak{F}_t] - B_5(\varsigma)\mathtt{E}^u[\Phi(\varsigma)|\mathfrak{F}_t]\Big)(v(\varsigma) - u(\varsigma)) \\
& + (\mathcal{A}_5(\varsigma)\Psi(\varsigma) - B_5(\varsigma)\Phi(\varsigma))(v(\varsigma) - u(\varsigma))\Big)d\varsigma\Big] \\
= \ & 0.
\end{aligned} \tag{35}$$

Hence, (31) is an optimal control.

With the help of (31) and the filtering estimates for optimal trajectories, we are looking for an explicit observable optimal control. For this, considering the terminal condition of (32), we get

$$\Psi(\varsigma) = \Pi_1(\varsigma)(\Phi(\varsigma) - \mathtt{E}[\Phi(\varsigma)]) + \Pi_2(\varsigma)\mathtt{E}[\Phi(\varsigma)] + \pi_1(\varsigma)(p(\varsigma) - \mathtt{E}[p(\varsigma)]) + \pi_2(\varsigma)\mathtt{E}[p(\varsigma)] + \chi(\varsigma), \tag{36}$$

where $\pi_1(\cdot), \pi_2(\cdot)$ satisfies the following Riccati equations(37) and (38):

$$\begin{cases} \dot{\pi}_1(\varsigma) + (2\mathcal{A}_1(\varsigma) + \mathcal{A}_3(\varsigma))\pi_1(\varsigma) + S_5^{-1}(\varsigma)\mathcal{A}_5^2(\varsigma)\pi_1^2(\varsigma) + S_1(\varsigma) = 0, \\ \pi_1(T_f) = M_1, \end{cases} \tag{37}$$

$$\begin{cases} \dot{\pi}_2(\varsigma) + (2\mathcal{A}_2(\varsigma) + \mathcal{A}_1 - \mathcal{A}_3(\varsigma) - \mathcal{A}_4(\varsigma))\pi_2(\varsigma) - S_5^{-1}(\varsigma)\mathcal{A}_5^2(\varsigma)\pi_2^2(\varsigma) + S_1(\varsigma) + S_2(\varsigma) = 0, \\ \pi_2(T_f) = M_1 + M_2, \end{cases} \tag{38}$$

Further the functions $\Pi_1(\cdot), \Pi_2(\cdot), \chi(\cdot)$ satisfies the following ordinary differential equations(39)$\sim$(41):

$$\begin{cases} \dot{\Pi}_1(\varsigma) + (\mathcal{A}_1(\varsigma) + B_3(\varsigma) - \pi_1(\varsigma)\mathcal{A}_5(\varsigma)S_5^{-1}\mathcal{A}_5(\varsigma))\Pi_1(\varsigma) + \mathcal{A}_3(\varsigma)\Pi^2{}_1(\varsigma) + B_1(\varsigma) = 0, \\ \Pi_1(T_f) = -C_1, \end{cases} \tag{39}$$

$$\begin{cases} \dot{\Pi}_2(\varsigma) + (B_3(\varsigma) + B_4(\varsigma) + \mathcal{A}_1(\varsigma) - \mathcal{A}_3(\varsigma) - \mathcal{A}_4(\varsigma) - \pi_2(\varsigma)\mathcal{A}_5(\varsigma)S_5^{-1}\mathcal{A}_5(\varsigma))\Pi_2(\varsigma) + B_1(\varsigma) \\ +B_2 + B_5(\varsigma) = 0, \\ \Pi_2(T_f) = -(C_1 + C_2), \end{cases} \tag{40}$$

and

$$\begin{cases} \dot{\chi}(\varsigma) + (\mathcal{A}_1(\varsigma) + \mathcal{A}_2(\varsigma) - \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_2(\varsigma))\chi(\varsigma) = 0, \\ \chi(T_f) = 0. \end{cases} \tag{41}$$

From the classical Riccati equation theory, it is obvious that the Riccati equations (37),(38) admit unique solutions and all the three ordinary differential equations (39),(40), and (41) which also have unique solutions respectively.

Next, we know that

$$\begin{aligned} d\Phi(\varsigma) = \Big(&(B_3(\varsigma) - \mathcal{A}_3(\varsigma)\Pi_1(\varsigma))\Phi(\varsigma) + (B_4(\varsigma) + \mathcal{A}_3(\varsigma)\Pi_1(\varsigma) - \mathcal{A}_3(\varsigma)\Pi_2(\varsigma) - \mathcal{A}_4(\varsigma)\Pi_2(\varsigma))\mathtt{E}[\Phi(\varsigma)] \\ &- \mathcal{A}_3(\varsigma)\pi_1(\varsigma)p(\varsigma) + (\mathcal{A}_3(\varsigma)\pi_1(\varsigma) - \mathcal{A}_3(\varsigma)\pi_2(\varsigma) - \mathcal{A}_4(\varsigma)\pi_2(\varsigma))\mathtt{E}[p(\varsigma)] - S_3(\varsigma)q(\varsigma) - S_4(\varsigma)\mathtt{E}[q(\varsigma)]\Big)d\varsigma, \\ \Phi(0) = &-N_1 q(0) - N_2\mathtt{E}[q(0)]. \end{aligned} \tag{42}$$

Our aim is to establish the filtering estimate $\hat{\Phi}(\varsigma)$ of $\Phi(\cdot)$ under $\mathfrak{F}_\varsigma^{\mathcal{W}}$.

$$\text{(i.e) } \hat{\Phi}(\varsigma) := \mathtt{E}^u[\ \Phi(\varsigma)|\mathfrak{F}_\varsigma^{\mathcal{W}}],\ 0 \le \varsigma \le T_f.$$

Then the observable optimal control is obtained, using (31) and (36) as follows:

$$\begin{aligned} u(\varsigma) = -S_5^{-1}(\varsigma)\Big(&\mathcal{A}_5(\varsigma)\Pi_1(\varsigma)(\hat{p}(\varsigma) - \mathtt{E}[\hat{p}(\varsigma)]) + (\mathcal{A}_5(\varsigma)\Pi_2(\varsigma) - B_5(\varsigma))\mathtt{E}[\hat{p}(\varsigma)] + \mathcal{A}_5(\varsigma)\pi_1(\varsigma)(\hat{p}(\varsigma) - \mathtt{E}[\hat{p}(\varsigma)]) \\ &+ \mathcal{A}_5(\varsigma)\pi_2(\varsigma)\mathtt{E}[\hat{p}(\varsigma)] + \mathcal{A}_5(\varsigma)\chi(\varsigma)\Big),\ 0 \le \varsigma \le T_f. \end{aligned} \tag{43}$$

Taking conditional expectation on (42), we get

$$
\begin{aligned}
d\hat{\Phi}(\varsigma) &= \Big((B_3(\varsigma) - \mathcal{A}_3(\varsigma)\Pi_1(\varsigma))\hat{\Phi}(\varsigma) + (B_4(\varsigma) + \mathcal{A}_3(\varsigma)\Pi_1(\varsigma) - \mathcal{A}_3(\varsigma)\Pi_2(\varsigma) - \mathcal{A}_4(\varsigma)\Pi_2(\varsigma))\mathtt{E}[\hat{\Phi}(\varsigma)] \\
&\quad - \mathcal{A}_3(\varsigma)\pi_1(\varsigma)\hat{p}(\varsigma) + (\mathcal{A}_3(\varsigma)\pi_1(\varsigma) - \mathcal{A}_3(\varsigma)\pi_2(\varsigma) - \mathcal{A}_4(\varsigma)\pi_2(\varsigma))\mathtt{E}[\hat{p}(\varsigma)] - S_3(\varsigma)\hat{q}(\varsigma) - S_4(\varsigma)\mathtt{E}[\hat{q}(\varsigma)] \Big) d\varsigma, \\
\hat{\Phi}(0) &= -N_1\hat{q}(0) - N_2\mathtt{E}[\hat{q}(0)],
\end{aligned}
\tag{44}
$$

where the filtering estimates of the optimal paths under $\mathfrak{F}_\varsigma^{\mathcal{W}}$ are defined as follows:

$$
\hat{p}(\varsigma) := \mathtt{E}^u[p_u(\varsigma)|\mathfrak{F}_\varsigma^{\mathcal{W}}],\ \hat{q}(\varsigma) := \mathtt{E}^u[q_u(\varsigma)|\mathfrak{F}_\varsigma^{\mathcal{W}}],\ \hat{r}(\varsigma) := \mathtt{E}^u[r_u(\varsigma)|\mathfrak{F}_\varsigma^{\mathcal{W}}],\ 0 \le \varsigma \le T_f.
$$

Now, we use the observable optimal control (43), in the proposed LQ system (27) and taking the conditional expectation $\mathtt{E}[\ \cdot|\mathfrak{F}_\varsigma^{\mathcal{W}}]$ on both sides to get

$$
\begin{aligned}
d\hat{p}_u(\varsigma) &= \Big((\mathcal{A}_1(\varsigma) - \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_1(\varsigma))\hat{p}_u(\varsigma) + (\mathcal{A}_2(\varsigma) + \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_1(\varsigma) \\
&\quad - \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_2(\varsigma))\mathtt{E}[\hat{p}_u(\varsigma)] + \mathcal{A}_3(\varsigma)\hat{q}_u(\varsigma) + \mathcal{A}_4(\varsigma)\mathtt{E}[\hat{q}_u(\varsigma)] - \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\Pi_1(\varsigma)\hat{\Phi}(\varsigma) \\
&\quad + (\mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\Pi_1(\varsigma) - \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\Pi_2(\varsigma) + \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)B_5(\varsigma))\mathtt{E}[\hat{\Phi}(\varsigma)] \\
&\quad - \mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\chi(\varsigma) \Big) d\varsigma, \\
-d\hat{q}_u(\varsigma) &= \Big((B_1(\varsigma) - B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_1(\varsigma))\hat{p}_u(\varsigma) + (B_2(\varsigma) + B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_1(\varsigma) \\
&\quad - B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\pi_2(\varsigma))\mathtt{E}[\hat{p}_u(\varsigma)] + B_3(\varsigma)\hat{q}_u(\varsigma) + B_4(\varsigma)\mathtt{E}[\hat{q}_u(\varsigma)] - B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\Pi_1(\varsigma)\hat{\Phi}(\varsigma) \\
&\quad + B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\Pi_1(\varsigma) - B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\Pi_2(\varsigma) + B_5(\varsigma)S_5^{-1}(\varsigma)B_5(\varsigma))\mathtt{E}[\hat{\Phi}(\varsigma)] \\
&\quad - B_5(\varsigma)S_5^{-1}(\varsigma)\mathcal{A}_5(\varsigma)\chi(\varsigma) \Big) d\varsigma, \\
\hat{p}(0) &= p_0, \\
\hat{q}_u(T_f) &= C_1\hat{p}_u(T_f) + C_2\mathtt{E}[\hat{p}_u(T_f)].
\end{aligned}
\tag{45}
$$

Clearly (44) and (45) admits unique solution. Therefore (43) is an observable optimal control.

Now, we are going to obtain the filtering estimate $\hat{r}_u(\varsigma)$. From the terminal condition of (27), we can get

$$
q_u(\varsigma) = \Sigma_1(\varsigma)(p_u(\varsigma) - \mathtt{E}[p_u(\varsigma)]) + \Sigma_2(\varsigma)\mathtt{E}[p_u(\varsigma)] + \Delta(\varsigma),\ \text{a.e. } \varsigma \in [0, T_f], \tag{46}
$$

where $\Sigma_1(\cdot), \Sigma_2(\cdot)$ are the solutions of the Riccati equations which are provided below

$$\begin{cases}\dot{\Sigma}_1(\varsigma) + (\mathcal{A}_1(\varsigma) - \mathcal{A}_5^2(\varsigma)S_5^{-1}(\varsigma)\pi_1(\varsigma) + B_1(\varsigma))\Sigma_1(\varsigma) + \mathcal{A}_3(\varsigma)\Sigma_1^2(\varsigma) + B_1(\varsigma) = 0, \\ \Sigma_1(T_f) = C_1, \end{cases} \tag{47}$$

$$\begin{cases}\dot{\Sigma}_2(\varsigma) + (\mathcal{A}_1(\varsigma) + \mathcal{A}_2(\varsigma) - \mathcal{A}_5^2(\varsigma)S_5^{-1}(\varsigma)\pi_2(\varsigma) + B_3(\varsigma) + B_4(\varsigma))\Sigma_2(\varsigma) + (\mathcal{A}_3(\varsigma) + \mathcal{A}_4(\varsigma))\Sigma_2^2(\varsigma) \\ +B_1(\varsigma) + B_2(\varsigma) = 0, \\ \Sigma_2(T_f) = C_1 + C_2 \end{cases} \tag{48}$$

and $\Delta(\cdot)$ satisfies the following ordinary differential equation

$$\begin{cases}\dot{\Delta}(\varsigma) + (B_3(\varsigma) + B_4(\varsigma))\Delta(\varsigma) - S_5^{-1}(\varsigma)\Sigma_1(\varsigma)\mathcal{A}_5^2(\varsigma)\Pi_1(\varsigma)(\hat{\Phi}(\varsigma) - \mathtt{E}[\hat{\Phi}(\varsigma)]) \\ -\Sigma_2(\varsigma)\mathcal{A}_5(\varsigma)S_5^{-1}(\varsigma)(\mathcal{A}_5(\varsigma)\Pi_2(\varsigma) - B_5(\varsigma))\mathtt{E}[\hat{\Phi}(\varsigma)] - \Sigma_2(\varsigma)\mathcal{A}_5^2(\varsigma)S_5^{-1}(\varsigma)\chi(\varsigma) = 0, \\ \Delta(T_f) = 0. \end{cases} \tag{49}$$

From the theory of classical BSDE, it is simple to derive

$$(\hat{r}^j)_u(\varsigma) = r^j(\varsigma) = D^j(\varsigma)\Sigma_1(\varsigma) \text{ a.e.}\varsigma \in [0, T_f], \forall i = 1, 2, 3, \ldots. \tag{50}$$

Therefore, the following result is obtained.

Theorem 4.1. *For the proposed LQ PO-OCP of fully-coupled forward-backward mean-field SDE* (27)*, the expression* (43) *gives the optimal control $u(\cdot)$ which is observable, where $\hat{\Phi}(\cdot)$ is the solution of* (44) *and $\pi_1(\varsigma), \pi_2(\varsigma), \Pi_1(\cdot), \Pi_2(\cdot)$ are the solutions of the Riccati equations*(37)*-*(40)*. Moreover the filtering estimates $(\hat{p}(\cdot), \hat{q}(\cdot), \hat{r}(\cdot))$ of optimal trajectories $(p(\cdot), q(\cdot), r(\cdot))$ are given by* (45)*, where $\Sigma_1(\cdot), \Sigma_2(\cdot)$ are the solutions of Riccati equation*(47),(48)*.*

5. Conclusion and Future works

This research focuses on establishing the maximum principle for the PO-OCP. The controlled state process is governed by FBSDE of mean-field type, with the influence of Teugels martingales. Through the transformation of the partial observation problem into an equivalent form with complete information, we have effectively derived the stochastic maximum principle for optimal control. This conversion has played a crucial role in overcoming the complexities associated with partial observation, enabling us to establish a robust framework for tackling the problem of optimal control. To validate our theoretical findings, we have explicitly solved a PO-LQ problem. This explicit solution serves as concrete evidence of the effectiveness and practical applicability of the proposed approach to deal with partially observed systems. These findings open up new avenues for addressing complex financial decision-making scenarios under partial information constraints, enhancing the applicability of the proposed methodology in the field of finance. The authors' future focus will be on establishing the stochastic maximum principle for mean-field type FBSDEs with mixed initial and terminal conditions.

Statements and Declarations

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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