

# Two equivalent multi-sensor Kalman filters with variable delays and intermittent measurements

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**Abstract**—Kalman filtering of measurement data from multiple sensors with time-varying delays and missing measurements is considered in this work. Two existing approaches to Kalman filtering with delays are extended by removing some assumptions in order to have equivalent filtering methods and making comparisons between them. The computational loads of the two methods are compared in terms of the average number of floating point operations required by each method for different system dimensionalities and delay upper bounds. The results show that the superiority of the methods over each other depends on the comparison conditions.

**Index Terms**—Kalman filter, time-varying delays, intermittent measurements, data packet losses, multirate sampling.

## I. INTRODUCTION

There is a wide variety of filtering applications where time-varying delays, missing measurement samples, and multiplicity of sensors are issues in many of them. Delays and missing measurements can be for example due to processing time or communication constraints [1], [2]. Kalman filtering with only random data packet losses is considered in a number of works including [3], [4], [5], [6] where the dimensionality of filter is the same as the dynamical system. However, the filter becomes more complex in the presence of delays.

There are three main approaches to filter design with delayed measurement information. The first approach is to revisit the past values of the recursive Kalman filtering variables upon receiving a delayed measurement sample. To reduce the complexity of this approach, the covariance matrix is calculated regardless of delay occurrences in [7], [8] which causes some estimation errors. These error can be avoided by storing the past filtering variables in finite length buffers as suggested in [9] where data packet losses are also taken into account. Some researchers attempt to combine the recursive filter equations for the previous time steps instead of using buffers which can increase the complexity of filter [7], [10], [11]. This complexity can be reduced to some extent by making simplifying assumptions for example about order or availability of data samples in [12], [13], [14], [15]. The second approach is based on augmentation of the state vector with the delayed measurements. This converts the delayed system to a delay-free system such that the ordinary Kalman filter can be applied [2], [16], [17], [7]. The third approach applies  $H_\infty$  analysis of time delay systems to design filters with bounded error covariance [18], [19], [20], [21]. The result is not a Kalman filter in the sense that it does not produce the

optimal estimation. It is relatively straightforward to design robust filters based on this approach [22], [23], [24].

This work aims to extend the first two Kalman filtering approaches mentioned above into equivalent methods such that a comparison between them is made possible. The extensions allow for tackling intermittent measurements from multiple sensors subject to time-varying delays. For the first approach, the filtering method in [9], which is found to be a more general formulation among the existing results, is extended to the case of multi-sensor measurements while modifying the underlying notation. The result is presented as a self-contained optimal filtering algorithm. The formulation of the second approach is also extended by adding the capabilities to handle missing measurements and simultaneous or miss-ordered arrival of measurement samples from the previous time steps. Then, the computational loads of the two methods are compared for different values of delay upper bound and system's dimensionality by evaluating the average count of floating point operations (flops) required for each case. The results show that each method can overtake the other one in a subset of conditions.

*Notation:* The set of integers  $\{a, a+1, \dots, b\}$  is denoted by  $\{a..b\}$ . For an ordered set  $S$ , a set of elements  $M_i$  for  $i \in S$  is denoted as  $\{M_i\}_{i \in S}$  or  $\{M_i\}_{i=a}^b$  if  $S = \{a..b\}$ . Vertical and diagonal concatenations of matrices  $\{M_i\}_{i=a}^b$  are denoted by  $\text{cat}\{M_i\}_{i=a}^b$  and  $\text{diag}\{M_i\}_{i=a}^b$  respectively. The expected value of a random matrix  $V$  is denoted by  $\mathbb{E}\{V\}$ . A matrix with zero elements is simply denoted as 0 if its dimensions can be inferred from the containing formula. An empty matrix is denoted by  $0_0$  which is allowed to be concatenated with an arbitrary matrix  $M$  as  $\text{cat}\{M, 0_0\} = \text{diag}\{M, 0_0\} = M$ .

## II. PROBLEM STATEMENT

The problem under consideration in this work is to optimally estimate the state of a dynamical system described by

$$x_k = F_k x_{k-1} + B_k u_k + w_k \quad (1a)$$

$$y_k^i = H_k^i x_k + v_k^i \quad (1b)$$

in which  $k$  is the discrete time step,  $x_k \in \mathbb{R}^{n_x}$  is the state vector,  $u_k \in \mathbb{R}^{n_u}$  is the input vector,  $y_k^i \in \mathbb{R}^{n_i}$  is the  $i$ th measured output vector for  $i \in \{1..m\}$  with  $\sum_{i=1}^m n_i = n_y$ ,  $w_k \in \mathbb{R}^{n_w}$  and  $v_k^i \in \mathbb{R}^{n_i}$  are uncorrelated, zero-mean white Gaussian noise vectors with covariance matrices  $R_k^i = \mathbb{E}\{v_k^i v_k^{iT}\}$  and  $Q_k = \mathbb{E}\{w_k w_k^T\}$ .

The  $i$ th output sample  $y_k^i$  is received by the estimator after a time-varying delay defined as

$$d_k^i = t - k \quad \text{if the estimator receives } y_k^i \text{ at } t \geq k. \quad (2)$$

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It is assumed that the measurement samples are time-stamped such that, the delay for each data sample can be calculated from its time-stamp by the estimator. Missing measurements are also assumed to be probable for example due to multirate sampling or communication errors. A missing measurement can be considered as a sample received with infinite delay. If a measurement sample is not missed, it is assumed to be received after a delay bounded by  $d_{max}$  such that

$$d_k^i \in \{0..d_{max}\} \cup \{\infty\} \quad \forall k \geq 0 \quad (3)$$

The optimal estimation of the state  $x_k$  is given by

$$\hat{x}_{k|k} = \mathbb{E}\{x_k | \mathcal{I}_k\} \quad (4a)$$

$$\mathcal{I}_k = \{y_j^i \mid j + d_j^i \leq k\} \quad (4b)$$

in which  $\mathcal{I}_k$  is the information set which is available to the estimator at the time step  $k$ .

If there are no delays or missing measurements (i.e.  $d_k^i = 0$ ), then the optimal estimation  $\hat{x}_{k|k}$  is calculated by the means of the ordinary Kalman filter in the appendix. In presence of the delays or missing measurements, the ordinary Kalman filter cannot be applied. In the following two sections, two equivalent approaches are presented for extension of Kalman filtering to measured data from multiple sensors, with variable delays and missing measurements according to (3).

*Remark 1:* The problem formulation in this section is capable for tackling multirate sampling systems. For this purpose, it is only needed to virtually assume that  $d_k^i = \infty$  if no sample is taken from the  $i$ th sensor at the time step  $k$ .

### III. BACKWARD RENOVATION KALMAN FILTER

In the first method, if  $y_k^i$  is received at the time step  $t$ , then the state estimator has to repeat the recursive calculations of filter from  $k$  to  $t$ . To formulate this method, the following notation is introduced for the variables  $y_k^i$ ,  $v_k^i$ ,  $H_k^i$ ,  $R_k^i$ .

$$\tilde{\varphi}_k^{i,t} = \begin{cases} \varphi_k^i & \text{if } y_k^i \text{ is received until } t, \\ 0_0 & \text{otherwise,} \end{cases} \quad (5)$$

$$\varphi \in \{y, v, H, R\}.$$

The set of measurements that are sampled at  $k$  and received until  $t$  are concatenated into a vector  $\tilde{y}_k^t$  defined as

$$\tilde{y}_k^t = \text{cat}\{\tilde{y}_k^{i,t}\}_{i=1}^m \quad (6)$$

which can be written as

$$\tilde{y}_k^t = H_k^t x_k + \tilde{v}_k^t, \quad (7a)$$

$$H_k^t = \text{cat}\{\tilde{H}_k^{i,t}\}_{i=1}^m, \quad (7b)$$

$$\tilde{v}_k^t = \text{cat}\{\tilde{v}_k^{i,t}\}_{i=1}^m, \quad (7c)$$

in which  $\tilde{v}_k^t$  is a new zero-mean noise vector with covariance

$$R_k^t = \mathbb{E}\{\tilde{v}_k^t \tilde{v}_k^{tT}\} = \text{diag}\{\tilde{R}_k^{i,t}\}_{i=1}^m. \quad (8)$$

Defining the information set  $\mathcal{I}_k^t$  as

$$\mathcal{I}_k^t = \{\tilde{y}_j^{i,t} \mid j \leq k, 1 \leq i \leq m\}, \quad (9)$$

then we have  $\mathcal{I}_k^k = \mathcal{I}_k$  according to (4b) and (5) for  $\varphi = y$ .

The optimal estimation of  $x_k$  in (24) and the corresponding error covariance given the information set  $\mathcal{I}_h^t$  for time steps  $t$  and  $h$  satisfying  $t \geq k \geq h$  are written as

$$\hat{x}_{k|h}^t = \mathbb{E}\{x_k \mid \mathcal{I}_h^t\} \quad (10a)$$

$$P_{k|h}^t = \mathbb{E}\{(x_k - \hat{x}_{k|h}^t)(x_k - \hat{x}_{k|h}^t)^T \mid \mathcal{I}_h^t\} \quad (10b)$$

Since  $\mathcal{I}_k^k = \mathcal{I}_k$ , the optimal estimation in (4) satisfies

$$\hat{x}_{k|k} = \hat{x}_{k|k}^k \quad (11)$$

For a given time step  $t$ , the ordinary Kalman filter equations in the appendix can be applied to write

$$\hat{x}_{k|k-1}^t = F_k \hat{x}_{k-1|k-1}^t + B_k u_k, \quad (12a)$$

$$P_{k|k-1}^t = F_k P_{k-1|k-1}^t F_k^T + Q_k, \quad (12b)$$

$$K_k^t = P_{k|k-1}^t H_k^{tT} (H_k^t P_{k|k-1}^t H_k^{tT} + R_k^t)^{-1} \quad (12c)$$

$$\hat{x}_{k|k}^t = \hat{x}_{k|k-1}^t + K_k^t (\tilde{y}_k^t - H_k^t \hat{x}_{k|k-1}^t), \quad (12d)$$

$$P_{k|k}^t = (I - K_k^t H_k^t) P_{k|k-1}^t. \quad (12e)$$

By the definition in (5), the variables  $\tilde{\varphi}_k^{i,t}$  for  $\varphi \in \{y, H, R\}$  remain constant with respect to  $t$  if  $t \geq k + d_{max}$ . Hence, the equations (12) are the same for  $t \geq k + d_{max}$  such that

$$k < t - d_{max} \implies \begin{cases} \hat{x}_{k|k}^t = \hat{x}_{k|k}^{t-1} \\ P_{k|k}^t = P_{k|k}^{t-1} \end{cases}. \quad (13)$$

Therefore, at every time step  $t$  the estimator only needs to recalculate  $\hat{x}_{k|k}^t$  and  $P_{k|k}^t$  for  $k \in \{t - d_{max}..t\}$  by repeating the recursive filtering calculations in (12). For this purpose, it is needed that the estimator is equipped with data buffers of length  $d_{max}$  to store  $\hat{x}_{k|k}^t$ ,  $P_{k|k}^t$  for  $k \in \{t - d_{max}..t\}$ .

The estimator also needs to keep track of  $y_k^i$ ,  $v_k^i$ ,  $H_k^i$ ,  $R_k^i$  to update  $\tilde{\varphi}_k^{i,t}$  in (5) for  $\varphi \in \{y, H, R\}$  that are required for repeating the recursive calculations. For this purpose, the set  $\mathcal{J}_t$  which indexes the samples received at  $t$  is defined as

$$\mathcal{J}_t = \{(j, i) \mid d_{t-j}^i = j\} \quad (14)$$

which according to (3) satisfies

$$\mathcal{J}_t \subseteq \{0..d_{max}\} \times \{1..m\}. \quad (15)$$

The equations in this section can be converted to the Algorithm 1 which gets the newly arrived information at every time step  $t$  given by  $\{y_{t-j}^i, H_{t-j}^i, R_{t-j}^i\}_{(j,i) \in \mathcal{J}_t}$  and calculates  $\hat{x}_{t|t}$  in (4). For this purpose, the algorithm stores  $\hat{x}_{k|k}^t$  and  $P_{k|k}^t$  for  $k \in \{t - d_{max}..t\}$  in buffers  $\bar{x}$ ,  $\bar{P}$  with finite length  $\ell \in \{0..d_{max}\}$ . The additional buffers  $\bar{y}$ ,  $\bar{H}$ ,  $\bar{R}$  are also used to store the received samples of  $y_k^i$  and the corresponding  $H_k^i$  and  $R_k^i$  for  $k \in \{t - d_{max}..t\}$  and  $i \in \{1..m\}$ . These value are used for repeating the recursive filter calculations between lines 8 and 13 of the algorithm. The non-stored auxiliary variables are denoted using non-italic names. In particular,  $x$  and  $P$  stand for  $\hat{x}_{k|k-1}^t$  and  $P_{k|k-1}^t$ . The value of  $\ell$  starts from zero and grows step by step up to  $d_{max}$  after which the buffer lengths remain constant. At  $t = 0$  the buffers must be initialized as

$$(\ell, \bar{x}_0, \bar{P}_0) \leftarrow (0, \hat{x}_0, P_{0|0}) \quad (16a)$$

$$(\bar{y}_0^i, \bar{H}_0^i, \bar{R}_0^i) \leftarrow (0_0, 0_0, 0_0) \quad \forall i \in \{1..m\} \quad (16b)$$

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1 function BRF( $t, \mathcal{J}_t, \{y_{t-j}^i, H_{t-j}^i, R_{t-j}^i\}_{(j,i) \in \mathcal{J}_t},$ 
    $\ell, \bar{x}_{0..\ell}, \bar{P}_{0..\ell}, \bar{y}_{0..\ell}^{1..m}, \bar{H}_{0..\ell}^{1..m}, \bar{R}_{0..\ell}^{1..m}$ )
2    $\ell \leftarrow \ell + 1$ ;
3   for every  $i \in \{1..m\}$  do
4      $(\bar{y}_\ell^i, \bar{H}_\ell^i, \bar{R}_\ell^i) \leftarrow (0_0, 0_0, 0_0)$ ;
5   for every  $(j, i) \in \mathcal{J}_t$  do
6      $(\bar{y}_{t-j}^i, \bar{H}_{t-j}^i, \bar{R}_{t-j}^i) \leftarrow (y_{t-j}^i, H_{t-j}^i, R_{t-j}^i)$ ;
7    $s \leftarrow \max\{j | (j, i) \in \mathcal{J}_t\} \cup \{0\}$ ;
8   for every  $j \in \{\ell - s.. \ell\}$  do
9      $k \leftarrow t - \ell + j$ ;
10     $H = \text{cat}\{\bar{H}_j^i\}_{i=1}^m$ ;
11     $P \leftarrow F_k \bar{P}_{j-1} F_k^T + Q_k$ ;
12     $K \leftarrow \text{PHT}(H P H^T + \text{diag}\{\bar{R}_j^i\}_{i=1}^m)^{-1}$ ;
13     $x \leftarrow F_k \bar{x}_{j-1} + B u_k$ ;
14     $\bar{P}_i \leftarrow (I - K H) P$ ;
15     $\bar{x}_i \leftarrow x + K(\text{cat}\{\bar{y}_j^i\}_{i=1}^m - H x)$ ;
16  end
17  if  $\ell > d_{max}$  then
18     $(\bar{x}_{0..\ell-1}, \bar{P}_{0..\ell-1}, \bar{y}_{0..\ell-1}^{1..m}, \bar{H}_{0..\ell-1}^{1..m}, \bar{R}_{0..\ell-1}^{1..m})$ 
19     $\leftarrow (\bar{x}_{1..\ell}, \bar{P}_{1..\ell}, \bar{y}_{1..\ell}^{1..m}, \bar{H}_{1..\ell}^{1..m}, \bar{R}_{1..\ell}^{1..m})$ ;
20     $\ell \leftarrow d_{max}$ ;
21  end
22   $\hat{x}_{t|t} \leftarrow \bar{x}_\ell$ ;
23  return  $(\hat{x}_{t|t}, \ell, \bar{x}_{0..\ell}, \bar{P}_{0..\ell}, \bar{y}_{0..\ell}^{1..m}, \bar{H}_{0..\ell}^{1..m}, \bar{R}_{0..\ell}^{1..m})$ ;
24 end

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**Algorithm 1:** Backward renovation filtering algorithm

#### IV. AUGMENTED SYSTEM FILTERING

In the second method for obtaining the estimation of state in (4), the system (24) is augmented as

$$\xi_k = \bar{F}_k \xi_{k-1} + \bar{B}_k u_k + \bar{w}_k, \quad (17a)$$

$$\xi_k = \text{cat}\{x_k, y_k, \dots, y_{k-d_{max}}\} \quad (17b)$$

$$y_k = \text{cat}\{y_k^i\}_{i=1}^m, \quad (17c)$$

$$\bar{w}_k = \text{cat}\{w_k, H_k w_k + v_k, 0\} \quad (17d)$$

$$H_k = \text{cat}\{H_k^i\}_{i=1}^m \quad (17e)$$

with the matrix coefficients

$$\bar{F}_k = \begin{bmatrix} F_k & 0 & \dots & 0 & 0 \\ H_k F_k & 0 & \dots & 0 & 0 \\ 0 & I_{n_y} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_{n_y} & 0 \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} B_k \\ H_k B_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (18)$$

and the covariance matrix for  $\bar{w}_k$  given by

$$\mathbb{E}\{\bar{w}_k \bar{w}_k^T\} = \text{diag}\left\{\begin{bmatrix} Q_k & Q_k H_k^T \\ H_k Q_k & R_k + H_k Q_k H_k^T \end{bmatrix}, 0\right\}. \quad (19)$$

The set of samples that are received by the estimator at  $t$  can be concatenated to a vector  $y_k^s$  defined as

$$y_k^s = \text{cat}\{y_{k-j}^i\}_{(j,i) \in \mathcal{J}_k} \quad (20)$$

with  $\mathcal{J}_k$  in (14) which can be written as

$$y_k^s = \bar{H}_k \xi_k, \quad (21a)$$

$$\bar{H}_k = [0 \quad \text{diag}\{\bar{H}_i\}_{(j,i) \in \mathcal{J}_k}] \quad (21b)$$

$$\bar{I}_i = [0_{n_i \times \sum_{j=1}^{i-1} n_j} \quad I_{n_i} \quad 0_{n_i \times \sum_{j=i+1}^m n_j}] \quad (21c)$$

The system (17a) with output  $y_k^s$  in (21a) is a delay-free system and the ordinary Kalman filter in the appendix can be applied to estimate  $\xi_k$  in (17b) which includes  $x_k$ . As a result, the estimation  $\hat{x}_{k|k}$  in (4) is obtained as

$$\hat{x}_{k|k} = [I \quad 0 \quad \dots \quad 0] \hat{\xi}_{k|k} \quad (22)$$

#### V. NUMERICAL EXAMPLE

In this section the proposed filtering methods are applied to an example system and their computational loads are compared. For this end, the time-dicretized state space model for a chain of  $n_x$  integrators is considered as

$$x_{k+1} = F x_k + B u_k + w_k \quad (23a)$$

$$y_k = H x_k + v_k \quad (23b)$$

$$F = e^{F_c h}, \quad B = \int_0^h e^{F_c \tau} B_c d\tau$$

$$F_c = \begin{bmatrix} 0 & I_{n_x-1} \\ 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = [1 \quad 0]$$

in which  $x_k \in \mathbb{R}^{n_x}$  is the vector of integrator outputs,  $u_k \in \mathbb{R}$  is the input,  $h = 0.05$ ,  $w_k \in \mathbb{R}^{n_x}$  and  $v_k \in \mathbb{R}$  are uncorrelated Gaussian white noises with  $\mathbb{E}\{w_k w_k^T\} = 0.1I$  and  $\mathbb{E}\{v_k^2\} = 0.1$ . To stabilize the system, a linear quadratic (LQR) state feedback controller  $u_k = K x_k$  is applied which minimizes the quadratic cost function  $J = \sum_0^\infty (x_k^T x_k + u_k^2)$ .

The backward renovation filter (BRF) in section III and the augmented system filter (AGF) in section IV are equivalent and produce the same result, since they both generate the state estimation in (4). The simulation results for the state  $x_k$  of the system (23) and the corresponding estimation  $\hat{x}_{k|k}$  using either of the two filters are plotted in Fig. 1a through Fig. 1c. The delays  $d_k$  are generated randomly with  $P\{d_k = \infty\} = 0.05$ , and equal probabilities  $P\{d_k = i\}$  for  $i \in \{0..d_{max}\}$  with  $d_{max} = 20$ . The delay values are plotted in Fig. 1d in which the missing measurements are indicated by a  $\triangle$  symbol at the corresponding time step with zero height.

The computational loads of the two filtering methods are compared in terms of the average count of floating point operations (flops) per time step during a simulation with 300 time steps. The flop counts are calculated by summing the flop counts for individual matrix operations required by each method. The comparison is made over  $n_x \in \{3, 10, 30\}$ ,  $d_{max} \in \{5, 10, 20, 50\}$  and the results are presented in Table I. According to the results, the load of computations increases with  $n_x$  and  $d_{max}$  for both methods. However, the increase of flop count with respect to  $n_x$  is much faster in the case of BRF method. On the other hand, the flop count increases more rapidly with the increase of  $d_{max}$  in the case of the AGF

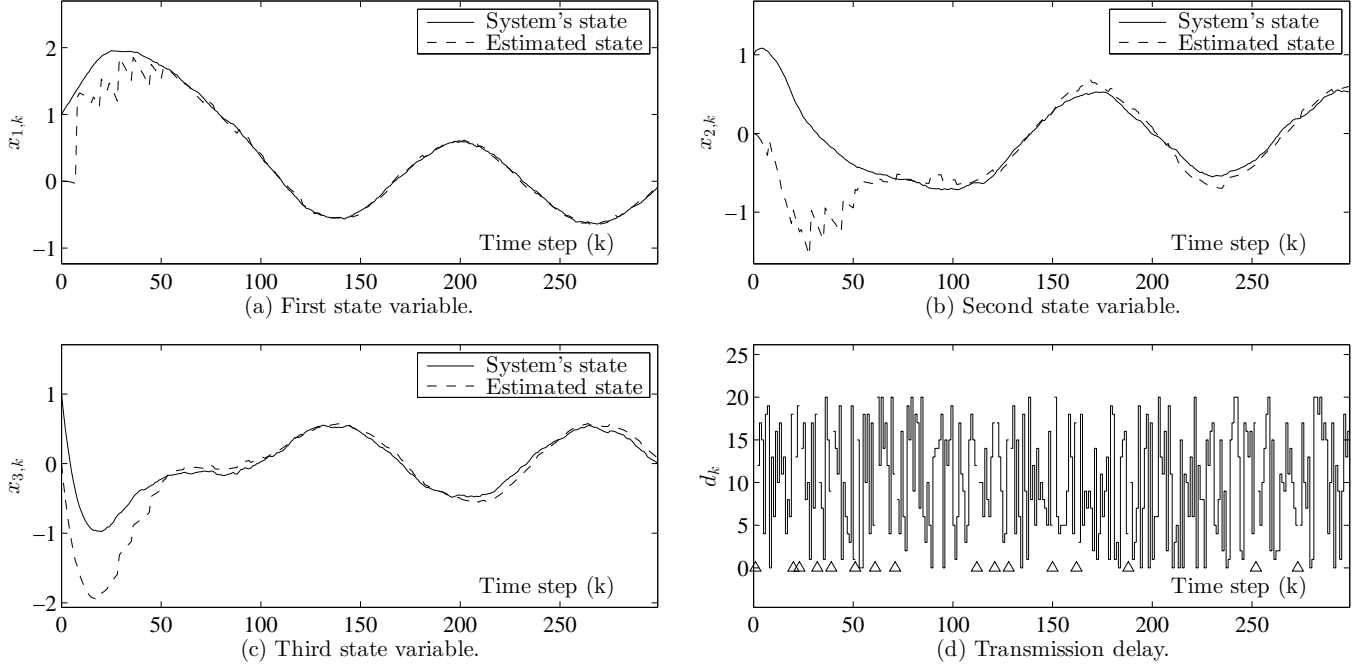


Fig. 1. Simulation of the methods for Kalman filtering with delays.

TABLE I  
FLOP COUNTS OF BACKWARD RENOVATION FILTER (BRF) AND AUGMENTED SYSTEM FILTER (AGF) FOR DIFFERENT CONDITIONS.

$n_x$	Method	Flop count			
		$d_{max} = 5$	$d_{max} = 10$	$d_{max} = 20$	$d_{max} = 50$
3	BRF	$1.52 \times 10^5$	$2.33 \times 10^5$	$3.89 \times 10^5$	$6.76 \times 10^5$
	AGF	$6.86 \times 10^5$	$2.23 \times 10^6$	$9.94 \times 10^6$	$5.63 \times 10^7$
10	BRF	$4.59 \times 10^6$	$7.11 \times 10^6$	$1.21 \times 10^7$	$2.17 \times 10^7$
	AGF	$3.21 \times 10^6$	$6.83 \times 10^6$	$2.06 \times 10^7$	$8.66 \times 10^7$
30	BRF	$1.18 \times 10^8$	$1.89 \times 10^8$	$3.08 \times 10^8$	$5.53 \times 10^8$
	AGF	$3.17 \times 10^7$	$4.61 \times 10^7$	$8.67 \times 10^7$	$2.28 \times 10^8$

method. Therefore, it is advisable to use the BRF method in the case of long delays and to use the AGF method for systems with larger dimensionality.

## VI. CONCLUSION

Two existing approaches to Kalman filtering with delays have been extended to equivalent methods for optimal estimation in presence of multiple effects. These effects include time-varying delays in multiple measurement channels, missing measurements (e.g. due to packet losses or multirate sampling), and miss-ordered arrival of data. The extension of first approach which is based on recalculation of the past filtering variables has been presented as an algorithm using finite length memory buffers. The second approach which is based on system augmentation has been also extended to tackle missing measurements and the other effects. The computational loads

of the resulting equivalent filtering methods were compared in terms of the flop counts for filtering integrator chains of various lengths. The comparison results suggest that the first method performs better for long delays while the second method can be more efficient for larger systems.

## APPENDIX A

### KALMAN FILTER FOR LINEAR SYSTEMS WITH VARYING OUTPUT DIMENSIONALITY

Consider a dynamical system system described by

$$x_k = F_k x_{k-1} + B_k u_k + w_k \quad (24a)$$

$$y_k = H_k x_k + v_k^i \quad (24b)$$

with  $x_k \in \mathbb{R}^{n_x}$ ,  $u_k \in \mathbb{R}^{n_u}$ , and  $y_k \in \mathbb{R}^{n_k}$  where  $n_k$  varies with  $k$ , and uncorrelated, zero-mean white Gaussian noise vectors  $w_k \in \mathbb{R}^{n_w}$  and  $v_k \in \mathbb{R}^{n_k}$  with  $R_k = \mathbb{E}\{v_k^i v_k^{iT}\}$  and  $Q_k = \mathbb{E}\{w_k w_k^T\}$ . The optimal estimation of  $x_k$  is defined as

$$\hat{x}_{k|k} = \mathbb{E}\{x_k | \mathcal{I}_k\} \quad (25)$$

with  $\mathcal{I}_k = \{y_i | i \leq k\}$ . Given  $\hat{x}_{0|0}$  and  $P_{0|0}$ , it is straightforward to show that the estimation  $\hat{x}_{k|k}$  can be calculated recursively according to

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k u_k, \quad (26a)$$

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k, \quad (26b)$$

$$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}, \quad (26c)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - H_k \hat{x}_{k|k-1}), \quad (26d)$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}. \quad (26e)$$

If  $n_k = 0$  for some  $k$ , then  $H_k$  and  $K_k$  in (26) become  $0 \times n_x$  and  $n_x \times 0$  empty matrices respectively, and the product  $K_k H_k$  in (26e) is defined to be a  $n_x \times n_x$  zero matrix.

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