Quantitative and Qualitative Analysis of a Quantum Multisingular Problem by Computational Method and Heatmaps

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Abstract

The physical phenomena with uncontrollable singularities pose challenges in solving related differential equations. In this work, we intend to investigate the quantitative and qualitative aspects of a multi-singular integro-differential equation with the help of quantum fractional operators by presenting numerical algorithms. Quantum calculus enables us to use numerical algorithms and software. The α-ψ-contraction, a new technique of fixed point theory, plays a significant role in proving the existence of the solution. To interpret tables with quantum values quickly and easily, we use heatmaps. We also presented three numerical examples to illustrate the accuracy and efficiency of our main results.

(2010) Mathematics Subject Classifications: 34A08; 34A12.
Keywords: Fixed point theory; α-ψ-contraction; Quantum calculus; Multi-singular equation; Fractional calculus.
Acknowledgments:The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF2/PSAU/2022/01/22923).

1 Introduction

Scientists in natural and mathematical fields constantly seek new approaches to model complex phenomena. Fractional operators are a prime example of this. Researchers in biological, physical, and engineering sciences are giving special attention to fractional differential equations ([1, 2, 3, 4, 5, 6, 7, 8]), as new laboratory data confirm their greater efficiency [9]. Over time, as fractional calculus has advanced, various new operators have been introduced and

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generalized. For example, we can mention Caputo, ψ-Caputo, Caputo-Fabrizio, Hadamard, Hilfer, Riemann-Liouville and Atangana-Baleanu. Some of the contributions made with the mentioned operators are: ([10, 11, 12, 13, 14, 15, 16, 17, 18]). Ongoing research on these operators presents new findings daily. George et. al. [19] recently demonstrated that the choice of the ψ function affects the boundedness of the solution set of the pantograph equation with the ψ-Caputo fractional operator.

Modeling natural phenomena often requires solving differential equations with singular points, which can be complicated and challenging. Therefore, the study of such equations to find analytical and numerical solutions has recently gained attention. In 2010, Agarwal et. al. [20] investigated positive solution for the following singular dirichlet problem

$$\begin{cases} \mathfrak{D}^{\gamma}\mathbf{v}(\kappa) + \mathbf{h}(\kappa, \mathbf{v}(\kappa), \mathfrak{D}^{\tau}\mathbf{v}(\kappa)) = 0, \quad \kappa \in (0,1), \\ \mathbf{v}(0) = \mathbf{v}(1) = 0, \end{cases}$$

where $\gamma \in (1,2)$, $0 < \tau \leq \gamma - 1$, $\mathfrak{D}^{\gamma}$ denotes Riemann-Liouville deivative of fractional order γ and $\mathbf{h}(\kappa, u, v)$ is singular at $t = 0$. In 2011, Feng et. al [21] studied the existence of solution for the following singular system using the krasnoselskii's and Leray-Schauder theorem

$$\begin{cases} \mathfrak{D}^{\gamma}\mathbf{v}(\kappa) + \mathbf{h}(\kappa, \mathbf{u}(\kappa)) = 0, \quad \kappa \in (0,1), \\ \mathfrak{D}^{\pounds}\mathbf{u}(\kappa) + \mathbf{t}(\kappa, \mathbf{v}(\kappa)) = 0, \quad \kappa \in (0,1), \\ \mathbf{v}(0) = \mathbf{v}'(0) = \mathbf{v}(1) = 0, \\ \mathbf{u}(0) = \mathbf{u}'(0) = \mathbf{u}(1) = 0, \end{cases}$$

such that $\gamma, \pounds \in (2,3]$, $\mathfrak{D}^{\gamma}, \mathfrak{D}^{\pounds}$ are fractional Riemann-Liouville derivatives and $\mathbf{h}, \mathbf{t}$ are realvalued continouos functions on $[0,\infty) \times \mathbb{R}$. In 2013, Vong [22] examined the posetive solution for the following singular problem:

$$\begin{cases} {}^{c}\mathfrak{D}^{\gamma}\mathbf{v}(\kappa) + \mathbf{h}(\kappa, \mathbf{v}(\kappa)) = 0, \quad \kappa \in [0,1], \\ \mathbf{v}(1) = \int_0^1 \mathbf{v}(p)\,\mathrm{d}p, \\ \mathbf{v}'(0) = \cdots = \mathbf{v}^{(n-1)}(0) = 0, \\ \int_0^1 \,\mathrm{d}p < 1, \end{cases}$$

using upper and lower solutions method with Schauder fixed point, where $n \geq 2$, $n-1 < \gamma < n$ and $\mathbf{h}$ have singularity at $\kappa = 1$. In 2014, Nyamoradi et. al. [23] reviewed the existence and uniqueness for the following singular BVP:

$$\begin{cases} -\mathfrak{D}^{\gamma}\mathbf{v}(\kappa) + \mathcal{M}\mathbf{h}(\kappa, \mathbf{v}(\kappa)) + \sum_{i=1}^{m} \mathcal{N}_i \mathfrak{J}^{\sigma_i}\mathbf{t}_i(\kappa, \mathbf{v}(\kappa)), \quad \kappa \in (0,1), \\ \mathfrak{D}^{\tau}\mathbf{v}(0) = 0, \\ \mathfrak{D}^{\tau}\mathbf{v}(1) = b\mathfrak{D}^{\frac{\gamma-\tau-1}{2}}\big(\mathfrak{D}^{\tau}\mathbf{v}(\kappa)\big)|_{\kappa=\ell}, \quad \ell \in (0, \frac{1}{2}], \end{cases}$$

where $1 < \gamma \leq 2$, $\tau_i \in (0,1)$, $b \in [0,\infty)$, furthere $\mathcal{M}, \mathcal{N}_i$ are real constant and $\mathfrak{D}^{\gamma}$ is the fractional Riemann-Liouville derivative. In 2022, Malekpour et. al. [24] studied a new class of singular equation namely multi-singular pointwised defined system as follows:

$$\begin{cases} {}^c\mathfrak{D}^{\gamma_1}\mathbf{v}(\kappa) + \mathbf{h}_1\big(\kappa, \mathbf{v}(\kappa), \mathbf{u}(\kappa), \mathbf{v}'(\kappa), \mathbf{u}'(\kappa), {}^c\mathfrak{D}^{\tau_1}\mathbf{v}(\kappa), \\ {}^c\mathfrak{D}^{\tau_2}\mathbf{u}(\kappa), \int_0^{\kappa} g_1(\lambda)\mathbf{v}(\lambda)\,\mathrm{d}\lambda, \int_0^{\kappa} g_2(\lambda)\mathbf{v}(\lambda)\,\mathrm{d}\lambda\big) = 0, \\ \\ {}^c\mathfrak{D}^{\gamma_2}\mathbf{v}(\kappa) + \mathbf{h}_2\big(\kappa, \mathbf{v}(\kappa), \mathbf{u}(\kappa), \mathbf{v}'(\kappa), \mathbf{u}'(\kappa), {}^c\mathfrak{D}^{\tau_1}\mathbf{v}(\kappa), \\ {}^c\mathfrak{D}^{\tau_2}\mathbf{u}(\kappa), \int_0^{\kappa} g_1(\lambda)\mathbf{v}(\lambda)\,\mathrm{d}\lambda, \int_0^{\kappa} g_2(\lambda)\mathbf{v}(\lambda)\,\mathrm{d}\lambda\big) = 0, \end{cases}$$

under boundary conditions:

$$\begin{cases} {}^c\mathfrak{D}^{\sigma_1}\mathbf{v}(\omega_1) = \ell_1, \qquad {}^c\mathfrak{D}^{\sigma_2}\mathbf{v}(\omega_2) = \ell_2, \\ \\ \mathbf{v}(1) = \mathbf{v}^{(j)}(0) = 0, \quad \mathbf{u}(1) = \mathbf{u}^{(j)}(0) = 0, \quad j \geq 2 \end{cases}$$

where $\gamma_i \geq 2$, $\omega_i, \sigma_i \in (0,1)$, $\ell_i \geq 0$, $\mathbf{h}_i, g_i \in \mathbf{L}^1$ such that $\mathbf{h}_i$ is a singular at some points in $[0,1]$ for $i = 1,2$. The reader can refer to [25, 26, 27] to learn about the rest of the research done in this field.

Computers have always struggled with finding numerical solutions and computational algorithms for singular equations. In this case, we are seeking a numerical method to solve the multi-singular integrodifferential equation. We are using quantum operators to provide the necessary space for software packages. In 1910, Frank Hilton Jackson introduced the quantum derivative operator [28, 29]. In his definition of derivative, namely $(\mathcal{D}_q\mathbf{v})(\kappa) = \mathbf{v}(\kappa) - \mathbf{v}(q\kappa)/(1-q)\kappa$, he removed the concept of limit, and this caused a discrete space to be prepared for the analysis of problems. In his quantum calculus for every real number κ defined the q-analogue of κ as: $[\kappa]_q = \frac{1-q^{\kappa}}{1-q} = 1 + q + \cdots + q^{\kappa-1}$. Moreover, q-analogue of the power function $(\kappa - p)^n$ for $n \geq 1$ defined as: $(\kappa - p)_q^{(n)} = \prod_{j=0}^{n-1}(\kappa - pq^j)$ and $(\kappa - p)_q^{(0)} = 1$, where $\kappa, p \in \mathbb{R}$. In addition, Let $\kappa \in \mathbb{R} - \{0, -1, -2, \dots\}$, then the quantum gamma function formulated as: $\Gamma_q(\kappa) = \frac{(1-q)^{(\kappa-1)}}{(1-q)^{\kappa-1}}$. Also, it is worth mentioning that $\Gamma_q(\kappa+1) = [\kappa]_q\Gamma_q(\kappa)$ holds true. In the preliminaries section, we present an algorithm for calculating the quantum gamma function. The characteristics of quantum operators have been investigated in detail by Kac and Cheung in [30]. Later in 2007, the properties of fractional operator in q-calculus was developed in works [31] and [32]. In 2011, Ferreira [33] investigated positive solution for the following q-fractional BVP:

$$\begin{cases} \mathfrak{D}_q^{\gamma}\mathbf{v}(\kappa) = -\mathbf{h}(\kappa, \mathbf{v}(\kappa)), \quad \kappa \in (0,1), \\ \\ \mathbf{v}(0) = \mathfrak{D}_q\mathbf{v}(0) = 0, \\ \\ \mathfrak{D}_q\mathbf{v}(1) = c \geq 0, \end{cases}$$

where $\gamma \in (2,3]$, $c \in \mathbb{R}$, $\mathfrak{D}_q^{\gamma}$ denotes the q-Riemann-Liouville derivative of fractional order γ and $\mathbf{h} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is nonnegative continuous function . In 2015, Xinhui Li et.al. [34]

studied BVP of fractional q-difference Schringer equation:

$$\begin{cases} \mathfrak{D}_q^{\gamma}\mathbf{v}(\kappa) + \dfrac{\mathbf{m}}{\mathbf{p}}\big(\mathbf{E} - \mathbf{u}(\kappa)\big)\mathbf{v}(\kappa) = 0, \quad \kappa \in (0,1), \\ \mathbf{v}(0) = \mathfrak{D}_q\mathbf{v}(0) = \mathfrak{D}_q\mathbf{v}(1) = 0, \end{cases}$$

where $\gamma \in (2,3)$, $\mathbf{m}, \mathbf{E}$ are the mass and energy particle respectively, and $\mathbf{p}$ is the Plank constant . Many articles have been published in the field of quantum differential problems in recent years, for information about which you can refer to [35, 36, 37, 38, 39, 40].

Here, by getting motivation from works [20, 22, 24, 33] and also filling the gap in numerical methods for studing singular equations, we are going to examine the following uncontrolled multi-singular pointwise defined equation in q-calculus:

$$^{c}\mathfrak{D}_q^{\gamma}\mathbf{v}(\kappa) + \mathbf{h}\big(\kappa, \mathbf{v}(\kappa), \mathbf{v}'(\kappa), {}^{c}\mathfrak{D}_q^{\tau}\mathbf{v}(\kappa), \mathfrak{I}_q^{\sigma}\mathbf{v}(\kappa)\big) = 0, \quad \kappa \in \mathcal{K} = [0,1] \tag{1.1}$$

under the following boundary conditions:

$$\begin{cases} \mathbf{v}'(0) = \mathbf{v}(\omega), \\ \mathbf{v}(1) = \int_0^{\ell} \mathbf{v}(p)\,\mathrm{d}p, \\ \mathbf{v}^{(2)}(0) = \cdots = \mathbf{v}^{(n-1)}(0) = 0, \quad n = [\gamma] + 1, \end{cases} \tag{1.2}$$

where $\gamma \geq 3$, $\sigma > 1$, $\tau, \omega, \ell \in (0,1)$, $\mathbf{v} \in \mathbf{C}^1[0,1]$, $\mathbf{h} : \mathcal{K} \times \mathbb{R}^4 \to \mathbb{R}$ is function where $\mathbf{h}(\kappa, ., ., ., .)$ singular at some point $\kappa \in [0,1]$ and ${}^{c}\mathfrak{D}_q^{\gamma}$ denote the q-Caputo derivative of fractional order γ. The function $\mathbf{h}$ is multi-singular when it is singular at more than one point. Note that we will continue to do all our calculations on the time scale, namely $TS_{\kappa_0} = \{\kappa_0, \kappa_0 q, \kappa_0 q^2, ...\} \cup \{0\}$, where $\kappa_0 \in \mathbb{R}$, and $q \in (0,1)$.

2 Preliminaries

Notation 2.1. *It should be noted that throughout this work:* $\|.\|$ *is the sup norm of* $\mathbf{C}[0,1]$, $\|.\|_1$ *is the norm of* $\mathbf{L}^1[0,1]$ *and* $\|\mathbf{v}\|_* = \max\big\{\|\mathbf{v}\|, \|\mathbf{v}'\|\big\}$ *is the norm of* $\mathbf{Y} = \mathbf{C}^1[0,1]$.

Notation 2.2. *Assume that the function* $\psi : [0,\infty) \to [0,\infty)$ *for all* $\kappa > 0$ *be such that* $\sum_{n=1}^{\infty} \psi^n(\kappa) < \infty$. *Then we denote the family of nondecreasing functions* ψ *by* Ψ. *Notice that* $\forall \kappa > 0$ *we have* $\psi(\kappa) < \kappa$ *(see [41]).*

Definition 2.3. *[22] consider the follwing equation*

$$^{c}\mathfrak{D}_q^{\gamma}\mathbf{v}(\kappa) + \mathbf{h}(\kappa) = 0, \qquad \kappa \in [0,1]. \tag{2.1}$$

If there exiset s set such as $\mathcal{P} \subset [0,1]$ *such that* $\mu(\mathcal{P}^c) = 0$ *and Eq.* (2.1) *hold on* $\mathcal{P}$, *then we call Eq.* (2.1) *pointwise defined equation where* μ *is the measure function.*

Definition 2.4. *[29] Let $\kappa \in \mathbb{R} - \{0, -1, -2, \dots\}$, then the quantum gamma function formulated as follows*

$$\Gamma_q(\kappa) = \frac{(1-q)^{(\kappa-1)}}{(1-q)^{\kappa-1}},$$

also, it is worth mentioning that $\Gamma_q(\kappa+1) = [\kappa]_q \Gamma_q(\kappa)$ holds true. we present an algorithm for calculating the quantum gamma function. Moreover, we computed for some values of q in Tables 1 and 2, also their heatmaps presented in Figuers 1 and 2.

Algorithm 1 The proposed procedure To calculate $\Gamma_q(\kappa)$.

function quantum gamma $= qG(q,\kappa,r)$
$t = 1$;
for $j = 0 : r$
$t = t * (1 - q^{(j+1)})/(1 - q^{(\kappa+j)})$;
end
$qG = t/(1-q)^{(\kappa-1)}$;
end

r	$q = 0.08$	$q = 0.17$	$q = 0.33$	$q = 0.49$	$q = 0.66$	$q = 0.81$
			$v = 3.75$			
1	1.1498	1.3475	1.8340	2.7443	5.4843	18.2050
2	1.1492	1.3410	1.7711	2.4622	4.3021	12.1461
3	1.1492	1.3398	1.7511	2.3392	3.7104	9.1158
4	1.1492	1.3397	1.7445	2.2822	3.3808	7.3785
5	1.1492	1.3396	1.7424	2.2550	3.1853	6.2897
6	1.1492	1.3396	1.7417	2.2418	3.0649	5.5639
7	1.1492	1.3396	1.7414	2.2354	2.9888	5.0579
8	1.1492	1.3396	1.7414	2.2323	2.9401	4.6934
9	1.1492	1.3396	1.7413	2.2307	2.9085	4.4241
...	...	...	...	...	...	...
13	1.1492	1.3396	1.7413	2.2294	2.8599	3.8528
14	1.1492	1.3396	1.7413	2.2293	2.8560	3.7792
...	...	...	...	...	...	...
26	1.1492	1.3396	1.7413	2.2293	2.8487	3.5085
27	1.1492	1.3396	1.7413	2.2293	2.8486	3.5042
..	...	...	...	...	...	...
39	1.1492	1.3396	1.7413	2.2293	2.8486	3.4877
40	1.1492	1.3396	1.7413	2.2293	2.8486	3.4874
...	...	...	...	...	...	...
52	1.1492	1.3396	1.7413	2.2293	2.8486	3.4864
53	1.1492	1.3396	1.7413	2.2293	2.8486	3.4863

Table 1: Numerical results for $\Gamma_q(3.75)$ for different value of q

r	$q = 0.08$	$q = 0.17$	$q = 0.33$	$q = 0.49$	$q = 0.66$	$q = 0.81$
			$v = 1.6$			
1	0.9796	0.9672	0.9686	1.0111	1.1428	1.4660
2	0.9792	0.9641	0.9513	0.9662	1.0494	1.2920
3	0.9792	0.9636	0.9458	0.9461	0.9978	1.1855
4	0.9792	0.9635	0.9440	0.9366	0.9673	1.1147
5	0.9792	0.9635	0.9434	0.9321	0.9484	1.0649
6	0.9792	0.9635	0.9432	0.9298	0.9365	1.0287
7	0.9792	0.9635	0.9432	0.9288	0.9288	1.0016
8	0.9792	0.9635	0.9432	0.9282	0.9239	0.9810
9	0.9792	0.9635	0.9432	0.9280	0.9206	0.9652
10	0.9792	0.9635	0.9431	0.9279	0.9185	0.9528
...	...	...	...	...	...	...
14	0.9792	0.9635	0.9431	0.9277	0.9152	0.9245
15	0.9792	0.9635	0.9431	0.9277	0.9150	0.9206
...	...	...	...	...	...	...
25	0.9792	0.9635	0.9431	0.9277	0.9145	0.9064
26	0.9792	0.9635	0.9431	0.9277	0.9144	0.9060
..	...	...	...	...	...	...
34	0.9792	0.9635	0.9431	0.9277	0.9144	0.9048
35	0.9792	0.9635	0.9431	0.9277	0.9144	0.9047
...	...	...	...	...	...	...
40	0.9792	0.9635	0.9431	0.9277	0.9144	0.9046
41	0.9792	0.9635	0.9431	0.9277	0.9144	0.9045

Table 2: Numerical results for $\Gamma_q(1.6)$ for different value of q

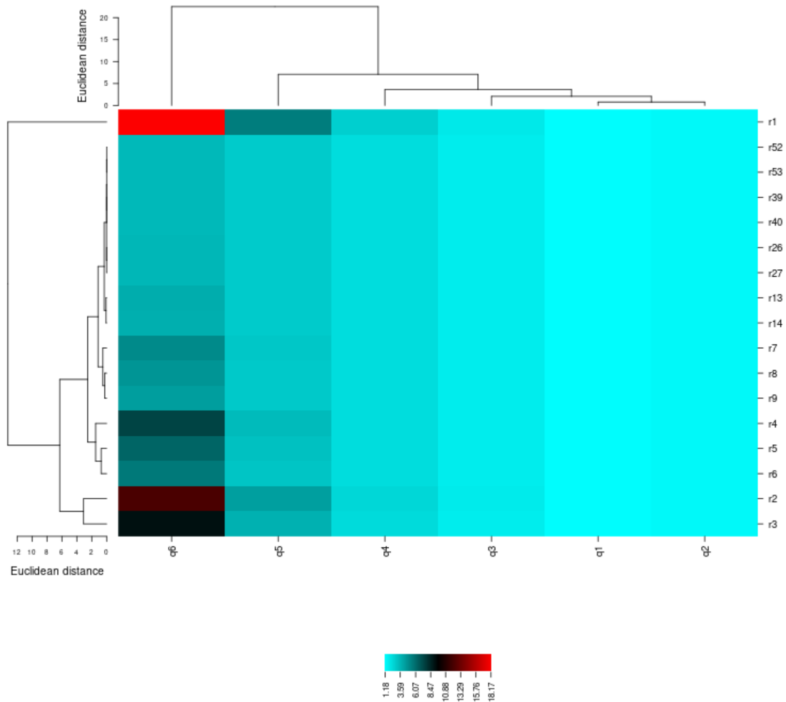

Figure 1: The heatmap of Table 1.

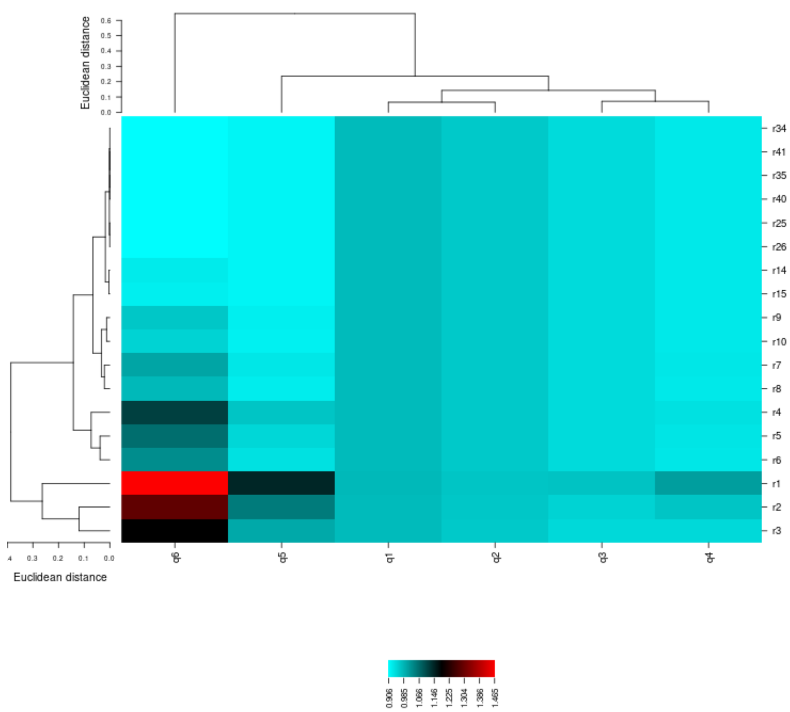

Figure 2: The heatmap of Table 2.

Definition 2.5. *[42] Suppose that* $\mathbf{v}(\kappa) : [0, \infty] \to \mathbb{R}$*, be a continuous function, then its fractional Riemann-Liouville quantum integral and its fractional Caputo quantum derivative are expressed respectively by*

$$\mathfrak{I}_q^{\gamma}\mathbf{v}(\kappa) = \frac{1}{\Gamma_q(\gamma)} \int_0^{\kappa} (\kappa - qp)^{\eta-1}\mathbf{v}(p)\, \mathrm{d}_q p,$$

and

$$^{c}\mathfrak{D}^{\gamma}\mathbf{v}(\kappa) = \frac{1}{\Gamma_q(n-\gamma)} \int_0^{\kappa} (\kappa - qp)^{n-\gamma-1}\mathfrak{D}_q^{n}\mathbf{v}(p)\, \mathrm{d}_q p, \quad n = [\gamma] + 1.$$

Lemma 2.6. *[43] assume that* $n = [\gamma] + 1$*, then the following relation holds true*

$$({}^{C}\mathfrak{I}_q^{\eta}\, {}^{C}\mathfrak{D}_q^{\eta}\mathbf{v})(\kappa) = \mathbf{v}(\kappa) - \sum_{j=0}^{n-1} \frac{\mathbf{v}^{j}}{\Gamma_q(j+1)}(\mathfrak{D}_q^{j}\mathbf{v})(0),$$

which is deduced from it, the general solution for ${}^{C}\mathfrak{D}_q^{\eta}\mathbf{v}(\kappa) = 0$*, expressed by*

$$\mathbf{v}(\kappa) = r_0 + r_1\kappa + r_2\kappa^2 + \cdots + r_{n-1}\kappa^{n-1},$$

where $r_0, \ldots, r_{n-1} \in \mathbb{R}$.

Definition 2.7. *[44] Consider the two maps* $\boldsymbol{\alpha} : \mathbf{Y} \times \mathbf{Y} \to [0, \infty)$ *and* $\mathcal{T} : \mathbf{Y} \to \mathbf{Y}$*. The function* $\mathcal{T}$ *is called* $\boldsymbol{\alpha}$*-admissible if* $\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) \geq 1$ *yelids that* $\boldsymbol{\alpha}(\mathcal{T}\mathbf{v}, \mathcal{T}\mathbf{u}) \geq 1$*. Moreover, let* $\psi \in \Psi$*,* $\mathcal{T} : \mathbf{Y} \to \mathbf{Y}$*, and* $(\mathbf{Y}, d_{\mathbf{Y}})$ *be a complete metric space. If for every* $\mathbf{v}, \mathbf{u} \in \mathbf{Y}$ *we have* $\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u})d_{\mathbf{Y}}(\mathcal{T}\mathbf{v}, \mathcal{T}\mathbf{u}) \leq \psi(d_{\mathbf{Y}}(\mathbf{v}, \mathbf{u}))$*, then the map* $\mathcal{T}$ *is called* $\boldsymbol{\alpha}$*-*ψ*-contraction.*

Lemma 2.8. *[44] Suppose that* $\boldsymbol{\alpha} : \mathbf{Y} \times \mathbf{Y} \to [0, \infty)$*,* $\psi \in \Psi$*,* $(\mathbf{Y}, d_{\mathbf{Y}})$ *be a complete metric space and* $\mathcal{T} : \mathbf{Y} \to \mathbf{Y}$ *is an* $\boldsymbol{\alpha}$*-*ψ*-contraction. If* $\mathcal{T}$ *be continuous and* $\exists \mathbf{v}_0 \in \mathbf{Y}$ *where* $\boldsymbol{\alpha}(\mathbf{v}_0, \mathcal{T}\mathbf{v}_0) \geq 0$*, then* $\mathcal{T}$ *has a fixed point.*

In the following, we present two key lemmas about the quantum Green's function and its properties.

Lemma 2.9. *Consider the following fractional quantum pointwise defined problem*

$$^{c}\mathfrak{D}_q^{\gamma}\mathbf{v}(\kappa) + \mathbf{u}_0 = 0 \tag{2.2}$$

with boundary conditions

$$\begin{cases} \mathbf{v}'(0) = \mathbf{v}(\omega), \\ \mathbf{v}(1) = \int_0^{\ell} \mathbf{v}(p)\, \mathrm{d}p, \\ \mathbf{v}^{(j)}(0) \quad \textit{for} \quad j = 2, \ldots, [\gamma], \end{cases} \tag{2.3}$$

such that $\mathbf{u}_0 \in \mathbf{L}^1[0, 1]$, $0 < \ell, \omega < 1$ *and* $\gamma \in [2, 3)$*. Then the function* $\mathbf{v}(\kappa) = \int_0^1 \mathcal{G}_q(\kappa, p)\mathbf{u}_0(p)\, \mathrm{d}_q p$ *is a solution for problem mentioned in* (2.2)*-*(2.3) *such that*

$$\mathcal{G}_q(\kappa, p) = \mathcal{L}_q(\kappa, p) + \frac{1}{1-\ell} \int_0^{\ell} \mathcal{L}_q(\kappa, p)\, \mathrm{d}p$$

and

$$\mathcal{L}_q(\kappa,p)=\frac{(\omega-qp)^{\gamma-1}+(1-qp)^{\gamma-1}-\kappa(\omega-qp)^{\gamma-1}-(\kappa-qp)^{\gamma-1}}{\Gamma_q(\gamma)},$$

where $0\le qp\le\kappa\le 1$ *and* $qp\le\omega$

$$\mathcal{L}_q(\kappa,p)=\frac{(1-qp)^{\gamma-1}-(\kappa-qp)^{\gamma-1}}{\Gamma_q(\gamma)},$$

where $0\le\omega\le qp\le\kappa\le 1$

$$\mathcal{L}_q(\kappa,p)=\frac{(\omega-qp)^{\gamma-1}+(1-qp)^{\gamma-1}-\kappa(\omega-qp)^{\gamma-1}}{\Gamma_q(\gamma)},$$

where $0\le\kappa\le qp\le\omega\le 1$

$$\mathcal{L}_q(\kappa,p)=\frac{(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)},$$

where $0\le\kappa\le qp\le 1$ *and* $\omega\le qp$.

Proof. Let the equation ${}^c\mathfrak{D}_q^\gamma\mathbf{v}(\kappa)+\mathbf{u}_0(\kappa)=0$ satisfy for all $\kappa\in\mathcal{P}$ such that $\mathcal{P}\subset[0,1]$ and measure of $\mathcal{P}^c$ is zero. Now, we take $\mathbf{u}\in\mathbf{L}^1[0,1]\cap\mathbf{C}^1[0,1]$ where $\mathbf{u}=\mathbf{u}_0$ on $\mathcal{P}$. If $\mathbf{v}_0$ be a solution for Eq. (2.2)-(2.3), then for all $\kappa\in[0,1]$ put $\mathbf{u}(\kappa)=-{}^c\mathfrak{D}_q^\gamma\mathbf{v}_0(\kappa)$. Moreover, we can write

$$\begin{aligned}
\mathfrak{I}_q^\gamma\big(\mathbf{u}_0(\kappa)\big)&=\frac{1}{\Gamma_q(\gamma)}\int_0^\kappa(\kappa-qp)^{\gamma-1}\mathbf{u}_0(p)\,\mathrm{d}_qp\\
&=\frac{1}{\Gamma_q(\gamma)}\Big(\int_{[0,\kappa]\cap\mathcal{E}}(\kappa-qp)^{\gamma-1}\mathbf{u}_0(p)\,\mathrm{d}_qp+\int_{[0,\kappa]\cap\mathcal{E}^c}(\kappa-qp)^{\gamma-1}\mathbf{u}_0(p)\,\mathrm{d}_qp\Big)\\
&=\frac{1}{\Gamma_q(\gamma)}\int_{[0,\kappa]\cap\mathcal{E}}(\kappa-qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_qp\\
&=\frac{1}{\Gamma_q(\gamma)}\Big(\int_{[0,\kappa]\cap\mathcal{E}}(\kappa-qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_qp+\int_{[0,\kappa]\cap\mathcal{E}^c}(\kappa-qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_qp\Big)\\
&=\frac{1}{\Gamma_q(\gamma)}\int_0^1(\kappa-qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_qp=\mathfrak{I}_q^\gamma\big(\mathbf{u}(\kappa)\big).
\end{aligned}$$

Now, let $\kappa\in\mathcal{P}^c\setminus\{0\}$. Take $\{\kappa_n\}$ in $\mathcal{P}$ where $\kappa_n\to\kappa^-$, so

$$\begin{aligned}
\mathfrak{I}_q^\gamma\big(\mathbf{u}_0(\kappa)\big)&=\frac{1}{\Gamma_q(\gamma)}\int_0^\kappa(\kappa-qp)^{\gamma-1}\mathbf{u}_0(p)\,\mathrm{d}_qp\\
&=\lim_{n\to\infty}\frac{1}{\Gamma_q(\gamma)}\int_0^{\kappa_n}(\kappa_n-qp)^{\gamma-1}\mathbf{u}_0(p)\,\mathrm{d}_qp=\lim_{n\to\infty}\mathfrak{I}_q^\gamma\big(\mathbf{u}_0(\kappa_n)\big)\\
&=\lim_{n\to\infty}\mathfrak{I}_q^\gamma\big(\mathbf{u}(\kappa_n)\big)=\lim_{n\to\infty}\frac{1}{\Gamma_q(\gamma)}\int_0^{\kappa_n}(\kappa_n-qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_qp
\end{aligned}$$

$$= \frac{1}{\Gamma_q(\gamma)} \int_0^\kappa (\kappa - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q = \mathfrak{I}_q^\gamma\big(\mathbf{u}(\kappa)\big).$$

If $\kappa = 0 \in \mathcal{P}^c$, then $\mathfrak{I}_q^\gamma(\mathbf{u}_0(\kappa)) = \mathfrak{I}_q^\gamma(\mathbf{u}(\kappa)) = 0$, hence for all $\kappa \in [0,1]$ we have $\mathfrak{I}_q^\gamma(\mathbf{u}_0(\kappa)) = \mathfrak{I}_q^\gamma(\mathbf{u}(\kappa)) = 0$. Therefore, for all $\kappa \in [0,1]$ we get

$$\mathfrak{I}_q^\gamma({}^c\mathfrak{D}_q^\gamma \mathbf{v}(\kappa)) = \mathfrak{I}_q^\gamma(-\mathbf{u}_0(\kappa)),$$

such that ${}^c\mathfrak{D}_q^\gamma \mathbf{v}(\kappa) + \mathbf{u}_0(\kappa) = 0$ for all $\kappa \in \mathcal{P}$. Thus on $[0,1]$ we have

$$\mathfrak{I}_q^\gamma({}^c\mathfrak{D}_q^\gamma \mathbf{v}(\kappa)) = \mathfrak{I}_q^\gamma(-\mathbf{u}(\kappa)),$$

In view of Lemma 1.1 we have

$$\mathbf{v}(\kappa) = -\frac{1}{\Gamma_q(\gamma)} \int_0^\kappa (\kappa - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p + r_0 + r_1 \kappa.$$

which from boundary condition (2.3) deduced that

$$r_1 = -\frac{1}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p,$$

so

$$\mathbf{v}(\kappa) = -\frac{1}{\Gamma_q(\gamma)} \int_0^\kappa (\kappa - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p + r_0 - \frac{\kappa}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p.$$

But on the other hand, we have

$$\int_0^\ell \mathbf{v}(p)\, \mathrm{d}p = \mathbf{v}(1) = \frac{-1}{\Gamma_q(\gamma)} \int_0^1 (1 - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p + r_0 + -\frac{1}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p$$

which yelids that

$$r_0 = \int_0^\ell \mathbf{v}(p)\, \mathrm{d}p + \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1 - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p + \frac{1}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p.$$

Therefore,

$$\begin{aligned}
\mathbf{v}(\kappa) = {} & \frac{-1}{\Gamma_q(\gamma)} \int_0^\kappa (\kappa - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p + \int_0^\ell \mathbf{v}(p)\, \mathrm{d}p + \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1 - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p \\
& + \frac{1}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p - \frac{\kappa}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p.
\end{aligned}$$

Now, for simplicity set

$$\begin{aligned}
\varphi(\kappa) = {} & \frac{-1}{\Gamma_q(\gamma)} \int_0^\kappa (\kappa - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p - \frac{\kappa}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p \\
& + \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1 - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p + \frac{1}{\Gamma_q(\gamma)} \int_0^\omega (\omega - qp)^{\gamma-1} \mathbf{u}(p)\, \mathrm{d}_q p,
\end{aligned}$$

then, it is clear that $\mathbf{v}(\kappa) = \varphi(\kappa) + \int_0^\ell \mathbf{v}(p)\,\mathrm{d}p$. Now, if $\omega \leq \kappa$ then, we get

$$\begin{aligned}
\varphi(\kappa) &= \frac{-1}{\Gamma_q(\gamma)}\Big(\int_0^\omega + \int_\omega^\kappa\Big)(\kappa - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p - \frac{\kappa}{\Gamma_q(\gamma)}\int_0^\omega (\omega - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p \\
&+ \frac{1}{\Gamma_q(\gamma)}\Big(\int_0^\omega + \int_\omega^\kappa + \int_\kappa^1\Big)(1 - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p + \frac{1}{\Gamma_q(\gamma)}\int_0^\omega (\omega - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p \\
&= \int_0^\omega \frac{(\omega - p)^{\gamma-1} + (1 - p)^{\gamma-1} - \kappa(\omega - p)^{\gamma-1} - (\kappa - p)^{\gamma-1}}{\Gamma_q(\gamma)}\mathbf{u}(p)\,\mathrm{d}_q p \\
&+ \int_\omega^\kappa \frac{(1 - p)^{\gamma-1} - (\kappa - p)^{\gamma-1}}{\Gamma_q(\gamma)}\mathbf{u}(p)\,\mathrm{d}_q p + \int_\kappa^1 \frac{(1 - p)^{\gamma-1}}{\Gamma_q(\gamma)}\mathbf{u}(p)\,\mathrm{d}_q p.
\end{aligned}$$

But, if $\omega \geq \kappa$ then, we obtain

$$\begin{aligned}
\varphi(\kappa) &= \frac{-1}{\Gamma_q(\gamma)}\int_0^\kappa (\kappa - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p - \frac{\kappa}{\Gamma_q(\gamma)}\Big(\int_0^\kappa + \int_\kappa^\omega\Big)(\omega - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p \\
&+ \frac{1}{\Gamma_q(\gamma)}\Big(\int_0^\kappa + \int_\kappa^\omega + \int_\omega^1\Big)(1 - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p \\
&+ \frac{1}{\Gamma_q(\gamma)}\Big(\int_0^\kappa + \int_\kappa^\omega\Big)(\omega - qp)^{\gamma-1}\mathbf{u}(p)\,\mathrm{d}_q p \\
&= \int_0^\kappa \frac{(\omega - qp)^{\gamma-1} + (1 - qp)^{\gamma-1} - \kappa(\omega - qp)^{\gamma-1} - (\kappa - qp)^{\gamma-1}}{\Gamma_q(\gamma)}\mathbf{u}(p)\,\mathrm{d}_q p \\
&+ \int_\kappa^\omega \frac{(\omega - qp)^{\gamma-1} + (1 - qp)^{\gamma-1} - \kappa(\omega - qp)^{\gamma-1}}{\Gamma_q(\gamma)}\mathbf{u}(p)\,\mathrm{d}_q p + \int_\omega^1 \frac{(1 - qp)^{\gamma-1}}{\Gamma_q(\gamma)}\mathbf{u}(p)\,\mathrm{d}_q p.
\end{aligned}$$

Hence,

$$\varphi(\kappa) = \int_0^1 \mathcal{L}_q(\kappa, p)\mathbf{u}(p)\,\mathrm{d}_q p,$$

so,

$$\mathbf{v}(\kappa) = \int_0^1 \mathcal{L}_q(\kappa, p)\mathbf{u}(p)\,\mathrm{d}_q p + \int_0^\ell \mathbf{v}(p)\,\mathrm{d}p. \tag{2.4}$$

By rewrite the Eq.(2.4), we have

$$\begin{aligned}
\int_0^\ell \mathbf{v}(\kappa)\,\mathrm{d}\kappa &= \int_0^\ell \int_0^1 \mathcal{L}_q(\kappa, p)\mathbf{u}(p)\,\mathrm{d}_q p\,\mathrm{d}\kappa + \int_0^\ell \int_0^\ell \mathbf{v}(p)\,\mathrm{d}p\,\mathrm{d}\kappa \\
&= \int_0^1 \Big(\int_0^\ell \mathcal{L}_q(\kappa, p)\,\mathrm{d}\kappa\Big)\mathbf{u}(p)\,\mathrm{d}_q p + \ell\int_0^\ell \mathbf{v}(p)\,\mathrm{d}p,
\end{aligned}$$

which implies that

$$(1 - \ell)\int_0^\ell \mathbf{v}(\kappa)\,\mathrm{d}\kappa = \int_0^1 \Big(\int_0^\ell \mathcal{L}_q(\kappa, p)\,\mathrm{d}\kappa\Big)\mathbf{u}(p)\,\mathrm{d}_q p$$

and so

$$\int_0^\ell \mathbf{v}(\kappa)\,\mathrm{d}\kappa = \int_0^1 \frac{1}{1-\ell}\Big(\int_0^\ell \mathcal{L}_q(\kappa,p)\,\mathrm{d}\kappa\Big)\mathbf{u}(p)\,\mathrm{d}_q p.$$

Thus,

$$\begin{aligned}
\mathbf{v}(\kappa) &= \int_0^1 \mathcal{L}_q(\kappa,p)\mathbf{u}(p)\,\mathrm{d}_q p + \int_0^1 \frac{1}{1-\ell}\Big(\int_0^\ell \mathcal{L}_q(\kappa,p)\,\mathrm{d}\kappa\Big)\mathbf{u}(p)\,\mathrm{d}_q p\\
&= \int_0^1 \Big(\mathcal{L}_q(\kappa,p) + \frac{1}{1-\ell}\int_0^\ell \mathcal{L}_q(\kappa,p)\,\mathrm{d}\kappa\Big)\mathbf{u}(p)\,\mathrm{d}_q p\\
&= \int_0^1 \mathcal{G}_q(\kappa,p)\mathbf{u}(p)\,\mathrm{d}_q p = \int_0^1 \mathcal{G}_q(\kappa,p)\mathbf{u}_0(p)\,\mathrm{d}_q p,
\end{aligned}$$

and with this our proof is complete. □

Lemma 2.10. *Assume that $\mathcal{G}_q(\kappa,p)$ be the one defined in Lemma 2.9. Then for all $\kappa,p\in[0,1]$, the following statments are satisfy*

I. $\mathcal{G}_q(\kappa,p)\geq 0$.

II. $\mathcal{G}_q(\kappa,p)\leq \mathcal{A}_q(\gamma,\ell)(1-qp)^{\gamma-1}$ *such that* $\mathcal{A}_q(\gamma,\ell)=\dfrac{3}{(1-\ell)\Gamma_q(\gamma)}$.

III. $\Big|\dfrac{\partial\mathcal{G}_q}{\partial\kappa}(\kappa,p)\Big|\leq \mathcal{B}_q(\gamma,\ell)(1-qp)^{\gamma-1}$ *such that* $\mathcal{B}_q(\gamma,\ell)=\dfrac{2}{(1-\ell)\Gamma_q(\gamma-1)}$.

IV. $\mathcal{G}_q(\kappa,p)\geq \dfrac{(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)}\Big(\dfrac{2-\gamma^2}{2(1-\gamma)}-\kappa\Big)\geq 0$.

Proof. **I.** If $0\leq qp\leq\kappa\leq 1$ and $qp\leq\omega$, then

$$(\omega-qp)^{\gamma-1}\geq \kappa(\omega-qp)^{\gamma-1} \quad\text{and}\quad (1-qp)^{\gamma-1}\geq -(\kappa-qp)^{\gamma-1},$$

hence,

$$(\omega-qp)^{\gamma-1}+(1-qp)^{\gamma-1}-\kappa(\omega-qp)^{\gamma-1}-(\kappa-qp)^{\gamma-1}\geq 0,$$

and ao $\mathcal{L}_q(\kappa,p)\geq 0$. Thus $\mathcal{G}_q(\kappa,p)\geq 0$. The proof in other case is easy and we skip writing it.

II. One can see that for all $\kappa,p\in[0,1]$ we have $\mathcal{L}_q(\kappa,p)\leq\dfrac{3(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)}$, and so

$$\begin{aligned}
\mathcal{G}_q(\kappa,p) &\leq \frac{3(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)} + \frac{1}{1-\ell}\int_0^\ell \frac{3(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)}\,\mathrm{d}\kappa\\
&= \frac{3(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)} + \frac{\ell}{1-\ell}\times\frac{3(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)} = \frac{3(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)}\Big(1+\frac{\ell}{1-\ell}\Big)\\
&= \frac{3(1-qp)^{\gamma-1}}{(1-\ell)\Gamma_q(\gamma)} = \mathcal{A}_q(\gamma,\ell)(1-qp)^{\gamma-1}.
\end{aligned}$$

III. Since

$$\frac{\partial \mathcal{L}_q}{\partial \kappa}(\kappa,p)=\begin{cases}\dfrac{-(\kappa-qp)^{\gamma-1}-(\omega-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}, & 0\le p\le\kappa\le 1,\ p\le\omega,\\[2mm] \dfrac{-(\kappa-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}, & 0\le\omega\le p\le\kappa\le 1,\\[2mm] \dfrac{-(\omega-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}, & 0\le\kappa\le p\le\omega\le 1,\\[2mm] 0, & 0\le\kappa\le p\le 1,\ \omega\le p.\end{cases}$$

Therefore, we have

$$\Big|\frac{\partial \mathcal{L}_q}{\partial \kappa}\Big|\le\frac{(\kappa-qp)^{\gamma-1}+(\omega-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}\le\frac{2(1-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)},$$

hence

$$\begin{aligned}\Big|\frac{\partial \mathcal{G}_q}{\partial \kappa}\Big|&\le\frac{2(1-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}+\frac{\ell}{1-\ell}\times\frac{2(1-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}=\frac{2(1-qp)^{\gamma-1}}{\Gamma_q(\gamma-1)}\Big(1+\frac{\ell}{1-\ell}\Big)\\&=\frac{2(1-qp)^{\gamma-1}}{(1-\ell)\Gamma_q(\gamma-1)}=\mathcal{B}_q(\gamma,\ell)(1-qp)^{\gamma-1}.\end{aligned}$$

IV If $0<qp<\kappa<1$ and $qp\le\omega$, then

$$\kappa(1-qp)>0\qquad\text{and}\qquad\kappa(1-qp)-qp+\kappa>0.$$

Thus, $\kappa-p<\kappa(1-qp)$ and $\frac{1-qp}{\kappa-qp}>\frac{1}{\kappa}$. Since $\frac{1}{\kappa}>1$ and $\gamma\ge 2$, so we have

$$\Big(\frac{1-qp}{\kappa-qp}\Big)^{\gamma-1}>\Big(\frac{1}{\kappa}\Big)^{\gamma}>\frac{1}{\kappa},$$

and

$$\begin{aligned}&(\omega-qp)^{\gamma-1}+(1-qp)^{\gamma-1}-\kappa(\omega-qp)^{\gamma-1}-(\kappa-qp)^{\gamma-1}\\&=(1-\kappa)(\omega-qp)^{\gamma-1}+(1-qp)^{\gamma-1}-(\kappa-qp)^{\gamma-1}\\&>(1-\kappa)(\omega-qp)^{\gamma-1}+(1-qp)^{\gamma-1}-\kappa(1-qp)^{\gamma-1}\\&=(1-\kappa)\Big((1-qp)^{\gamma-1}+(\omega-qp)^{\gamma-1}\Big)\\&\ge(1-\kappa)(1-qp)^{\gamma-1},\end{aligned}$$

which implies that

$$\mathcal{L}_q(\kappa,p)>(1-\kappa)(1-qp)^{\gamma-1}.$$

If $0 < qp \le \omega < \kappa < 1$ then

$$\begin{aligned}(1-qp)^{\gamma-1} - (\kappa - qp)^{\gamma-1} &> (1-qp)^{\gamma-1} - \kappa(1-qp)^{\gamma-1}\\ &= (1-\kappa)(1-qp)^{\gamma-1}\end{aligned}$$

and

$$\mathcal{L}_q(\kappa, p) > \frac{(1-\kappa)(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)}.$$

Thusly we obtain

$$\begin{aligned}\mathcal{G}_q(\kappa, p) &\ge \frac{1}{\Gamma_q(\gamma)}\Big[(1-\kappa)(1-qp)^{\gamma-1} + \frac{1}{1-\ell}\int_0^\ell (1-\kappa)(1-qp)^{\gamma-1}\,\mathrm{d}\kappa\Big]\\ &= \frac{1}{\Gamma_q(\gamma)}\Big[(1-\kappa)(1-qp)^{\gamma-1} + \frac{(1-qp)^{\gamma-1}}{1-\ell} \times \ell(1-\frac{\ell}{2})\Big]\\ &= \frac{(1-qp)^{\gamma-1}}{\Gamma_q(\gamma)}\Big(\frac{2-\gamma^2}{2(1-\gamma)} - \kappa\Big) \ge 0\end{aligned}$$

□

3 Main Results

Theorem 3.1. *Let $\mathbf{h} : \mathcal{K} \times (\mathbf{C}[0,1])^4 \to \mathbb{R}$ be a singular function at some point $\kappa \in \mathcal{K} = [0,1]$ such that for all $\mathbf{v}_1, \ldots, \mathbf{v}_4, \mathbf{u}_1, \ldots, \mathbf{u}_4 \in \mathbf{Y}$ the following inequality hold*

$$\Big|\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) - \mathbf{h}(\kappa, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)\Big| \le \sum_{i=1}^{4} s_i(\kappa)\|\mathbf{v}_i - \mathbf{u}_i\|$$

where $s_1, \ldots, s_4 \in \mathbf{L}^1[0,1]$ are nonnegative realvalued maps. Moreover, let $\exists t_0 \in \mathbb{N}$ such that there exists $\mathbf{Z}_1, \ldots, \mathbf{Z}_{t_0} \in \mathbf{L}^1[0,1]$ and $\mathbf{X}_1, \ldots, \mathbf{X}_{t_0} : \mathbb{R}^4 \to [0,\infty)$ where $\mathbf{Z}_i$ are nonnegative and $\mathbf{X}_i$ are nondecreasing and nonnegative functions for all $i = 1, \ldots, t_0$. Also, let $\beta_q = \min\big\{1, \Gamma_q(2-\tau), \Gamma_q(\sigma+1)\big\}$, $\mathcal{C}_q(\gamma,\ell) = \max\big\{\mathcal{A}_q(\gamma,\ell), \mathcal{B}_q(\gamma,\ell)\big\}$ and assume that the following are holds true

- *For all $(\mathbf{v}_1, \ldots, \mathbf{v}_4) \in \mathbf{Y}^4$ and almost $\kappa \in [0,1]$*

$$\Big|\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)\Big| \le \sum_{i=1}^{t_0} \mathbf{Z}_i(\kappa)\mathbf{X}_i(\mathbf{v}_1, \ldots, \mathbf{v}_4).$$

- $\lim_{y\to\infty} \dfrac{\mathbf{X}_i(y,y,y,y)}{y} = \rho$ *where $\rho \in \mathbb{R}^+$ and for some $\nu > 0$*

$$0 \le \rho \le \frac{\beta_q}{\mathcal{C}_q(\gamma,\ell)\sum_{i=1}^{t_0}\|\mathbf{Z}_i\| + \nu}.$$

- $\mathcal{C}_q(\gamma,\ell)\Big(\hat{s_1}+\hat{s_2}+\dfrac{\hat{s_3}}{\Gamma_q(2-\tau)}+\dfrac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big)<1$ *where* $\hat{s_i}=\int_0^1(1-qp)^{\gamma-1}s_i\,\mathrm{d}_qp$.

Then, the quantum multi-singular integrodifferential problem mentioned in (1.1)-(1.2) *has a solution.*

Proof. At first, we define the operator $\mathcal{T}:\mathbf{Y}\to\mathbf{Y}$ as follows:

$$\mathcal{T}(\kappa)=\int_0^1\mathcal{G}_q(\kappa,p)\mathbf{h}\big(\kappa,\mathbf{v}(\kappa),\mathbf{v}'(\kappa),{}^c\mathfrak{D}_q^{\tau}\mathbf{v}(\kappa),\mathfrak{I}_q^{\sigma}\mathbf{v}(\kappa)\big).$$

Now in three steps, we show that this operator has a fixed point which is the same solution to the desired problem.

step1. we shall show that $\mathcal{T}$ is continuous. Suppose that $\mathbf{v}_1,\mathbf{v}_2\in\mathbf{Y}$ and $\kappa\in[0,1]$. Afterwards, we can write

$$\begin{aligned}|\mathcal{T}_{\mathbf{v}_1}(\kappa)-\mathcal{T}_{\mathbf{v}_2}(\kappa)|\le&\int_0^1\mathcal{G}_q(\kappa,p)\Big|\mathbf{h}\big(p,\mathbf{v}_1(p),\mathbf{v}_1'(p),{}^c\mathfrak{D}_q^{\tau}\mathbf{v}_1(p),\mathfrak{I}_q^{\sigma}\mathbf{v}_1(p)\big)\\&-\mathbf{h}\big(p,\mathbf{v}_2(p),\mathbf{v}_2'(p),{}^c\mathfrak{D}_q^{\tau}\mathbf{v}_2(p),\mathfrak{I}_q^{\sigma}\mathbf{v}_2(p)\big)\Big|\,\mathrm{d}_qp\\&\le\int_0^1\mathcal{A}_q(\gamma,\ell)(1-qp)^{\gamma-1}\Big(s_1(p)\|\mathbf{v}_1-\mathbf{v}_2\|+s_2(p)\|\mathbf{v}_1'-\mathbf{v}_2'\|\\&+s_3(p)\|{}^c\mathfrak{D}_q^{\tau}\mathbf{v}_1-{}^c\mathfrak{D}_q^{\tau}\mathbf{v}_2\|+s_4(p)\|\mathfrak{I}_q^{\sigma}\mathbf{v}_1-\mathfrak{I}_q^{\sigma}\mathbf{v}_2\|\Big)\,\mathrm{d}_qp.\end{aligned}$$

But,

$$|\mathfrak{I}_q^{\sigma}\mathbf{v}(\kappa)|\le\frac{1}{\Gamma_q(\sigma)}\int_0^{\kappa}(\kappa-qp)^{\sigma}|\mathbf{v}(p)|\,\mathrm{d}_qp\le\frac{\|\mathbf{v}\|}{\Gamma_q(\sigma)}\times\frac{1}{\sigma}\big[(\kappa-qp)^{\sigma}\big|_{\kappa}^{0}\big]=\frac{\|\mathbf{v}\|}{\Gamma_q(\sigma+1)}\kappa^{\sigma},$$

which implies that $\|\mathfrak{I}_q^{\sigma}\mathbf{v}\|\le\dfrac{\|\mathbf{v}\|}{\Gamma_q(\sigma+1)}$. Therefore,

$$\|\mathfrak{I}_q^{\sigma}\mathbf{v}_1-\mathfrak{I}_q^{\sigma}\mathbf{v}_2\|=\|\mathfrak{I}_q^{\sigma}(\mathbf{v}_1-\mathbf{v}_2)\|\le\frac{\|\mathbf{v}_1-\mathbf{v}_2\|}{\Gamma_q(\sigma+1)}.$$

Also, similar to the above processes for all $\mathbf{v}_1,\mathbf{v}_2\in\mathbf{C}^1[0,1]$, we deduce that

$$\|{}^c\mathfrak{D}_q^{\tau}\mathbf{v}_1-{}^c\mathfrak{D}_q^{\tau}\mathbf{v}_2\|\le\frac{\|\mathbf{v}_1-\mathbf{v}_2\|}{\Gamma_q(2-\tau)}.$$

Hence,

$$\begin{aligned}&|\mathcal{T}_{\mathbf{v}_1}(\kappa)-\mathcal{T}_{\mathbf{v}_2}(\kappa)|\\&\le\mathcal{A}_q(\gamma,\ell)\int_0^1\Big(s_1(p)\|\mathbf{v}_1-\mathbf{v}_2\|+s_2(p)\|\mathbf{v}_1'-\mathbf{v}_2'\|\end{aligned}$$

$$+ s_3(p)\frac{\|\mathbf{v}_1-\mathbf{v}_2\|}{\Gamma_q(2-\tau)} + s_4(p)\frac{\|\mathbf{v}_1-\mathbf{v}_2\|}{\Gamma_q(\sigma+1)}\Big)(1-qp)^{\gamma-1}\,\mathrm{d}_qp$$

$$= \mathcal{A}_q(\gamma,\ell)\int_0^1 \Big[\big(s_1(p)+\frac{s_4(p)}{\Gamma_q(\sigma+1)}\big)(1-qp)^{\gamma-1}\|\mathbf{v}_1-\mathbf{v}_2\|$$

$$+ \big(s_2(p)+\frac{s_3(p)}{\Gamma_q(2-\tau)}\big)(1-qp)^{\gamma-1}\|\mathbf{v}_1'-\mathbf{v}_2'\|\Big]\,\mathrm{d}_qp$$

$$\leq \mathcal{A}_q(\gamma,\ell)\|\mathbf{v}_1-\mathbf{v}_2\|_*\int_0^1\Big[s_1(p)+s_2(p)+\frac{s_3(p)}{\Gamma_q(2-\tau)}+\frac{s_4(p)}{\Gamma_q(\sigma+1)}\Big](1-qp)^{\gamma-1}\,\mathrm{d}_qp$$

$$= \mathcal{A}_q(\gamma,\ell)\Big(\hat{s_1}+\hat{s_2}+\frac{\hat{s_3}}{\Gamma_q(2-\tau)}+\frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big)\|\mathbf{v}_1-\mathbf{v}_2\|_*.$$

As well as, we have

$$|\mathcal{T}_{\mathbf{v}_1}'(\kappa)-\mathcal{T}_{\mathbf{v}_2}'(\kappa)| \leq \int_0^1\Big|\frac{\partial\mathcal{G}_q(\kappa,p)}{\partial\kappa}\Big|\Big|\mathbf{h}\big(p,\mathbf{v}_1(p),\mathbf{v}_1'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}_1(p),\mathfrak{I}_q^\sigma\mathbf{v}_1(p)\big)$$

$$- \mathbf{h}\big(p,\mathbf{v}_2(p),\mathbf{v}_2'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}_2(p),\mathfrak{I}_q^\sigma\mathbf{v}_2(p)\big)\Big|\,\mathrm{d}_qp$$

$$\leq \int_0^1\mathcal{B}_q(\gamma,\ell)(1-qp)^{\gamma-1}\Big(s_1(p)\|\mathbf{v}_1-\mathbf{v}_2\|+s_2(p)\|\mathbf{v}_1'-\mathbf{v}_2'\|$$

$$+ s_3(p)\|{}^c\mathfrak{D}_q^\tau\mathbf{v}_1-{}^c\mathfrak{D}_q^\tau\mathbf{v}_2\|+s_4(p)\|\mathfrak{I}_q^\sigma\mathbf{v}_1-\mathfrak{I}_q^\sigma\mathbf{v}_2\|\Big)\,\mathrm{d}_qp$$

$$\leq \mathcal{B}_q(\gamma,\ell)\int_0^1\Big(s_1(p)\|\mathbf{v}_1-\mathbf{v}_2\|+s_2(p)\|\mathbf{v}_1'-\mathbf{v}_2'\|$$

$$+ s_3(p)\frac{\|\mathbf{v}_1-\mathbf{v}_2\|}{\Gamma_q(2-\tau)}+s_4(p)\frac{\|\mathbf{v}_1-\mathbf{v}_2\|}{\Gamma_q(\sigma+1)}\Big)(1-qp)^{\gamma-1}\,\mathrm{d}_qp$$

$$= \mathcal{B}_q(\gamma,\ell)\Big(\hat{s_1}+\hat{s_2}+\frac{\hat{s_3}}{\Gamma_q(2-\tau)}+\frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big)\|\mathbf{v}_1-\mathbf{v}_2\|_*.$$

It follows from above inequalities

$$\|\mathcal{T}_{\mathbf{v}_1}-\mathcal{T}_{\mathbf{v}_2}\|_* = \max\big\{\|\mathcal{T}_{\mathbf{v}_1}(\kappa)-\mathcal{T}_{\mathbf{v}_2}\|,\|\mathcal{T}_{\mathbf{v}_1}'-\mathcal{T}_{\mathbf{v}_2}'\|\big\}$$

$$\leq \mathcal{C}_q(\gamma,\ell)\Big(\hat{s_1}+\hat{s_2}+\frac{\hat{s_3}}{\Gamma_q(2-\tau)}+\frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big)\|\mathbf{v}_1-\mathbf{v}_2\|_*.$$

Accordingly $\|\mathcal{T}_{\mathbf{v}_1}-\mathcal{T}_{\mathbf{v}_2}\|_* \to 0$ as $\|\mathbf{v}_1-\mathbf{v}_2\|_* \to 0$, thus $\mathcal{T}$ is continuous.

step2. We claim that $\mathcal{T}$ is α-admisable. At first, since

$$\rho \times \frac{\mathcal{C}_q(\gamma,\ell)\sum_{i=1}^{t_0}\|\mathbf{Z}_i\|}{\beta_q} < 1,$$

then, we can choose λ where

$$(\rho+\lambda)\frac{\mathcal{C}_q(\gamma,\ell)\sum_{i=1}^{t_0}\|\mathbf{Z}_i\|}{\beta_q}<1.$$

According to $\frac{\mathbf{X}(y,y,y,y)}{y}\to\rho$ as $y\to\infty$, then $\exists\zeta=\nu_\lambda>0$ such that for all $y>\zeta=\nu_\lambda$ we have

$$\frac{\mathbf{X}(y,y,y,y)}{y}<\rho+\lambda. \tag{3.1}$$

Now, let $\mathbf{W}=\big\{\mathbf{v}\in\mathbf{Y}:\|\mathbf{v}\|_*<\zeta\big\}$ and define the map $\boldsymbol{\alpha}:\mathbf{Y}^2\to[0,\infty)$as follows:

$$\boldsymbol{\alpha}(\mathbf{v},\mathbf{u})=\begin{cases}1, & \mathbf{v},\mathbf{u}\in\mathbf{W}\\ 0, & \mathbf{v},\mathbf{u}\notin\mathbf{W}.\end{cases}$$

If $\boldsymbol{\alpha}(\mathbf{v},\mathbf{u})\geq 1$; then, $\|\mathbf{v}\|_*\leq\zeta$ and $\|\mathbf{u}\|_*\leq\zeta$. suppose that $\kappa\in[0,1]$, then we get

$$\begin{aligned}
|\mathcal{T}_{\mathbf{v}}(\kappa)| &\leq \int_0^1 \mathcal{G}_q(\kappa,p)\Big|\mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}(p),\mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\Big|\,\mathrm{d}_qp\\
&\leq \mathcal{A}_q(\gamma,\ell)\int_0^1(1-qp)^{\gamma-1}\sum_{i=0}^{t_0}\mathbf{Z}_i(p)\mathbf{X}_i\big(\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}(p),\mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\,\mathrm{d}_qp\\
&\leq \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\int_0^1(1-qp)^{\gamma-1}\sum_{i=0}^{t_0}\mathbf{Z}_i(p)\mathbf{X}_i\Big(\|\mathbf{v}\|,\|\mathbf{v}'\|,\frac{\|\mathbf{v}'\|}{\Gamma_q(2-\tau)},\frac{\|\mathbf{v}\|}{\Gamma_q(1+\sigma)}\Big)\,\mathrm{d}_qp\\
&\leq \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\mathbf{X}_i\Big(\zeta,\zeta,\frac{\zeta}{\Gamma_q(2-\tau)},\frac{\zeta}{\Gamma_q(1+\sigma)}\Big)\Big(\int_0^1\mathbf{Z}_i(p)\sup(1-qp)^{\gamma-1}\,\mathrm{d}_qp\Big)\\
&\leq \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\mathbf{X}_i\Big(\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q}\Big)\|\mathbf{Z}_i\|.
\end{aligned}$$

It follows from (3.1) and $\dfrac{\zeta}{\beta}>\zeta$ that $\mathbf{X}_i(\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q})<(\rho+\lambda)\frac{\zeta}{\beta_q}$ and so

$$\begin{aligned}
|\mathcal{T}_{\mathbf{v}}(\kappa)| &\leq \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\frac{\zeta}{\beta_q}(\rho+\lambda)\|\mathbf{Z}_i\|\\
&=(\rho+\lambda)\Big(\frac{\mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\|\mathbf{Z}_i\|}{\beta_q}\Big)\times\zeta\\
&<\zeta,
\end{aligned}$$

therefore, $\|\mathcal{T}_{\mathbf{v}}\|\leq\zeta$. In addition, we can write

$$|\mathcal{T}'_{\mathbf{v}}(\kappa)|\leq\int_0^1\Big|\frac{\partial\mathcal{G}_q(\kappa,p)}{\partial\kappa}\Big|\Big|\mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}(p),\mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\Big|\,\mathrm{d}_qp$$

$$\leq \mathcal{B}_q(\gamma,\ell)\int_0^1 (1-qp)^{\gamma-1}\sum_{i=0}^{t_0}\mathbf{Z}_i(p)\mathbf{X}_i\big(\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}(p),\mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\,\mathrm{d}_q p$$

$$\leq \mathcal{B}_q(\gamma,\ell)\sum_{i=0}^{t_0}\mathbf{X}_i\Big(\zeta,\zeta,\frac{\zeta}{\Gamma_q(2-\tau)},\frac{\zeta}{\Gamma_q(1+\sigma)}\Big)\Big(\int_0^1 \sup\mathbf{Z}_i(p)(1-qp)^{\gamma-1}\,\mathrm{d}_q p\Big)$$

$$\leq \mathcal{B}_q(\gamma,\ell)\sum_{i=0}^{t_0}\mathbf{X}_i\Big(\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q},\frac{\zeta}{\beta_q}\Big)\|\mathbf{Z}_i\|$$

$$=(\rho+\lambda)\Big(\frac{\mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\|\mathbf{Z}_i\|}{\beta_q}\Big)\zeta<\zeta$$

and so

$$\|\mathcal{T}_{\mathbf{v}}\|_*=\max\big\{\|\mathcal{T}_{\mathbf{v}}\|,\|\mathcal{T}'_{\mathbf{v}}\|\big\}\leq\zeta.$$

Therefore, we draw conclusions that $\mathcal{T}_{\mathbf{v}}\in\mathbf{W}$ and $\mathcal{T}_{\mathbf{u}}\in\mathbf{W}$, which implies that $\boldsymbol{\alpha}(\mathcal{T}_{\mathbf{v}},\mathcal{T}_{\mathbf{u}})\geq 1$. hence, $\mathcal{T}$ is $\boldsymbol{\alpha}$-admissible.

Step3. In view of $\mathbf{W}\neq\emptyset$, afterwards $\exists\mathbf{v}_0\in\mathbf{W}$ such that $\mathcal{T}_{\mathbf{v}_0}\in\mathbf{W}$. Thus, $\boldsymbol{\alpha}(\mathbf{v}_0,\mathcal{T}_{\mathbf{v}_0})\geq 1$. For convenience, set

$$\mathcal{J}_q(\gamma,\ell):=\Big[\hat{s_1}+\hat{s_2}+\frac{\hat{s_3}}{\Gamma_q(2-\tau)}+\frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big]\mathcal{C}_q(\gamma,\ell)<1,$$

and for all $\kappa\geq 0$ put $\psi(\kappa)=\mathcal{J}_q(\gamma,\ell)\kappa$. Since

$$\sum_{n=1}^{\infty}\psi^n(\kappa)=\sum_{n=1}^{\infty}\mathcal{J}_q^n(\gamma,\ell)\kappa=\frac{\mathcal{J}_q(\gamma,\ell)}{1-\mathcal{J}_q(\gamma,\ell)}\kappa<\infty,$$

and $\psi:[0,\infty)\to[0,\infty)$ is nondecreasing, we have $\psi\in\Psi$. Now, we obtain

$$\begin{aligned}|\mathcal{T}_{\mathbf{v}}(\kappa)-\mathcal{T}_{\mathbf{u}}(\kappa)|&\leq\int_0^1\mathcal{G}_q(\kappa,p)\Big|\mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}(p),\mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\\&\quad-\mathbf{h}\big(p,\mathbf{u}(p),\mathbf{u}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{u}(p),\mathfrak{I}_q^\sigma\mathbf{u}(p)\big)\Big|\,\mathrm{d}_q p\\&\leq\mathcal{A}_q(\gamma,\ell)\Big[\hat{s_1}+\hat{s_2}+\frac{\hat{s_3}}{\Gamma_q(2-\tau)}+\frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big]\|\mathbf{v}-\mathbf{u}\|_*,\end{aligned}$$

which yelids

$$\|\mathcal{T}_{\mathbf{v}}-\mathcal{T}_{\mathbf{u}}\|\leq\mathcal{A}_q(\gamma,\ell)\Big[\hat{s_1}+\hat{s_2}+\frac{\hat{s_3}}{\Gamma_q(2-\tau)}+\frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big]\|\mathbf{v}-\mathbf{u}\|_*.$$

Also, in a similar way

$$|\mathcal{T}'_{\mathbf{v}}(\kappa)-\mathcal{T}'_{\mathbf{u}}(\kappa)|\leq\int_0^1\Big|\frac{\partial\mathcal{G}_q(\kappa,p)}{\partial\kappa}\Big|\Big|\mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau\mathbf{v}(p),\mathfrak{I}_q^\sigma\mathbf{v}(p)\big)$$

$$- \mathbf{h}\big(p, \mathbf{u}(p), \mathbf{u}'(p), {}^{c}\mathfrak{D}_q^{\tau}\mathbf{u}(p), \mathfrak{I}_q^{\sigma}\mathbf{u}(p)\big)\Big| \,\mathrm{d}_q p$$

$$\leq \mathcal{B}_q(\gamma, \ell)\Big[\hat{s_1} + \hat{s_2} + \frac{\hat{s_3}}{\Gamma_q(2-\tau)} + \frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big]\|\mathbf{v} - \mathbf{u}\|_*,$$

which yelids

$$\|\mathcal{T}'_{\mathbf{v}} - \mathcal{T}'_{\mathbf{u}}\| \leq \mathcal{B}_q(\gamma, \ell)\Big[\hat{s_1} + \hat{s_2} + \frac{\hat{s_3}}{\Gamma_q(2-\tau)} + \frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big]\|\mathbf{v} - \mathbf{u}\|_*.$$

Thus for all $\mathbf{v}, \mathbf{u} \in \mathbf{W}$ we obtain

$$\|\mathcal{T}_{\mathbf{v}} - \mathcal{T}_{\mathbf{u}}\|_* \leq \mathcal{C}_q(\gamma, \ell)\Big[\hat{s_1} + \hat{s_2} + \frac{\hat{s_3}}{\Gamma_q(2-\tau)} + \frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big]\|\mathbf{v} - \mathbf{u}\|_*$$

$$= \mathcal{J}_q(\gamma, \ell)\|\mathbf{v} - \mathbf{u}\|_* = \psi(\|\mathbf{v} - \mathbf{u}\|_*).$$

Hence for all $\mathbf{v}, \mathbf{u} \in \mathbf{Y}$ we get

$$\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u})\|\mathcal{T}_{\mathbf{v}} - \mathcal{T}_{\mathbf{u}}\|_* \leq \psi(d(\mathbf{v}, \mathbf{u})).$$

In view of Lemma 2.8, the operator $\mathcal{T}$ has a fixed point which is the solution to the problem mentioned in (1.1)-(1.2). □

Theorem 3.2. *Assume that the function* $\mathbf{h} : \mathcal{K} \times (\mathbf{C}[0,1])^4 \to \mathbb{R}$ *defined for all* $\mathbf{v}_1, \ldots, \mathbf{v}_4 \in \mathbf{C}[0,1]$ *and almost all* $\kappa \in [0,1]$*. In addition, let* $\exists t_1 \in \mathbb{N}$ *and there exists nondecreasing maps* $\mathbf{X}_1, \ldots, \mathbf{X}_{t_1} : \mathbb{R}^4 \to \mathbb{R}$ *such that* $\mathbf{X}_i(y, y, y, y) \geq 0$ *and* $\lim_{y \to 0^+} \frac{\mathbf{X}_i(y,y,y,y)}{y} = \rho_i$ *for all* $y \geq 0$, $i = 1, \ldots, t_1$ *and some* $0 \leq \rho_i < 1$*. Moreover, for all* $\mathbf{v}_1, \ldots, \mathbf{v}_4, \mathbf{u}_1, \ldots, \mathbf{u}_4 \in \mathbf{Y}$ *there exists* $s_1, \ldots, s_{t_1} : [0,1] \to [0, \infty)$ *s.t.*

$$\Big|\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) - \mathbf{h}(\kappa, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)\Big|$$

$$\leq \sum_{i=1}^{t_1} s_i(\kappa)\mathbf{X}_i(\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2, \mathbf{v}_3 - \mathbf{u}_3, \mathbf{v}_4 - \mathbf{u}_4)$$

where $\mathbf{v}_j \geq \mathbf{u}_j \geq 0$*. If*

$$\mathcal{C}_q(\gamma, \ell)\sum_{i=0}^{t_1} \|(1 - qp)^{\gamma - 1}s_i\|_1 \leq 1.$$

Then, the quantum multi-singular integrodifferential problem mentioned in (1.1)-(1.2) *has a solution.*

Proof. we present our proof in three steps.

Step1. Assume that $\lambda > 0$ be given and put

$$\mathbf{q}_1 := \mathcal{C}_q(\gamma, \ell)\sum_{i=0}^{t_1} \|(1 - \kappa)^{\gamma - 1}s_i\|_1 + 1.$$

Moreover, acording to $\mathbf{X}_i(\frac{y}{\beta},\dots,\frac{y}{\beta}) \to 0$ as $y \to 0^+$ then for each $i = 1,\dots,t_1$ there exists $\zeta_i > 0$ such that $\mathbf{X}_i(\frac{y}{\beta},\dots,\frac{y}{\beta}) < \frac{\lambda}{\mathbf{q}_1}$ where $0 < y \le \zeta_i$. If $\zeta = \min\{\zeta_i : 1 \le i \le t_1\}$, then for all $0 < y \le \zeta$ we get $\mathbf{X}_i(\frac{y}{\beta},\dots,\frac{y}{\beta}) < \frac{\lambda}{\mathbf{q}_1}$. If $\mathbf{v}_n \to \mathbf{v}$ then $\exists n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$ we have $\|\mathbf{v}_n - \mathbf{v}\|_* < \zeta$. Afterwards for all $\kappa \in [0,1]$ and $n \ge n_0$ we have

$$
\begin{aligned}
&|\mathcal{T}_{\mathbf{v}_n}(\kappa) - \mathcal{T}_{\mathbf{v}}(\kappa)| \\
&\le \int_0^1 \mathcal{G}_q(\kappa,p)\Big|\mathbf{h}\big(p,\mathbf{v}_n(p),\mathbf{v}_n'(p),{}^c\mathfrak{D}_q^\tau \mathbf{v}_n(p),\mathfrak{I}_q^\sigma \mathbf{v}_n(p)\big) \\
&- \mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau \mathbf{v}(p),\mathfrak{I}_q^\sigma \mathbf{v}(p)\big)\Big|\,\mathrm{d}_q p \\
&\le \int_0^1 \mathcal{G}_q(\kappa,p)\sum_{i=0}^{t_1} s_i(p)\mathbf{X}_i\big(\|\mathbf{v}_n-\mathbf{v}\|,\|(\mathbf{v}_n-\mathbf{v})'\|,\|{}^c\mathfrak{D}_q^\tau(\mathbf{v}_n-\mathbf{v})\|,\|\mathfrak{I}_q^\sigma(\mathbf{v}_n-\mathbf{v})\|\big)\,\mathrm{d}_q p \\
&\le \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_1}\mathbf{X}_i\Big(\zeta,\zeta,\frac{\zeta}{\Gamma_q(2-\tau)},\frac{\zeta}{\Gamma_q(1+\sigma)}\Big)\Big(\int_0^1 s_i(p)(1-qp)^{\gamma-1}\,\mathrm{d}_q p\Big) \\
&\le \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_1}\mathbf{X}_i\Big(\frac{\zeta}{\beta},\frac{\zeta}{\beta},\frac{\zeta}{\beta},\frac{\zeta}{\beta}\Big)\Big(\int_0^1 s_i(p)(1-qp)^{\gamma-1}\,\mathrm{d}_q p\Big) \\
&\le \mathcal{A}_q(\gamma,\ell)\frac{\lambda}{t_*}\sum_{i=0}^{t_1}\|(1-\kappa)^{\gamma-1}s_i\|_1\zeta < \lambda,
\end{aligned}
$$

so for $n \ge n_0$ we obtain $\|\mathcal{T}_{\mathbf{v}_n} - \mathcal{T}_{\mathbf{v}}\| < \lambda$. Also, similary $\forall\kappa \in [0,1]$ and $n \ge n_0$ we get

$$
\begin{aligned}
|\mathcal{T}_{\mathbf{v}_n}'(\kappa) - \mathcal{T}_{\mathbf{v}}'(\kappa)| &\le \int_0^1 \Big|\frac{\partial \mathcal{G}_q(\kappa,p)}{\partial\kappa}\Big|\Big|\mathbf{h}\big(p,\mathbf{v}_n(p),\mathbf{v}_n'(p),{}^c\mathfrak{D}_q^\tau \mathbf{v}_n(p),\mathfrak{I}_q^\sigma \mathbf{v}_n(p)\big) \\
&\quad - \mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^c\mathfrak{D}_q^\tau \mathbf{v}(p),\mathfrak{I}_q^\sigma \mathbf{v}(p)\big)\Big|\,\mathrm{d}_q p \\
&\le \mathcal{B}_q(\gamma,\ell)\frac{\lambda}{t_*}\sum_{i=0}^{t_1}\|(1-\kappa)^{\gamma-1}s_i\|_1 < \lambda,
\end{aligned}
$$

and so $\|\mathcal{T}_{\mathbf{v}_n}' - \mathcal{T}_{\mathbf{v}}'\| \le \lambda$. Thus

$$\|\mathcal{T}_{\mathbf{v}_n}' - \mathcal{T}_{\mathbf{v}}'\|_* = \max\big\{\|\mathcal{T}_{\mathbf{v}_n} - \mathcal{T}_{\mathbf{v}}\|, \|\mathcal{T}_{\mathbf{v}_n}' - \mathcal{T}_{\mathbf{v}}'\|\big\} \le \lambda,$$

which implies that $\mathcal{T}_{\mathbf{v}_n} \to \mathcal{T}_{\mathbf{v}}$ as $\mathbf{v}_n \to \mathbf{v}$. Therefotre, $\mathcal{T}$ is continuous.

Step2. Notice that, for all $i = 1,\dots,t_1$ we have $\lim_{y\to 0^+}\frac{\mathbf{X}_i(y,y,y,y)}{y} = \rho_i < 1$; So $\forall\lambda_i > 0$, $\exists\zeta_i := \zeta(\lambda_i) > 0$ where $0 < \frac{y}{\beta} < \zeta_i$ and it follows that

$$\mathbf{X}_i(\frac{y}{\beta},\dots,\frac{y}{\beta}) \le (\rho_i + \lambda_i)\frac{y}{\beta}$$

where $\beta = \min\big\{\Gamma_q(2-\tau), \Gamma_q(1+\sigma)\big\}$. Now assume that λ_i^0 be such that $\rho_i + \lambda_i^0 < 1$ and $\zeta_i^0 := \zeta(\lambda_i^0)$. Put:

$$\zeta = \min\{\zeta_1^0, \dots, \zeta_{t_1}^0\}, \quad \rho = \max\{\rho_1, \dots, \rho_{t_1}\} \quad \text{and} \quad \lambda_0 = \min\{\lambda_1^0, \dots, \lambda_{t_1}^0\},$$

which for all $0 < \frac{y}{\beta} < \zeta_i$ imlies that

$$\rho + \lambda_0 < 1 \quad \text{and} \quad \mathbf{X}_i(\frac{y}{\beta}, \dots, \frac{y}{\beta}) \le (\rho + \lambda_0)\frac{y}{\beta}.$$

Moreover,

$$\mathbf{X}_i(\frac{y}{\beta}, \dots, \frac{y}{\beta}) \le (\rho + \lambda_0)\frac{\zeta}{\beta}\beta \le (\rho + \lambda_0)\zeta \le (\rho + \lambda_0)\lambda_0.$$

Define the map $\boldsymbol{\alpha} : \mathbf{Y}^2 \to [0, \infty)$ as follows:

$$\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) = \begin{cases} 1, & \|\mathbf{v} - \mathbf{u}\|_* \le \zeta \\ 0, & \text{otherwise.} \end{cases}$$

Now, let $\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) \ge 1$ then

$$\begin{aligned}
&|\mathcal{T}_{\mathbf{v}}(\kappa) - \mathcal{T}_{\mathbf{u}}(\kappa)| \\
&\le \int_0^1 \mathcal{G}_q(\kappa, p)\Big|\mathbf{h}\big(p, \mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau \mathbf{v}(p), \mathfrak{I}_q^\sigma \mathbf{v}(p)\big) \\
&- \mathbf{h}\big(p, \mathbf{u}(p), \mathbf{u}'(p), {}^c\mathfrak{D}_q^\tau \mathbf{u}(p), \mathfrak{I}_q^\sigma \mathbf{u}(p)\big)\Big| \,\mathrm{d}_q p \\
&\le \int_0^1 \mathcal{G}_q(\kappa, p) \sum_{i=0}^{t_1} s_i(p)\big|\mathbf{X}_i\big((\mathbf{v} - \mathbf{u}), (\mathbf{v} - \mathbf{u})', {}^c\mathfrak{D}_q^\tau(\mathbf{v} - \mathbf{u}), \mathfrak{I}_q^\sigma(\mathbf{v} - \mathbf{u})\big)\big| \,\mathrm{d}_q p \\
&\le \mathcal{A}_q(\gamma, \ell) \int_0^1 (1 - qp)^{\gamma-1} \sum_{i=0}^{t_1} s_i(p)\big|\mathbf{X}_i\big(\|\mathbf{v} - \mathbf{u}\|, \|(\mathbf{v} - \mathbf{u})'\|, \|{}^c\mathfrak{D}_q^\tau(\mathbf{v} - \mathbf{u})\|, \|\mathfrak{I}_q^\sigma(\mathbf{v} - \mathbf{u})\|\big)\big| \,\mathrm{d}_q p \\
&\le \mathcal{A}_q(\gamma, \ell) \sum_{i=0}^{t_1} \mathbf{X}_i\Big(\zeta, \zeta, \frac{\zeta}{\Gamma_q(2-\tau)}, \frac{\zeta}{\Gamma_q(1+\sigma)}\Big)\Big(\int_0^1 s_i(p)(1 - qp)^{\gamma-1} \,\mathrm{d}_q p\Big) \\
&\le \mathcal{A}_q(\gamma, \ell) \sum_{i=0}^{t_1} \mathbf{X}_i\Big(\frac{\zeta}{\beta}, \frac{\zeta}{\beta}, \frac{\zeta}{\beta}, \frac{\zeta}{\beta}\Big)\Big(\int_0^1 s_i(p)(1 - qp)^{\gamma-1} \,\mathrm{d}_q p\Big) \\
&\le \mathcal{A}_q(\gamma, \ell) \sum_{i=0}^{t_1} \|(1 - qp)^{\gamma-1} s_i\|_1 (\rho + \lambda)\zeta \\
&\le \mathcal{A}_q(\gamma, \ell) \sum_{i=0}^{t_1} \|(1 - qp)^{\gamma-1} s_i\|_1 \zeta \le \zeta
\end{aligned}$$

so, $\|\mathcal{T}_{\mathbf{v}} - \mathcal{T}_{\mathbf{u}}\| \leq \zeta$ which yelids that $\boldsymbol{\alpha}(\mathcal{T}_{\mathbf{v}}, \mathcal{T}_{\mathbf{u}}) = 1$. Now, put

$$\mathbf{q}_2 = \mathcal{C}_q(\gamma, \ell) \sum_{i=0}^{t_1} \|(1-\kappa)^{\gamma-1} s_i\|_1 (\rho + \lambda_0)$$

then follows from assumption that $\mathbf{q}_2 < 1$. If $\psi(\kappa) = \mathbf{q}_2 \kappa$ then $\psi \in \Psi$. If $\|\mathbf{v} - \mathbf{u}\| \leq \zeta$, then we have

$$\|\mathcal{T}_{\mathbf{v}} - \mathcal{T}_{\mathbf{u}}\| \leq \mathcal{A}_q(\gamma, \ell) \sum_{i=0}^{t_1} \|(1-\kappa)^{\gamma-1} s_i\|_1 (\rho + \lambda_0) \|\mathbf{v} - \mathbf{u}\|_* \leq \mathbf{q}_2 \|\mathbf{v} - \mathbf{u}\|_*$$

and

$$\|\mathcal{T}'_{\mathbf{v}} - \mathcal{T}'_{\mathbf{u}}\| \leq \mathcal{B}_q(\gamma, \ell) \sum_{i=0}^{t_1} \|(1-\kappa)^{\gamma-1} s_i\|_1 (\rho + \lambda_0) \|\mathbf{v} - \mathbf{u}\|_* \leq \mathbf{q}_2 \|\mathbf{v} - \mathbf{u}\|_*.$$

Hence, $\|\mathcal{T}_{\mathbf{v}} - \mathcal{T}_{\mathbf{u}}\|_* \leq \mathbf{q}_2 \|\mathbf{v} - \mathbf{u}\|_* = \psi(\|\mathbf{v} - \mathbf{u}\|_*)$ and so for all $\mathbf{v}, \mathbf{u} \in \mathbf{Y}$ we get

$$\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) \|\mathcal{T}_{\mathbf{v}} - \mathcal{T}_{\mathbf{u}}\|_* \leq \psi(\|\mathbf{v} - \mathbf{u}\|_*).$$

step3. Now, we claim that $\exists \mathbf{v}_0 \in \mathbf{Y}$ such that $\boldsymbol{\alpha}(\mathbf{v}_0, \mathcal{T}_{\mathbf{v}_0}) = 1$. To achieve this goal we shall show that $\|\mathcal{T}_{\mathbf{v}_0} - \mathbf{v}_0\| \leq \zeta$ for some $\mathbf{v}_0 \in \mathbf{Y}$. According to $\mathbf{X}_i(y, y, y, y) \to 0$ as $y \to 0^+$, then $\forall \pounds > 0$ there exists $n = n(\lambda)$ where $0 < y \leq \frac{\pounds}{n}$ which implies that $\mathbf{X}_i(y, y, y, y) \leq \lambda$ for all $1 \leq i \leq t_1$. Therefore, $\mathbf{X}_i(\frac{y}{n}, \frac{y}{n}, \frac{y}{n}, \frac{y}{n}) \leq \lambda$. Put

$$\pounds := \max\left\{1, \frac{1}{\Gamma_q(1+\sigma)}, \frac{1}{\Gamma_q(2-\tau)}\right\} = \max\left\{\frac{1}{\Gamma_q(1+\sigma)}, \frac{1}{\Gamma_q(2-\tau)}\right\}$$

and take λ_m such that

$$\sum_{i=0}^{t_1} \|(1-\kappa)^{\gamma-1} s_i\|_1 \lambda_m \mathcal{C}_q(\gamma, \ell) < \zeta.$$

Put $\mathbf{b}_1 = n(\lambda_m)$ and take $\mathbf{b}_2 \in \mathbb{N}$ where

$$\sum_{i=0}^{t_1} \|(1-\kappa)^{\gamma-1} s_i\|_1 \lambda_m \mathcal{C}_q(\gamma, \ell) \leq \zeta - \frac{1}{\mathbf{b}_2}.$$

Now, if $\mathbf{b}_* = \max\{\mathbf{b}_1, \mathbf{b}_2\}$ then for all $1 \leq i \leq t_1$ we have $\mathbf{X}_i(\frac{y}{\mathbf{b}_*}, \frac{y}{\mathbf{b}_*}, \frac{y}{\mathbf{b}_*}, \frac{y}{\mathbf{b}_*}) \leq \lambda_m$. Define the map $\mathbf{v}_0$ as follows:

$$\mathbf{v}_0(\kappa) = \begin{cases} 0, & \kappa \leq \frac{1}{\mathbf{b}_*+1}, \\ \dfrac{6\mathbf{b}_*^2}{6\mathbf{b}_*^2 + 5\mathbf{b}_* + 2}\left(\dfrac{\kappa^3}{3} - \dfrac{2\mathbf{b}_* + 1}{2\mathbf{b}_*(\mathbf{b}_* + 1)}\kappa^2 + \dfrac{\kappa}{\mathbf{b}_*(\mathbf{b}_* + 1)}\right) + \dfrac{1}{\mathbf{b}_* + 2}, & \frac{1}{\mathbf{b}_*+1} < \kappa < \frac{1}{\mathbf{b}_*}, \\ \dfrac{1}{\mathbf{b}_*}, & \kappa \geq \frac{1}{\mathbf{b}_*}, \end{cases}$$

It is clear that $\mathbf{v}_0 \in \mathbf{C}[0,1]$ and for all $\kappa \in [0,1]$ we have $0 \le \mathbf{v}_0 \le \frac{1}{\mathbf{b}_*}$ and $\mathbf{v}_0(\frac{1}{\mathbf{b}_*+1}) = 0$. In addition, we get

$$\mathbf{v}_0'(\kappa) = \begin{cases} 0, & \kappa \le \frac{1}{\mathbf{b}_*+1}, \\ \dfrac{6\mathbf{b}_*^2}{6\mathbf{b}_*^2+5\mathbf{b}_*+2}\Big(\kappa^2 - \dfrac{2\mathbf{b}_*+1}{\mathbf{b}_*(\mathbf{b}_*+1)}\kappa + \dfrac{1}{\mathbf{b}_*(\mathbf{b}_*+1)}\Big), & \frac{1}{\mathbf{b}_*+1} < \kappa < \frac{1}{\mathbf{b}_*}, \\ \dfrac{1}{\mathbf{b}_*}, & \kappa \ge \frac{1}{\mathbf{b}_*}. \end{cases}$$

Hence, $\mathbf{v}_0' \in \mathbf{C}[0,1]$ and $\mathbf{v}_0'(\frac{1}{\mathbf{b}_*+1}) = \mathbf{v}_0'(\frac{1}{\mathbf{b}_*}) = 0$, which implies that $\mathbf{v}_0 \in \mathbf{C}^1[0,1]$. Moreover, we obtain

$$\begin{aligned} \mathbf{v}_0'(\kappa) &\le \frac{6\mathbf{b}_*^2}{6n_*^2+5\mathbf{b}_*+2}\Big(\frac{1}{\mathbf{b}_*^2} - \frac{2\mathbf{b}_*+1}{2\mathbf{b}_*(\mathbf{b}_*+1)^2} + \frac{1}{\mathbf{b}_*^4(\mathbf{b}_*+1)}\Big) \\ &\le 1 \times \frac{1}{\mathbf{b}_*} \times \Big(\frac{1}{\mathbf{b}_*} - \frac{2\mathbf{b}_*}{(\mathbf{b}_*+1)^2} + \frac{1}{\mathbf{b}_*^3(\mathbf{b}_*+1)}\Big) \le \frac{1}{\mathbf{b}_*}, \end{aligned}$$

which yields $\|\mathbf{v}_{\mathbf{b}_*}\| \le \dfrac{1}{\mathbf{b}_*}$. So, we can write

$$\begin{aligned} &|\mathcal{T}_{\mathbf{v}_0}(\kappa) - \mathbf{v}_0(\kappa)| \\ &= \Big| \int_0^1 \mathcal{G}_q(\kappa,p)\mathbf{h}\big(p, \mathbf{v}_n(p), \mathbf{v}_n'(p), {}^c\mathfrak{D}_q^\tau \mathbf{v}_n(p), \mathfrak{I}_q^\sigma \mathbf{v}_n(p)\big)\, \mathrm{d}_q p - \mathbf{v}_0(p)\Big| \\ &\le \int_0^1 \mathcal{G}_q(\kappa,p)|\mathbf{h}\big(p, \mathbf{v}_n(p), \mathbf{v}_n'(p), {}^c\mathfrak{D}_q^\tau \mathbf{v}_n(p), \mathfrak{I}_q^\sigma \mathbf{v}_n(p)\big)\, \mathrm{d}_q p| + \frac{1}{\mathbf{b}_*} \\ &\le \mathcal{A}_q(\gamma,\ell)\int_0^1 (1-qp)^{\gamma-1}\sum_{i=0}^{t_0} s_i(p)\mathbf{X}_i\Big(\|\mathbf{v_0}\|, \|\mathbf{v_0}'\|, \frac{\|\mathbf{v_0}'\|}{\Gamma_q(2-\tau)}, \frac{\|\mathbf{v_0}\|}{\Gamma_q(1+\sigma)}\Big)\, \mathrm{d}_q p + \frac{1}{\mathbf{b}_*} \\ &\le \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\Big[\Big(\int_0^1 (1-qp)^{\gamma-1}s_i(p)\,\mathrm{d}_q p\Big)\mathbf{X}_i\Big(\frac{£}{\mathbf{b}_*}, \frac{£}{\mathbf{b}_*}, \frac{£}{\mathbf{b}_*}, \frac{£}{\mathbf{b}_*}\Big)\Big] + \frac{1}{\mathbf{b}_*} \\ &\le \mathcal{A}_q(\gamma,\ell)\sum_{i=0}^{t_0}\|(1-\kappa)^{\gamma-1}s_i\|\lambda_1 + \frac{1}{\mathbf{b}_*} \le \zeta \end{aligned}$$

and so $\|\mathcal{T}_{\mathbf{v}_0} - \mathbf{v}_0\| \le \zeta$. In a same way

$$\begin{aligned} |\mathcal{T}_{\mathbf{v}_0}'(\kappa) - \mathbf{v}_0'(\kappa)| &\le \int_0^1 \Big|\frac{\partial \mathcal{G}_q(\kappa,p)}{\partial\kappa}\Big||\mathbf{h}\big(p, \mathbf{v}_n(p), \mathbf{v}_n'(p), {}^c\mathfrak{D}_q^\tau \mathbf{v}_n(p), \mathfrak{I}_q^\sigma \mathbf{v}_n(p)\big)\, \mathrm{d}_q p| + \frac{1}{\mathbf{b}_*} \\ &\le \mathcal{B}_q(\gamma,\ell)\sum_{i=0}^{t_0}\|(1-\kappa)^{\gamma-1}s_i\|\lambda_1 + \frac{1}{\mathbf{b}_*} \le \zeta. \end{aligned}$$

It follows $\|(\mathcal{T}_{\mathbf{v}_0}-\mathbf{v}_0)'\| \leq \zeta$. Thus,

$$\|\mathcal{T}_{\mathbf{v}_0}-\mathbf{v}_0\|_* = \max\left\{\|\mathcal{T}_{\mathbf{v}_0}-\mathbf{v}_0\|, \|(\mathcal{T}_{\mathbf{v}_0}-\mathbf{v}_0)'\|\right\} \leq \lambda,$$

which implies that $\boldsymbol{\alpha}(\mathbf{v}_0, \mathcal{T}_{\mathbf{v}_0}) = 1$. Thanks to Lemma 2.8, the multi-singular problem mentioned in (1.1)-(1.2) has a solution. $\square$

Theorem 3.3. *Let $\mathbf{E}^c$ be a null subset of $[0,1]$, namely its measure is zero, and for all $\mathbf{v}_1, \dots, \mathbf{v}_4 \in \mathbf{Y}$ and $\kappa \in \mathbf{E}$ the function $\mathbf{h} : [0,1]\times\mathbf{Y}^4 \to [0,\infty)$ be such that $\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) < \infty$ and continuous with respect to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Moreover, let $\exists t_1 \in \mathbb{N}$ such that there exists maps $s_i : [0,1] \to [0,\infty)$, $s_1, \dots, s_{t_1} \in \mathbf{L}^1[0,1]$ and $\mathbf{X}_1, \dots, \mathbf{X}_{t_0} : \mathbb{R}^4 \to [0,\infty)$, $\mathbf{S} : \mathbb{R}^4 \to [0,\infty)$ such that*

$$\|\mathbf{S}\|_1^* := \sup_{\mathbf{v}\in\mathbf{C}[0,1]} \int_0^1 \mathbf{S}\big(\mathbf{v}(\kappa), \mathbf{v}(\kappa), \mathbf{v}(\kappa), \mathbf{v}(\kappa)\big)\, \mathrm{d}_q\kappa < \infty$$

and

$$\|\mathbf{X}_i\|_\infty := \sup_{y\in\mathbb{R}} \left\{\mathbf{X}_i(y,y,y,y)\right\} < \infty,$$

also

$$|\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)| \leq \sum_{i=1}^{t_1} s_i(\kappa)\mathbf{X}_i(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) + \mathbf{S}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4).$$

In addition, let there exists maps $\psi : \mathbb{R}^4 \to [0,\infty)$, s.t $\|\psi\|^* := \min\{\psi(\mathbf{v}_1, \dots, \mathbf{v}_4)\}$ and $z : [0,1] \to [0,\infty)$ where

$$\|z\|_1^{\mathbf{L}} := \int_0^1 (1-\kappa)^{\gamma-1} z(\kappa)\, \mathrm{d}_q\kappa$$

and for all $\mathbf{v}_1, \dots, \mathbf{v}_4 \in \mathbf{Y}$ and slomst $\kappa \in [0,1]$

$$\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \geq z(\kappa)\psi(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4).$$

Also, let there exists $g_1, g_2, g_3, g_4 \in \mathbf{L}^1[0,1]$ where

$$\sum_{i=1}^{4} \|g_i\|_1 < \frac{1}{\mathcal{C}_q(\gamma, \ell)},$$

and $\Xi : [0,\infty) \to [0,\infty)$, $\Xi_\beta \in \Psi$ such that for all $\lambda > 0$ we set $\Xi_\lambda(\mathbf{v}) := \Xi(\frac{\mathbf{v}}{\lambda})$ and

$$|\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) - \mathbf{h}(\kappa, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)| \leq \sum_{i=0}^{4} g_i(\kappa)\Xi(\|\mathbf{v}_i - \mathbf{u}_i\|)$$

with $\|\mathbf{v}_i\|, \|\mathbf{u}_i\| \in [\zeta_1, \zeta_2]$ where

$$\zeta_1 \leq \frac{\|\psi\|^*\|z\|_1^{\mathbf{L}}(4-\gamma^2-2\gamma)}{2\Gamma_q(\gamma)(1-\gamma)} \quad \text{and} \quad \zeta_2 \geq \mathcal{C}_q(\gamma, \ell)\Big[\sum_{i=1}^{t_1} \|\mathbf{X}_i\|_\infty \|s_i\|_1 + \|\mathbf{S}\|_1^*\Big].$$

Then, the quantum pointwise multi-singular integrodifferential problem mentioned in (1.1)-(1.2) has a solution.

Proof. Assume that $\{\mathbf{v}_n\}$ be s sequence in $\mathbf{Y}$ such that $\|\mathbf{v}_n - \mathbf{v}\|_* \to 0$, which implies that $\mathbf{v}_n \to \mathbf{v}$ and $\mathbf{v}'_n \to \mathbf{v}'$. in view of the following inequalities:

$$\|{}^c\mathfrak{D}_q^\tau(\mathbf{v}_n - \mathbf{v})\| \leq \frac{\|(\mathbf{v}_n - \mathbf{v})'\|}{\Gamma_q(2-\tau)} \quad \text{and} \quad \|\mathfrak{I}_q^\sigma(\mathbf{v}_n - \mathbf{v})\| \leq \frac{\|\mathbf{v}_n - \mathbf{v}\|}{\Gamma_q(1+\sigma)}$$

we obtain ${}^c\mathfrak{D}_q^\tau\mathbf{v}_n \to^c \mathfrak{D}_q^\tau\mathbf{v}$ and $\mathfrak{I}_q^\sigma\mathbf{v}_n \to \mathfrak{I}_q^\sigma\mathbf{v}$. It follows from the continuity of $\mathbf{h}(\kappa, \mathbf{v}_1, \dots, \mathbf{v}_4)$ that $\mathbf{h}(\kappa, \mathbf{v}_n, \mathbf{v}'_n, {}^c\mathfrak{D}_q^\tau\mathbf{v}_n, \mathfrak{I}_q^\sigma\mathbf{v}_n) \to \mathbf{h}(\kappa, \mathbf{v}, \mathbf{v}', {}^c\mathfrak{D}_q^\tau\mathbf{v}, \mathfrak{I}_q^\sigma\mathbf{v})$. suppose that $\mathbf{v} \to \mathbf{Y}$ and $\kappa \in \mathcal{K} = [0,1]$ then, we can write

$$\begin{aligned}|\mathcal{T}_{\mathbf{v}}(\kappa)| \leq \mathcal{A}_q(\gamma,\ell)\Big[\int_0^1 (1-qp)^{\gamma-1}\Big(\sum_{i=1}^{t_1} s_i(p)\mathbf{X}_i\big(\mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau\mathbf{v}(p), \mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\\ + \mathbf{S}\big(\mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau\mathbf{v}(p), \mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\Big)\,\mathrm{d}_q p\Big].\end{aligned}$$

Put, $\mathbf{v}_*(p) := \max\big\{\mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau\mathbf{v}(p), \mathfrak{I}_q^\sigma\mathbf{v}(p)\big\}$, then $\mathbf{v}_* \in \mathbf{C}[0,1]$ and afterwards for all $\kappa \in [0,1]$ we have

$$\begin{aligned}|\mathcal{T}_{\mathbf{v}}(\kappa)| &\leq \mathcal{A}_q(\gamma,\ell)\Big[\sum_{i=1}^{t_1}\mathbf{X}_i\Big(\|\mathbf{v}\|, \|\mathbf{v}'\|, \frac{\|\mathbf{v}'\|}{\Gamma_q(2-\tau)}, \frac{\|\mathbf{v}\|}{\Gamma_q(1+\sigma)}\Big)\Big] \times \int_0^1 s_i(p)\,\mathrm{d}_q p\\ &+ \int_0^1 \mathbf{S}\big(\mathbf{v}_*(p), \mathbf{v}_*(p), \mathbf{v}_*(p), \mathbf{v}_*(p)\big)\,\mathrm{d}_q p\\ &\leq \mathcal{A}_q(\gamma,\ell)\Big(\sum_{i=1}^{t_1}\|\mathbf{X}_i\|_\infty\|s_i\|_1 + \|\mathbf{S}\|_1^*\Big).\end{aligned}$$

In a same way, we get

$$|\mathcal{T}_{\mathbf{v}'}(\kappa)| \leq \mathcal{B}_q(\gamma,\ell)\Big(\sum_{i=1}^{t_1}\|\mathbf{X}_i\|_\infty\|s_i\|_1 + \|\mathbf{S}\|_1^*\Big).$$

Hence, for all $\mathbf{v} \in \mathbf{Y}$ we obtain

$$\|\mathcal{T}_{\mathbf{v}}\|_* \leq \mathcal{C}_q(\gamma,\ell)\Big(\sum_{i=1}^{t_1}\|\mathbf{X}_i\|_\infty\|s_i\|_1 + \|\mathbf{S}\|_1^*\Big) < \infty. \tag{3.2}$$

In view of the Lebesgue dominated convergence theorem, for all $\kappa \in [0,1]$ we deduce that

$$\begin{aligned}\mathcal{T}_{\mathbf{v}_n}(\kappa) &= \int_0^1 \mathcal{G}_q(\kappa,p)\mathbf{h}\big(p, \mathbf{v}_n(p), \mathbf{v}'_n(p), {}^c\mathfrak{D}_q^\tau\mathbf{v}_n(p), \mathfrak{I}_q^\sigma\mathbf{v}_n(p)\big)\,\mathrm{d}_q p\\ &\to \int_0^1 \mathcal{G}_q(\kappa,p)\mathbf{h}\big(p, \mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau\mathbf{v}(p), \mathfrak{I}_q^\sigma\mathbf{v}(p)\big)\,\mathrm{d}_q p = \mathcal{T}_{\mathbf{v}}(\kappa).\end{aligned}$$

Thus $\mathcal{T}$ is continuous.

Define the map $\boldsymbol{\alpha} : \mathbf{Y}^2 \to [0, \infty)$ as follows:

$$\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) = \begin{cases} 1, & \|\mathbf{v}\|_*, \|\mathbf{u}\|_* \in [\zeta_1, \zeta_2] \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) \geq 1$, then $\|\mathbf{v}\|_*, \|\mathbf{u}\|_* \in [\zeta_1, \zeta_2]$ and afterwards for all $\kappa \in [0, 1]$ we can write

$$\begin{aligned}
|\mathcal{T}_{\mathbf{v}}(\kappa)| &= \Big| \int_0^1 \mathcal{G}_q(\kappa, p)\mathbf{h}\big(p, \mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau \mathbf{v}(p), \mathfrak{I}_q^\sigma \mathbf{v}(p)\big)\, \mathrm{d}_q p \Big| \\
&\geq \frac{1}{\Gamma_q(\gamma)} \int_0^1 (1 - qp)^{\gamma - 1} \Big(\frac{2 - \ell^2}{2(1 - \ell)} - \kappa \Big) z(p) \\
&\times \psi\big(p, \mathbf{v}(p), \mathbf{v}'(p), {}^c\mathfrak{D}_q^\tau \mathbf{v}(p), \mathfrak{I}_q^\sigma \mathbf{v}(p)\big)\, \mathrm{d}_q p \\
&\geq \|\psi\|^* \Big[\frac{-\kappa}{\Gamma_q(\gamma)} \int_0^1 (1 - qp)^{\gamma - 1} z(p)\, \mathrm{d}_q p \\
&+ \frac{2 - \ell^2}{2(1 - \ell)} \int_0^1 (1 - qp)^{\gamma - 1} z(p)\, \mathrm{d}_q p \Big] \\
&\geq \|\psi\|^* \|z\|_1^{\mathbf{L}} \Big(\frac{1}{\Gamma_q(\gamma)} + \frac{2 - \ell^2}{2(1 - \ell)} \Big) \\
&= \|\psi\|^* \|z\|_1^{\mathbf{L}} \Big(\frac{4 - \ell^2 - 2\ell}{2\Gamma_q(\gamma)(1 - \ell)} \Big),
\end{aligned}$$

which implies that

$$\|\mathcal{T}_{\mathbf{v}}\| \geq \frac{\|\psi\|^* \|z\|_1^{\mathbf{L}} (4 - \ell^2 - 2\ell)}{2\Gamma_q(\gamma)(1 - \ell)},$$

and

$$\|\mathcal{T}_{\mathbf{v}}\|_* := \{\|\mathcal{T}_{\mathbf{v}}\|, \|\mathcal{T}'_{\mathbf{v}}\|\} \geq \frac{\|\psi\|^* \|z\|_1^{\mathbf{L}} (4 - \ell^2 - 2\ell)}{2\Gamma_q(\gamma)(1 - \ell)} \geq \zeta_1.$$

It follows from Eq. (3.2)

$$\|\mathcal{T}_{\mathbf{v}}\|_* \leq \mathcal{C}_q(\gamma, \ell) \Big(\sum_{i=1}^{t_1} \|\mathbf{X}_i\|_\infty \|s_i\|_1 + \|\mathbf{S}\|_1^* \Big) \leq \zeta_2,$$

therefore, $\boldsymbol{\alpha}(\mathcal{T}_{\mathbf{v}}, \mathcal{T}_{\mathbf{u}}) \geq 1$. On the other hands, if $\mathbf{v}_0 \in [\zeta_1, \zeta_2]$ it is obviously $\boldsymbol{\alpha}(\mathbf{v}_0, \mathcal{T}_{\mathbf{v}_0}) \geq 1$. Now, let $\mathbf{v}, \mathbf{u} \in [\zeta_1, \zeta_2]$, then we have

$$\boldsymbol{\alpha}(\mathbf{v}, \mathbf{u}) \big| \mathcal{T}_{\mathbf{v}}(\kappa) - \mathcal{T}_{\mathbf{u}}(\kappa) \big|$$

$$\leq \int_0^1 \mathcal{G}_q(\kappa,p)\Big|\mathbf{h}\big(p,\mathbf{v}(p),\mathbf{v}'(p),{}^{c}\mathfrak{D}_q^{\tau}\mathbf{v}(p),\mathfrak{I}_q^{\sigma}\mathbf{v}(p)\big)$$

$$-\mathbf{h}\big(p,\mathbf{u}(p),\mathbf{u}'(p),{}^{c}\mathfrak{D}_q^{\tau}\mathbf{u}(p),\mathfrak{I}_q^{\sigma}\mathbf{u}(p)\big)\Big|\,\mathrm{d}_q p$$

$$\leq \mathcal{A}_q(\gamma,\ell)\int_0^1(1-qp)^{\gamma-1}\Big[g_1(p)\Xi(|\mathbf{v}-\mathbf{u}|)+g_2(p)\Xi(|\mathbf{v}'-\mathbf{u}'|)$$

$$+g_3(p)\Xi(|{}^{c}\mathfrak{D}_q^{\tau}\mathbf{v}-{}^{c}\mathfrak{D}_q^{\tau}\mathbf{u}|)+g_4(p)\Xi(|\mathfrak{I}_q^{\sigma}\mathbf{v}-\mathfrak{I}_q^{\sigma}\mathbf{u}|)\Big]\,\mathrm{d}_q p$$

$$\leq \mathcal{A}_q(\gamma,\ell)\int_0^1(1-qp)^{\gamma-1}\Big[g_1(p)\Xi(\|\mathbf{v}-\mathbf{u}\|)+g_2(p)\Xi(\|\mathbf{v}'-\mathbf{u}'\|)$$

$$+g_3(p)\Xi\Big(\frac{\|\mathbf{v}'-\mathbf{u}'\|}{\Gamma_q(2-\tau)}\Big)+g_4(p)\Xi\Big(\frac{\|\mathbf{v}-\mathbf{u}\|}{\Gamma_q(\sigma+1)}\Big)\Big]\,\mathrm{d}_q p$$

$$\leq \mathcal{A}_q(\gamma,\ell)\int_0^1(1-qp)^{\gamma-1}\Xi\Big(\frac{\|\mathbf{v}-\mathbf{u}\|_*}{\beta}\Big)\sum_{i=1}^{4}g_i(p)\,\mathrm{d}_q p$$

$$\leq \mathcal{A}_q(\gamma,\ell)\Xi\Big(\frac{\|\mathbf{v}-\mathbf{u}\|_*}{\beta}\Big)\sum_{i=1}^{4}\int_0^1(1-qp)^{\gamma-1}g_i(p)\,\mathrm{d}_q p$$

$$\leq \sum_{i=1}^{4}\|g_i\|_1\Xi\Big(\frac{\|\mathbf{v}-\mathbf{u}\|_*}{\beta}\Big)$$

$$\leq \Xi_\beta(\|\mathbf{v}-\mathbf{u}\|_*)$$

which for all $\mathbf{v},\mathbf{u}\in\mathbf{Y}$ implies that

$$\boldsymbol{\alpha}(\mathbf{v},\mathbf{u})\|\mathcal{T}_{\mathbf{v}}-\mathcal{T}_{\mathbf{u}}\|\leq \Xi_\beta(\|\mathbf{v}-\mathbf{u}\|_*).$$

Moreover, for all $\mathbf{v},\mathbf{u}\in\mathbf{Y}$ we get

$$\boldsymbol{\alpha}(\mathbf{v},\mathbf{u})\|\mathcal{T}'_{\mathbf{v}}-\mathcal{T}'_{\mathbf{u}}\|\leq \Xi_\beta(\|\mathbf{v}-\mathbf{u}\|_*),$$

Hence for all $\mathbf{v},\mathbf{u}\in\mathbf{Y}$ we have

$$\boldsymbol{\alpha}(\mathbf{v},\mathbf{u})\|\mathcal{T}_{\mathbf{v}}-\mathcal{T}_{\mathbf{u}}\|_*\leq \Xi_\beta(\|\mathbf{v}-\mathbf{u}\|_*).$$

Thanks to Lemma 2.8, the operator $\mathcal{T}$ has a fixed point which is a solution for our quantum multi-singular problem mentioned in (1.1)-(1.2). □

4 Examples

Example 4.1. *Regard the following quantum multi-singular fractional problem*

$${}^{c}\mathfrak{D}_q^{\frac{15}{4}}\mathbf{v}(\kappa)+\frac{1}{75\sqrt{\pi}\big(\mathbf{f}(\kappa)\big)^{\frac{1}{5}}}+\Big(\|\mathbf{v}\|+\|\mathbf{v}'\|+\|{}^{c}\mathfrak{D}_q^{\frac{4}{5}}\mathbf{v}\|+\|\mathfrak{I}_q^{\frac{3}{8}}\mathbf{v}\|\Big)=0, \tag{4.1}$$

under boundary conditions

$$\begin{cases} \mathbf{v}'(0) = \mathbf{v}(\frac{2}{3}), \\ \mathbf{v}(1) = \int_0^{\frac{1}{2}} \mathbf{v}(p)\, \mathrm{d}p, \\ \mathbf{v}''(0) = 0, \end{cases} \tag{4.2}$$

such taht

$$\mathbf{f}(\kappa) = \begin{cases} 0, & \kappa \in [0,1] \cap Q, \\ \kappa, & \kappa \in (0,1) \cap Q^c. \end{cases}$$

In this case put: $\gamma = \frac{15}{4}$, $\tau = \frac{4}{5}$, $\sigma = \frac{3}{8}$, $\omega = \frac{2}{3}$, $\ell = \frac{1}{2}$. *Also,* $s_1(\kappa) = s_2(\kappa) = s_3(\kappa) = s_4(\kappa) = \mathbf{Z}_1(\kappa) = \mathbf{Z}_2(\kappa) = \mathbf{Z}_3(\kappa) = \mathbf{Z}_4(\kappa) = \dfrac{1}{\kappa^{\frac{1}{5}}}$ *which implies that* $\|s\|_1 = \|\mathbf{Z}\|_1 = \dfrac{1}{60\sqrt{\pi}}$, *and for* $i = 1, \ldots, 4$ *we set* $\mathbf{X}_i(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \|\mathbf{v}_i\|$. *Then we have*

$$\mathcal{A}_q(\gamma, \ell) = \mathcal{A}_q(\frac{15}{4}, \frac{1}{2}) = \frac{3}{(1 - \frac{1}{2})\Gamma_q(\frac{15}{4})} = \frac{3}{\frac{1}{2}\Gamma_q(\frac{15}{4})},$$

and

$$\mathcal{B}_q(\gamma, \ell) = \mathcal{B}_q(\frac{15}{4}, \frac{1}{2}) = \frac{2}{(1 - \frac{1}{2})\Gamma_q(\frac{15}{4} - 1)} = \frac{2}{\frac{1}{2}\Gamma_q(\frac{11}{4})},$$

hence

$$\mathcal{C}_q(\gamma, \ell) = \mathcal{C}_q(\frac{15}{4}, \frac{1}{2}) = \max\left\{\mathcal{A}_q(\frac{15}{4}, \frac{1}{2}), \mathcal{B}_q(\frac{15}{4}, \frac{1}{2})\right\} = \max\left\{\frac{3}{\frac{1}{2}\Gamma_q(\frac{15}{4})}, \frac{2}{\frac{1}{2}\Gamma_q(\frac{11}{4})}\right\},$$

which their numerical values are presented in Table 3. Also, the heatmap of Table 3 is presented in Figure 3.

	$q = 0.08$	$q = 0.17$	$q = 0.33$	$q = 0.49$	$q = 0.66$	$q = 0.81$
$\mathcal{A}_q(\frac{15}{4}, \frac{1}{2})$	5.2212	4.4789	3.4456	2.6915	2.1063	1.7210
$\mathcal{B}_q(\frac{15}{4}, \frac{1}{2})$	3.7799	3.5700	3.2659	3.0235	2.8126	2.6559
$\mathcal{C}_q(\frac{15}{4}, \frac{1}{2})$	5.2212	4.4789	3.4456	3.0235	2.8126	2.6559

Table 3: Numerical resual for $\mathcal{A}_q(\frac{15}{4}, \frac{1}{2})$, $\mathcal{B}_q(\frac{15}{4}, \frac{1}{2})$ and $\mathcal{C}_q(\frac{15}{4}, \frac{1}{2})$.

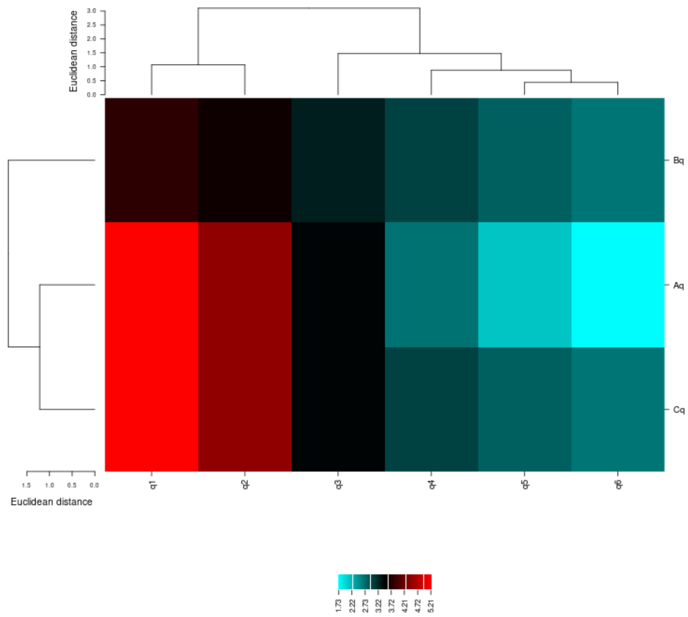

Figure 3: The heatmap of Table 3.

Moreover,

$$\beta_q = \min\left\{\Gamma_q(2-\tau), \Gamma_q(1+\sigma)\right\} = \min\left\{\Gamma_q(\frac{6}{5}), \Gamma_q(\frac{11}{8})\right\},$$

and its numerical values are given in Table 4.

	$q = 0.08$	$q = 0.17$	$q = 0.33$	$q = 0.49$	$q = 0.66$	$q = 0.81$
$\Gamma_q(\frac{6}{5})$	0.9802	0.9679	0.9529	0.9419	0.9326	0.9257
$\Gamma_q(\frac{11}{8})$	0.9754	0.9587	0.9379	0.9225	0.9094	0.8996
β_q	0.9802	0.9679	0.9529	0.9419	0.9326	0.9257

Table 4: Numerical resual for $\Gamma_q(\frac{6}{5})$, $\Gamma_q(\frac{11}{8})$ and β_q.

Consider the map $\mathbf{h}$ *as follows:*

$$\mathbf{h}\Big(\kappa, \mathbf{v}(\kappa), \mathbf{v}'(\kappa), {}^{c}\mathfrak{D}_q^{\tau}\mathbf{v}(\kappa), \mathfrak{I}_q^{\sigma}\mathbf{v}(\kappa)\Big) := \frac{1}{75\sqrt{\pi}\big(\mathbf{f}(\kappa)\big)^{\frac{1}{5}}} + \Big(\|\mathbf{v}\| + \|\mathbf{v}'\| + \|{}^{c}\mathfrak{D}_q^{\frac{4}{5}}\mathbf{v}\| + \|\mathfrak{I}_q^{\frac{3}{8}}\mathbf{v}\|\Big),$$

then we can write

$$\big|\mathbf{h}(\kappa, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) - \mathbf{h}(\kappa, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)\big| = \frac{\|\mathbf{v}_1\| - \|\mathbf{u}_1\| + \cdots + \|\mathbf{v}_4\| - \|\mathbf{u}_4\|}{75\sqrt{\pi}\big(\mathbf{f}(\kappa)\big)^{\frac{1}{5}}}$$

$$\leq \frac{\|\mathbf{v}_1-\mathbf{u}_1\|+\cdots+\|\mathbf{v}_4-\mathbf{u}_4\|}{75\sqrt{\pi}\big(\mathbf{f}(\kappa)\big)^{\frac{1}{5}}}$$

$$= \frac{1}{75\sqrt{\pi}\kappa^{\frac{1}{5}}}\sum_{i=1}^{4}\|\mathbf{v}_i-\mathbf{u}_i\|$$

$$= \sum_{i=1}^{4} s_i(\kappa)\|\mathbf{v}_i-\mathbf{u}_i\|,$$

and

$$\begin{aligned}\big|\mathbf{h}(\kappa,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)\big| &= \frac{\|\mathbf{v}_1\|+\cdots+\|\mathbf{v}_4\|}{75\sqrt{\pi}\big(\mathbf{f}(\kappa)\big)^{\frac{1}{5}}}\\ &= \frac{\|\mathbf{v}_1\|+\cdots+\|\mathbf{v}_4\|}{75\sqrt{\pi}\kappa^{\frac{1}{5}}}\\ &= \sum_{i=1}^{4}\mathbf{Z}(\kappa)\mathbf{X}_i(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4).\end{aligned}$$

In addition, we have $\lim_{y\to\infty}\dfrac{\mathbf{X}(y,y,y,y)}{y}=1=\rho$ *and*

$$\mathfrak{A}_q := \frac{\beta_q}{\mathcal{C}_q(\gamma,\ell)\sum_{i=1}^{4}\|\mathbf{Z}\|_1} = \frac{\beta_q}{\mathcal{C}_q(\frac{15}{4},\frac{1}{2})\times\frac{4}{60\sqrt{\pi}}} > 1 \tag{4.3}$$

where the establishment of inequality (4.3) *is shown in Table 5.*

q	$\mathfrak{A}_q > 1$
0.08	1.5888
0.17	1.8288
0.33	2.3382
0.49	2.6364
0.66	2.8061
0.81	2.9497

Table 5: Numerical resual for $\mathfrak{A}_q$.

Now, take $\nu > 0$ such that

$$\frac{\beta_q}{\mathcal{C}_q(\frac{15}{4}, \frac{1}{2}) \times \frac{1}{15\sqrt{\pi}} + \nu} \geq 1$$

On other hands,

$$\hat{s} = \int_0^1 (1 - qp)^{\frac{15}{4}-1} s(p) \, \mathrm{d}_q p \leq \int_0^1 s(p) \, \mathrm{d}_q p = \|s\|_1 = \frac{1}{60\sqrt{\pi}},$$

which implies that

$$\begin{aligned} \mathfrak{A}_q^* =& \mathcal{C}_q(\gamma, \ell)\Big(\hat{s_1} + \hat{s_2} + \frac{\hat{s_3}}{\Gamma_q(2-\tau)} + \frac{\hat{s_4}}{\Gamma_q(\sigma+1)}\Big) \\ &= \mathcal{C}_q(\frac{15}{4}, \frac{1}{2})\Big(\hat{s} + \hat{s} + \frac{\hat{s}}{\Gamma_q(\frac{6}{5})} + \frac{\hat{s}}{\Gamma_q(\frac{11}{8})}\Big) \\ &\leq \mathcal{C}_q(\frac{15}{4}, \frac{1}{2})\Big(1 + 1 + \frac{1}{\Gamma_q(\frac{6}{5})} + \frac{1}{\Gamma_q(\frac{11}{8})}\Big)\frac{1}{60\sqrt{\pi}} < 1 \end{aligned}$$

and the data related to the correctness of this inequality are given in Table 6. By using Theorem 3.1 our problem which formulated in (4.1)-(4.2) *has a solution.*

q	$\mathfrak{A}_q^* < 1$
0.08	0.6240
0.17	0.5393
0.33	0.4189
0.49	0.3703
0.66	0.3466
0.81	0.3289

Table 6: Numerical resual for $\mathfrak{A}_q^*$.

Example 4.2. *Regard the following quantum multi-singular fractional problem*

$$^{c}\mathfrak{D}_q^{\frac{13}{5}} \mathbf{v}(\kappa) + \frac{0.06}{\kappa^{\frac{1}{3}}(\kappa - \frac{1}{4})^{\frac{1}{6}}}\Big(1 - (\frac{4}{5})^{\frac{2}{5}\left(\mathbf{v} + \mathbf{v}' + {}^{c}\mathfrak{D}_q^{\frac{3}{20}} \mathbf{v} + \mathfrak{I}_q^{\frac{4}{7}} \mathbf{v}\right)}\Big) = 0, \tag{4.4}$$

under boundary conditions

$$\begin{cases} \mathbf{v}'(0) = \mathbf{v}(\frac{3}{4}), \\ \mathbf{v}(1) = \int_0^{\frac{1}{8}} \mathbf{v}(p) \, \mathrm{d}p. \end{cases} \tag{4.5}$$

In this case put: $\gamma = \frac{13}{5}$, $\tau = \frac{3}{20}$, $\sigma = \frac{4}{7}$, $\omega = \frac{3}{4}$, $\ell = \frac{1}{8}$ *and* $t_1 = 1$. *Then we have*

$$\mathcal{A}_q(\gamma, \ell) = \mathcal{A}_q(\frac{13}{5}, \frac{1}{8}) = \frac{3}{(1-\frac{1}{8})\Gamma_q(\frac{13}{5})} = \frac{3}{\frac{7}{8}\Gamma_q(\frac{13}{5})},$$

and

$$\mathcal{B}_q(\gamma, \ell) = \mathcal{B}_q(\frac{13}{5}, \frac{1}{8}) = \frac{2}{(1-\frac{1}{8})\Gamma_q(\frac{13}{5}-1)} = \frac{2}{\frac{7}{8}\Gamma_q(\frac{8}{5})},$$

hence

$$\mathcal{C}_q(\gamma, \ell) = \mathcal{C}_q(\frac{13}{5}, \frac{1}{8}) = \max\Big\{\mathcal{A}_q(\frac{13}{5}, \frac{1}{8}), \mathcal{B}_q(\frac{13}{5}, \frac{1}{8})\Big\} = \max\Big\{\frac{3}{\frac{7}{8}\Gamma_q(\frac{13}{5})}, \frac{2}{\frac{7}{8}\Gamma_q(\frac{8}{5})}\Big\},$$

which their numerical values are presented in Table 7. Also, the heatmap of Table 7 is presented in Figure 4.

	$q = 0.08$	$q = 0.17$	$q = 0.33$	$q = 0.49$	$q = 0.66$	$q = 0.81$
$\mathcal{A}_q(\frac{13}{5}, \frac{1}{8})$	3.2791	3.1378	2.9333	2.7692	2.6250	2.5165
$\mathcal{B}_q(\frac{13}{5}, \frac{1}{8})$	2.3344	2.3724	2.4235	2.4638	2.4996	2.5271
$\mathcal{C}_q(\frac{13}{5}, \frac{1}{8})$	3.2791	3.1378	2.9333	2.7692	2.6250	2.5271

Table 7: Numerical resual for $\mathcal{A}_q(\frac{13}{5}, \frac{1}{8})$, $\mathcal{B}_q(\frac{13}{5}, \frac{1}{8})$ and $\mathcal{C}_q(\frac{13}{5}, \frac{1}{8})$ in Example 4.2.

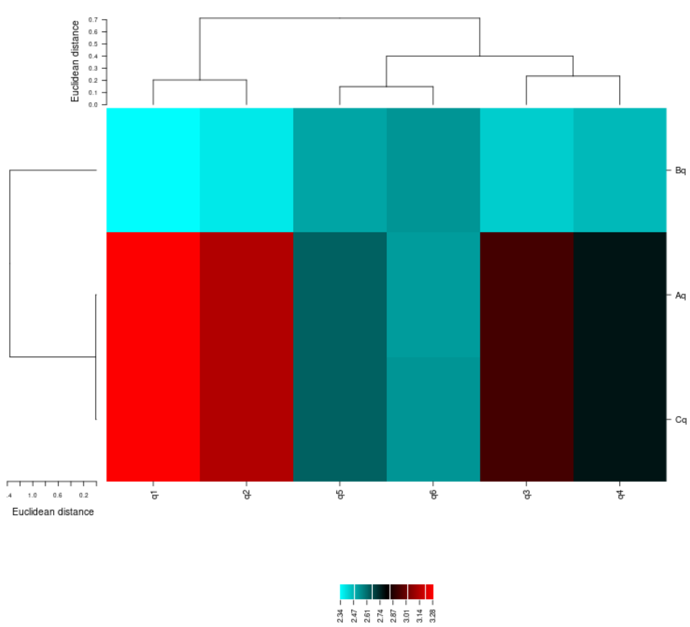

Figure 4: The heatmap of Table 7 in Example 4.2.

Consider the map $\mathbf{h}$ *as follows:*

$$\mathbf{h}\big(\kappa,\mathbf{v}_1,\dots,\mathbf{v}_4\big):=\frac{0.06}{\kappa^{\frac{1}{3}}(\kappa-\frac{1}{4})^{\frac{1}{6}}}\Big(1-(\frac{4}{5})^{\frac{2}{5}(\mathbf{v}_1+\cdots+\mathbf{v}_4)}\Big),$$

further,

$$s(\kappa)=\frac{0.06}{\kappa^{\frac{1}{3}}(\kappa-\frac{1}{4})^{\frac{1}{6}}}\quad \textit{and}\quad \mathbf{X}(\mathbf{v}_1,\dots,\mathbf{v}_4)=1-(\frac{4}{5})^{\frac{2}{5}(\mathbf{v}_1+\cdots+\mathbf{v}_4)}.$$

It is obvious that for all $y\geq 0$ *we have* $\mathbf{X}(y,y,y,y)\geq 0$ *and the function* $\mathbf{X}$ *is nondecreasing. Now, assume that for* $i=1,\dots,4$ *we have* $0\leq \mathbf{u}_i\leq \mathbf{v}_i$ *such that* $(\mathbf{v}_1,\dots,\mathbf{v}_4),(\mathbf{u}_1,\dots,\mathbf{u}_4)\in \mathbf{Y}^4$*. Since*

$$(\frac{4}{5})^{\mathbf{u}_i}\leq(\frac{4}{5})^{\mathbf{v}_i},$$

then

$$(\frac{4}{5})^{\mathbf{u}_i}\Big((\frac{4}{5})^{\mathbf{u}_i}-(\frac{4}{5})^{\mathbf{v}_i}\Big)\leq(\frac{4}{5})^{\mathbf{u}_i}-(\frac{4}{5})^{\mathbf{v}_i},$$

which implies that

$$(\frac{4}{5})^{\mathbf{u}_i}\Big((\frac{4}{5})^{\mathbf{u}_i}-(\frac{4}{5})^{\mathbf{v}_i}\Big)\leq(\frac{4}{5})^{\mathbf{u}_i}\Big(1-(\frac{4}{5})^{\mathbf{v}_i-\mathbf{u}_i}\Big).$$

Hence,

$$\Big(1-(\frac{4}{5})^{\mathbf{v}_i}\Big)-\Big(1-(\frac{4}{5})^{\mathbf{u}_i}\Big)\leq 1-(\frac{4}{5})^{\mathbf{v}_i-\mathbf{u}_i}.$$

From the placement of $\mathbf{v}_i$ *and* $\mathbf{u}_i$ *with* $\frac{2}{5}\sum_{i=1}^{4}\mathbf{v}_i$ *and* $\frac{2}{5}\sum_{i=1}^{4}\mathbf{u}_i$*, follows that:*

$$\Big(1-(\frac{4}{5})^{\frac{2}{5}\sum_{i=1}^{4}\mathbf{v}_i}\Big)-\Big(1-(\frac{4}{5})^{\frac{2}{5}\sum_{i=1}^{4}\mathbf{u}_i}\Big)\leq 1-(\frac{4}{5})^{\frac{2}{5}\sum_{i=1}^{4}\mathbf{v}_i-\mathbf{u}_i}.$$

Thus,

$$\mathbf{X}(\mathbf{v}_1,\dots,\mathbf{v}_4)-\mathbf{X}(\mathbf{u}_1,\dots,\mathbf{u}_4)\leq\mathbf{X}(\mathbf{v}_1-\mathbf{u}_1,\dots,\mathbf{v}_4-\mathbf{u}_4)$$

as well as

$$\mathbf{h}(\kappa,\mathbf{v}_1,\dots,\mathbf{v}_4)-\mathbf{h}(\kappa,\mathbf{u}_1,\dots,\mathbf{u}_4)\leq s(\kappa)\mathbf{X}(\mathbf{v}_1-\mathbf{u}_1,\dots,\mathbf{v}_4-\mathbf{u}_4).$$

and

$$\lim_{y\to 0^+}\frac{\mathbf{X}(y,y,y,y)}{y}=\lim_{y\to 0^+}\frac{1-(\frac{4}{5})^{4\times\frac{2}{5}y}}{y}=-4\frac{2}{5}\ln\frac{4}{5}=0.3570<1$$

Now all the assumptions of Theorem 3.2 are satisfied, so our problem, namely (4.4)-(4.5)*, has a solution.*

Example 4.3. *Regard the following quantum multi-singular fractional problem*

$$ {}^{c}\mathfrak{D}_q^{\frac{17}{6}}\mathbf{v}(\kappa)+\frac{0.02}{\mathbf{f}(\kappa)}\mathbf{X}\big(\mathbf{v}(\kappa)+\mathbf{v}'(\kappa)+{}^{c}\,\mathfrak{D}_q^{\frac{7}{12}}\mathbf{v}(\kappa)+\mathfrak{I}_q^{\frac{11}{13}}\mathbf{v}(\kappa)\big)+2=0, \tag{4.6}$$

under boundary conditions

$$\begin{cases} \mathbf{v}'(0) = \mathbf{v}(\frac{2}{5}), \\ \mathbf{v}(1) = \int_0^{\frac{2}{9}} \mathbf{v}(p)\,\mathrm{d}p. \end{cases} \tag{4.7}$$

such that

$$\mathbf{f}(\kappa) = \begin{cases} 0, & \kappa \in [0,1] \cap Q, \\ \sqrt{\kappa}, & \kappa \in (0,1) \cap Q^c. \end{cases}$$

In this case put: $\gamma = \frac{17}{6}$, $\tau = \frac{7}{12}$, $\sigma = \frac{11}{13}$, $\omega = \frac{2}{5}$, $\ell = \frac{2}{9}$ *and* $t_1 = 1$. *Then we obtain*

$$\mathcal{A}_q(\gamma, \ell) = \mathcal{A}_q(\frac{17}{6}, \frac{2}{9}) = \frac{3}{(1 - \frac{2}{9})\Gamma_q(\frac{17}{6})} = \frac{3}{\frac{7}{9}\Gamma_q(\frac{17}{6})},$$

and

$$\mathcal{B}_q(\gamma, \ell) = \mathcal{B}_q(\frac{17}{6}, \frac{2}{9}) = \frac{2}{(1 - \frac{2}{9})\Gamma_q(\frac{17}{6} - 1)} = \frac{2}{\frac{7}{9}\Gamma_q(\frac{11}{6})},$$

hence

$$\mathcal{C}_q(\gamma, \ell) = \mathcal{C}_q(\frac{17}{6}, \frac{2}{9}) = \max\Big\{\mathcal{A}_q(\frac{17}{6}, \frac{2}{9}), \mathcal{B}_q(\frac{17}{6}, \frac{2}{9})\Big\} = \max\Big\{\frac{3}{\frac{7}{9}\Gamma_q(\frac{17}{6})}, \frac{2}{\frac{7}{9}\Gamma_q(\frac{11}{6})}\Big\},$$

which their numerical values are presented in Table 8. Also, the heatmap of Table 8 is presented in Figure 5.

	$q = 0.08$	$q = 0.17$	$q = 0.33$	$q = 0.49$	$q = 0.66$	$q = 0.81$
$\mathcal{A}_q(\frac{17}{6}, \frac{2}{9})$	3.6204	3.3937	3.0657	2.8053	2.5803	2.4142
$\mathcal{B}_q(\frac{17}{6}, \frac{2}{9})$	2.5979	2.6201	2.6508	2.6754	2.6975	2.7145
$\mathcal{C}_q(\frac{17}{6}, \frac{2}{9})$	3.6204	3.3937	3.0657	2.8053	2.6975	2.7145

Table 8: Numerical resual for $\mathcal{A}_q(\frac{17}{6}, \frac{2}{9})$, $\mathcal{B}_q(\frac{17}{6}, \frac{2}{9})$ and $\mathcal{C}_q(\frac{17}{6}, \frac{2}{9})$ in Example 4.3.

Consider the map $\mathbf{X} : \mathbb{R}^4 \to [0, \infty)$ *as follows:*

$$\mathbf{X}(\mathbf{v}_1, \ldots, \mathbf{v}_4) = \begin{cases} \frac{1}{2} \sum_{i=1}^{4} \frac{\|\mathbf{v}_i\|}{1 + \|\mathbf{v}_1\|}, & \mathbf{v}_i, \ldots, \mathbf{v}_4 \in [0, 23], \\ |\sin(\mathbf{v}_1 + \cdots + \mathbf{v}_4)|, & \mathbf{v}_1, \ldots, \mathbf{v}_4 \in (-\infty, 0], \\ -\frac{23}{48}\Big(\frac{1}{4} \sum_{i=1}^{4} \mathbf{v}_i - 24\Big), & \mathbf{v}_i, \ldots, \mathbf{v}_4 \in [23, 24]. \end{cases}$$

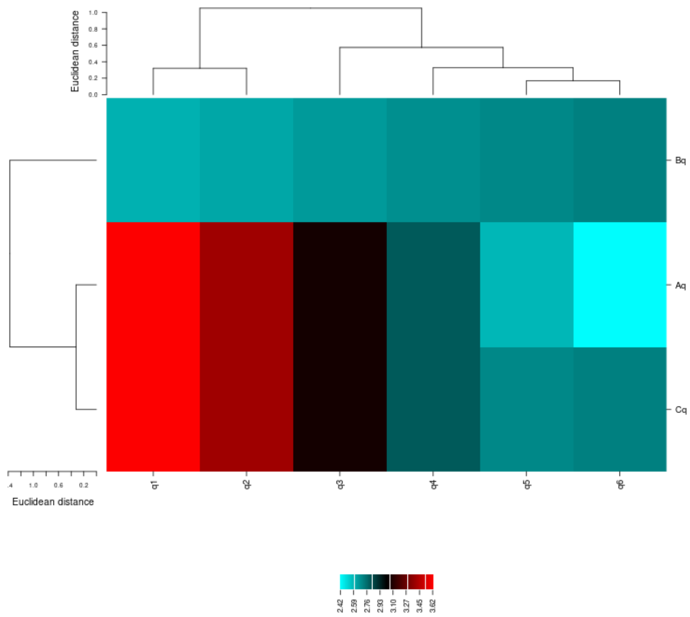

Figure 5: The heatmap of Table 8 in Example 4.3.

which implies that $\|\mathbf{X}\|_\infty = 1$*. Moreover, define* $\psi(\mathbf{v}_1, \dots, \mathbf{v}_4) := \mathbf{X}(\mathbf{v}_1, \dots, \mathbf{v}_4)$*, also put* $g(\kappa) = z(\kappa) = s(\kappa) = \frac{0.02}{\sqrt{\kappa}}$ *and* $\Xi(\kappa) = \frac{\kappa}{2}$*. If we set* $\mathbf{S}(\mathbf{v}_1, \dots, \mathbf{v}_4) = 2$ *and*

$$\mathbf{h}(\kappa, \mathbf{v}_1, \dots, \mathbf{v}_4) := \frac{0.02}{\mathbf{f}(\kappa)} \mathbf{X}(\mathbf{v}_1, \dots, \mathbf{v}_4) + 2,$$

further, we take $\zeta_1 = 0$ *and* $\zeta_2 = 23$*. Then for all* $\mathbf{v}_1, \dots, \mathbf{v}_4 \in \mathbf{Y}$ *and* $\kappa \in \mathbb{Q}^c \cap [0, 1]$ *we get*

$$\mathbf{h}(\kappa, \mathbf{v}_1, \dots, \mathbf{v}_4) < \infty$$

and $\mathbf{h}(\kappa, \mathbf{v}_1, \dots, \mathbf{v}_4)$ *is continuous respect to its components. Therefore, we have*

$$\begin{aligned} \mathbf{h}(\kappa, \mathbf{v}_1, \dots, \mathbf{v}_4) &= \frac{1}{\mathbf{f}(\kappa)} \mathbf{X}(\mathbf{v}_1, \dots, \mathbf{v}_4) + 2 \\ &\geq z(\kappa) \psi(\mathbf{v}_1, \dots, \mathbf{v}_4) \end{aligned}$$

and

$$\begin{aligned} |\mathbf{h}(\kappa, \mathbf{v}_1, \dots, \mathbf{v}_4) - \mathbf{h}(\kappa, \mathbf{u}_1, \dots, \mathbf{u}_4)| &\leq \sum_{i=1}^{4} \frac{1}{2} \times \frac{0.02}{\sqrt{\kappa}} \|\mathbf{v}_i - \mathbf{u}_i\| \\ &= \sum_{i=1}^{4} g(\kappa) \Xi(\|\mathbf{v}_i - \mathbf{u}_i\|). \end{aligned}$$

It is worth noting that $\Xi_\beta \in \Psi$ *and*

$$\sum_{i=1}^{4} \|g\|_1 = 4 \times 0.02 = 0.08 < \frac{1}{\mathcal{C}_q(\frac{17}{6}, \frac{2}{9})}.$$

In addition

$$\frac{\|\psi\|^*\|z\|_1^{\mathbf{L}}(4-\gamma^2-2\gamma)}{2\Gamma_q(\gamma)(1-\gamma)} = \frac{1\times 0.02\times(4-\frac{289}{36}-\frac{17}{3})}{2\Gamma_q(\frac{17}{6})(1-\frac{17}{6})}$$

$$= \frac{0.0969}{1.8333\Gamma_q(\frac{17}{6})} \geq 0 = \zeta_1,$$

and

$$\mathcal{C}_q(\gamma,\ell)\Big[\sum_{i=1}^{t_1}\|\mathbf{X}_i\|_\infty\|s_i\|_1 + \|\mathbf{S}\|_1^*\Big] = \mathcal{C}_q(\frac{17}{6},\frac{2}{9})[1\times 0.02+2]$$

$$= \mathcal{C}_q(\frac{17}{6},\frac{2}{9})\times 2.02 \leq 23 = \zeta_2.$$

Now, all the assumptions of Theorem 3.3 are satisfied, so our problem, namely (4.6)-(4.7), *has a solution.*

5 Conclusion

Today, computer capabilities and software packages are essential for solving complex problems. Therefore, physical phenomenon models must be comprehensible to computers, particularly if they have certain complications like singularity. Our study presents numerical algorithms that investigate the quantitative and qualitative aspects of a pointwise defined multisingular differential equation using quantum fractional operators. We ensured the existence of the solution in three different conditions by using the α-ψ-contraction theorem. Other researchers can also use our method to examine well-known equations with singularities and strong singularities.

Declarations

Availability of Data and Materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Funding
Not Applicable.

Authors' contributions
Conceptualization, M.G., S.R.; Formal analysis, R.G., F.S.A., M.G., S.R.; Funding acquisition, R.G.; Methodology, R.G., F.S.A., M.G., S.R.; Software, M.G., S.R.; All authors read and approved the final manuscript. The authors declare that the study was realized in collaboration with equal responsibility.

Competing interests
The authors declare that they have no competing interests.

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