First-order periodic coupled systems with generalized impulse conditions

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Abstract

We present some existence and localization results for periodic solutions of impulsive first-order coupled non-linear systems of two equations, without requiring periodicity for the nonlinearities. The arguments are based on Schauder's Fixed Point Theorem together with the upper and lower solution method, where the upper and lower solutions are not necessarily well-ordered. In addition, results on equi-regulated functions are required for the impulsive analysis. An application to a Wilson-Cowan system of two strongly coupled neurons illustrates one of the main results.

Keywords: *Impulsive nonlinear systems; Upper and lower solutions; Periodic solutions; Existence and localization of solutions; Equi-regulated functions; Wilson-Cowan model.*

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1 Introduction

We study the following first-order coupled nonlinear system,

$$\begin{cases} z'(t) = f\left(t, z(t), w(t)\right) \\ w'(t) = g\left(t, z(t), w(t)\right) \end{cases}, \tag{1}$$

a.e. t in $[0,T], T>0$, and the $f,g:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ L^1-Carathéodory functions, with the periodic boundary conditions

$$\begin{aligned} z(0) &= z(T), \\ w(0) &= w(T), \end{aligned} \tag{2}$$

subject to impulses given by

$$\begin{aligned} \Delta z(t_k) &= I_k(t_k, z(t_k), w(t_k)), \\ \Delta w(\tau_l) &= J_l(\tau_l, z(\tau_l), w(\tau_l)), \end{aligned} \tag{3}$$

with $k=1,...,n-1$, $l=1,...,m-1$, $n>2$, $m>2$, $\Delta z(t_k)=z(t_k^+)-z(t_k^-)$, $\Delta w(\tau_l)=w(\tau_l^+)-w(\tau_l^-)$, $I_k, J_l \in C([0,T]\times\mathbb{R}^2,\mathbb{R})$, and the time instants t_k, τ_l such that $0=t_0<t_1<...<t_{n-1}<t_n=T$ and $0=\tau_0<\tau_1<...<\tau_{m-1}<\tau_m=T$, where

$$u(t_k^-)=\lim_{t\to t_k^-} u(t) \qquad \text{and} \qquad u(t_k^+)=\lim_{t\to t_k^+} u(t).$$

The study of non-linear systems is well documented and widely found in the literature, in the fields of Ecology [1], Biology [2], Celestial Mechanics [3], Neuroscience [4], among others. In particular, the search for periodic solutions in such systems arises, for example, in [5, 6, 7, 8, 9].

The complexity of non-linear systems can increase when their evolution is described by sudden changes, that is, impulses, [10, 11, 12, 13, 14], especially when they are state-dependent [15, 16, 17].

Moreover, when the nonlinearities are non-periodic, the search for periodic solutions in impulsive non-linear systems is presented as a greater challenge. We present a method to overcome this hurdle.

Our method consists of proving the existence of at least a periodic solution of an impulsive first-order non-linear coupled system, with non-periodic nonlinearities, in the interval $[0,T]$ and localizing it in a strip bounded by a lower and an upper solution. In this problem, both the system nonlinearities and the impulses have explicit time and variable dependence.

Following a translation technique suggested in [18], we obtain the existence and the localization of an impulsive periodic solution. In such method, for C^1 lower and upper functions, there is a change of sign in the nonlinearities (see Definition 4). However, with less regularity (PC^1 lower and upper functions, as in Definition 9), the change of sign in the nonlinearities can be overcome in the presence of well-ordered lower and upper solutions.

So, for impulsive first-order coupled systems, some novelties are obtained: 1) sufficient conditions for the existence of periodic solutions are given, even when the nonlinearities have no periodicity at all; 2) lower and upper solutions do not need to be well-ordered; 3) the sum of all jumps must be null (so the problem must have more than one instant of impulse); 4) there exist non-negative periodic impulsive solutions despite the eventual change of sign in the nonlinearities.

The main results require some monotonicity relations in the nonlinearities and in the impulsive bounding functions. Furthermore, results on equi-regulated functions are essential to deal with the discontinuities at the instants of impulse. A similar problem is studied in [19, 20, 21], and with similar techniques in [18, 22].

As an example, we apply this method to a variant of a Wilson-Cowan system of strongly coupled neurons [23], using one of the most commonly used activation functions in neural networks [24] as the impulsive function, defined in certain time instants.

This work is organized as follows. In Section 2 we present the required definitions and auxiliary theorems. Section 3 contains one of the main results, together with the respective proof of the existence and localization of at least one solution of problem (1), (2), (3), together with an numerical example. In Section 4 we adapt the solvability conditions of Section 3 by allowing the sign of the nonlinearities to remain constant and, for the effect, recovering the order of the upper and lower solutions. In Section 5 we present a numerical result by applying our technique to a variant of a Wilson-Cowan system of strongly coupled neurons.

2 Definitions

In this work we consider the space of piecewise continuous functions in $[0,T]$, $(PC[0,T])^2$, with the norm $\|(z,w)\| = \max\{\|z\|, \|w\|\}$, with $\|u\| = \sup_{t\in[0,T]} |u(t)|$. So, $(PC[0,T])^2$ is a Banach space.

We consider $\mathcal{G}$ as the space of regulated functions [25],

$$\mathcal{G} = \{u : u(t^-) \in \mathbb{R}, \forall t \in (0,T], u(s^+) \in \mathbb{R}, \forall s \in [0,T)\}.$$

A main result for regulated functions is given by Theorem 1:

Theorem 1. *[26] A given subset B of the space $\mathcal{G}$ of regulated functions is relatively compact if and only if*

- *B is the set of equi-regulated functions, i.e. for every $\epsilon > 0$ there is a division $\xi_0 < \cdots < \xi_p$ of the interval $[0,T]$ such that, for every $v \in B$, $j \in \{1,\ldots,p\}$ and every $t,s \in (\xi_{j-1},\xi_j)$ we have $|v(t) - v(s)| < \epsilon$;*
- *the set $\{v(t) : v \in B\} \subset \mathbb{R}$ is bounded for each $t \in [0,T]$.*

Define the set $D = \{\xi_1, \dots, \xi_{p-1}\}$ such that $0 = \xi_0 < \xi_1 < \cdots < \xi_{p-1} < \xi_p = T$. Let PC_D be the space of piecewise continuous functions on $[0, T]$, given by

$$PC_D := \{u \in C([0,T]\backslash D) : u(t_k^-) = u(t_k), u(t_k^+) \in \mathbb{R}\}.$$

Define

$$\begin{aligned} D_z &= \{t_1, \dots, t_n\} \\ D_w &= \{\tau_1, \dots, \tau_m\} \end{aligned} \tag{4}$$

as the sets of instants of impulse of functions z and w, respectively.

It is clear that $PC_{D_z}[0,T] \times PC_{D_w}[0,T]$, endowed with the norm $\|\cdot\|$, is a Banach space.

The relation between equi-regulation and compactness on PC_D is based on Theorem 1:

Corollary 2. *[26] A subset B of the space PC_D is relatively compact if and only if:*

- *for a given $t \in [0,T]$, the set $\{u(t) : u \in B\} \subset \mathbb{R}$ is bounded;*
- *the set $\{u(t) : u \in B\}$ is equi-regulated.*

The nonlinearities in (1) are assumed to be L^1-Carathéodory functions, *i.e.*,

Definition 3. *A function $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, is a L^1-Carathéodory if it verifies*

(i) *for each $(y_1, y_2) \in \mathbb{R}^2$, $t \mapsto h(t, y_1, y_2)$ is measurable on $[0,T]$;*

(ii) *for almost every $t \in [0,T]$, $(y_1, y_2) \mapsto h(t, y_1, y_2)$ is continuous in $\mathbb{R}^2$;*

(iii) *for each $L > 0$, there exists a positive function $\psi_L \in L^1[0,T]$, such that, for $\max\{\|y_i\|, i = 1, 2\} < L$,*

$$|h(t, y_1, y_2)| \leq \psi_L(t), \ \ a.e. \ t \in [0,T]. \tag{5}$$

Our method consists of localizing existing solutions of (1), (2), (3) using bounding functions with translations. Because the upper bound is shifted towards its maximum and the lower bound is shifted towards its minimum, there is no requirement for the order of the upper and lower solutions, as stated by Definition 4:

Definition 4. *Consider the C^1-functions $\alpha_i, \beta_i : [0,T] \to \mathbb{R}$, $i = 1,2$. The functions (α_1, α_2) are lower solutions of the periodic problem (1), (2), (3) if*

$$\begin{cases} \alpha_1'(t) \leq f\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) \\ \alpha_2'(t) \leq g\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) \end{cases}, \tag{6}$$

with

$$\alpha_i^0(t) := \alpha_i(t) - \|\alpha_i\|, \; i = 1,2, \tag{7}$$

with

$$\alpha_i(0) \leq \alpha_i(T), \; i = 1,2, \tag{8}$$

and

$$\begin{cases} \Delta\alpha_1(t_k) > I_k(t_k, \alpha_1^0(t_k), \alpha_2^0(t_k)) \\ \Delta\alpha_2(\tau_l) > J_l(\tau_l, \alpha_1^0(\tau_l), \alpha_2^0(\tau_l)) \end{cases}. \tag{9}$$

The functions (β_1, β_2) are upper solutions of the periodic problem (1), (2), (3) if

$$\begin{cases} \beta_1'(t) \geq f\left(t, \beta_1^0(t), \beta_2^0(t)\right) \\ \beta_2'(t) \geq g\left(t, \beta_1^0(t), \beta_2^0(t)\right) \end{cases}, \tag{10}$$

with

$$\beta_i^0(t) := \beta_i(t) + \|\beta_i\|, \; i = 1,2,$$

with

$$\beta_i(0) \geq \beta_i(T), \; i = 1,2, \tag{11}$$

and

$$\begin{cases} \Delta\beta_1(t_k) < I_k(t_k, \beta_1^0(t_k), \beta_2^0(t_k)) \\ \Delta\beta_2(\tau_l) < J_l(\tau_l, \beta_1^0(\tau_l), \beta_2^0(\tau_l)) \end{cases}.$$

The existence tool is the well known Schauder's Fixed Point Theorem:

Theorem 5. *[27] Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P as at least one fixed point in Y.*

3 Existence and localization theorem

Our main result consists of a proof of the existence of at least one solution for problem (1), (2), (3), as well as its respective localization within a strip, bounded by upper and lower solutions under the conditions of Definition 4. We present the following theorem:

Theorem 6. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (1), (2), (3), respectively.*
Assume that f, g are L^1-Carathéodory functions on the set

$$\{(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2 : \alpha_1^0(t) \le y_1 \le \beta_1^0(t),\ \alpha_2^0(t) \le y_2 \le \beta_2^0(t)\},$$

with

$$f(t, y_1, \alpha_2^0(t)) \le f(t, y_1, y_2) \le f(t, y_1, \beta_2^0(t)), \tag{12}$$

for fixed $t \in [0, T]$, $y_1 \in \mathbb{R}$, and $\alpha_2^0(t) \le y_2 \le \beta_2^0(t)$, and with

$$g(t, \alpha_1^0(t), y_2) \le g(t, y_1, y_2) \le g(t, \beta_1^0(t), y_2),$$

for fixed $t \in [0, T]$, $y_2 \in \mathbb{R}$, and $\alpha_1^0(t) \le y_1 \le \beta_1^0(t)$.
Assume that the impulse functions I_k and J_l verify

$$I_k(t_k, y_1, \alpha_2^0) \ge I_k(t_k, y_1, y_2) \ge I_k(t_k, y_1, \beta_2^0), \tag{13}$$

for some fixed $k \in \{1, \dots n-1\}$, $y_1 \in \mathbb{R}$, and $\alpha_2^0(t) \le y_2 \le \beta_2^0(t)$, and that

$$\sum_{k=1}^{n-1} I_k(t_k, y_1, y_2) = 0. \tag{14}$$

Assume that the impulse functions J_l verify

$$J_l(\tau_l, \alpha_1^0, y_2) \ge J_l(\tau_l, y_1, y_2) \ge J_l(\tau_l, \beta_1^0, y_2),$$

for some fixed $l \in \{1, \dots m-1\}$, $y_2 \in \mathbb{R}$ and $\alpha_1^0(t) \le y_1 \le \beta_1^0(t)$, and that

$$\sum_{l=1}^{m-1} J_l(\tau_l, y_1, y_2) = 0. \tag{15}$$

Then, problem (1), (2), (3) has, at least, a solution $(z, w) \in (PC^1[0, T])^2$ such that

$$\begin{aligned}&\alpha_1^0(t) \le z(t) \le \beta_1^0(t),\\ &\alpha_2^0(t) \le w(t) \le \beta_2^0(t), \forall t \in [0, T].\end{aligned}$$

Remark 7. *Conditions (14) and (15) imply, respectively, that $n > 2$ and $m > 2$.*

Proof. For $i = 1, 2$, define the continuous functions $\delta_i^0 : [0,T] \times \mathbb{R}$, given by

$$\delta_1^0(t,z) = \begin{cases} \alpha_1^0(t) & \text{if} \quad z < \alpha_1^0(t) \\ z & \text{if} \quad \alpha_1^0(t) \leq z \leq \beta_1^0(t) \\ \beta_1^0(t) & \text{if} \quad z > \beta_1^0(t) \end{cases}$$

and (16)

$$\delta_2^0(t,w) = \begin{cases} \alpha_2^0(t) & \text{if} \quad w < \alpha_2^0(t) \\ w & \text{if} \quad \alpha_2^0(t) \leq w \leq \beta_2^0(t) \\ \beta_2^0(t) & \text{if} \quad w > \beta_2^0(t) \end{cases},$$

and consider the modified problem, composed by

$$\begin{cases} z'(t) + z(t) = f\left(t, \delta_1^0(t, z(t)), \delta_2^0(t, w(t))\right) + \delta_1^0(t, z(t)), & t \neq t_k, \\ w'(t) + w(t) = g\left(t, \delta_1^0(t, z(t)), \delta_2^0(t, w(t))\right) + \delta_2^0(t, w(t)), & t \neq \tau_l, \end{cases} \tag{17}$$

together with the boundary conditions (2) and the truncated impulse conditions,

$$\begin{array}{c} \Delta z(t_k) = I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))), \\ \Delta w(\tau_l) = J_l(\tau_l, \delta_1^0(\tau_l, z(\tau_l)), \delta_2^0(\tau_l, w(\tau_l))). \end{array} \tag{18}$$

Step 1: *Integral form of the problem (17), (2), (18).*

The integral form of (17), (2) considering the impulses (18) is given by

$$\begin{cases} z(t) = \displaystyle\sum_{k: t > t_k} I_k(t_k, \delta_1^0, \delta_2^0) + \\ \qquad\qquad + e^{-t}\Big(\dfrac{e^{-T}}{1 - e^{-T}} \displaystyle\int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big) \\ w(t) = \displaystyle\sum_{l: t > \tau_l} J_l(\tau_l, \delta_1^0, \delta_2^0) + \\ \qquad\qquad + e^{-t}\Big(\dfrac{e^{-T}}{1 - e^{-T}} \displaystyle\int_0^T e^s q_2(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_2(s, \delta_1^0, \delta_2^0)\, ds\Big), \end{cases} \tag{19}$$

where

$$\begin{array}{c} I_k(t_k, \delta_1^0, \delta_2^0) = I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))), \\ J_l(\tau_l, \delta_1^0, \delta_2^0) = J_l(\tau_l, \delta_1^0(\tau_l, z(\tau_l)), \delta_2^0(\tau_l, w(\tau_l))), \end{array}$$

and

$$q_1(s,\delta_1^0,\delta_2^0) := f(s,\delta_1^0(s,z(s)),\delta_2^0(s,w(s))) + \delta_1^0(s,z(s)),$$
$$q_2(s,\delta_1^0,\delta_2^0) := g(s,\delta_1^0(s,z(s)),\delta_2^0(s,w(s))) + \delta_2^0(s,w(s)).$$

Define the operator $T : (PC[0,T])^2 \to (PC[0,T])^2$ such that

$$T(z,w)(t) = (T_1(z,w)(t), T_2(z,w)(t)), \tag{20}$$

with

$$\begin{cases} T_1(z,w)(t) = \displaystyle\sum_{k:t>t_k} I_k(t_k,\delta_1^0,\delta_2^0)+ \\ \qquad + e^{-t}\Big(\dfrac{e^{-T}}{1-e^{-T}}\displaystyle\int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big) \\ T_2(z,w)(t) = \displaystyle\sum_{l:t>\tau_l} J_l(\tau_l,\delta_1^0,\delta_2^0)+ \\ \qquad + e^{-t}\Big(\dfrac{e^{-T}}{1-e^{-T}}\displaystyle\int_0^T e^s q_2(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_2(s,\delta_1^0,\delta_2^0)\,ds\Big). \end{cases} \tag{21}$$

The norm of the operator T, defined in (20), is given by

$$\begin{aligned} &\|T(z,w)\| = \max\big\{\|T_1(z,w)\|,\ \|T_2(z,w)\|\big\} = \\ &\max\big\{\sup_{t\in[0,T]}|T_1(z,w)(t)|,\ \sup_{t\in[0,T]}|T_2(z,w)(t)|\big\}. \end{aligned} \tag{22}$$

Step 2: *T has a fixed point.*

The conditions for Theorem 5 require the existence of a nonempty, bounded, closed and convex subset $B \subset (PC[0,T])^2$ such that $TB \subset B$.

As f, g are L^1-Carathéodory functions, by Definition 3, there are positive $L^1[0,T]$ functions $\psi_{iL}, i = 1,2$, such that

$$\begin{array}{l} |f(t,\delta_1^0(t,z),\delta_2^0(t,w))| \le \psi_{1L}(t) \\ |g(t,\delta_1^0(t,z),\delta_2^0(t,w))| \le \psi_{2L}(t) \end{array}, \quad a.e.\ t\in[0,T] \tag{23}$$

with

$$L := \max\{|\alpha_1^0(t)|, |\alpha_2^0(t)|, \beta_1^0(t), \beta_2^0(t)\}. \tag{24}$$

We may thus consider the closed ball of radius K,

$$B := \{(z,w) \in (PC[0,T])^2 : \|(z,w)\| \le K\}, \tag{25}$$

with K given by

$$K = \max \left\{ \begin{array}{l} M_I(n-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T (\psi_{1L}(s)\, ds + LT \Big), \\ M_J(m-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T (\psi_{2L}(s)\, ds + LT \Big) \end{array} \right\}, \tag{26}$$

and

$$\begin{aligned} M_I &:= \max_{1 \le k \le n-1} \{ |I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k)))| \}, \\ M_J &:= \max_{1 \le l \le m-1} \{ |J_l(\tau_l, \delta_1^0(\tau_l, z(\tau_l)), \delta_2^0(\tau_l, w(\tau_l)))| \}. \end{aligned}$$

For $t \in [0, T]$,

$$\begin{aligned} &|T_1(z,w)(t)| = \\ &= \Big| \sum_{k:t>t_k} I_k(t_k, \delta_1^0, \delta_2^0) + e^{-t} \Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Big) \Big| \\ &\le \Big| \sum_{k:t>t_k} I_k(t_k, \delta_1^0, \delta_2^0) \Big| + \Big| e^{-t} \Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Big) \Big| \\ &\le \sum_{k:t>t_k} \Big| I_k(t_k, \delta_1^0, \delta_2^0) \Big| + \Big| \Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Big) \Big| \\ &\le \sum_{k:t>t_k} M_I + \Big| \frac{1}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Big| \\ &\le M_I(n-1) + \frac{e^T}{1-e^{-T}} \Big| \int_0^T q_1(s, \delta_1^0, \delta_2^0)\, ds \Big| \\ &\le M_I(n-1) + \frac{e^T}{1-e^{-T}} \int_0^T |q_1(s, \delta_1^0, \delta_2^0)|\, ds \\ &\le M_I(n-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T |f(s, \delta_1^0, \delta_2^0)| ds + LT \Big) \\ &\le M_I(n-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T \psi_{1L}(s) ds + LT \Big). \end{aligned}$$

From (26), we have

$$|T_1(z,w)(t)| \le M_I(n-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T (\psi_{1L}(s)\, ds + LT \Big) \le K, \quad \forall t \in [0, T].$$

Similarly,

$$|T_2(z,w)(t)| \le M_J(m-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T (\psi_{2L}(s)\, ds + LT \Big) \le K, \quad \forall t \in [0, T].$$

Since T_1 and T_2 are uniformly bounded, so is T and, by (25) and (26), $TB \subseteq B$.

Consider a, b, with $a < b$, without loss of generality, and let $[a, b] \subseteq (t_k, t_{k+1})$ for some $k \in \{0, \dots, n-1\}$.

Then,

$$
\begin{aligned}
&|T_1(z,w)(a) - T_1(z,w)(b)| = \\
&\quad = \Bigg| \sum_{k:a>t_k} I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))) + \\
&\quad + e^{-a}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^a e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big) - \\
&\quad - \sum_{k:b>t_k} I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))) - \\
&\quad - e^{-b}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^b e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big)\Bigg| = \\
&\quad = \Bigg| \big(e^{-a} - e^{-b}\big)\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^a e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big) - \\
&\qquad - e^{-b} \int_a^b e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Bigg| \xrightarrow[a \to b]{} 0,
\end{aligned} \tag{27}
$$

proving that T_1 is equi-regulated.

Similarly, $|T_2(z,w)(a) - T_2(z,w)(b)| \xrightarrow[a \to b]{} 0$. Therefore, T is equi-regulated.

By Corollary 2, T is relatively compact. Then, by Theorem 5, T has a fixed point $(z^*(t), w^*(t)) \in (PC[0,T])^2$, which is solution of (17), (2), (18).

Step 3: *The pair $(z^*(t), w^*(t))$, solution of (17), (2), (18), is a solution of the initial problem, (1), (2), (3).*

To prove that $(z^*, w^*) \in (PC[0,T])^2$ is a solution of the original problem (1), (2), (3) it is enough to prove that

$$
\alpha_1^0(t) \le z^*(t) \le \beta_1^0(t), \quad \alpha_2^0(t) \le w^*(t) \le \beta_2^0(t), \quad \forall t \in [0,T]. \tag{28}
$$

In the first inequality, suppose, by contradiction, that there exists $t \in [0,T]$ such that

$$
z^*(t) < \alpha_1^0(t),
$$

and define

$$
\inf_{t \in [0,T]} (z^* - \alpha_1^0)(t) := z^*(t_0) - \alpha_1^0(t_0) < 0. \tag{29}
$$

We now consider different cases for t_0.

If $t_0 \in]t_k, t_{k+1}[$ for some $k \in \{0, \ldots, n\}$, then, by (29)

$$(z^* - \alpha_1^0)'(t_0) = 0. \tag{30}$$

However, by (29), (12) and (6),

$$\begin{aligned}
(z^*)'(t_0) &= f(t_0, \delta_1^0(t_0, z^*(t_0)), \delta_2^0(t_0, w(t_0))) + \delta_1^0(t_0, z^*(t_0)) - z^*(t_0) \\
&= f(t_0, \alpha_1^0(t_0), \delta_2^0(t_0, w(t_0))) + \alpha_1^0(t_0) - z^*(t_0) \\
&> f(t_0, \alpha_1^0(t_0), \delta_2^0(t_0, w(t_0))) \\
&\geq f(t_0, \alpha_1^0(t_0), \alpha_2^0(t_0)) \\
&\geq {\alpha_1}'(t_0),
\end{aligned}$$

contradicting (30).

If $t_0 = t_k$ for some $k \in \{1, \ldots, n-1\}$, then either $t_0 = t_k^+$ or $t_0 = t_k^-$. If $t_0 = t_k^+$, then $t_0 \in]t_k, t_{k+1}[$, so the previous reasoning must be applied. If, instead, $t_0 = t_k^-$, then we consider

$$\min_{t \in [0,T]} (z^* - \alpha_1^0)(t) := z^*(t_0) - \alpha_1^0(t_0) < 0, \tag{31}$$

and thus, the impulse on t_k is necessarily non-negative. By (31), (7), (18), (13) and (9) the following contradiction holds,

$$\begin{aligned}
0 &\leq \Delta(z^* - \alpha_1^0)(t_k) \\
&= I_k(t_k, \delta_1^0(t_k, z^*(t_k)), \delta_2^0(t_k, w(t_k))) - \Delta\alpha_1^0(t_k) \\
&= I_k(t_k, \alpha_1^0(t_k), \delta_2^0(t_k, w(t_k))) - \Delta\alpha_1^0(t_k) \\
&= I_k(t_k, \alpha_1^0(t_k), \delta_2^0(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\
&\leq I_k(t_k, \alpha_1^0(t_k), \alpha_2^0(t_k)) - \Delta\alpha_1(t_k) < 0.
\end{aligned} \tag{32}$$

Finally, if $t_0 = 0$, we can consider (31). By (2), (7) and (8), then

$$\begin{aligned}
z^*(0) - \alpha_1^0(0) =& z^*(T) - (\alpha_1(0) - \|\alpha\|) \\
=& z^*(T) - \alpha_1(0) + \|\alpha\| \\
\geq& z^*(T) - \alpha_1(T) + \|\alpha\| \\
=& z^*(T) - (\alpha_1(T) - \|\alpha\|) \\
=& z^*(T) - \alpha_1^0(T).
\end{aligned}$$

Then, by (31),

$$z^*(0) - \alpha_1^0(0) = z^*(T) - \alpha_1^0(T),$$

and

$$(z^*)'(T) - (\alpha_1)'(T) \leq 0. \tag{33}$$

Then, by (31), (12) and (6), the following contradiction with 33 holds,

$$\begin{aligned}(z^*)'(T) &= f(T, \delta_1^0(T, z^*(T)), \delta_2^0(T, w(T))) + \delta_1^0(T, z^*(T)) - z^*(T)\\ &= f(T, \alpha_1^0(T), \delta_2^0(T, w(T))) + \alpha_1^0(T) - z^*(T)\\ &> f(T, \alpha_1^0(T), \delta_2^0(T, w(T)))\\ &\geq f(T, \alpha_1^0(T), \alpha_2^0(T))\\ &\geq \alpha_1'(T).\end{aligned}$$

Therefore, $z^*(t) \geq \alpha_1^0(t), \forall t \in [0, T]$.

The same arguments can be applied to prove the other inequalities in (28).

□

Example 8. *Consider the following system, for $t \in [0, 1]$,*

$$\begin{cases} z'(t) = a_1 z^3(t) + a_2 w(t) + a_3 t, & a_2 > 0,\\ w'(t) = b_1 w^3(t) + b_2 z(t) + b_3 t, & b_2 > 0, \end{cases} \tag{34}$$

together with the periodic boundary conditions (2), and the impulse conditions

$$\begin{cases} \Delta z(t_k) = S_{\epsilon_1}(c_1 z(t_k) - c_2 w(t_k) + c_3) - 1/2, & c_2 > 0,\\ \Delta w(\tau_l) = S_{\epsilon_2}(-d_1 z(\tau_l) + d_2 w(\tau_l) + d_3) - 1/2, & d_1 > 0, \end{cases} \tag{35}$$

where $S : \mathbb{R} \to \mathbb{R}^+$ is the sigmoid function, defined as

$$S_\epsilon(x) = \frac{1}{1 + e^{-\epsilon x}}, \qquad \epsilon > 0. \tag{36}$$

As a numerical example, we consider the normalized period, $T = 1$, and the parameter set,

$$\begin{array}{cccc} a_1 = -5 & a_2 = 0.1 & a_3 = 2 & \\ b_1 = -0.5 & b_2 = 1 & b_3 = 10 & \\ c_1 = 6 & c_2 = 2 & c_3 = 1 & \epsilon_1 > 0\\ d_1 = 3 & d_2 = 6 & d_3 = 1 & \epsilon_2 > 0 \end{array}$$

and we shall consider impulses at $t_1 = 1/2$, $t_2 = 3/4$ and $\tau_l = 1/3$, $\tau_2 = 2/3$, that is, we shall consider the problem

$$\begin{cases} z'(t) = -5z^3(t) + 0.1w(t) + 2t\\ w'(t) = -0.5w^3(t) + z(t) + 10t \end{cases}, \tag{37}$$

together with the boundary conditions

$$\begin{aligned} z(0) &= z(1) \\ w(0) &= w(1) \end{aligned} \tag{38}$$

and the impulses

$$\left\{ \begin{array}{lr} \Delta x_1(t_1) = S_{\epsilon_1}(6x_1(t_1) - 2x_2(t_1) + 1) - 1/2, & \epsilon_1 > 0 \\ \Delta x_1(t_2) = S_{\epsilon_1}(6x_1(t_2) - 2x_2(t_2) + 1) - 1/2, & \epsilon_1 > 0 \\ \Delta x_2(\tau_1) = S_{\epsilon_2}(-3x_1(\tau_1) + 6x_2(\tau_1) + 1) - 1/2, & \epsilon_2 > 0 \\ \Delta x_2(\tau_2) = S_{\epsilon_2}(-3x_1(\tau_2) + 6x_2(\tau_2) + 1) - 1/2, & \epsilon_2 > 0 \end{array} \right. , \tag{39}$$

It is clear that (37), (38), (39) is a particular case of problem (1), (2), (3), with

$$\left\{ \begin{array}{l} f(t, z(t), w(t)) = -5z^3(t) + 0.1w(t) + 2t \\ g(t, z(t), w(t)) = -0.5w^3(t) + z(t) + 10t \end{array} \right. ,$$

and

$$\left\{ \begin{array}{l} I_1(t_1, x_1(t_1), x_2(t_1)) = S_{\epsilon_1}(6x_1(t_1) - 2x_2(t_1) + 1) - 1/2 \\ I_2(t_2, x_1(t_2), x_2(t_2)) = S_{\epsilon_1}(6x_1(t_2) - 2x_2(t_2) + 1) - 1/2 \\ J_1(\tau_1, x_1(\tau_1), x_2(\tau_1)) = S_{\epsilon_2}(-3x_1(\tau_1) + 6x_2(\tau_1) + 1) - 1/2 \\ J_2(\tau_2, x_1(\tau_2), x_2(\tau_2)) = S_{\epsilon_2}(-3x_1(\tau_2) + 6x_2(\tau_2) + 1) - 1/2 \end{array} \right. .$$

The functions $\alpha_i, \beta_i : [0,1] \to \mathbb{R}$, $i = 1, 2$, given by

$$\begin{aligned} \alpha_1(t) &= t, & \alpha_2(t) &= 2t - 1, \\ \beta_1(t) &= 1 - t, & \beta_2(t) &= 2 - t. \end{aligned}$$

are, respectively, lower and upper solutions of problem (37), (38), (39), according to Definition 4, with

$$\begin{aligned} \alpha_1^0(t) &= t - 1, & \alpha_2^0(t) &= 2t - 2, \\ \beta_1^0(t) &= 2 - t, & \beta_2^0(t) &= 4 - t. \end{aligned}$$

We observe that the inequalities required by Definition 4 are verified on the interval $[0,1]$ for the nonlinearities,

$$1 = \alpha_1'(t) \leq f\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) = -5t^3 + 15t^2 - 12.80t + 4.8,$$

$$2 = \alpha_2'(t) \leq g\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) = -4t^3 + 12t^2 - t + 3,$$

$$-1 = \beta_1'(t) \geq f\left(t, \beta_1^0(t), \beta_2^0(t)\right) = 5t^3 - 30t^2 + 61.90t - 39.6,$$

$$-1 = \beta_2'(t) \geq g\left(t, \beta_1^0(t), \beta_2^0(t)\right) = 0.5t^3 - 6t^2 + 33t + 34.$$

As for the impulses, we notice that, for $\epsilon > 0$, given a function $u(t)$, the following monotonicity relations hold,

$$\begin{aligned} S_\epsilon(u(t)) - 1/2 > 0 \Rightarrow u(t) > 0, \\ S_\epsilon(u(t)) - 1/2 < 0 \Rightarrow u(t) < 0. \end{aligned}$$

Therefore, in order to verify the inequalities of Definition 4, i.e.,

$$\begin{aligned} 0 = \Delta\alpha_1(t_1) > I_1\left(t_1, \alpha_1^0(t_1), \alpha_2^0(t_1)\right), \\ 0 = \Delta\alpha_1(t_2) > I_2\left(t_2, \alpha_1^0(t_2), \alpha_2^0(t_2)\right), \\ 0 = \Delta\alpha_2(\tau_1) > J_1\left(\tau_1, \alpha_1^0(\tau_1), \alpha_2^0(\tau_1)\right), \\ 0 = \Delta\alpha_2(\tau_2) > J_2\left(\tau_2, \alpha_1^0(\tau_2), \alpha_2^0(\tau_2)\right), \end{aligned}$$

$$\begin{aligned} 0 = \Delta\beta_1(t_1) < I_1\left(t_1, \beta_1^0(t_1), \beta_2^0(t_1)\right), \\ 0 = \Delta\beta_1(t_2) < I_2\left(t_2, \beta_1^0(t_2), \beta_2^0(t_2)\right), \\ 0 = \Delta\beta_2(\tau_1) < J_1\left(\tau_1, \beta_1^0(\tau_1), \beta_2^0(\tau_1)\right), \\ 0 = \Delta\beta_2(\tau_2) < J_2\left(\tau_2, \beta_1^0(\tau_2), \beta_2^0(\tau_2)\right), \end{aligned}$$

we only need to show that, for every $\epsilon_i > 0$,

$$\begin{aligned} c_1\alpha_1^0(t_1) - c_2\alpha_2^0(t_1) + c_3 = -2 < 0, \\ c_1\alpha_1^0(t_2) - c_2\alpha_2^0(t_2) + c_3 = -3/2 < 0, \\ -d_1\alpha_1^0(\tau_1) + d_2\alpha_2^0(\tau_1) + d_3 = -10/3 < 0, \\ -d_1\alpha_1^0(\tau_2) + d_2\alpha_2^0(\tau_2) + d_3 = -8/3 < 0, \end{aligned}$$

$$\begin{aligned} c_1\beta_1^0(t_1) - c_2\beta_2^0(t_1) + c_3 = 1/2 > 0, \\ c_1\beta_1^0(t_2) - c_2\beta_2^0(t_2) + c_3 = 1/4 > 0, \\ -d_1\beta_1^0(\tau_1) + d_2\beta_2^0(\tau_1) + d_3 = 2 > 0, \\ -d_1\beta_1^0(\tau_2) + d_2\beta_2^0(\tau_2) + d_3 = 2 > 0. \end{aligned}$$

As all the assumptions of Theorem 6 are verified, then there is at least a non-trivial periodic solution (x_1^*, x_2^*) of problem (37), (38), (39), moreover,

$$\begin{array}{c} t - 1 \leq x_1^*(t) \leq 2 - t, \\ 2t - 2, \leq x_2^*(t) \leq 4 - t, \end{array} , \forall t \in [0, 1],$$

Although $\alpha_i(t)$ and $\beta_i(t)$ are not necessarily well ordered, the order is recovered with the translations, as shown in Figure 1.

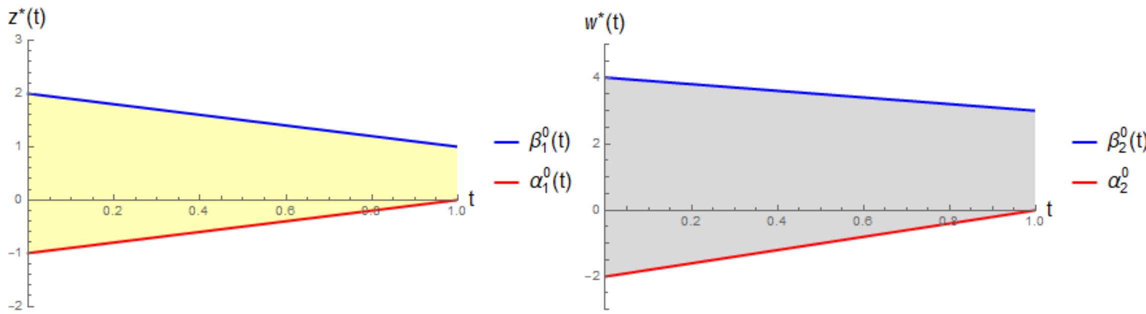


Figure 1: (x_1^*, x_2^*)-solution localization, in $[0,1]$.

4 Ordered lower and upper solutions and sign of nonlinearities

In Section 3, Theorem 6 localizes an existing solution of problem (1), (2), (3) in a strip bounded by upper and lower solutions with translations, $\alpha_i^0(t) \leq 0$ and $\beta_i^0(t) \geq 0$ (see Definition 4). However, the differential inequalities (6) and (10), together with the boundary conditions (8) and (11), require the change of sign in the nonlinearities.

In this section, it is required less regularity to lower and upper solutions. However, it is necessary to impose an order relation between them, $\alpha_i(t) \leq \beta_i(t), i = 1, 2$, in order to define a method that does not require the sign of the nonlinearities to change.

Definition 9. *Consider the PC^1-functions $\alpha_i, \beta_i : [0,T] \to \mathbb{R}$, $i = 1, 2$. The functions (α_1, α_2) are lower solutions of the periodic problem (1), (2), (3) if*

$$\begin{cases} \alpha_1'(t) \leq f(t, \alpha_1(t), \alpha_2(t)) \\ \alpha_2'(t) \leq g(t, \alpha_1(t), \alpha_2(t)) \end{cases}, \tag{40}$$

with

$$\alpha_i(0) \leq \alpha_i(T), \ i = 1, 2, \tag{41}$$

and

$$\begin{cases} \Delta\alpha_1(t_k) > I_k(t_k, \alpha_1(t_k), \alpha_2(t_k)) \\ \Delta\alpha_2(\tau_l) > J_l(\tau_l, \alpha_1(\tau_l), \alpha_2(\tau_l)) \end{cases}. \tag{42}$$

The functions (β_1, β_2) are upper solutions of the periodic problem (1), (2), (3) if

$$\begin{cases} \beta_1'(t) \geq f(t, \beta_1(t), \beta_2(t)) \\ \beta_2'(t) \geq g(t, \beta_1(t), \beta_2(t)) \end{cases}, \tag{43}$$

with

$$\beta_i(0) \geq \beta_i(T),\ \ i = 1, 2,$$

and

$$\begin{cases} \Delta\beta_1(t_k) < I_k(t_k, \beta_1(t_k), \beta_2(t_k)) \\ \Delta\beta_2(\tau_l) < J_l(\tau_l, \beta_1(\tau_l), \beta_2(\tau_l)) \end{cases}.$$

Theorem 10. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (1), (2), (3), respectively, according to Definition 9, such that*

$$\alpha_i(t) \leq \beta_i(t), \quad i = 1, 2, \quad \forall t \in [0, T],$$

Assume that f, g are L^1-Carathéodory functions on the set

$$\{(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2 : \alpha_1(t) \leq y_1 \leq \beta_1(t),\ \alpha_2(t) \leq y_2 \leq \beta_2(t)\},$$

with

$$f(t, y_1, \alpha_2(t)) \leq f(t, y_1, y_2) \leq f(t, y_1, \beta_2(t)), \tag{44}$$

for fixed $t \in [0, T]$, $y_1 \in \mathbb{R}$, and $\alpha_2(t) \leq y_2 \leq \beta_2(t)$, and with

$$g(t, \alpha_1(t), y_2) \leq g(t, y_1, y_2) \leq g(t, \beta_1(t), y_2),$$

for fixed $t \in [0, T]$, $y_2 \in \mathbb{R}$, and $\alpha_1(t) \leq y_1 \leq \beta_1(t)$.
Assume that the impulse functions I_k and J_l verify

$$I_k(t_k, y_1, \alpha_2) \geq I_k(t_k, y_1, y_2) \geq I_k(t_k, y_1, \beta_2), \tag{45}$$

for some fixed $k \in \{1, \ldots n-1\}$, $y_1 \in \mathbb{R}$, and $\alpha_2(t) \leq y_2 \leq \beta_2(t)$, and that

$$\sum_{k=1}^{n-1} I_k(t_k, y_1, y_2) = 0.$$

Assume that the impulse functions J_l verify

$$J_l(\tau_l, \alpha_1, y_2) \geq J_l(\tau_l, y_1, y_2) \geq J_l(\tau_l, \beta_1, y_2),$$

for some fixed $l \in \{1, \ldots m-1\}$, $y_2 \in \mathbb{R}$ and $\alpha_1(t) \leq y_1 \leq \beta_1(t)$, and that

$$\sum_{l=1}^{m-1} J_l(\tau_l, y_1, y_2) = 0.$$

Then, problem (1), (2), (3) has, at least, a solution $(z, w) \in \left(PC^1[0, T]\right)^2$ such that

$$\begin{aligned} &\alpha_1(t) \leq z(t) \leq \beta_1(t), \\ &\alpha_2(t) \leq w(t) \leq \beta_2(t), \forall t \in [0, T]. \end{aligned}$$

Proof. For $i = 1, 2$, define the continuous functions $\delta_i : [0, T] \times \mathbb{R}$, given by

$$\delta_1(t, z) = \begin{cases} \alpha_1(t) & \text{if} \quad z < \alpha_1(t) \\ z & \text{if} \quad \alpha_1(t) \le z \le \beta_1(t) \\ \beta_1(t) & \text{if} \quad z > \beta_1(t) \end{cases}$$

and (46)

$$\delta_2(t, w) = \begin{cases} \alpha_2(t) & \text{if} \quad w < \alpha_2(t) \\ w & \text{if} \quad \alpha_2(t) \le w \le \beta_2(t) \\ \beta_2(t) & \text{if} \quad w > \beta_2(t) \end{cases},$$

and consider the modified problem, composed by

$$\begin{cases} z'(t) + z(t) = f\left(t, \delta_1(t, z(t)), \delta_2(t, w(t))\right) + \delta_1(t, z(t)), & t \ne t_k, \\ w'(t) + w(t) = g\left(t, \delta_1(t, z(t)), \delta_2(t, w(t))\right) + \delta_2(t, w(t)), & t \ne \tau_l, \end{cases} \tag{47}$$

together with the boundary conditions (2) and the truncated impulse conditions,

$$\begin{array}{c} \Delta z(t_k) = I_k(t_k, \delta_1(t_k, z(t_k)), \delta_2(t_k, w(t_k))), \\ \Delta w(\tau_l) = J_l(\tau_l, \delta_1(\tau_l, z(\tau_l)), \delta_2(\tau_l, w(\tau_l))). \end{array} \tag{48}$$

Problem (47), (2), (48) can be addressed using the arguments for the proof of Theorem 6. The corresponding operator has a fixed point $(\bar{z}, \bar{w})$. The delicate step, however, is to show that $(\bar{z}, \bar{w})$ is a solution of (47), (2), (48), such that

$$\alpha_1(t) \le \bar{z}(t) \le \beta_1(t), \quad \alpha_2(t) \le \bar{w}(t) \le \beta_2(t), \quad \forall t \in [0, T]. \tag{49}$$

In the first inequality, suppose, by contradiction, that there exists $t \in [0, T]$ such that

$$\bar{z}(t) < \alpha_1(t),$$

and define

$$\inf_{t \in [0,T]} (\bar{z} - \alpha_1)(t) := \bar{z}(t_0) - \alpha_1(t_0) < 0. \tag{50}$$

We now consider different cases for t_0.

If $t_0 \in]t_k, t_{k+1}[$ for some $k \in \{0, \dots, n\}$, then, by (50)

$$(\bar{z} - \alpha_1)'(t_0) = 0. \tag{51}$$

However, by (50), (44) and (40),

$$\begin{aligned} (\bar{z})'(t_0) &= f(t_0, \delta_1(t_0, \bar{z}(t_0)), \delta_2(t_0, w(t_0))) + \delta_1(t_0, \bar{z}(t_0)) - \bar{z}(t_0) \\ &= f(t_0, \alpha_1(t_0), \delta_2(t_0, w(t_0))) + \alpha_1(t_0) - \bar{z}(t_0) \\ &> f(t_0, \alpha_1(t_0), \delta_2(t_0, w(t_0))) \\ &\ge f(t_0, \alpha_1(t_0), \alpha_2(t_0)) \\ &\ge {\alpha_1}'(t_0), \end{aligned}$$

contradicting (51).

If $t_0 = t_k$ for some $k \in \{1, \dots, n-1\}$, then either $t_0 = t_k^+$ or $t_0 = t_k^-$. If $t_0 = t_k^+$, then $t_0 \in]t_k, t_{k+1}[$, so the previous reasoning must be applied. If, instead, $t_0 = t_k^-$, then we consider

$$\min_{t\in[0,T]} (\bar{z} - \alpha_1)(t) := \bar{z}(t_0) - \alpha_1(t_0) < 0, \tag{52}$$

and thus, the impulse on t_k is necessarily non-negative. By (52), (48), (45) and (42) the following contradiction holds,

$$\begin{aligned} 0 &\le \Delta(\bar{z} - \alpha_1)(t_k) \\ &= I_k(t_k, \delta_1(t_k, \bar{z}(t_k)), \delta_2(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\ &= I_k(t_k, \alpha_1(t_k), \delta_2(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\ &= I_k(t_k, \alpha_1(t_k), \delta_2(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\ &\le I_k(t_k, \alpha_1(t_k), \alpha_2(t_k)) - \Delta\alpha_1(t_k) < 0. \end{aligned} \tag{53}$$

Finally, if $t_0 = 0$, we can consider (52). By (2) and (41), then

$$\bar{z}(0) - \alpha_1(0) = \bar{z}(T) - \alpha_1(0) \ge \bar{z}(T) - \alpha_1(T).$$

Then, by (52),

$$\bar{z}(0) - \alpha_1(0) = \bar{z}(T) - \alpha_1(T),$$

and

$$(\bar{z})'(T) - (\alpha_1)'(T) \le 0. \tag{54}$$

Then, by (52), (44) and (40), the following contradiction with 54 holds,

$$\begin{aligned} (\bar{z})'(T) &= f(T, \delta_1(T, \bar{z}(T)), \delta_2(T, w(T))) + \delta_1(T, \bar{z}(T)) - \bar{z}(T) \\ &= f(T, \alpha_1(T), \delta_2(T, w(T))) + \alpha_1(T) - \bar{z}(T) \\ &> f(T, \alpha_1(T), \delta_2(T, w(T))) \\ &\ge f(T, \alpha_1(T), \alpha_2(T)) \\ &\ge \alpha_1'(T). \end{aligned}$$

Therefore, $\bar{z}(t) \ge \alpha_1(t), \forall t \in [0, T]$.

The same arguments can be applied to prove the other inequalities in (49).

□

5 Application to a system of two strongly connected Wilson-Cowan neural oscillators

In the work [23] the authors study the dynamics, synchronization and control of chaos in a system of strongly connected Wilson-Cowan neural oscillators, and present the respective mathematical model,

$$\begin{gathered} x_i'(t) = -\phi_i x_i(t) + S\Big(\rho_i + \sum_{j=1}^{n} \theta_{ij} x_j\Big), \\ x_i \in \mathbb{R}, \quad i, j = 1, \dots, n, \quad \phi_i \geq 0, \end{gathered} \tag{55}$$

where ϕ_i is the internal decay rate of the i^{th} neuron, ρ_i, θ_{ij} are the input parameters of the activation function $S : \mathbb{R} \to \mathbb{R}^+$ of neuron i, acting on coupled neuron j. From the list of most commonly used activation functions suggested in [24], we choose $S(x) = \tanh(x)$.

Motivated by these works, we adapted (55) to a system of two neurons with the following form,

$$\left\{ \begin{array}{l} x_1'(t) = -a_1 x_1(t) + e^{a_2 x_2(t)} + a_3 t \\ x_2'(t) = -b_1 x_2(t) + e^{b_2 x_1(t)} + b_3 t \end{array} \right. , \tag{56}$$

with $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$, together with the periodic boundary conditions

$$x_i(0) = x_i(T), \qquad i = 1, 2, \tag{57}$$

and the impulses given by

$$\left\{ \begin{array}{l} \Delta x_1(t_k) = \tanh(c_1 x_1(t_k) - c_2 x_2(t_k) + c_3) \\ \Delta x_2(\tau_l) = \tanh(-d_1 x_1(\tau_l) + d_2 x_2(\tau_l) + d_3) \end{array} \right. , \tag{58}$$

with $c_2 > 0$, $d_1 > 0$.

The quantity x_i denotes the activation state of the i^{th} neuron, a_1, b_1 the respective internal decay rate, a_2, b_2 are the weights of the non-linear components, and $a_3 t, b_3 t$ are the time-dependent external inputs, with $a_3, b_3 \in \mathbb{R}$.

The quantities Δx_i denote the instantaneous jumps of the i^{th} neuron at the respective instants of impulse, modelled by the function S, together with the weights of each variables, where $c_1, d_2, c_3, d_3 \in \mathbb{R}$.

As a numerical example, we consider the normalized period, $T = 1$, and the parameter set,

$$\begin{array}{ccc} a_1 = 0.1 & a_2 = 0.2 & a_3 = -0.5 \\ b_1 = 0.1 & b_2 = 0.1 & b_3 = 0.1 \\ c_1 = 5 & c_2 = 5 & c_3 = -3 \\ d_1 = 0.5 & d_2 = 1.5 & d_3 = -5 \end{array}$$

and we shall consider impulses at $t_1 = 1/3$, $t_2 = 2/3$, and $\tau_l = 1/4$, $\tau_2 = 1/2$, $\tau_3 = 3/4$. In short, we consider the numerical problem

$$\begin{cases} x_1'(t) = -0.1x_1(t) + e^{0.2x_2(t)} - 0.5t \\ x_2'(t) = -0.1x_2(t) + e^{0.1x_1(t)} + 0.1t \end{cases}, \tag{59}$$

together with the boundary conditions

$$x_i(0) = x_i(1), \tag{60}$$

and the impulses

$$\begin{cases} \Delta x_1(t_1) = \tanh(5x_1(t_1) - 5x_2(t_1) - 3) \\ \Delta x_1(t_2) = \tanh(5x_1(t_2) - 5x_2(t_2) - 3) \\ \Delta x_2(\tau_1) = \tanh(-0.5x_1(\tau_1) + 1.5x_2(\tau_1) - 5) \\ \Delta x_2(\tau_2) = \tanh(-0.5x_1(\tau_2) + 1.5x_2(\tau_2) - 5) \\ \Delta x_2(\tau_3) = \tanh(-0.5x_1(\tau_3) + 1.5x_2(\tau_3) - 5). \end{cases} \tag{61}$$

It is clear that (59), (60), (61) is a particular case of problem (1), (2), (3), with

$$\begin{cases} f(t, x_1(t), x_2(t)) = -0.1x_1(t) + e^{0.2x_2(t)} - 0.5t \\ g(t, x_1(t), x_2(t)) = -0.1x_2(t) + e^{0.1x_1(t)} + 0.1t \end{cases},$$

and

$$\begin{cases} I_1(t_1, x_1(t_1), x_2(t_1)) = \tanh(5x_1(t_1) - 5x_2(t_1) - 3) \\ I_2(t_2, x_1(t_2), x_2(t_2)) = \tanh(5x_1(t_2) - 5x_2(t_2) - 3) \\ J_1(\tau_1, x_1(\tau_1), x_2(\tau_1)) = \tanh(-0.5x_1(\tau_1) + 1.5x_2(\tau_1) - 5) \\ J_2(\tau_2, x_1(\tau_2), x_2(\tau_2)) = \tanh(-0.5x_1(\tau_2) + 1.5x_2(\tau_2) - 5) \\ J_3(\tau_3, x_1(\tau_3), x_2(\tau_3)) = \tanh(-0.5x_1(\tau_3) + 1.5x_2(\tau_3) - 5). \end{cases}$$

The functions $\alpha_i, \beta_i : [0,1] \to \mathbb{R}$, $i = 1, 2$, given by

$$\alpha_1(t) = \begin{cases} -t + 1/3, & 0 \le t \le 1/3 \\ -t + 1, & 1/3 < t \le 2/3 \\ -t + 5/3, & 2/3 < t \le 1 \end{cases} \qquad \beta_1(t) = \begin{cases} 4t + 5, & 0 \le t \le 1/3 \\ 4t + 3, & 1/3 < t \le 2/3 \\ 4t + 1, & 2/3 < t \le 1 \end{cases},$$

$$\alpha_2(t) = \begin{cases} -t + 1/4, & 0 \le t \le 1/4 \\ -t + 3/4, & 1/4 < t \le 1/2 \\ -t + 5/4, & 1/2 < t \le 3/4 \\ -t + 7/4, & 3/4 < t \le 1 \end{cases}, \qquad \beta_2(t) = \begin{cases} 3t + 5, & 0 \le t \le 1/4 \\ 3t + 4, & 1/4 < t \le 1/2 \\ 3t + 3, & 1/2 < t \le 3/4 \\ 3t + 2, & 3/4 < t \le 1 \end{cases},$$

are, respectively, lower and upper solutions of problem (59), (60), (61), according to Definition 9. In fact, the four differential inequalities (equations (40), (43)) are verified in the interval $[0, 1]$, as shown in Figure 3.

As for the impulses, we verify the inequalities of Definition 9,

$$\begin{aligned}
2/3 = \Delta\alpha_1(t_1) &> I_1\left(t_1, \alpha_1(t_1), \alpha_2(t_1)\right) = -0.999923, \\
2/3 = \Delta\alpha_1(t_2) &> I_2\left(t_2, \alpha_1(t_2), \alpha_2(t_2)\right) = -0.999593, \\
1/2 = \Delta\alpha_2(\tau_1) &> J_1\left(\tau_1, \alpha_1(\tau_1), \alpha_2(\tau_1)\right) = -0.999916, \\
1/2 = \Delta\alpha_2(\tau_2) &> J_2\left(\tau_2, \alpha_1(\tau_2), \alpha_2(\tau_2)\right) = -0.999883, \\
1/2 = \Delta\alpha_2(\tau_3) &> J_3\left(\tau_3, \alpha_1(\tau_3), \alpha_2(\tau_3)\right) = -0.999837,
\end{aligned}$$

$$\begin{aligned}
-2 = \Delta\beta_1(t_1) &< I_1\left(t_1, \beta_1(t_1), \beta_2(t_1)\right) = 0.998694, \\
-2 = \Delta\beta_1(t_2) &< I_2\left(t_2, \beta_1(t_2), \beta_2(t_2)\right) = 0.321513, \\
-1 = \Delta\beta_2(\tau_1) &< J_1\left(\tau_1, \beta_1(\tau_1), \beta_2(\tau_1)\right) = 0.5546, \\
-1 = \Delta\beta_2(\tau_2) &< J_2\left(\tau_2, \beta_1(\tau_2), \beta_2(\tau_2)\right) = 0.635149, \\
-1 = \Delta\beta_2(\tau_3) &< J_3\left(\tau_3, \beta_1(\tau_3), \beta_2(\tau_3)\right) = 0.703906,
\end{aligned}$$

As all the assumptions of Theorem 10 are verified, then there is at least a non-trivial non-negative periodic solution (x_1^*, x_2^*) of problem (59), (60), (61), moreover,

$$\begin{array}{l} \alpha_1(t) \leq x_1^*(t) \leq \beta_1(t), \\ \alpha_2(t), \leq x_2^*(t) \leq \beta_2(t), \end{array} , \forall t \in [0,1],$$

as shown in Figure 2.

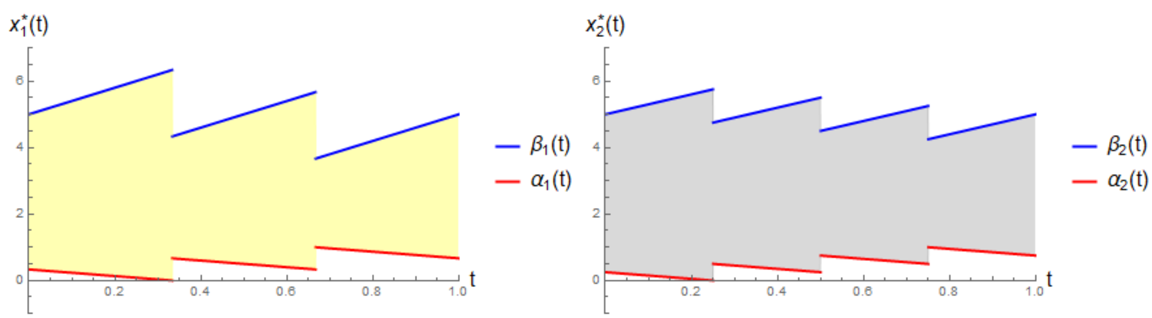
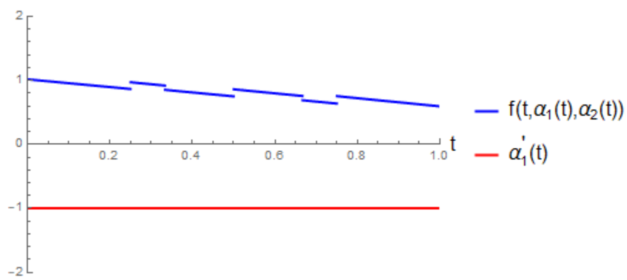


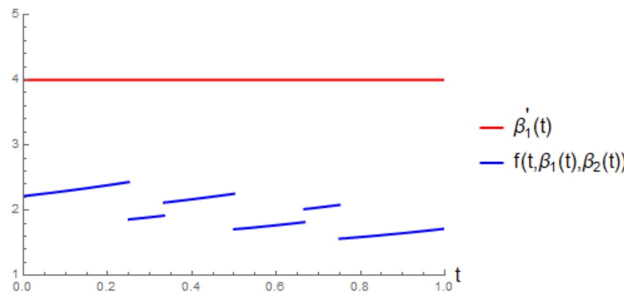
Figure 2: (x_1^*, x_2^*)-solution localization, in $[0,1]$.

6 Conclusions

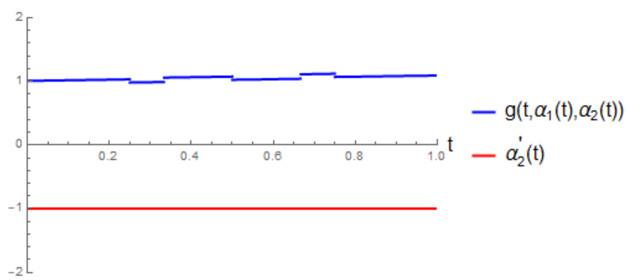
In the literature, the lower and upper solutions are typically well-ordered, that is, the lower function lies below the upper one, and some monotonicity conditions



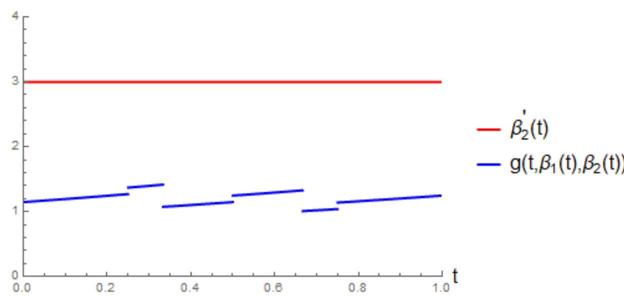
$\alpha_1'(t) \leq f(t, \alpha_1(t), \alpha_2(t))$



$\beta_1'(t) \geq f(t, \beta_1(t), \beta_2(t))$



$\alpha_2'(t) \leq g(t, \alpha_1(t), \alpha_2(t))$



$\beta_2'(t) \geq g(t, \beta_1(t), \beta_2(t))$

Figure 3: Relation between the nonlinearities and the lower and upper solutions.

are imposed to the nonlinearities. In this work, we overcome both restrictions, in order to ease the search for functions that verify the required properties of lower and upper solutions. We overcome the first restriction by applying a translation such that, regardless of the order relation between the lower and upper solutions, the shifted functions are always well-ordered.

The existence result of Section 3 obligates the nonlinearities to change sign in the interval $[0, T]$. We overcome that restriction by requiring less regularity to the lower and upper functions in Section 4, but recovering a relation of order between them.

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