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RESEARCH ARTICLE

Wave propagation for the three-dimensional isentropic compressible Navier-Stokes/Allen-Cahn system

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Abstract

In this paper, we study the Cauchy problem for the three-dimensional isentropic compressible Navier-Stokes/Allen-Cahn system, which describes the phase transitions in two-component patterns interacting with a compressible fluid. We establish the existence and space-time pointwise behaviors of global solutions to this non-conserved system. In order to control the source term consisting of the phase variable, we make use of the Green's function and space-time weighted estimates to prove that the phase variable only contains the diffusion wave whose amplitude decays exponentially in time, so as to show that the density and momentum of the fluid obey the generalized Huygens' principle.

KEYWORDS

Navier-Stokes/Allen-Cahn system, Cauchy problem, Green's function, pointwise estimates.

1 | INTRODUCTION

In this paper, we investigate the isentropic compressible Navier-Stokes/Allen-Cahn (NSAC) system, which was proposed by Blesgen[1] to describe the interface phase transitions in a fluid mixture, given by

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P = \operatorname{div}(2\tilde{\nu}\mathbb{D}(u) + \tilde{\eta}\operatorname{div}u\mathrm{I}_3) - \epsilon\operatorname{div}\left(\nabla\chi\otimes\nabla\chi - \frac{1}{2}|\nabla\chi|^2\mathrm{I}_3\right), \\ \partial_t(\rho\chi) + \operatorname{div}(\rho u\chi) = -\tilde{\mu}, \\ \rho\tilde{\mu} = \dfrac{\rho}{\epsilon}(\chi^3-\chi) - \epsilon\Delta\chi, \end{cases} \tag{1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}_+$. The total density, the velocity, and the phase field of this diffusion interface model are denoted by $\rho=\rho(x,t)$, $u=u(x,t)=(u_1,u_2,u_3)(x,t)$ and $\chi=\chi(x,t)$, respectively. The constant $\tilde{\mu}$ represents the chemical potential, and $\epsilon>0$ is the interface thickness between the phases. The fluid pressure $P=P(\rho)$ is assumed to be a smooth function of ρ satisfying for any $\rho>0$ that $P'(\rho)>0$, and $\mathbb{D}u=\frac{1}{2}\nabla u+\frac{1}{2}(\nabla u)^\top$ is the deformation tensor. The symbol I_3 denotes the 3×3 unit matrix. The constant viscosity coefficients $\tilde{\nu}$ and $\tilde{\eta}$ satisfy

$$\tilde{\nu}>0,\quad 2\tilde{\nu}+3\tilde{\eta}\geq 0. \tag{2}$$

We impose (1) with the following initial conditions

$$(\rho,u,\chi)(x,0)=(\rho_0,u_0,\chi_0)(x), \tag{3}$$

and the far-field states

$$\lim_{|x|\to+\infty}(\rho_0,u_0)(x)=(\bar\rho,0),\quad \lim_{|x|\to+\infty}|\chi_0(x)|=1, \tag{4}$$

where $\bar\rho>0$ is a positive constant.

Regarding the compressible NSAC system, there is significant progress on the global existence of solutions and related topics, such as the dynamical behaviors of solutions. More precisely, considering the one-dimensional case, Ding-Li-Luo[10] obtained the global solutions for the initial boundary value problem with positive density. Ding-Li-Tang[11] established the global existence of strong solutions to the NSAC system with free boundary. Chen et al.[5,6] obtained the global strong solutions for the non-isentropic NSAC system with degenerate heat-conductivity, and Yan-Ding-Li[21] proved the global existence of strong solutions when the phase variable is viscosity-dependent. As for the case that the initial density contains vacuum, Li et al.[16] obtained the existence of global weak solutions, and Chen-Guo[2] established the global existence of classical solutions. Chen-Zhu[4] stated the blow-up criterion and obtained the global existence of strong solutions to the initial boundary value problem. Besides, Luo-Yin-Zhu[19,20] investigated the nonlinear stability of the rarefaction wave and the composite wave consisting of two rarefaction waves and a viscous contact wave to the Cauchy problem. When considering the multi-dimensional case, Feireisl et al.[14] first established the global existence of weak solutions in a bounded domain for the adiabatic exponent γ of pressure satisfying $\gamma>6$. Later, Chen-Wen-Zhu[3] extended Feireisl's result to $\gamma>2$. Kotschote[15] proved the local existence and uniqueness of strong solutions to the non-isentropic NSAC system with general initial data in a bounded domain. On the other hand, Zhao[23] and Chen-Hong-Shi[7] obtained the global well-posedness and time decay rates for Cauchy problem with different far-field states of the phase variable, provided that the initial density is bounded and away from zero. Chen-Tang[9] and Chen-Li-Tang[8] investigated the global existence and optimal time decay rates of the three-dimensional compressible NSAC system for the isentropic and non-isentropic cases respectively. Moreover, Fei et al.[13] considered the sharp interface limit of a matrix-valued Allen-Cahn equation, and showed that the sharp interface system is a two-phase flow system where the interface evolves according to the motion by mean curvature. However, as far as we know, only a few results are available for the space-time pointwise behaviors of solutions to the immiscible two-phase flow.

The motivation of this paper is to obtain the global existence and space-time pointwise behaviors of classical solutions to the isentropic compressible NSAC system so as to observe the influence of phase transition phenomena on the compressible fluid, and understand the wave propagation of the immiscible two-phase flow. It should be pointed out that the phase field variable χ is introduced to identify the two components of the mixture, and in this paper we consider the case that χ has the constant equilibrium states $\chi=\pm1$, which describes the phenomenon of phase transition in the immiscible two-phase flow. Under this assumption, we define a new variable $\varphi=\chi^2$ and rewrite the system (1)–(4) into the density-momentum formulation with $m=\rho u$ below

$$\begin{cases}\partial_t\rho+\mathrm{div}m=0,\\ \partial_t m+\mathrm{div}\left(\dfrac{m\otimes m}{\rho}\right)+\nabla P=\tilde\nu\Delta\left(\dfrac{m}{\rho}\right)+(\tilde\nu+\tilde\eta)\nabla\mathrm{div}\left(\dfrac{m}{\rho}\right)-\epsilon\mathrm{div}\left(\dfrac{2\nabla\varphi\otimes\nabla\varphi-|\nabla\varphi|^2\mathrm{I}_3}{8\varphi}\right),\\ \partial_t\varphi+\dfrac{m\cdot\nabla\varphi}{\rho}=-\dfrac{2\varphi(\varphi-1)}{\rho\epsilon}+\dfrac{\epsilon\Delta\varphi}{\rho^2}-\dfrac{\epsilon|\nabla\varphi|^2}{2\rho^2\varphi},\end{cases} \tag{5}$$

subjected to the initial data

$$(\rho,m,\varphi)(x,0)=(\rho_0,m_0,\varphi_0)(x):=(\rho_0,\rho_0u_0,\chi_0^2), \tag{6}$$

and the far-field states

$$\lim_{|x|\to+\infty}(\rho_0,m_0,\varphi_0)(x)=(\bar\rho,0,1). \tag{7}$$

From now on, we mainly study the Cauchy problem (5)–(7), and establish the space-time pointwise behaviors of global solutions to this problem as follows.

Theorem 1. *Suppose that the initial data (ρ_0,m_0,φ_0) satisfies*

$$\rho_0-\bar\rho\in H^4(\mathbb{R}^3)\cap L^1(\mathbb{R}^3),\ m_0\in H^4(\mathbb{R}^3)\cap L^1(\mathbb{R}^3),\ \varphi_0-1\in H^5(\mathbb{R}^3), \tag{8}$$

and there exists a small positive constant $\varepsilon_0>0$ such that

$$\varepsilon_0:=\|(\rho_0-\bar\rho,m_0)\|_{H^4}+\|\varphi_0-1\|_{H^5}. \tag{9}$$

Then the Cauchy problem (5)–(7) *admits a unique global classical solution* (ρ, m, φ) *satisfying*

$$\|(\rho-\bar{\rho}, m)(t)\|_{H^4} + \|(\varphi-1)(t)\|_{H^5} \leq C\varepsilon_0, \quad t>0, \tag{10}$$

and it holds for $0 \leq |k| \leq 4$ *that*

$$\|D_x^k(\rho-\bar{\rho}, m)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{|k|}{2}}, \ \|(\varphi-1)(t)\|_{H^5} \leq Ce^{-\frac{t}{\epsilon\bar{\rho}}}, \quad t>0. \tag{11}$$

If the initial data (ρ_0, m_0, φ_0) *further satisfies*

$$\begin{aligned} |D^\alpha(\rho_0-\bar{\rho}, m_0)(x)| &\leq \varepsilon_0(1+|x|^2)^{-r_1}, \quad r_1 > \frac{21}{10}, \\ |D^\beta(\varphi_0-1)(x)| &\leq \varepsilon_0(1+|x|^2)^{-r_2}, \quad\quad r_2 > \frac{3}{2}, \end{aligned} \tag{12}$$

for $|\alpha| \leq 1$, $|\beta| \leq 2$, *then the solution* (ρ, m, φ) *has the space-time pointwise behaviors*

$$\begin{aligned} |D_x^\alpha(\rho-\bar{\rho})(x,t)| &\leq C\varepsilon_0(1+t)^{-\frac{4+|\alpha|}{2}} \left\{ \left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} \right\}, \\ |D_x^\alpha m(x,t)| &\leq C\varepsilon_0(1+t)^{-\frac{3+|\alpha|}{2}} \left\{ (1+t)^{-\frac{1}{2}} \left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} \right\}, \\ |D_x^\beta(\varphi-1)(x,t)| &\leq C\varepsilon_0 e^{-\frac{2t}{\epsilon\bar{\rho}}} \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+, \end{aligned} \tag{13}$$

where $c_0 = \sqrt{P'(\bar{\rho})}$ *and* $C > 0$ *is a positive constant independent of* (x, t).

In addition, we can derive the following detailed descriptions of the global solution (ρ, m, φ) and its L^p $(p > 1)$ time decay rates.

Corollary 1. *Let* $(\bar{\rho}, \bar{m}, \bar{\varphi})(x, t)$ *be the solution to the linearized system of* (5). *Under the assumptions in Theorem* 1, *there exists a positive constant* $C > 0$ *independent of time such that*

$$\|(\rho-\bar{\rho}, m-\bar{m})(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}}, \ \|(\varphi-\bar{\varphi})(t)\|_{L^2} \leq Ce^{-\frac{t}{\epsilon\bar{\rho}}}. \tag{14}$$

Furthermore, the following L^p*-norm estimates hold*

$$\|(\rho-\bar{\rho})(t)\|_{L^p} \leq C(1+t)^{-(2-\frac{5}{2p})}, \quad 1<p\leq+\infty, \tag{15}$$

$$\|m(t)\|_{L^p} \leq \begin{cases} C(1+t)^{-(2-\frac{5}{2p})}, & 1<p<2, \\ C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, & 2\leq p\leq+\infty, \end{cases} \tag{16}$$

$$\|(\varphi-1)(t)\|_{L^p} \leq Ce^{-\frac{t}{\epsilon\bar{\rho}}}, \quad 1<p\leq+\infty. \tag{17}$$

Remark 1. It should be noted that the estimate (13) in Theorem 1 implies that the density and momentum of the compressible NSAC system have the same space-time pointwise behaviors as the compressible Navier-Stokes system in[17,18]. Thanks to the damping structure of the phase variable derived from the constant equilibrium states, we are able to obtain the exponential time decay rates of the phase variable, which will not impact the generalized Huygens' principle of the mixture fluid. Moreover, the estimate (13) shows that the phase variable contains the diffusion wave only.

Remark 2. We mention that the global existence and optimal L^2-norm time decay rates of classical solutions have been proved by Chen-Tang in[9] and here we only restate the results (10)–(11) in terms of the density-momentum formulation. Indeed, we extend the L^2-norm time decay rates obtained in[8,9] to the L^p-norm $(1 < p \leq +\infty)$ time decay rates (15)–(17) of the global solution.

Throughout this paper, we denote by C a general positive constant that may vary at different formulas. We use the standard notations L^p and $W^{k,p}$ to denote the usual Lebesgue and Sobolev space on $\mathbb{R}^3$, with the norm $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$ respectively. In particular, when $p = 2$, we denote $W^{k,2}$ by H^k with the norm $\|\cdot\|_{H^k}$.

The rest of this paper is organized as follows. In Section 2, we reformulate the system (5) and then obtain the pointwise estimates of Green's function for the reformulated system. In Section 3, we establish the space-time decay rates of solutions to the Cauchy problem (5)–(7), and finally give the proof of main theorem.

2 | POINTWISE ESTIMATES OF GREEN'S FUNCTION

In this section, motivated by[9,18,22], we first define

$$\nu = \frac{\tilde{\nu}}{\bar{\rho}},\ \eta = \frac{\tilde{\eta}}{\bar{\rho}},\ a = \frac{2}{\epsilon\bar{\rho}},\ b = \frac{\epsilon}{\bar{\rho}^2},\ c_0 = \sqrt{P'(\bar{\rho})}, \tag{18}$$

and introduce the new variables

$$n = \frac{\rho-\bar{\rho}}{\bar{\rho}},\ \ w = \frac{m}{c_0\bar{\rho}},\ \ \phi = \varphi - 1. \tag{19}$$

Then the system (5) can be rewritten as

$$\begin{cases} \partial_t n + c_0 \mathrm{div} w = 0, \\ \partial_t w + c_0 \nabla n = \nu\Delta w + (\nu+\eta)\nabla \mathrm{div} w + F_1, \\ \partial_t \phi + a\phi = b\Delta\phi + F_2, \end{cases} \tag{20}$$

with the following initial data

$$(n, w, \phi)(x, 0) = (n_0, w_0, \phi_0)(x) := \left(\frac{\rho_0-\bar{\rho}}{\bar{\rho}}, \frac{m_0}{c_0\bar{\rho}}, \varphi_0 - 1\right), \tag{21}$$

and the nonlinear term F_i $(i = 1, 2)$ satisfies

$$\begin{aligned} F_1 =& - c_0 \mathrm{div}\left(\frac{w\otimes w}{1+n}\right) + c_0\nabla n - \frac{\nabla P(\bar{\rho}(1+n))}{c_0\bar{\rho}} - \nu\Delta\left(\frac{nw}{1+n}\right) - (\nu+\eta)\nabla\mathrm{div}\left(\frac{nw}{1+n}\right) \\ & - \frac{\epsilon}{c_0\bar{\rho}}\mathrm{div}\left(\frac{2\nabla\phi\otimes\nabla\phi - |\nabla\phi|^2 \mathrm{I}_3}{8(1+\phi)}\right), \\ F_2 =& - \frac{c_0 w\cdot\nabla\phi}{1+n} + \frac{a\phi(n-\phi)}{1+n} - \frac{bn(n+2)\Delta\phi}{(1+n)^2} - \frac{b|\nabla\phi|^2}{2(1+n)^2(1+\phi)}. \end{aligned} \tag{22}$$

Set $U = (n, w, \phi)^\top$, $U_0 = (n_0, w_0, \phi_0)^\top$, $F = (0, F_1, F_2)^\top$ and

$$\mathcal{L} = \begin{pmatrix} 0 & -c_0\mathrm{div} & 0 \\ -c_0\nabla & \nu\Delta + (\nu+\eta)\nabla\mathrm{div} & 0 \\ 0 & 0 & -a + b\Delta \end{pmatrix}, \tag{23}$$

the Cauchy problem (20)–(22) can be reformulated into the vector form

$$\begin{cases} \partial_t U - \mathcal{L}U = F, \\ U|_{t=0} = U_0. \end{cases} \tag{24}$$

Let us introduce a semigroup generated by $\mathcal{L}$. For $U \in L^2$, we set

$$S(t)U = \mathfrak{F}^{-1}\left(e^{t\hat{\mathcal{L}}(\xi)}\hat{U}(\xi, t)\right) = G(\cdot, t) * U(\cdot, t), \tag{25}$$

where $\mathfrak{F}^{-1}$ represents the inverse Fourier transform, $\hat{\mathcal{L}}(\xi)$ is a linear operator satisfying

$$\hat{\mathcal{L}}(\xi) = \begin{pmatrix} 0 & -ic_0\xi & 0 \\ -ic_0\xi^\top & -\nu|\xi|^2\mathrm{I}_3 - (\nu+\eta)\xi^\top\xi & 0 \\ 0 & 0 & -a - b|\xi|^2 \end{pmatrix}_{5\times 5}, \tag{26}$$

and the Green's function $G(x,t)$ is defined by

$$G(x,t) = \mathfrak{F}^{-1}\left(e^{t\hat{\mathcal{L}}(\xi)}\right)(x,t). \tag{27}$$

After a direct calculation, we have the following expressions of the Green's function.

Lemma 1. (i) *The set of eigenvalues for $\hat{\mathcal{L}}(\xi)$ consists of $\lambda_j(\xi)$ $(j=1,2,3,4)$, where*

$$\begin{aligned}
&\lambda_1(\xi) = -\nu|\xi|^2 \ \ (double),\\
&\lambda_2(\xi) = -\frac{\mu}{2}|\xi|^2 + \frac{1}{2}\sqrt{\mu^2|\xi|^4 - 4c_0^2|\xi|^2},\ \mu := 2\nu + \eta,\\
&\lambda_3(\xi) = -\frac{\mu}{2}|\xi|^2 - \frac{1}{2}\sqrt{\mu^2|\xi|^4 - 4c_0^2|\xi|^2},\\
&\lambda_4(\xi) = -a - b|\xi|^2.
\end{aligned} \tag{28}$$

(ii) $e^{t\hat{\mathcal{L}}(\xi)}$ *has the spectral resolution*

$$e^{t\hat{\mathcal{L}}(\xi)} = \sum_{j=1}^{4} e^{\lambda_j(\xi)t} P_j(\xi). \tag{29}$$

Here $P_j(\xi)$ $(j=1,2,3,4)$ is the eigenprojection related to $\lambda_j(\xi)$ satisfying

$$\begin{aligned}
&P_1(\xi) = \begin{pmatrix} 0 & 0 & 0\\ 0 & \mathrm{I}_3 - \frac{\xi^\top\xi}{|\xi|^2} & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad
P_2(\xi) = \begin{pmatrix} -\frac{\lambda_3}{\lambda_2-\lambda_3} & -\frac{ic_0\xi}{\lambda_2-\lambda_3} & 0\\ -\frac{ic_0\xi^\top}{\lambda_2-\lambda_3} & \frac{\lambda_2}{\lambda_2-\lambda_3}\frac{\xi^\top\xi}{|\xi|^2} & 0\\ 0 & 0 & 0\end{pmatrix},\\
&P_3(\xi) = \begin{pmatrix} \frac{\lambda_2}{\lambda_2-\lambda_3} & \frac{ic_0\xi}{\lambda_2-\lambda_3} & 0\\ \frac{ic\xi^\top}{\lambda_2-\lambda_3} & -\frac{\lambda_3}{\lambda_2-\lambda_3}\frac{\xi^\top\xi}{|\xi|^2} & 0\\ 0 & 0 & 0\end{pmatrix}, \quad
P_4(\xi) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},
\end{aligned} \tag{30}$$

which implies for $j,k=1,2,3$ that

$$\hat{G}(\xi,t) = e^{t\hat{\mathcal{L}}(\xi)} = \begin{pmatrix} \hat{G}_{00} & \hat{G}_{0k} & 0\\ \hat{G}_{j0} & \hat{G}_{jk} & 0\\ 0 & 0 & \hat{G}_{44}\end{pmatrix}, \tag{31}$$

where

$$\begin{aligned}
&\hat{G}_{00}(\xi,t) = \frac{\lambda_2 e^{\lambda_3 t} - \lambda_3 e^{\lambda_2 t}}{\lambda_2 - \lambda_3},\\
&\hat{G}_{j0}(\xi,t) = \hat{G}_{0j}(\xi,t) = -\frac{e^{\lambda_2 t} - e^{\lambda_3 t}}{\lambda_2 - \lambda_3} ic_0\xi_j,\\
&\hat{G}_{jk}(\xi,t) = e^{-\nu|\xi|^2 t}\delta_{jk} + \left(\frac{\lambda_2 e^{\lambda_2 t} - \lambda_3 e^{\lambda_3 t}}{\lambda_2 - \lambda_3} - e^{-\nu|\xi|^2 t}\right)\frac{\xi_j\xi_k}{|\xi|^2},\\
&\hat{G}_{44}(\xi,t) = e^{-(a+b|\xi|^2)t}.
\end{aligned} \tag{32}$$

To verify the spectrum structure, it is necessary to study the asymptotic behaviors of eigenvalues. Hence, we present the following lemma, which can be easily deduced by Taylor's expansion.

Lemma 2. *Assume that the constant $r>0$ is sufficiently small, then the eigenvalues satisfy*
(i) *for low frequency part $|\xi| \le r \ll 1$,*

$$\begin{aligned}
&\lambda_1(\xi) = -\nu|\xi|^2 \ \ (double), && \lambda_2(\xi) = ic_0|\xi| - \frac{\mu}{2}|\xi|^2 + O(|\xi|^3),\\
&\lambda_3(\xi) = -ic_0|\xi| - \frac{\mu}{2}|\xi|^2 + O(|\xi|^3), && \lambda_4(\xi) = -a - b|\xi|^2.
\end{aligned} \tag{33}$$

(ii) *for high frequency part* $|\xi| \geq \frac{1}{r} \gg 1$,

$$\begin{aligned} &\lambda_1(\xi) = -\nu|\xi|^2 \ \ (double), &&\lambda_2(\xi) = -\frac{c_0^2}{\mu} + O(|\xi|^{-2}), \\ &\lambda_3(\xi) = -\mu|\xi|^2 + \frac{c_0^2}{\mu} + O(|\xi|^{-2}), &&\lambda_4(\xi) = -b|\xi|^2 - a, \end{aligned} \tag{34}$$

and there exists a positive constant $R_1 = \min\left\{\frac{\nu}{r^2}, \frac{c_0^2}{\mu}, a\right\}$ *such that*

$$Re\lambda_j \leq -R_1, \ j = 1, 2, 3, 4. \tag{35}$$

(iii) *for medium frequency part* $r < |\xi| < \frac{1}{r}$, *there exists a constant* $R_2 = \min\{\nu r^2, \frac{c_0^2}{\mu}, a\} > 0$ *such that*

$$Re\lambda_j \leq -R_2, \ j = 1, 2, 3, 4. \tag{36}$$

Here $\mu = 2\nu + \eta$ *and the constants* $\nu, \ \eta, \ a, \ b, \ c_0$ *are given by* (18).

We conclude from Lemma 2 that the semigroup has different characters in different frequency parts and thus we decompose $\hat{G}(\xi, t)$ into three parts as

$$\begin{aligned} \hat{G}(\xi, t) &= \hat{G}^\ell(\xi, t) + \hat{G}^m(\xi, t) + \hat{G}^h(\xi, t) \\ &:= \chi^\ell(\xi)\hat{G}(\xi, t) + \chi^m(\xi)\hat{G}(\xi, t) + \chi^h(\xi)\hat{G}(\xi, t), \end{aligned} \tag{37}$$

where χ^ℓ, χ^m and χ^h are smooth cut-off functions satisfying

$$\chi^\ell(\xi) = \begin{cases} 1, \ |\xi| < r/2, \\ 0, \ |\xi| > r, \end{cases} \quad \chi^h(\xi) = \begin{cases} 1, \ |\xi| > 1/r + 1, \\ 0, \ |\xi| < 1/r, \end{cases} \quad \chi^m(\xi) = 1 - \chi^\ell(\xi) - \chi^h(\xi). \tag{38}$$

The pointwise estimates of Green's function are stated as follows.

Proposition 1. *Let* $G(x, t)$ *be the Green's function to* (24) *and* α *be a multi-index, then the Green's function can be decomposed into*

$$D_x^\alpha G(x, t) = D_x^\alpha \left[g(x, t) + G_S^\alpha(x, t)\right] + G_R^\alpha(x, t), \tag{39}$$

where $g(x, t) = (g_{jk})(x, t)$ $(j, k = 0, 1, ..., 4)$ *and* $G_S^\alpha(x, t)$ *are the leading long waves and short waves, respectively. Each component* g_{jk} *has the following estimates for* $j, k = 1, 2, 3$ *that*

$$\begin{aligned} &\left|D^\alpha g_{00}(x, t)\right| \leq Ct^{-\frac{3+|\alpha|}{2}}(1 + t)^{-\frac{1}{2}}e^{-\frac{(|x|-c_0t)^2}{Ct}}, \\ &\left|D^\alpha (g_{j0}, g_{0j})(x, t)\right| \leq Ct^{-\frac{3+|\alpha|}{2}}(1 + t)^{-\frac{1}{2}}e^{-\frac{(|x|-c_0t)^2}{Ct}}, \\ &\left|D^\alpha g_{jk}(x, t)\right| \leq Ct^{-\frac{3+|\alpha|}{2}}\left(e^{-\frac{|x|^2}{Ct}} + (1 + t)^{-\frac{1}{2}}e^{-\frac{(|x|-c_0t)^2}{Ct}} + \chi_{\{|x|\leq c_0t\}}\left(1 + \frac{|x|^2}{t}\right)^{-\frac{3+|\alpha|}{2}}\right), \\ &\left|D^\alpha g_{44}(x, t)\right| \leq Ct^{-\frac{3+|\alpha|}{2}}e^{-at}e^{-\frac{|x|^2}{Ct}}. \end{aligned} \tag{40}$$

The leading short waves term $G_S^\alpha(x, t)$ *holds*

$$G_S^\alpha(x, t) = e^{-\frac{c_0^2}{\mu}t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\delta(x) + S(x, t), \tag{41}$$

where $S(x, t)$ *satisfies for some generic constants* $\zeta > 0$ *and* $R > 0$ *that*

$$\left|S(x, t)\right| \leq e^{-\zeta t}\mathcal{S}(x) \ with \ \mathcal{S}(x) \in L^1(\mathbb{R}^3), \ and \ \mathcal{S}(x) = \begin{cases} C|x|^{-2}, & |x| \leq R, \\ C(N)|x|^{-N}, & |x| > R, \end{cases} \tag{42}$$

for any given positive integer $N > 0$.

The remainder $G_R^\alpha(x,t)$ holds the following estimates

$$\left|G_R^\alpha(x,t)\right| \le C\left((1+t)^{-\frac{3+|\alpha|}{2}}(1+t)^{-1}e^{-\frac{(|x|-c_0t)^2}{Ct}}+e^{-\frac{|x|+t}{C}}\right). \tag{43}$$

Proof. We divide the domain into two parts: the finite Mach number region $\{|x| \le Mc_0t\}$ and outside finite Mach number region $\{|x| > Mc_0t\}$ with suitable positive constant $M > 0$. Inside the finite Mach number region, the decomposition of long and short waves and the complex analysis is fully applied as used in [12,17,18]. It should be pointed out that the extra term of Green's function corresponding to the phase variable can be expressed as

$$G_{44}(x,t) = g_{44}(x,t) = \mathfrak{F}^{-1}\left(e^{-(a+b|\xi|^2)t}\right) = e^{-at}h(x,bt), \tag{44}$$

with $h(x,t) = (4\pi t)^{-\frac{3}{2}}e^{-\frac{|x|^2}{4t}}$ the heat kernel. Hence, there exists a positive constant $C > 0$ such that

$$\left|D^\alpha g_{44}(x,t)\right| \le Ct^{-\frac{3+|\alpha|}{2}}e^{-at}e^{-\frac{|x|^2}{Ct}}, \tag{45}$$

and then we conclude (40)–(42). Furthermore, it holds

$$\begin{aligned}
&\left|D_x^\alpha\left(G^\ell - g\right)(x,t)\right| \le C(1+t)^{-\frac{3+|\alpha|}{2}}(1+t)^{-1}e^{-\frac{(|x|-c_0t)^2}{Ct}},\\
&\left|D_x^\alpha G^m(x,t)\right| \le C(1+t)^{-\frac{3+|\alpha|}{2}}e^{-\frac{t}{C}},\\
&\left|D_x^\alpha\left(G^h - G_S^\alpha\right)(x,t)\right| \le C\left(t^{-\frac{3+|\alpha|}{2}}e^{-\frac{t}{C}}+e^{-\frac{t}{C}}\right).
\end{aligned} \tag{46}$$

As for outside finite Mach number region, we consider the initial value problem (IVP) below

$$\begin{cases}
\partial_t n + c_0\mathrm{div}w = 0,\\
\partial_t w + c_0\nabla n - \nu\Delta w - (\nu+\eta)\nabla\mathrm{div}w = 0,\\
\partial_t\phi + a\phi - b\Delta\phi = 0,\\
(n,w,\phi)(x,0) = (n_0,w_0,\phi_0)(x).
\end{cases} \tag{47}$$

Multiplying $e^{|x|-Mc_0t}$ by $\left(n(47)_1 + w\cdot(47)_2 + \phi(47)_3\right)$ and then integrating by parts over $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(n^2+|w|^2+\phi^2\right)dx\\
=&-\frac{Mc_0}{2}\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(n^2+|w|^2+\phi^2\right)dx - \int_{\mathbb{R}^3} e^{|x|-Mc_0t}\mathrm{div}(c_0nw)dx\\
&+\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(\nu\Delta w\cdot w + (\nu+\eta)\nabla\mathrm{div}w\cdot w\right)dx - \int_{\mathbb{R}^3} e^{|x|-Mc_0t}\phi\left(a\phi - b\Delta\phi\right)dx\\
=&-\frac{Mc_0}{2}\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(n^2+|w|^2+\phi^2\right)dx + c_0\int_{\mathbb{R}^3} e^{|x|-Mc_0t}nw\cdot\frac{x}{|x|}dx\\
&-\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(\nu|\nabla w|^2 + (\nu+\eta)|\mathrm{div}w|^2\right)dx - \int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(\nu w\cdot\nabla w + (\nu+\eta)w\mathrm{div}w\right)\cdot\frac{x}{|x|}dx\\
&-a\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\phi^2dx - b\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(|\nabla\phi|^2 + \phi\nabla\phi\cdot\frac{x}{|x|}\right)dx,
\end{aligned} \tag{48}$$

which, by applying the Schwarz inequality, yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(n^2+|w|^2+\phi^2\right)dx\\
\le&-\frac{1}{2}(Mc_0-c_0)\int_{\mathbb{R}^3} e^{|x|-Mc_0t}n^2dx - \frac{1}{2}(Mc_0-c_0-\mu)\int_{\mathbb{R}^3} e^{|x|-Mc_0t}|w|^2dx - \frac{1}{2}(Mc_0-2a-b)\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\phi^2dx\\
&-\frac{1}{2}\int_{\mathbb{R}^3} e^{|x|-Mc_0t}\left(\nu|\nabla w|^2 + (\nu+\eta)|\mathrm{div}w|^2\right)dx - \frac{b}{2}\int_{\mathbb{R}^3} e^{|x|-Mc_0t}|\nabla\phi|^2dx.
\end{aligned} \tag{49}$$

Here we take $M > 0$ sufficiently large satisfies

$$\begin{cases} Mc_0 - c_0 - \mu > 0, \\ Mc_0 - 2a - b > 0, \end{cases} \tag{50}$$

and so the weighted L^2-norm of (n, w, ϕ) is non-increasing in time, which implies there exists a positive constant $C > 0$ depending only on (n_0, w_0, ϕ_0) such that

$$\int_{\mathbb{R}^3} e^{|x|-Mc_0 t} \left(n^2 + |w|^2 + \phi^2\right) dx \leq C. \tag{51}$$

We can apply the same method as above for $D_x^\alpha(n, w, \phi)(x, t)$ with $|\alpha| > 0$. Due to the Sobolev embedding theorem, we conclude that

$$\sup_{(x,t)\in\mathbb{R}^3\times\mathbb{R}_+} e^{\frac{1}{2}(|x|-Mc_0 t)} \left|(n, w, \phi)(x, t)\right| \leq C. \tag{52}$$

Note that for $|x| > (2M + 1)c_0 t$,

$$|x| - Mc_0 t > \frac{|x|}{2} + \left(\frac{|x|}{2} - Mc_0 t\right) > \frac{|x|}{2} + \frac{c_0 t}{2}, \tag{53}$$

which, together with (46), gives rise to (43).

Thus, we complete the proof of Proposition 1. □

3 | POINTWISE ESTIMATES OF NONLINEAR SYSTEM

In this section, we will study the initial value problem (24). By Duhamel's principle, the solution can be expressed as

$$\begin{aligned} D_x^\alpha U(x, t) &= D^\alpha \int_{\mathbb{R}^3} G(x - y, t) U_0(y) dy + D^\alpha \int_0^t \int_{\mathbb{R}^3} G(x - y, t - s) F(y, s) dy ds \\ &\triangleq \mathcal{I}^\alpha(x, t) + \mathcal{N}^\alpha(x, t). \end{aligned} \tag{54}$$

We notice that the nonlinear term F_1 contains the derivative of U and is therefore separated as

$$\begin{aligned} F_1 &= \operatorname{div}\left(-c_0\left(\frac{w \otimes w}{1 + n}\right) - \nu\nabla\left(\frac{nw}{1 + n}\right) - \frac{\epsilon}{c_0\bar{\rho}}\left(\frac{2\nabla\phi \otimes \nabla\phi - |\nabla\phi|^2 \mathrm{I}_3}{8(1 + \phi)}\right)\right) \\ &\quad + \nabla\left(c_0 n - \frac{P(\bar{\rho}(1 + n))}{c_0\bar{\rho}} - (\nu + \eta)\operatorname{div}\left(\frac{nw}{1 + n}\right)\right) \triangleq \operatorname{div} f_1 + \nabla f_2, \\ F_2 &= -\frac{c_0 w \cdot \nabla\phi}{1 + n} + \frac{a\phi(n - \phi)}{1 + n} - \frac{bn(n + 2)\Delta\phi}{(1 + n)^2} - \frac{b|\nabla\phi|^2}{2(1 + n)^2(1 + \phi)}. \end{aligned} \tag{55}$$

It is easy to obtain that each component of the nonlinear terms F_1 and F_2 satisfies

$$\begin{aligned} f_1 &= \mathcal{O}(1)\Big(|w|^2 + |n||Dw| + |\nabla n||w| + |n||Dn||w| + |D\phi|^2\Big), \\ f_2 &= \mathcal{O}(1)\Big(|n|^2 + |n||Dw| + |Dn||w| + |n||Dn||w|\Big), \\ F_1 &= \mathcal{O}(1)\Big(|w||Dw| + |Dn||w|^2 + |n||Dn| + |D^2 n||w| + |n||D^2 w| + |Dn||Dw| \\ &\qquad + |Dn|^2|w| + |n||Dn||Dw| + |n||D^2 n||w| + |D\phi|^3 + |D\phi||D^2\phi|\Big), \\ F_2 &= \mathcal{O}(1)\Big(|w||D\phi| + |n||\phi| + |\phi|^2 + |n||D^2\phi| + |n|^2|D^2\phi| + |D\phi|^2\Big). \end{aligned} \tag{56}$$

3.1 | Initial propagation

We first study the propagation of the initial data $\mathcal{I}^\alpha(x, t)$. Denote $(\bar{n}, \bar{w}, \bar{\phi})$ as the solution to the linearized system of (20), from (31) and (54), we have for any multi-index β that

$$D_x^\beta \bar{\phi}(x, t) = D^\beta \int_{\mathbb{R}^3} G_{44}(x - y, t)\phi_0(y) dy = \int_{\mathbb{R}^3} D^\beta g_{44}(x - y, t)\phi_0(y) dy. \tag{57}$$

Due to (40) and the assumption (12), it holds for $|\beta| \le 2$ that

$$
\begin{aligned}
\left|D_x^\beta \bar{\phi}(x,t)\right| &\le C\varepsilon_0(1+t)^{-\frac{3+|\beta|}{2}} e^{-at} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{Ct}} (1+|y|^2)^{-r_2} dy \\
&\le C\varepsilon_0(1+t)^{-\frac{3+|\beta|}{2}} e^{-at} \left(\int_{|y|\ge \frac{|x|}{2}} + \int_{|y|\le \frac{|x|}{2}} \right) e^{-\frac{|x-y|^2}{Ct}} (1+|y|^2)^{-r_2} dy \\
&\le C\varepsilon_0(1+t)^{-\frac{3+|\beta|}{2}} e^{-at} (1+t)^{\frac{3}{2}} \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} + C\varepsilon_0(1+t)^{-\frac{3+|\beta|}{2}} e^{-at} e^{-\frac{|x|^2}{Ct}} \\
&\le C\varepsilon_0(1+t)^{-\frac{|\beta|}{2}} e^{-at} \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}},
\end{aligned}
\tag{58}
$$

where we used the fact that $|y| \le \frac{|x|}{2} \Rightarrow |x-y| \ge \frac{|x|}{2}$.

Similarly, we also obtain the estimates of $\left|D_x^\alpha \bar{n}\right|$ and $\left|D_x^\alpha \bar{w}\right|$ for $|\alpha| \le 1$. Here we omit the details and one can refer to [17]. As a result, we have the following pointwise estimates for the linearized system.

Proposition 2. *For* $0 \le |\alpha| \le 1$, $0 \le |\beta| \le 2$, *it holds*

$$
\begin{aligned}
\left|D_x^\alpha \bar{n}(x,t)\right| &\le C\varepsilon_0(1+t)^{-\frac{|\alpha|}{2}} \left\{ (1+t)^{-2}\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + (1+t)^{-2}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} \right\}, \\
\left|D_x^\alpha \bar{w}(x,t)\right| &\le C\varepsilon_0(1+t)^{-\frac{|\alpha|}{2}} \left\{ (1+t)^{-2}\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + (1+t)^{-\frac{3}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} \right\}, \\
\left|D_x^\beta \bar{\phi}(x,t)\right| &\le C\varepsilon_0 e^{-at}(1+t)^{-\frac{|\beta|}{2}} \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}.
\end{aligned}
\tag{59}
$$

3.2 | Nonlinear coupling

Let $T > 0$, based on Proposition 2, we introduce the following ansatz

$$
\begin{aligned}
M(T) = \sup_{0\le t\le T} \Big\{ &\|n\psi_1^{-1}\|_{L^\infty} + \|w\psi_2^{-1}\|_{L^\infty} + (1+t)^{\frac{1}{2}}\|Dn\psi_1^{-1}\|_{L^\infty} + (1+t)^{\frac{1}{2}}\|Dw\psi_2^{-1}\|_{L^\infty} \\
&+ \sum_{|\beta|=0}^{2} \|D^\beta\phi\psi_3^{-1}\|_{L^\infty} + (1+t)^{\frac{5}{2}}\|D^2(n,w)\|_{L^\infty} \Big\},
\end{aligned}
\tag{60}
$$

where ψ_i $(i = 1,2,3)$ satisfies the following expression of waves

$$
\begin{aligned}
\psi_1(x,t) &= (1+t)^{-2}\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + (1+t)^{-2}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}, \\
\psi_2(x,t) &= (1+t)^{-2}\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + (1+t)^{-\frac{3}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}, \\
\psi_3(x,t) &= e^{-at}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}.
\end{aligned}
\tag{61}
$$

In what follows, we mainly prove that $M(T) \le C$. When the initial data (n_0, w_0, ϕ_0) satisfies (8)–(9), by (11) and Sobolev inequality, we know the ansatz on $D^2(n,w)$ is reasonable. Thus, we focus on the pointwise estimates for the low-order derivatives of solution in ansatz (60).

For the nonlinear term $\mathcal{N}^\alpha(x,t)$, we have

$$
\mathcal{N}^\alpha(x,t) = (\mathcal{N}_j^\alpha, \mathcal{N}_4^\alpha)(x,t), \quad j = 0,1,2,3,
\tag{62}
$$

where

$$
\begin{aligned}
\mathcal{N}_j^\alpha(x,t) &= D_x^\alpha \int_0^t \int_{\mathbb{R}^3} G_{jk}(x-y,t-s)F^k(y,s)dyds \\
&= \int_0^t \int_{\mathbb{R}^3} D^\alpha g_{jk}(x-y,t-s)F^k(y,s)dyds + \int_0^t \int_{\mathbb{R}^3} D^\alpha G_S^\alpha(x-y,t-s)F^k(y,s)dyds \\
&\quad + \int_0^t \int_{\mathbb{R}^3} G_R^\alpha(x-y,t-s)F^k(y,s)dyds \\
&\triangleq L_j^\alpha + S_j^\alpha + R_j^\alpha,
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
\mathcal{N}_4^\alpha(x,t) &= D_x^\alpha \int_0^t \int_{\mathbb{R}^3} G_{44}(x-y,t-s)F_2(y,s)dyds \\
&= \int_0^t \int_{\mathbb{R}^3} D^\alpha g_{44}(x-y,t-s)F_2(y,s)dyds.
\end{aligned}
\tag{64}
$$

Here g, G_S and G_R are defined in Proposition 1, and F^k denotes the k-th term of F. First, from the ansatz we have

$$
\begin{aligned}
&|n| \le M(T)\psi_1, && |w| \le M(T)\psi_2, && |\phi| \le M(T)\psi_3, \\
&|Dn| \le M(T)(1+t)^{-\frac{1}{2}}\psi_1, && |Dw| \le M(T)(1+t)^{-\frac{1}{2}}\psi_2, && |D\phi| \le M(T)\psi_3, \\
&|D^2(n,w)| \le M(T)(1+t)^{-\frac{5}{2}}, && |D^2\phi| \le M(T)\psi_3, &&
\end{aligned}
\tag{65}
$$

which together with (56) gives rise to

$$
\begin{aligned}
&\left|f_1 + f_2\right| \le CM^2(T)\left(\psi_1^2 + \psi_2^2 + (1+t)^{-\frac{1}{2}}\psi_1\psi_2 + \psi_3^2\right), \\
&\left|F_1\right| \le CM^2(T)\left((1+t)^{-\frac{1}{2}}\left(\psi_1^2 + \psi_2^2 + (1+t)^{-\frac{1}{2}}\psi_1\psi_2\right) + (1+t)^{-\frac{5}{2}}(\psi_1 + \psi_2) + \psi_3^2\right), \\
&\left|F_2\right| \le CM^2(T)(\psi_1 + \psi_2 + \psi_3)\psi_3.
\end{aligned}
\tag{66}
$$

Since the Green function contains the Huygens' wave of the form $t^{-\frac{3+|\alpha|}{2}}(1+t)^{-\frac{1}{2}}e^{-\frac{(|x|-c_0t)^2}{Ct}}$, the diffusion wave of the form $t^{-\frac{3+|\alpha|}{2}}e^{-\frac{|x|^2}{Ct}}$, and the Riesz wave of the form $t^{-\frac{3+|\alpha|}{2}}\chi_{\{|x|\le c_0t\}}\left(1+\frac{|x|^2}{t}\right)^{-\frac{3+|\alpha|}{2}}$, to estimate the interaction of these different waves, we should divide both the time t and the space x into several parts. In particular, we define

$$
\begin{aligned}
&X_1 = \left\{|x|^2 \le 1+t\right\}, \quad X_2 = \left\{(|x|-c_0t)^2 \le 1+t\right\}, \quad X_3 = \left\{|x| \ge c_0t + \sqrt{1+t}\right\}, \\
&X_4 = \left\{\sqrt{1+t} \le |x| \le \frac{c_0t}{2}\right\}, \quad X_5 = \left\{\frac{c_0t}{2} \le |x| \le c_0t - \sqrt{1+t}\right\},
\end{aligned}
\tag{67}
$$

and set

$$
t_0 = \begin{cases} \max\left\{\dfrac{t}{2},\ t - \dfrac{\sqrt{1+t}}{4}\right\}, & (x,t) \in X_1 \cup X_2 \cup X_3, \\ t - \min\left\{\dfrac{c_0t-|x|}{4c_0},\ \dfrac{|x|}{4c_0}\right\}, & (x,t) \in X_4 \cup X_5. \end{cases}
\tag{68}
$$

Moreover, for any multi-index α, we denote

$$
\begin{aligned}
&H_\alpha(x,t) = t^{-\frac{3+|\alpha|}{2}}(1+t)^{-\frac{1}{2}}e^{-\frac{(|x|-c_0t)^2}{Ct}}, && h_\alpha(x,t) = (1+t)^{-|\alpha|}\left(1 + \frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}}, \\
&D_\alpha(x,t) = t^{-\frac{3+|\alpha|}{2}}e^{-\frac{|x|^2}{Ct}}, && d_\alpha(x,t) = (1+t)^{-|\alpha|}\left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}, \\
&R_\alpha(x,t) = t^{-\frac{3+|\alpha|}{2}}\chi_{\{|x|\le c_0t\}}\left(1 + \frac{|x|^2}{t}\right)^{-\frac{3+|\alpha|}{2}}, &&
\end{aligned}
\tag{69}
$$

and give the following lemmas.

Lemma 3. (refer to[12]) *For $0 \le |\alpha| \le 1$, there exists a positive constant $C > 0$ independent of (x,t) such that*

$$
\begin{aligned}
(1)I_1^\alpha &= \int_0^{t_0} H_{1+\alpha}(x-y,t-s)h_2^2(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}\left(h_0+d_0\right),\\
(2)I_2^\alpha &= \int_0^{t_0} H_{1+\alpha}(x-y,t-s)d_{\frac{3}{2}}^2(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}\left(h_0+d_0\right),\\
(3)I_3^\alpha &= \int_0^{t_0} D_{1+\alpha}(x-y,t-s)h_2^2(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}\left(h_0+d_0\right),\\
(4)I_4^\alpha &= \int_0^{t_0} D_{1+\alpha}(x-y,t-s)d_{\frac{3}{2}}^2(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}d_0,\\
(5)I_5^\alpha &= \int_0^{t_0} R_{1+\alpha}(x-y,t-s)h_2^2(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}\left(h_0+d_0\right),\\
(6)I_6^\alpha &= \int_0^{t_0} R_{1+\alpha}(x-y,t-s)d_{\frac{3}{2}}^2(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}d_0.
\end{aligned}
$$

Lemma 4. (refer to[17]) *For $0 \le |\alpha| \le 1$, there exists a positive constant $C > 0$ independent of (x,t) such that*

$$
\begin{aligned}
(1)J_1^\alpha &= \int_{t_0}^{t} H_1(x-y,t-s)(1+s)^{-\frac{4+|\alpha|}{2}}h_2(y,s)dyds \le C(1+t)^{-\frac{6+|\alpha|}{2}}\left(h_0+d_0\right),\\
(2)J_2^\alpha &= \int_{t_0}^{t} H_1(x-y,t-s)(1+s)^{-\frac{3+|\alpha|}{2}}d_{\frac{3}{2}}(y,s)dyds \le C(1+t)^{-\frac{4+|\alpha|}{2}}\left((1+t)^{-\frac{1}{2}}h_0+d_0\right),\\
(3)J_3^\alpha &= \int_{t_0}^{t} D_1(x-y,t-s)(1+s)^{-\frac{4+|\alpha|}{2}}h_2(y,s)dyds \le C(1+t)^{-\frac{6+|\alpha|}{2}}\left(h_0+d_0\right),\\
(4)J_4^\alpha &= \int_{t_0}^{t} D_1(x-y,t-s)(1+s)^{-\frac{3+|\alpha|}{2}}d_{\frac{3}{2}}(y,s)dyds \le C(1+t)^{-\frac{5+|\alpha|}{2}}d_0,\\
(5)J_5^\alpha &= \int_{t_0}^{t} R_1(x-y,t-s)(1+s)^{-\frac{4+|\alpha|}{2}}h_2(y,s)dyds \le C(1+t)^{-\frac{6+|\alpha|}{2}}\left(h_0+d_0\right),\\
(6)J_6^\alpha &= \int_{t_0}^{t} R_1(x-y,t-s)(1+s)^{-\frac{3+|\alpha|}{2}}d_{\frac{3}{2}}(y,s)dyds \le C(1+t)^{-\frac{5+|\alpha|}{2}}d_0.
\end{aligned}
$$

To begin with, we devote to the estimates of $\mathcal{N}_j^\alpha$. It follows from (40), (63)–(66) that

$$
\begin{aligned}
\left|L_j^\alpha\right| &\le \left|\int_0^{t_0}\int_{\mathbb{R}^3}\left(H_{1+\alpha}+D_{1+\alpha}+R_{1+\alpha}\right)(x-y,t-s)(f_1+f_2)(y,s)dyds\right|\\
&\quad+\left|\int_{t_0}^{t}\int_{\mathbb{R}^3}\left(H_1+D_1+R_1\right)D^\alpha(f_1+f_2)(y,s)dyds\right|\\
&\le CM^2(T)\sum_{k=1}^{6}\left(I_k^\alpha+J_k^\alpha\right),
\end{aligned}
\tag{70}
$$

which, together with Lemmas 3–4, gives rise to

$$
\left|L_j^\alpha\right| \le CM^2(T)(1+t)^{-\frac{4+|\alpha|}{2}}\left\{\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}}+\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}\right\}. \tag{71}
$$

Next, we consider the convolution between the leading part in short wave G_S^α and F. In terms of (41), we have

$$
\begin{aligned}
\left|S_j^\alpha\right| &\le CM^2(T)\int_0^t\int_{\mathbb{R}^3}e^{-\beta(t-s)}\mathcal{S}(x-y)\left((1+s)^{-\frac{1}{2}}\psi_2^2(y,s)+(1+s)^{-\frac{5}{2}}\psi_2(y,s)\right)dyds\\
&\le CM^2(T)\left((1+t)^{-\frac{1}{2}}\psi_2^2(x,t)+(1+t)^{-\frac{5}{2}}\psi_2(x,t)\right)\\
&\le CM^2(T)(1+t)^{-\frac{7}{2}}\left(\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}}+\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}\right).
\end{aligned}
\tag{72}
$$

Since the convolution between G_R^α and F can be estimated as those for L_j^α, here we omit the details and give the conclusion as follows

$$\left|R_j^\alpha\right| \le CM^2(T)(1+t)^{-\frac{5+|\alpha|}{2}}\left\{\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}}+\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}\right\}. \tag{73}$$

On the other hand, due to (40), (64) and (66), it holds for any multi-index β that

$$\left|\mathcal{N}_4^\beta\right| \le CM^2(T)\int_0^t\int_{\mathbb{R}^3}(t-s)^{-\frac{3+|\beta|}{2}}e^{-a(t-s)}e^{-\frac{|x-y|^2}{C(t-s)}}(1+s)^{-\frac{3}{2}}e^{-as}\left(1+\frac{|y|^2}{1+s}\right)^{-\frac{3}{2}}dyds. \tag{74}$$

We consider the following decomposition

$$\begin{cases}|y| \ge \dfrac{|x|}{2},\\ |y| \le \dfrac{|x|}{2} \ \Rightarrow\ |x-y| \ge |x|-|y| \ge \dfrac{|x|}{2}.\end{cases} \tag{75}$$

For the first part, we have

$$\begin{aligned}&\int_0^t\int_{|y|\ge\frac{|x|}{2}}(t-s)^{-\frac{3+|\beta|}{2}}e^{-a(t-s)}e^{-\frac{|x-y|^2}{C(t-s)}}(1+s)^{-\frac{3}{2}}e^{-as}\left(1+\frac{|y|^2}{1+s}\right)^{-\frac{3}{2}}dyds\\ &\le C\int_0^t(t-s)^{-\frac{3+|\beta|}{2}}e^{-a(t-s)}(1+s)^{-\frac{3}{2}}e^{-as}\left(1+\frac{|x|^2}{1+s}\right)^{-\frac{3}{2}}\int_{|y|\ge\frac{|x|}{2}}e^{-\frac{|x-y|^2}{C(t-s)}}dyds\\ &\le C\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}\int_0^t(t-s)^{-\frac{|\beta|}{2}}e^{-a(t-s)}(1+s)^{-\frac{3}{2}}e^{-as}ds\\ &\le C(1+t)^{-\frac{|\beta|}{2}}e^{-at}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}},\end{aligned} \tag{76}$$

and the second part can be treated in a similar manner

$$\begin{aligned}&\int_0^t\int_{|y|\le\frac{|x|}{2}}(t-s)^{-\frac{3+|\beta|}{2}}e^{-a(t-s)}e^{-\frac{|x-y|^2}{C(t-s)}}(1+s)^{-\frac{3}{2}}e^{-as}\left(1+\frac{|y|^2}{1+s}\right)^{-\frac{3}{2}}dyds\\ &\le Ce^{-\frac{|x|^2}{Ct}}\int_0^t(t-s)^{-\frac{3+|\beta|}{2}}e^{-a(t-s)}e^{-as}ds\\ &\le Ce^{-at}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}.\end{aligned} \tag{77}$$

Accordingly, we have the following estimates of nonlinear couplings.

Proposition 3. *For multi-indexes α, β with $|\alpha| \le 1$, $|\beta| \le 2$, it holds*

$$\begin{aligned}|\mathcal{N}_0^\alpha| &\le CM^2(T)(1+t)^{-\frac{4+|\alpha|}{2}}\left\{\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}}+\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}\right\},\\ |\mathcal{N}_j^\alpha| &\le CM^2(T)(1+t)^{-\frac{4+|\alpha|}{2}}\left\{\left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}}+\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}}\right\},\ \ j=1,2,3,\\ |\mathcal{N}_4^\beta| &\le CM^2(T)e^{-at}\left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}},\end{aligned} \tag{78}$$

where $C>0$ is a positive constant independent of space and time.

So far, with the help of Propositions 2 and 3, we are ready to prove Theorem 1 below.

Proof of Theorem 1. It follows from (54), (59) and (78) that

$$M(T) \le C(\varepsilon_0 + M^2(T)), \tag{79}$$

which, combined with the smallness of ε_0 and the continuity of $M(T)$ leads to

$$M(T) \leq C\varepsilon_0. \tag{80}$$

Thus, based on the definition of $M(T)$ in (60), we conclude

$$\begin{aligned}
|D_x^\alpha n(x,t)| &\leq C\varepsilon_0(1+t)^{-\frac{4+|\alpha|}{2}} \left\{ \left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} \right\},\\
|D_x^\alpha w(x,t)| &\leq C\varepsilon_0(1+t)^{-\frac{3+|\alpha|}{2}} \left\{ (1+t)^{-\frac{1}{2}} \left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-\frac{3}{2}} + \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}} \right\},\\
|D_x^\beta \phi(x,t)| &\leq C\varepsilon_0 e^{-at} \left(1+\frac{|x|^2}{1+t}\right)^{-\frac{3}{2}},
\end{aligned} \tag{81}$$

which, together with (19), gives rise to (13) and the proof is completed. □

Proof of Corollary 1. In terms of (54), we are able to obtain for $|\alpha| \leq 1$ that

$$\begin{aligned}
\|D^\alpha (n-\bar{n})(t)\|_{L^2} &= \|\mathcal{N}_0^\alpha(t)\|_{L^2}\\
&\leq C\varepsilon_0(1+t)^{-\frac{4+|\alpha|}{2}} \left\{ \int_{\mathbb{R}^3} \left(1+\frac{(|x|-c_0t)^2}{1+t}\right)^{-3} + \left(1+\frac{|x|^2}{1+t}\right)^{-3} dx \right\}^{\frac{1}{2}}\\
&\leq C\varepsilon_0(1+t)^{-\frac{4+|\alpha|}{2}} \left\{ \int_{\mathbb{R}^3} \left(1+y^2\right)^{-3} (1+t)^{\frac{3}{2}} dy \right\}^{\frac{1}{2}}\\
&\leq C\varepsilon_0(1+t)^{-\frac{5}{4}-\frac{|\alpha|}{2}}.
\end{aligned} \tag{82}$$

Similarly, it holds

$$\|D^\alpha (w-\bar{w})(t)\|_{L^2} \leq C\varepsilon_0(1+t)^{-\frac{5}{4}-\frac{|\alpha|}{2}}, \tag{83}$$

and for $|\beta| \leq 2$ that

$$\|D^\beta (\phi-\bar{\phi})(t)\|_{L^2} \leq C\varepsilon_0 e^{-\frac{a}{2}t}. \tag{84}$$

This, together with (19), completes the proof of Corollary 1. □

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Conflict of interest

This work does not have any conflicts of interest.

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