Block Sparse Vector Recovery for Compressive Sensing via $\ell_1 - \alpha\ell_q$-minimization Model

Hongyan Shi * Shaohua Xie † Jiangtao Wang ‡

Abstract

This paper solves the problem of block sparse vector recovery using the block $\ell_1 - \alpha\ell_q$- minimization model. Based on the block restricted isometry property (B-RIP) condition, we obtain exact block sparse vector recovery result. We also obtain the theoretical bound for the block $\ell_1 - \alpha\ell_q$- minimization model when measurements are depraved by the noises.

Keywords: $\ell_1 - \ell_2$-minimization, Compressed Sensing, Block Sparse vector, RIP condition

1 Introduction

Compressed sensing is a sparse signal recovery technique. It restores high-dimensional signals from low-dimensional measurements. Mathematically, it can be expressed as

$$min\ \|x\|_0 \qquad subject \quad to \qquad Ax = y,$$

where $x \in R^n$ is the unknown signal to be recovered, $A \in^{m\times n}$ $(m < n)$ is the measurement matrix, $y \in R^m$ is the measurement value and $\|x\|_0$ counts the number of nonzero elements in the vector x. The above model is called ℓ_0-minimization model, which is NP-hard [1]. Fortunately, people have found that when x is a sparse signal, the ℓ_1-minimization model can effectively solve the ℓ_0-minimization model. The ℓ_1-minimization model is as follows:

$$min\ \|x\|_1 \qquad subject \quad to \qquad Ax = y,$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$. The existing literature suggests that when the measurement matrix satisfies certain properties, such as null space property [1], restricted isometry property(RIP), coherence [2], the solution of the ℓ_1-minimization model is that of ℓ_0-minimization model. We now provide the definition of RIP:

Definition 1 *The sth restricted isometry constant $\delta_s = \delta_s(A)$ of a matrix $A \in R^{m\times n}$ is the smallest $\delta > 0$ such that*

$$(1-\delta\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta)\|x\|_2^2$$

for all s-sparse vectors $x \in R^n$.

Although the ℓ_1-minimization model can effectively solve the ℓ_0-minimization model, they are not completely equivalent [3]. Therefore, in order to better solve the ℓ_0-minimization model, other models have emerged one after another. Among these models, the ℓ_p-minimization model is a well-known one. It can be expressed as

$$min\ \|x\|_p^p \qquad subject \quad to \qquad Ax = y,$$

where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$, $0 < p < 1$. [9] utilized the null space property of ℓ_p-minimization model to obtain a necessary and sufficient condition for ℓ_p-minimization model to recover sparse signals. [8], based on the RIP condition, obtained a sharp upper bound for ℓ_p-minimization model to recover sparse signals.

Another well-known alternative to the ℓ_1-minimization model is the $\ell_1-\ell_2$-minimization model. It can be expressed as

$$min\ \|x\|_1 - \|x\|_2 \qquad subject \quad to \qquad Ax = y,$$

where $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. [4] provides a necessary and sufficient condition for the $\ell_1 - \ell_2$-minimization to recover sparse signals from the perspective of null space. Based on the RIP condition, [5] proposed sufficient conditions for the $\ell_1 - \ell_2$-minimization model to recover sparse signals. In addition, both theoretically [7] and experimentally [5] show that it can recover sparse signals better than ℓ_1-minimization model.

There have been many theoretical results for the recovery of sparse signals. However, in real life, we will also encounter sparse signals with special structures, such as block sparse signal. Block signal divides unknown signals x into l block, i.e.

$$x = [\underbrace{x_1, \cdots, x_{d_1}}_{x[1]}, \underbrace{x_{d_1+1}, \cdots, x_{d_1+d_2}}_{x[2]}, \cdots, \underbrace{x_{N-d_l+1}, \cdots, x_N}_{x[l]}]^T.$$

If $\|x\|_{2,0} := \sum_{i=1}^l \iota(\|x[i]\|_2 > 0) \le k$, where $\iota(x)$ denotes an indicator function that $\iota(x) = 1$ or 0 according as $x > 0$ or otherwise, We call this x as k block sparse vector. In addition, we define $supp([x]) = \{i : i \in [l], \|x[i]\|_2 \ne 0\}$.

ℓ_2/ℓ_1-minimization model is a model for processing block sparse signals, which is a variant of the ℓ_1-minimization model. its mathematical model can be expressed as

$$min\ \|x\|_{2,I} \qquad subject \quad to \qquad Ax = y,$$

where $\|x\|_{2,I} = \sum_{i=1}^l \|x[i]\|_2$, $I = \{d_1, d_2, \cdots d_l\}$. Define $\|x\|_{2,2} = \sqrt{\sum_{i=1}^l \|x[i]\|_2^2}$, then like the traditional RIP condition, k-block sparse vectors also have block RIP conditions:

Definition 2 *Give a measurement matrix A with size $m \times n$, where $m < n$, one says that the measurement matrix A obeys the block RIP over $I = \{d_1, d_2, \cdots, d_l\}$ with constants $\delta_{k|I}$ if for every vector $x \in R^n$ wiht k bolck sparse over I such that*

$$(1-\delta_{k|I})\|x\|_{2,2}^2 \le \|Ax\|_{2,2}^2 \le (1+\delta_{k|I})\|x\|_{2,2}^2$$

holds. We say the smallest constant $\delta_{k|I}$ that meets the above inequality as the block RIC corresponding with the matrix A.

According to the block RIP condition, [6] provides a sharp sufficient condition so that k block sparse signals can be recovered by ℓ_2/ℓ_1-minimization model.

For the recovery problem of block sparse vectors, there are also variations of the traditional $\ell_1 - \ell_2$-minimization model, i.e. block $\ell_1 - \ell_2$-minimization model:

$$min\ \|x\|_{2,1} - \|x\|_{2,2} \qquad s.t. \qquad Ax = y.$$

*Department of Mathematics , Yang-En University, Ma-Jia Town, LuoJiang District, QuanZhou, 362014, China. (shihongyan@yeu.edu.cn).

†School of Applied Mathematics, Guangdong University of Technology Jieyang Institute of Technology, Jieyang, 522000, China(xsh1209490052@163.com)

‡School of Network Communication, Zhejiang Yuexiu University, 2801 QunXian Middle Road, shaoxing, 312000, China. (3068525606@qq.com).

[10] provides sufficient conditions for the block $\ell_1-\ell_2$-minimization model to recover block sparse vectors based on the block RIP condition, and designs an algorithms for the block $\ell_1-\ell_2$-minimization model based on the difference of convex function algorithm. Experimental results show that this model outperforms other models in recovering block sparse vectors.

Combining the ℓ_p-minimization model with the block $\ell_1-\ell_2$-minimization model, this paper proposes a new model to solve the problem of block sparse vector recovery. We call this new model block $\ell_1-\alpha\ell_q$-minimization model:

$$min\ \|x\|_{2,1}-\alpha\|x\|_{2,q} \qquad s.t. \qquad Ax=y \tag{1}$$

where $1<p\le 2$, $0\le\alpha\le 1$, $\|x\|_{2,q}=(\sum_{i=1}^{l}\|x[i]\|_2^q)^{\frac{1}{q}}$. Its noise model as following:

$$min\ \|x\|_{2,1}-\alpha\|x\|_{2,q} \qquad s.t. \qquad \|Ax-y\|_2\le\epsilon \tag{2}$$

where ϵ is a very small constant.

The main contributions of this paper are: (i) We propose a block RIP condition that fully guarantees that the block $\ell_1-\alpha\ell_q$-minimization model can accurately recover all block sparse vectors; (ii) We have demonstrated that this condition can also ensure that the noisy block $\ell_1-\alpha\ell_q$-minimization model can stably recover all block s-sparse vectors.

2 Main

In this section, we give out the main conclusions of this paper.

Lemma 1 *Let $x\in R^n$, $1<p\le 2$, $0\le\alpha\le 1$, then*
(i) $(l-\alpha l^{\frac{1}{q}})\min_{i\in[l]}\|x[i]\|_2\le\|x\|_{2,1}-\alpha\|x\|_{2,q}\le(l^{1-\frac{1}{q}}-\alpha)\|x\|_{2,q}$.
(ii) *If* $\|x\|_{2,0}=s$, *then* $(s-\alpha s^{\frac{1}{q}})\min_{i\in[l]}\|x[i]\|_2\le\|x\|_{2,1}-\alpha\|x\|_{2,q}\le(s^{1-\frac{1}{q}}-\alpha)\|x\|_{2,q}$

Proof *By Holder's inequality and the norm inequality, we have* $\|x\|_{2,1}\le l^{1-\frac{1}{q}}\|x\|_{2,q}$. *Thus the right hand side of* (i) *is established.*

Now we want to show the left hand side of (i) *is also true. For any* $z\in R^l$ *with* $z_i\ge 0$, *setting* $f(z)=\|z\|_1-\alpha\|z\|_q=\sum_{i=1}^{l}z_i-\alpha(\sum_{i=1}^{l}z_i^q)^{\frac{1}{q}}$, *we can find that*

$$\bigtriangledown_{z_i}f(z_i)=1-\alpha z_i^{q-1}(\sum_{k=1}^{l}z_k^q)^{\frac{1}{q}-1}\ge 0,$$

where $z_i^{q-1}(\sum_{k=1}^{l}z_k^q)^{\frac{1}{q}-1}=(\frac{z_i}{\|z\|_q})^{q-1}\le 1$. *Hence,* $f(z)$ *is a monotonic increasing function with respect to* z_i. *Consequently*

$$f(z)\ge f(\min_{i\in[l]}z_i,\dots\min_{i\in[l]}z_i)$$

Thus $(l-\alpha l^{\frac{1}{q}})\min_{i\in[l]}\|x[i]\|_2\le\|x\|_{2,1}-\alpha\|x\|_{2,q}$.

(ii) *Taking* $x=x_{supp([x])}$, *according to* (i), (ii) *is obvious.*

Theorem 1 *For $t>0$, let x be any vector with block sparsity of s satisfying*

$$a(t,s)=\frac{\lceil ts\rceil^{\frac{1}{2}}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}{s^{\frac{1}{2}}+\alpha s^{\frac{1}{q}-\frac{1}{2}}}>1, \tag{3}$$

and let $y=Ax$. Suppose A satisfies the condition

$$\delta_{\lceil ts\rceil|I}+a(t,s)\delta_{\lceil ts\rceil+s|I}<a(t,s)-1. \tag{4}$$

then x is the unique solution to (1).

Proof *Let $\overline{x}$ be any feasible solution satisfying the constraint $A\overline{x}=y$ yet with a smaller objective value, i.e.,*

$$\|\overline{x}\|_{2,1}-\alpha\|\overline{x}\|_{2,q}<\|x\|_{2,1}-\alpha\|x\|_{2,q} \tag{5}$$

Set $\overline{x}=x+h$ with $h\in kerA$ and we will show that $h=0$. Suppose $supp([x])=S$, and write $h=h_S+h_{\overline{S}}$. It follows from (5) *that*

$$\|x+h_S+h_{\overline{S}}\|_{2,1}-\alpha\|x+h_S+h_{\overline{S}}\|_{2,q}<\|x\|_{2,1}-\alpha\|x\|_{2,q} \tag{6}$$

Note that

$$\begin{aligned}&\|x+h_S+h_{\overline{S}}\|_{2,1}-\alpha\|x+h_S+h_{\overline{S}}\|_{2,q}\\&=\|x+h_S\|_{2,1}+\|h_{\overline{S}}\|_{2,1}-\alpha\|x+h_S+h_{\overline{S}}\|_{2,q}\\&\ge\|x+h_S\|_{2,1}+\|h_{\overline{S}}\|_{2,1}-\alpha\|x\|_{2,q}-\alpha\|h_S\|_{2,q}-\alpha\|h_{\overline{S}}\|_{2,q}\\&\ge\|x\|_{2,1}-\|h_S\|_{2,1}+\|h_{\overline{S}}\|_{2,1}\\&-\alpha\|x\|_{2,q}-\alpha\|h_S\|_{2,q}-\alpha\|h_{\overline{S}}\|_{2,q}.\end{aligned} \tag{7}$$

Combining (6) *and* (7), *we obtain*

$$\|h_S\|_{2,1}+\alpha\|h_S\|_{2,q}\ge\|h_{\overline{S}}\|_{2,1}-\alpha\|h_{\overline{S}}\|_{2,q} \tag{8}$$

Arrange the block indices in $\overline{S}$ in order of decreasing $\|h[i]\|_2$ of $h_{\overline{S}}$ and divide $h_{\overline{S}}$ into block subsets of size $\lceil ts\rceil$. Then $\overline{S}=S_1\cup S_2\cup\cdots\cup S_\gamma$, where each S_i contains $\lceil tk\rceil$ block indices probably except S_γ. Denoting $S_0=S\cup S_1$ and using the block RIP of A, we have

$$\begin{aligned}0=\|Ah\|_{2,2}&=\|Ah_{S_0}+\sum_{i=2}^{\gamma}Ah_{S_i}\|_{2,2}\\&\ge\|Ah_{S_0}\|_{2,2}-\|\sum_{i=2}^{\gamma}Ah_{S_i}\|_2\|_{2,2}\\&\ge\sqrt{1-\delta_{\lceil ts\rceil}+s|I}\|h_{S_0}\|_{2,2}-\sqrt{1+\delta_{\lceil ts\rceil}|I}\sum_{i=2}^{\gamma}\|h_{S_i}\|_{2,2}\end{aligned} \tag{9}$$

On the other hand, for any $r\in S_i$, $i\ge 2$,

$$\|h[r]\|_2\le\min_{t\in S_{i-1}}\|h[t]\|_2\le\frac{\|h_{S_{i-1}}\|_{2,1}-\alpha\|h_{S_{i-1}}\|_{2,q}}{\lceil ts\rceil-\alpha\lceil ts\rceil^{\frac{1}{q}}},$$

where the second inequality use Lemma 1 *of* (ii). *This further more yields that*

$$\begin{aligned}\|h_{S_i}\|_{2,2}&\le\sqrt{\lceil ts\rceil}\frac{\|h_{S_{i-1}}\|_{2,1}-\alpha\|h_{S_{i-1}}\|_{2,q}}{\lceil ts\rceil-\alpha\lceil ts\rceil^{\frac{1}{q}}}\\&=\frac{\|h_{S_{i-1}}\|_{2,1}-\alpha\|h_{S_{i-1}}\|_{2,q}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=2}^{\gamma}\|h_{S_i}\|_{2,2}&\le\sum_{i=1}^{\gamma-1}\frac{\|h_{S_i}\|_{2,1}-\alpha\|h_{S_i}\|_{2,q}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}\\&\le\frac{\sum_{i=1}^{\gamma}\|h_{S_i}\|_{2,1}-\sum_{i=1}^{\gamma}\alpha\|h_{S_i}\|_{2,q}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}\end{aligned} \tag{10}$$

Note that

$$\sum_{i=1}^{\gamma}\|h_{S_i}\|_{2,1}=\|h_{\overline{S}}\|_{2,1},\ \sum_{i=1}^{\gamma}\|h_{S_i}\|_{2,q}\ge\|h_{\overline{S}}\|_{2,q} \tag{11}$$

it follows from (10) *and* (11), *that*

$$\sum_{i=2}^{\gamma}\|h_{S_i}\|_{2,2}\le\frac{\|h_{\overline{S}}\|_{2,1}-\alpha\|h_{\overline{S}}\|_{2,q}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}. \tag{12}$$

Combining (8) *and* (12), *we get*

$$\sum_{i=2}^{\gamma}\|h_{S_i}\|_{2,2} \le \frac{\|h_{\overline{S}}\|_{2,1}-\alpha\|h_{\overline{S}}\|_{2,q}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}$$
$$\le \frac{\|h_S\|_{2,1}+\alpha\|h_S\|_{2,q}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}} \le \frac{(\sqrt{s}+\alpha s^{\frac{1}{q}-\frac{1}{2}})\|h_S\|_{2,2}}{\sqrt{\lceil ts\rceil}-\alpha\lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}} \quad (13)$$
$$= \frac{\|h_S\|_{2,2}}{\sqrt{a(t,s)}}$$

(9) *and* (13) *yields that*

$$0 \ge \sqrt{1-\delta_{s+\lceil ts\rceil|I}}\|h_{S_0}\|_{2,2} - \frac{\sqrt{1+\delta_{\lceil ts\rceil|I}}}{\sqrt{a(t,s)}}\|h_S\|_{2,2}$$
$$\ge \sqrt{1-\delta_{s+\lceil ts\rceil|I}}\|h_{S_0}\|_{2,2} - \frac{\sqrt{1+\delta_{\lceil ts\rceil|I}}}{\sqrt{a(t,s)}}\|h_{S_0}\|_{2,2}. \quad (14)$$

Since (4) *implies* $\sqrt{1-\delta_{s+\lceil ts\rceil|I}} - \frac{\sqrt{1+\delta_{\lceil ts\rceil|I}}}{\sqrt{a(t,s)}} > 0$, *then we have* $\|h_{S_0}\|_{2,2} = 0$, *and hence, we have* $h = 0$. □

Theorem 2 *Under the assumptions of Theorem 1 except that* $y = Ax + e$, *where* $e \in R^m$ *is any perturbation with* $\|e\|_2 \le \epsilon$, *we have that the solution* $\overline{x}$ *to* (2) *subject to* $\|\overline{x} - x\|_2 \le C\epsilon$ *for some constant* $C > 0$ *depengding on* $\delta_{s+\lceil ts\rceil|I}$ *and* $\delta_{\lceil ts\rceil|I}$.

Proof *Setting* $\overline{x} = x + h$, $supp([x]) = S$, *similar to the proof of Theorem 1, we have*

$$\sum_{i=2}^{\gamma}\|h_{S_i}\|_{2,2} \le \frac{\|h_S\|_{2,2}}{\sqrt{a(s,k)}}, \quad (15)$$

and

$$\|Ah\|_{2,2} \ge \left(\sqrt{1-\delta_{k+\lceil tk\rceil|I}} - \frac{\sqrt{1+\delta_{\lceil tk\rceil|I}}}{\sqrt{a(t,s)}}\right)\|h_{S_0}\|_{2,2} \quad (16)$$

therefor

$$\|h\|_{2,2} = \sqrt{\|h_{S_0}\|_{2,2}^2 + \sum_{i=2}^{l}\|h_{S_i}\|_{2,2}^2}$$
$$\le \sqrt{\|h_{S_0}\|_{2,2}^2 + \frac{\|h_S\|_{2,2}^2}{a(t,s)}} \le \sqrt{1+\frac{1}{a(t,s)}}\|h_{S_0}\|_{2,2}. \quad (17)$$

In addition, we have

$$\|Ah\|_{2,2} = \|A\overline{x} - y - (Ax - y)\|_{2,2}$$
$$\le \|Ax - y\|_{2,2} + \|A\overline{x} - y\|_{2,2} \quad (18)$$
$$= \|Ax - y\|_2 + \|A\overline{x} - y\|_2 \le 2\epsilon.$$

it follows from (16) *and* (18)

$$\|h_{S_0}\|_{2,2} \le 2\left(\sqrt{1-\delta_{s+\lceil ts\rceil|I}} - \frac{\sqrt{1+\delta_{\lceil ts\rceil|I}}}{\sqrt{a(t,s)}}\right)^{-1}\epsilon. \quad (19)$$

(17) *and* (19) *yield that*

$$\|\overline{x} - x\|_2 = \|h\|_2 = \|h\|_{2,2}$$
$$\le 2\sqrt{1+\frac{1}{a(t,s)}}\left(\sqrt{1-\delta_{k+\lceil tk\rceil|I}} - \frac{\sqrt{1+\delta_{\lceil tk\rceil|I}}}{\sqrt{a(t,s)}}\right)^{-1}\epsilon := C\epsilon.$$

□

Remark 1 *It is worth noting that if* α *monotonically decreases, then* $a(t,s)$ *becomes larger. Therefore, the smaller* α, *the easier it is to achieve condition* (3), *but condition* (4) *may not necessarily be also easier to achieve.*

Corollary 1 *If* $\alpha = 1$ *in the* $\ell_1 - \alpha\ell_q$*-minimization model, the block* $\ell_1 - \alpha\ell_q$*-minimization model has a unique solution* x *with block sparsity* s *if the vector* x *satisfying*

$$a(t,s) = \frac{\lceil ts\rceil^{\frac{1}{2}} - \lceil ts\rceil^{\frac{1}{q}-\frac{1}{2}}}{s^{\frac{1}{2}} + s^{\frac{1}{q}-\frac{1}{2}}} > 1, \quad (20)$$

and matrix A *satisfies the condition*

$$\delta_{\lceil ts\rceil|I} + a(t,s)\delta_{\lceil ts\rceil+s|I} < a(t,s) - 1. \quad (21)$$

Corollary 2 *Under the condition* (20), (21) *and if* $\alpha = 1$ *in the minimization model* (2) *then the model* (2) *obeys* $\|\overline{x} - x\|_2 \le C\epsilon$ *for some constant* $C > 0$ *depengding on* $\delta_{s+\lceil ts\rceil|I}$ *and* $\delta_{\lceil ts\rceil|I}$.

3 Conclusion

From this paper, we find that based on some condition of block RIP, the block $\ell_1 - a\ell_q$-minimization model can exactly recover block s-sparse signals in noiseless cases and stably recover block s-sparse signals in the noise cases.

DECLARATIONS

The authors have no relevant financial or non-financial interests to disclose. All authors contributed to the study conception and proof. The authors declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted. This study focuses on theoretical analysis and does not involve ethical issues. All data generated or analysed during this study are included in this published article. The corresponding authors of this paper is Shaohua Xie.

References

[1] S. Foucart, H. Rauhut. A mathematical introduction to compressive sensing, Applied and Numerical Harmonic Analysis Series, New York: Birkh auser/Springer, 2013.

[2] D. Donoho. Stable recovery of sparse overcomplete representations in the presence of noise. IEEE Trans. Information Theory, 2006, 52(1):6-18.

[3] Y. Zhao. RSP-Based analysis for sparsest and lease ℓ_1 norm solutions to underdetermined linear systems. IEEE Transactions on Signal Processing, 2013, 61(22):5777-5788.

[4] N. Bi, W. Tang. A necessary and sufficient condition for sparse vector recovery via $\ell_1 - \ell_2$ minimization. Applied and Computational Harmonic Analysis, 2022, 56:337-350.

[5] Y. Lou, Y. Qi, J. Xin. Minimization of l(1-2) for compressed sensing. SIAM Journal on Scientific Computing, 2015, 37(1):A536-A563.

[6] J. Huang, J. Wang, W. Wang, et al. Sharp sufficient condition of block signal recovery via l_2/l_1-minimization. IET Signal Processing, 2017, 13(5):495-505.

[7] N. Bi, W. Tang. A necessary and sufficient condition for sparse vector recovery via $\ell_1 - \ell_2$ minimization, Applied and Computational Harmonic Analysis,2022, 56, :337350.

[8] R. Zhang, S. Li. A proof of conjecture on restricted isometry property constants. IEEE Transactions on Information Theory, 2018, 64(3), 1699-1705.

[9] S. Foucart, M. Lai. Sparsest solutions of undetermined linear systems via ℓ_p-minimization for $0 < q \le 1$

[10] S Xie, K Liang. k block sparse vector recovery via block L1-L2 minimization. Circuits, systems, and signal processing: CSSP (2023).