Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

Global and blow up solutions for a semilinear heat equation with variable reaction reaction on a general domain

Ricardo Castillo[1] | Miguel Loayza*[2]

[1]Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile
[2]Departamento de Matemática, Universidade Federal de Pernambuco, Pernambuco, Brasil

Correspondence
*Corresponding author name. Email: miguel.loayza@ufpe.br

Summary

We are concerned with the existence of global and blow-up solutions for the semilinear heat equation with variable exponent $u_t - \Delta u = h(t) f(u)^{p(x)}$ in $\Omega \times (0, T)$ with zero Dirichlet boundary condition and initial data in $C_0(\Omega)$. The scope of our analysis encompasses both bounded and unbounded domains, with $p(x) \in C(\Omega), 0 < p^- \leq p(x) \leq p^+$, $h \in C(0, \infty)$, and $f \in C[0, \infty)$. Our findings have significant implications, as they enhance the blow-up result discovered by Castillo and Loayza in Comput. Math. App. 74(3), 351-359 (2017) when $f(u) = u$.

KEYWORDS:
Semilinear heat equation, Global Solution, Blow up solution, Variable exponent, Arbitrary domain

1 | INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a domain (bounded or unbounded) with smooth boundary $\partial\Omega$. We consider the semilinear parabolic problem

$$\begin{cases} u_t - \Delta u = h(t) F(x, u) & \text{in } \Omega \times (0, T), \\ \quad\quad u = 0 & \text{on } \partial\Omega \times (0, T), \\ \quad u(0) = u_0 \geq 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $F(x, s) = f(s)^{p(x)}$, for $x \in \Omega$, $s \geq 0$, $f \in C[0, \infty)$ is a nondecreasing locally Lipschitz function, $h \in C(0, \infty)$, $p \in C(\Omega)$ is a bounded function such that

$$0 < p^- \leq p(x) \leq p^+ < \infty, \tag{2}$$

for all $x \in \Omega$, with $p^- = \inf_{x\in\Omega}\{p(x)\}$, $p^+ = \sup_{x\in\Omega}\{p(x)\}$, and $u_0 \in C_0(\Omega)$. Here, $C_0(\Omega)$ denotes the closure in $L^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω. Throughout the work we consider only nonnegative solutions in the sense of (11).

Problem (1) appears in several models of the applied sciences such as electrorheological fluids[22], thermo-rheological fluids[3], image processing[1,5], chemical reactions, heat transfer and population dynamics[12]. It has been considered for many authors. For example, when Ω is a bounded domain and $h(t) = 1$, blow up results for problem (1) were obtained in[13] for $F(x, s) = e^{p(x)s}$, and in[21] for $F(x, u) = a(x)u^{p(x)}$. When $\Omega = \mathbb{R}^N$, Fujita type results were obtained in[14] for $F(x, s) = s^{p(x)}$, $h(t) = 1$. Specifically, in the last case it was shown that:

- If $p^- > 1 + 2/N$, then problem (1) possesses global nontrivial solutions.
- If $1 < p^- < p^+ \leq 1 + 2/N$, then all nontrivial solutions to problem (1) blow up in finite time.
- If $p^- < 1+2/N < p^+$, then there are functions p such that problem (1) possesses global nontrivial solutions and functions p such that all nontrivial solutions blow up.

[0]**MSC:** 35B33, 35B44, 35K15, 35K55, 35K57

These results were extended for any domain Ω (bounded or unbounded); see Theorem 1.2 and Remark 1.3 of[9]. Specifically, they showed the following result.

Theorem 1. Suppose that $F(x,s) = s^{p(x)}$ for $s \geq 0$.

(i) If $p^+ \leq 1$, then all solutions of problem (1) are global.

(ii) If $p^+ > 1$ and

$$\limsup_{t\to\infty} \|S(t)u_0\|_\infty^{p^+-1} \int_0^t h(\sigma)d\sigma = \infty, \tag{3}$$

for every nonnegative $0 \neq u_0 \in C_0(\Omega)$, then every nontrivial solution of problem (1) either blow up in finite time or in infinite time. In the last case, we mean that the solution is global and $\limsup_{t\to\infty} \|u(t)\|_\infty = \infty$.

(iii) If $p^- > 1$ and there exists $w_0 \in C_0(\Omega)$, $w_0 \geq 0$, $w_0 \neq 0$ verifying

$$\int_0^\infty h(\sigma)\|S(t)w_0\|_\infty^{p^--1} < \infty, \tag{4}$$

then there exists a constant $\Lambda > 0$, depending on p^+ and p^-, so that if $0 < \lambda < \Lambda$, then the solution of (1), with initial data λw_0, is a nontrivial global solution.

Notice that the conditions (3) and (4) of Theorem 1 are expressed in terms of the asymptotic behavior of $\|S(t)u_0\|_\infty$, where $\{S(t)\}_{t\geq 0}$ denotes the heat semigroup. The first result of this type was given by Meier[19] for problem (1) in the case $F(x,s) = s^p, s \geq 0, p > 1$. It is important because the conditions are valid for any domain Ω, bounded or unbounded, and because it is sufficient to know the behavior of $\|S(t)u_0\|_\infty$ to decide whether the solution of problem (1) is global or not. For example, we know, in $\mathbb{R}^N$, that $\|S(t)u_0\|_\infty \sim t^{-N/2}$ for t near infinity and $u_0 \in C_0(\mathbb{R}^N), u_0 \neq 0$. Thus, assuming $h = 1$, condition (3) holds if $p^+ < 1 + 2/N$, while condition (4) holds if $p^- > 1 + 2/N$. This coincides with the results obtained in[14]. Similar results have been obtained for parabolic coupled system related to problem (1) in[7],[8] and[10].

The main objective of this work is to obtain Meier type results, similar to Theorem 1, for problem (1) considering $F(x,s) = f(s)^{p(x)}$, where $f \in C[0,\infty)$ is a locally Lipschitz and nondecreasing function, and $p \in C(\Omega)$ satisfies condition (2). We also analyze situations where $p(x) < 1$ or $p(x) > 1$ on subdomains of Ω. As a consequence of our results, we improve Theorem 1 (ii) and remove the possibility of the existence of solutions that blow up in infinite time, see Remark 2-(vi).

Our results depend on the conditions:

$$\int_\alpha^\infty \frac{d\sigma}{\min\{f(\sigma)^{p^-}, f(\sigma)^{p^+}\}} < \infty, \tag{5}$$

for some $\alpha \geq 0$ such that $f(\alpha) > 0$, and

$$\int_\tau^\infty \frac{d\sigma}{\max\{f(\sigma)^{p^-}, f(\sigma)^{p^+}\}} = \infty, \tag{6}$$

for all $\tau > 0$ with $f(\tau) > 0$.

Note that if $F(x,s) = f(s)$ and $h = 1$, condition (5) turns into

$$\int_w^\infty \frac{d\sigma}{f(\sigma)} < \infty, \tag{7}$$

which is well known as a necessary and sufficient condition for the existence of blow up solutions. Some examples of a function f satisfying condition (7) are $f(u) = u^q, f(u) = (1+u)[\ln(1+u)]^q, f(u) = e^{\alpha u} - 1$ for $q > 1$ and $\alpha > 0$.

In our first result we use condition (6) to get global solutions for problem (1).

Theorem 2. Assume that condition (6) holds with $p^- < 1$. Then for every $u_0 \in C_0(\Omega), u_0 \geq 0$ there exists a global solution of problem (1).

Moreover, u is a positive if

(i) $f(0) > 0$ or $u_0 \neq 0$ or

(ii) $u_0 = 0$ with the additional assumptions:

  (a) $f(0) = 0$, $f(s) > 0$ in $(0, \tau)$ for some $\tau > 0$.

  (b) $p(x) \leq \gamma < 1$ for some subdomain $\Omega' \subset \Omega$.

  (c) $\int_0^\tau \frac{d\sigma}{f(\sigma)^\gamma} < \infty$ for some $\tau > 0$.

  Moreover, $u(t) \geq \mu(t)\chi_{B(x_0,r)}$ on some interval $[0, \tau_1]$, $\tau_1 \leq \tau$, $r > 0$ such that $B_{r+2\delta}(x_0) \subset \Omega'$, $\delta > 0$, and $\mu \in C([0, \tau_1], [0, \infty))$ is a positive solution of the Cauchy problem:

$$x_t = \frac{c_N}{2^N} h(t) f^\gamma(x), \; x(0) = 0, \tag{8}$$

  where c_N is the constant given in Lemma 1. Here $\chi_{B_r(x_0)}$ denotes the characteristic function on the open ball centered at x_0 and radius $r > 0$.

Remark 1. Here are some comments about Theorem 2.

(i) Condition $f(0) = 0$ implies that $u = 0$ is a solution of problem (8) and assumption $\int_0^\tau d\sigma / f(\sigma)^\gamma < \infty$ guarantees the existence of a positive solution of problem (8).

(ii) The existence of a positive solution of (1) with $u_0 = 0$, for $f(s) = s, h = 1$, it was shown in[14] considering a subsolution of the form $w(t) = Ct^{1/(1-\gamma)}\varphi_1$ for an appropriate constant $C > 0$ and $\varphi_1 > 0$ the first eigenfunction of the Laplacian operator on $H_0^1(\Omega')$. Here, we use the subsolution $w = \mu(\cdot)\chi_r$ of problem $u_t - \Delta u = h(t)f(u)^\gamma$ in $\Omega' \times (0, \tau_1)$. This idea was used firstly in[17].

(iii) For $f(s) = s$, $p(x) = p \in (0, 1)$ constant, $h = 1$ and $\Omega = \mathbb{R}^N$, the function $u(t) = [(1 - p)t]^{1/(1-p)}, t > 0$, is the positive solution of problem (1) ($u_0 = 0$) which is obtained solving the Cauchy problem: $x_t = x^p, \; x(0) = 0$, see[2] and[11].

(iv) When $F(x, s) = s^{p(x)}, s \geq 0$ and $0 < \tau < 1$ we have

$$+\infty = \int_\tau^\infty \frac{d\sigma}{\max\{\sigma^{p^-}, \sigma^{p^+}\}} = \int_\tau^1 \frac{d\sigma}{s^{p^-}} + \int_1^\infty \frac{d\sigma}{s^{p^+}}$$

  if and only if $p^+ \leq 1$. Thus Theorem 2 coincides with Theorem 1(i).

In our second result we use condition (5) to obtain blow up solutions.

Theorem 3. (i) (Global existence) Let $\mathcal{F} : (0, m] \to [0, \infty)$ be defined by $\mathcal{F}(s) = \frac{1}{s}\max\{f(s)^{p^-}, f(s)^{p^+}\}$ for $s \in (0, m]$. Assume that $\mathcal{F}$ is a nondecreasing function and there exists $v_0 \in C_0(\Omega), 0 \neq v_0 \geq 0, \|v_0\|_\infty \leq m$ satisfying

$$\int_0^\infty h(\sigma)\mathcal{F}\left(\|S(\sigma)v_0\|_\infty\right) d\sigma < 1.$$

  Then there exists a constant $\delta > 0$ such that for $u_0 = \delta v_0$ the solution of problem (1) is a global solution.

(ii) (Nonglobal existence) Assume that $f(0) = 0$, condition (5) holds, $p^- \geq 1$ and the following assumptions are satisfied:

  (a) $f(s) > 0$ for all $s > 0$, and

$$f(S(t)v_0) \leq S(t)f(v_0), \tag{9}$$

  for all $0 \leq v_0 \in C_0(\Omega)$ and $t > 0$.

  (b) There exist $\tau > 0$ such that

$$\int_{\|S(\tau)u_0\|_\infty}^\infty \frac{d\sigma}{\min\{f(\sigma)^{p^-}, f(\sigma)^{p^+}\}} \leq 2^{-p^+} \int_0^\tau h(\sigma)d\sigma. \tag{10}$$

  Then the solution of problem (1) with initial condition $u_0 \geq 0, u_0 \neq 0$ blows up in finite time.

Remark 2. Here are some comments about Theorem 3.

(i) If $f(0) = 0$ and $p^- \geq 1$, then $\mathcal{F}$ is well defined, since f is locally Lipschitz, and if we assume additionally that f is a convex function we have that $\mathcal{F}$ is nondecreasing.

(ii) Condition $f(0) = 0$ is used in inequality (9) because the Dirichlet condition on the boundary must be satisfied.

(iii) Constant 2^{-p^+} in inequality (10) appears due to Jensen's inequality, see Lemma 2.

(iv) Condition (9) holds for any convex function f when $\Omega = \mathbb{R}^N$. This is a consequence of Jensen's inequality and the representation of the semigroup $S(t)u_0 = K_t \star u_0$, where $K_t = (4\pi t)^{-N/2} \exp(-|x|^2/(4t))$ is the heat kernel.

(v) When Ω is any domain, condition (9) holds for any twice differentiable and convex function with $f(0) = 0$. Indeed, if $v(t) = f(S(t)u_0)$ then

$$v_t - \Delta v = -f''(S(t)u_0)|\nabla S(t)u_0|^2 \leq 0$$

in $\Omega \times (0, \infty)$ and $v(t) = f(0) = 0$ on $\partial\Omega \times (0, \infty)$. Since $v(0) = f(u_0)$ we conclude by the maximum principle.

(vi) Theorem 3 improves Theorem 1(ii) if $p^- > 1$, $f(s) = s$ and condition (3) holds. Indeed, since $p^- > 1$ the condition (5) is verified. Thus, it is sufficient to check the condition (10). First, note that

$$\int_\alpha^\infty \frac{d\sigma}{\min\{\sigma^{p^-}, \sigma^{p^+}\}} \leq \frac{p^+ - p^-}{(p^+ - 1)(p^- - 1)} + \frac{\alpha^{1-p^+}}{p^+ - 1},$$

for every $\alpha > 0$. From condition (3) there exists $\tau > 0$ such that

$$\frac{p^+ - p^-}{(p^+ - 1)(p^- - 1)} \|u_0\|_\infty^{p^+-1} + \frac{1}{p^+ - 1} \leq \left(\frac{1}{2}\right)^{p^+} \|S(\tau)u_0\|_\infty^{p^+-1} \int_0^\tau h(\sigma) d\sigma.$$

Hence,

$$\begin{aligned} &\int_{\|S(\tau)u_0\|_\infty}^\infty \frac{d\sigma}{\min\{\sigma^{p^-}, \sigma^{p^+}\}} \\ &\leq \|S(\tau)u_0\|_\infty^{1-p^+} \left[\frac{p^+-p^-}{(p^+-1)(p^--1)} \|S(\tau)u_0\|_\infty^{p^+-1} + \frac{1}{p^+-1}\right] \\ &\leq \|S(\tau)u_0\|_\infty^{1-p^+} \left[\frac{p^+-p^-}{(p^+-1)(p^--1)} \|u_0\|_\infty^{p^+-1} + \frac{1}{p^+-1}\right] \\ &\leq 2^{-p^+} \int_0^\tau h(\sigma) d\sigma. \end{aligned}$$

By Theorem 3, u blows up in finite time.

In the proof of Theorem 3, we adapt the techniques used in [18]. It is worth noting that in that work, the authors utilized their findings to derive Fujita exponents for the problem (1) with $F(x, u) = (1 + u)(\ln(u + 1))^q$ and $F(x, u) = e^{\alpha u} - 1$. Theorem 3 can also be applied to obtain Fujita-type results for problem (1) with more complex source terms and on different domains Ω. This may include the logarithmic function with variable exponent $[(1 + u)(\ln(u + 1))^q]^{p(x)}$ and the exponential with variable exponent $[e^{\alpha u} - 1]^{p(x)}$.

It is important always to be aware that solutions may blow up in a finite time when dealing with large initial data. This was demonstrated in [14, Theorem 3.3] using Kaplan's argument [15]. Our next Theorem shows how this approach can be modified to present a similar result. We will focus on the scenario where $h = 1$ for simplicity.

Theorem 4. Suppose that $p^+ > 1$, $h = 1$ and there exists a bounded subdomain $\Omega' \subset \Omega$ such that $p(x) \geq \gamma > 1$ for all $x \in \Omega'$. Assume also that f is a convex function such that $\int_\tau^\infty d\sigma/f(\sigma)^\gamma < \infty$ for some $\tau > 0$ with $f(\tau) > 0$. Then there are solutions of problem (1) such that blow up in finite time.

Remark 3. Theorem 4 for $f(s) = s$ was established in [14, Theorem 3.3].

The rest of the paper is organized as follows. Section 2 is dedicated to analyze the existence of positive global solution and Theorem 2 is proved. Blow up for large initial data is shown in Section 3. Section 4 is devoted to the proof of Theorem 3.

2 | EXISTENCE AND UNIQUENESS

Solutions of problem (1) are understood in the following sense: given $u_0 \in C_0(\Omega)$, a function $u \in C([0,T), C_0(\mathbb{R}^N))$ is said to be a solution of problem (1) in $(0,T)$ if u is nonnegative and verifies the following equation

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)F(\cdot, u(\sigma))d\sigma \tag{11}$$

for all $t \in (0,T)$, where $F(x,u) = f(u)^{p(x)}$.

Since $f \in C[0,\infty)$ is a locally Lipschitz function, it is clear that if $p(x) \geq 1$, the nonlinear term $F(x,u)$, for $x \in \Omega$ fixed, is a locally Lipschitz function. Thus, using usual methods it is possible to show the existence of a unique local solution of (1) defined in some interval $[0,T]$. Moreover, this solution can be extended to a maximal interval $[0,T_{\max})$ and the blow up alternative occurs: either $T_{\max} = +\infty$ (we say that u is a global solution) or $T_{\max} < \infty$ and $\limsup_{t\to T_{\max}} \|u(t)\|_\infty = +\infty$. In the last case, we say that the solution blows up in a finite time, see for example[6], [14], [4] and[9].

When $p(x) < 1$ on some subdomain of Ω, the function $F(x,u)$ is not locally Lipschitz (for x fixed), and we can use an approximation method to find a solution; see problem (12). We give more details in the proof of Theorem 2 below.

The existence of a positive solution of problem (1) for $u_0 = 0$ is proved with the aid of the following result given in[16, Lemma 2.1].

Lemma 1. There exists a constant c_N, which depend only on N, such that for any $r, \delta > 0$ with $B_{r+2\delta} = B(0, r+2\delta) \subset \Omega$,

$$S(t)\chi_r \geq c_N \left(\frac{r}{r+\sqrt{t}}\right)^N \chi_{r+\sqrt{t}}$$

for all $0 < t \leq \delta^2$.

Proof of Theorem 2 *Local existence.* We use a standard approximation method, see for instance[20]. For every $\epsilon > 0$, let $F_\epsilon : \Omega \times [0,\infty) \to [0,\infty)$ be defined by

$$F_\epsilon(x,s) = \begin{cases} f(s)^{p(x)} & \text{if } s \geq \epsilon \text{ or } p(x) \geq 1, \\ f(\epsilon)^{p(x)-1} f(s) & \text{if } 0 \leq s < \epsilon \text{ and } p(x) < 1. \end{cases}$$

Note that since we are assuming $p^- < 1$ there exists a subdomain of Ω where $p(x) < 1$.

The function $F_\epsilon(x,\cdot)$ is locally Lipschitz for every $x \in \Omega$. Let u^ϵ be a solution of the problem

$$\begin{cases} u_t - \Delta u = h(t)F_\epsilon(x,u) & \text{in } \Omega \times (0,T), \\ u = \epsilon & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 + \epsilon & \text{in } \Omega, \end{cases} \tag{12}$$

defined on a maximal interval $[0, T^\epsilon_{\max})$. We know that the blow-up alternative occurs, that is, either $T^\epsilon_{\max} = \infty$ or $T^\epsilon_{\max} < \infty$ and $\limsup_{t\to T^\epsilon_{\max}} \|u^\epsilon(t)\|_\infty = \infty$. Since $u = \epsilon$ is a subsolution to problem (12), by a comparison principle we conclude that $u^\epsilon \geq \epsilon$. Note that if $\epsilon_1 < \epsilon_2$ then $F_{\epsilon_2}(\cdot, u^{\epsilon_2}) = F_{\epsilon_1}(\cdot, u^{\epsilon_2})$ and u^{ϵ_2} is a supersolution to problem (12) (with $\epsilon = \epsilon_1$). Hence, by a comparison principle we have $u^{\epsilon_1} \leq u^{\epsilon_2}$ in $[0, T^{\epsilon_2}_{\max})$. Thus, we can define $u = \lim_{\epsilon\to 0} u^\epsilon$ on $[0, T^{\epsilon_0}_{\max})$ for some $\epsilon_0 > 0$.

Global existence. By the existence part we observe that it is sufficient to show that $T^\epsilon_{\max} = \infty$ for some $\epsilon > 0$ sufficiently small. Since u^ϵ is a solution of problem (12) and $u^\epsilon(t) \geq \epsilon$ we obtain

$$u^\epsilon(t) = S(t)u(0) + \epsilon + \int_0^t h(\sigma)S(t-\sigma)[f(u^\epsilon(\sigma))]^{p(x)}d\sigma, \tag{13}$$

for $t \in (0, T^\epsilon_{\max})$. Hence

$$\|u^\epsilon(t)\|_\infty \leq \|u_0\|_\infty + \epsilon + \textstyle\int_0^t h(\sigma)\|[f(u^\epsilon(\sigma))]^{p(x)}\|_\infty d\sigma.$$

Using the fact that f is nondecreasing we have that $f(u^\epsilon(\sigma)) \leq f(\|u^\epsilon(\sigma)\|_\infty)$, and hence

$$\begin{aligned} \|[f(u^\epsilon(\sigma))]^{p(x)}\|_\infty &\leq \|[f\left(\|u^\epsilon(\sigma)\|_\infty\right)]^{p(x)}\|_\infty \\ &\leq \max\{[f(\|u^\epsilon(\sigma)\|_\infty)]^{p^-}, [f(\|u^\epsilon(\sigma)\|_\infty)]^{p^+}\}. \end{aligned}$$

Thus,

$$\|u^\epsilon(t)\|_\infty \le \|u_0\|_\infty + \epsilon + \int_0^t h(\sigma)\max\{[f(\|u^\epsilon(\sigma)\|_\infty)]^{p^-}, [f(\|u^\epsilon(\sigma)\|_\infty)]^{p^+}\}d\sigma.$$

Set

$$\begin{aligned}\Psi(t) &= \|u_0\|_\infty + \epsilon + \textstyle\int_0^t h(\sigma)\max\{[f(\|u^\epsilon(\sigma)\|_\infty]^{p^-}), [f(\|u^\epsilon(\sigma)\|_\infty]^{p^+})\}d\sigma \text{ and}\\ g_1(t) &= \max\{[f(t)]^{p^-}, [f(t)]^{p^+})\}.\end{aligned}$$

Then, $\|u^\epsilon(t)\|_\infty \le \Psi(t)$ and

$$\begin{aligned}\Psi'(t) &= h(t)\max\{[f(\|u^\epsilon(t)\|_\infty]^{p^-}), [f(\|u^\epsilon(t)\|_\infty]^{p^+})\}\\ &\le h(t)\max\{[f(\Psi(t))]^{p^-}, [f(\Psi(t))]^{p^+}\}.\end{aligned}$$

Fix $\tau \in (0, \min\{\epsilon, T^\epsilon_{\max}\})$ such that $f(\tau) > 0$ and condition (6) holds. Defining $H(t) = \int_\tau^t d\sigma/g_1(\sigma)$, for $t \ge \tau$, we obtain $(H\circ\Psi)'(t) \le h(t)$ for $t \in (0, T^\epsilon_{\max})$. Thus,

$$\int_\tau^{\|u^\epsilon(t)\|_\infty} \frac{d\sigma}{g_1(\sigma)} \le \int_\tau^{\Psi(t)} \frac{d\sigma}{g_1(\sigma)} \le \int_0^t h(\sigma)d\sigma + H(\Psi(0)), \tag{14}$$

for $t \in (0, T^\epsilon_{\max})$. From this inequality, we concluded that $T^\epsilon_{\max} = \infty$, since if $T^\epsilon_{\max} < \infty$ we have that $\limsup_{t\to T^\epsilon_{\max}} \|u^\epsilon(t)\|_\infty = +\infty$, which contradicts condition (6).

Existence of a positive solution. (i) If $u_0 \ge 0$ and $u_0 \ne 0$, the result follows from (11) and the strong maximum principle, since $u(t) \ge S(t)u_0 > 0$ for $t > 0$.

Assume now that $f(0) > 0$. Without loss of generality we may assume that $0 \in \Omega$ and $B_{r+\delta} \subset \Omega$ for some $r > 0$ and $\delta > 0$, where $B_{r+2\delta} = B_{r+2\delta}(0)$. Since u_0 and u are nonnegatives, and f is nondecreasing, from (11) we have

$$\begin{aligned}u(t) &\ge \textstyle\int_0^t h(\sigma)S(t-\sigma)[f(u(\sigma))]^{p(x)}d\sigma\\ &\ge \textstyle\int_0^t h(\sigma)S(t-\sigma)f(0)^{p(x)}d\sigma\\ &\ge \min\{f(0)^{p^-}, f(0)^{p^+}\}\textstyle\int_0^t h(\sigma)S(t-\sigma)\chi_r d\sigma,\end{aligned}$$

where $\chi_r = \chi_{B_r}$. Let $\varphi_{1,r} > 0$ be the first eigenfunction of the Laplacian operator on $H^1_0(B_r)$ associated to the first eigenvalue $\lambda_{1,r} > 0$. Since $\chi_r \ge C\varphi_{1,r}$ for some constant $C > 0$, we have that $S(t-\sigma)\chi_r \ge Ce^{-(t-\sigma)\lambda_{1,r}}\varphi_{1,r}$, and thus

$$u(t) \ge C\min\{f(0)^{p^-}, f(0)^{p^+}\}e^{-\lambda_{1,r}t}\varphi_{1,r}\int_0^t h(\sigma)d\sigma > 0$$

on $B_r(0) \times (0, \infty)$.

Using again (11) it is possible to show that $u(t) \ge S(t-s)u(s)$ for $t \ge s > 0$. Thus, since $0 \ne u(s) \ge 0$, by the strong maximum principle, we have that $u(t) > 0$ for $t \ge s > 0$. Letting $s \to 0$ we get the result.

(ii) When $u_0 = 0$, from (14) we have that

$$\|u^\epsilon(t)\|_\infty \le H^{-1}\left(\int_0^t h(\sigma)d\sigma + H(\epsilon)\right),$$

for $t \in (0, T^\epsilon_{\max})$. Thus, $f(u^{\epsilon_0}(t)) \le f(\|u^{\epsilon_0}(t)\|_\infty) \le 1$ for $t \in [0, T]$ with $T = T(\epsilon_0) > 0$ small and some $\epsilon_0 > 0$.

On the other hand, since $p^- < 1$, there exists a subdomain $\Omega' \subset \Omega$ so that $p(x) \le \gamma < 1$ for $x \in \Omega'$. Assume that $0 \in \Omega'$ and that the ball $B_{r+2\delta} \subset \Omega'$ for some $r, \delta > 0$. Since $\{u^\epsilon\}$ is nonincreasing in ϵ we have that $f(u^\epsilon(t)) \le f(u^{\epsilon_0}(t)) \le 1$ for $0 < \epsilon \le \epsilon_0$ and $0 \le t \le T$. Thus, from (13)

$$\begin{aligned}u^\epsilon(t) &\ge \textstyle\int_0^t h(\sigma)S(t-\sigma)\left\{[f(u^\epsilon(\sigma))]^{p(x)}\chi_r\right\}d\sigma\\ &\ge \textstyle\int_0^t h(\sigma)S(t-\sigma)\left\{[f(u^\epsilon(\sigma))]^{\gamma}\chi_r\right\}d\sigma.\end{aligned} \tag{15}$$

It is well known that condition $\int_0^\tau d\sigma/[f(\sigma)]^\gamma < \infty$ assures that the solution μ of the Cauchy problem (8) is continuous and positive in some interval $[0, \tau_1]$. Since $f(0) = 0$ and $\mu(0) = 0$, it is possible to choose $\tau_2 \in (0, \tau_1)$ so that $f(\mu(t)) \le 1$ for

$t \in (0, \tau_2)$. Thus by Lemma 1

$$\begin{aligned}
&\int_0^t h(\sigma)S(t-\sigma)[f(w(\sigma))]^\gamma \chi_r d\sigma \\
&= \int_0^t h(\sigma)S(t-\sigma)[f(\mu(\sigma)\chi_r)]^\gamma \chi_r d\sigma \\
&= \int_0^t h(\sigma)[f(\mu(\sigma))]^\gamma S(t-\sigma)\chi_r d\sigma \\
&\geq c_N \int_0^t h(\sigma)[f(\mu(\sigma))]^\gamma \left(\frac{r}{\sqrt{t-\sigma}+r}\right)^N \chi_{r+\sqrt{t-\sigma}}\, d\sigma \\
&\geq \frac{c_N}{2^N}\chi_r \int_0^t h(\sigma)[f(\mu(\sigma))]^\gamma d\sigma \\
&= \mu(t)\chi_r = w(t),
\end{aligned} \tag{16}$$

for $0 < t < \min\{\tau_2, r^2, \delta^2\} = \tau_3$.

Subtracting (16) of (15)

$$\begin{aligned}
&w(t) - u^\epsilon(t) \\
&\leq \int_0^t h(\sigma)S(t-\sigma)\{[f(w)]^\gamma - [f(u^\epsilon(\sigma))]^\gamma\}\chi_r d\sigma \\
&\leq \gamma \int_0^t h(\sigma)S(t-\sigma)[\theta f(w) + (1-\theta)f(u^\epsilon)]^{\gamma-1}(w-u^\epsilon)_+\chi_r d\sigma;\ \theta \in (0,1) \\
&\leq \gamma \int_0^t h(\sigma)S(t-\sigma)[f(u^\epsilon)]^{\gamma-1}(w-u^\epsilon)_+\chi_r d\sigma \\
&\leq \gamma [f(\epsilon)]^{\gamma-1} \int_0^t h(\sigma)S(t-\sigma)(w-u^\epsilon)_+\chi_r d\sigma,
\end{aligned}$$

where $a_+ = \max\{a, 0\}$ for all $a \in \mathbb{R}$. Thus,

$$[w(t) - u^\epsilon(t)]_+ \leq p^+[f(\epsilon)]^{p^+-1} \int_0^t h(\sigma)S(t-\sigma)(w-u^\epsilon)_+\chi_r d\sigma,$$

and

$$\|[w(t) - u^\epsilon(t)]_+\chi_r\|_\infty \leq p^+[f(\epsilon)]^{p^+-1} \int_0^t h(\sigma)\|[w-u^\epsilon]_+\chi_r\|_\infty d\sigma.$$

By Gronwall's inequality, $(w(t) - u^\epsilon(t))_+\chi_r = 0$, for $t \in (0, \tau_3)$, that is, $w(t) \leq u^\epsilon(t)$ on the ball B_r for $t \in (0, \tau_3)$. Letting, $\epsilon \to 0$ we conclude that $w(t) \leq u(t)$ on $B_r \times [0, \tau_3)$.

Since $w \geq 0$ and $w \not\equiv 0$, we can argue as in case (i) to conclude that u is positive.

3 | LARGE INITIAL DATA

For the existence of blow up solutions we need of the following result established in [14, Lemma 3.1].

Lemma 2. Let η be a positive measure in $\Omega \subset \mathbb{R}^N$ such that $\int_\Omega d\eta = 1$ and let $f \in L^{p^+}(\Omega, d\eta)$ with $1 \leq p^- \leq p(x) \leq p^+$ for all $x \in \Omega$. Then

$$\int_\Omega |f(x)|^{p(x)} d\eta(x) \geq 2^{-p+} \min\left\{\left(\int_\Omega |f(x)|d\eta(x)\right)^{p^-}, \left(\int_\Omega |f(x)|d\eta(x)\right)^{p^+}\right\}.$$

Proof of Theorem 4 Let $\varphi_1 > 0$ be the first eigenvalue associated to the first eigenvalue $\lambda_1 > 0$ of the Laplacian operator on $H_0^1(\Omega')$ such that $\int_{\Omega'} \varphi_1 = 1$. Let $\Theta(t) = \int_{\Omega'} u(t)\varphi_1 dx$. By Lemma 2 and Jensen's inequality

$$\begin{aligned}
\Theta' + \lambda_1\Theta &\geq \int_{\Omega'}[f(u(t))]^{p(x)}\varphi_1 dx \\
&\geq 2^{-p^+} \min\left\{\left(\int_{\Omega'} f(u(t))\varphi_1\right)^\gamma, \left(\int_{\Omega'} f(u(t))\varphi_1\right)^{p^+}\right\} \\
&\geq 2^{-p^+} \min\{[f(\Theta(t))]^\gamma, [f(\Theta(t))]^{p^+}\} \\
&\geq 2^{-p^+} f^\gamma(\Theta(t)),
\end{aligned}$$

if $f(\Theta(t)) \geq 1$. Since f^γ is a convex function and $\int_\tau^\infty \frac{d\sigma}{f(\sigma)^\gamma} < \infty$, we have that

$$\lim_{r\to\infty} \frac{f^\gamma(r) - f^\gamma(0)}{r} = +\infty.$$

Thus, there exists $M > 0$ such that $\frac{1}{2^{p^+}} f^\gamma(r) - \lambda_1 r > \frac{1}{2^{p^++1}} f^\gamma(r)$ for $r > M$. Therefore, $\Theta' > \frac{1}{2^{p^++1}} f^\gamma(\Theta)$ whenever $f(\Theta) \geq 1$ and $\Theta > M$. Taking $\Theta(0)$ such that $\Theta(0) > \max\{M, \alpha\}$, where $f(\alpha) > 1$, we have that the solution blows up.

4 | BLOW UP AND GLOBAL EXISTENCE

Proof of Theorem 3 (i) We apply an argument similar to the one used in[24]. Consider $\delta > 0$ such that

$$\delta < \frac{1}{\beta + 1}, \tag{17}$$

where $\beta > 0$ satisfies

$$\int_0^\infty h(\sigma)\mathcal{F}(\|S(\sigma)v_0\|_\infty)d\sigma < \frac{\beta}{\beta + 1},$$

for some $v_0 \in C_0(\Omega)$, $v_0 \geq 0$, $v_0 \neq 0$. Set $u_0 = \delta v_0 \in C_0(\Omega)$ and define the sequence $\{u^k\}_{k\geq 0}$ by $u^0(t) = S(t)u_0$ and

$$u^k(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)[f(u^{k-1}(\sigma))]^{p(x)}\, d\sigma,$$

for $k \in \mathbb{N}$ and $t \geq 0$.

We claim that

$$u^k(t) \leq (1+\beta)S(t)u_0, \tag{18}$$

for $k \geq 0$ and $t > 0$. To show this, we use induction on k. Estimate (18) is clear for $k = 0$, thus we assume that (18) holds for k. Note that condition (17) implies $\|(1+\beta)S(t)u_0\|_\infty \leq \|S(t)v_0\|_\infty \leq m$ for $t > 0$. Since $\mathcal{F}(0, m] \to [0, \infty)$ and f are nondecreasing functions, and $s\mathcal{F}(s) = \max\{f(s)^{p^-}, f(s)^{p^+}\}$ for $s \in (0, m]$ we have

$$\begin{aligned}
u^{k+1}(t) &= S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)[f(u^k(\sigma))]^{p(x)}\, d\sigma \\
&\leq S(t)u_0 + \int_0^t h(\sigma)S(t-\sigma)[f((1+\beta)S(\sigma)u_0)]^{p(x)}\, d\sigma \\
&\leq S(t)u_0 + \int_0^t h(\sigma)S(t-\sigma)[f(S(\sigma)v_0)]^{p(x)}\, d\sigma \\
&\leq S(t)u_0 + \int_0^t h(\sigma)S(t-\sigma)\max\{[f(S(\sigma)v_0)]^{p^-}, [f(S(\sigma)v_0)]^{p^+}\}d\sigma \\
&= S(t)u_0 + \int_0^t h(\sigma)S(t-\sigma)\mathcal{F}(S(\sigma)v_0)S(\sigma)v_0 d\sigma \\
&\leq S(t)u_0 + S(t)v_0 \int_0^t h(\sigma)\mathcal{F}(\|S(\sigma)v_0\|_\infty)\, d\sigma \\
&\leq S(t)u_0 + (1+\beta)S(t)u_0 \frac{\beta}{\beta+1} = (1+\beta)S(t)u_0.
\end{aligned}$$

Hence, claim (18) holds for $k + 1$.

On the other hand, using again induction on k, it is possible to that $u^{k+1} \leq u^k$ for all $k \in \mathbb{N}$. Thus, from monotone convergence theorem and estimate (18), we conclude that $u = \lim u_n$ is a global solution of (1).

Proof of Theorem 3 (ii) We argue by contradiction and assume that there exists a global solution $u \in C([0, \infty), C_0(\Omega)$ of problem (1) with initial condition $u_0 \neq 0$, that is

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)[f(u(\sigma))]^{p(x)}\, d\sigma,$$

for $t \geq 0$. Let $0 < t < s$. Then,

$$S(s-t)u(t) = S(s)u_0 + \int_0^t h(\sigma)S(s-\sigma)[f(u(\sigma))]^{p(x)}\, d\sigma. \tag{19}$$

Set $\Phi(t) = S(s)u_0 + \int_0^t h(\sigma)S(s-\sigma)[f(u(\sigma))]^{p(x)}\, d\sigma$, for $t \in [0, s]$. Then

$$\Phi'(t) = h(t)S(s-t)[f(u(t))]^{p(x)},$$

and from Lemma 2

$$\begin{aligned} S(s-t)[f(u(t))]^{p(x)} &= \int_\Omega K_\Omega(x, y; s-t)[f(u(t, y))]^{p(y)} dy \\ &\geq 2^{-p^+} \min\left\{ \frac{[S(s-t)f(u(t))]^{p^-}}{a(s-t,x)^{p^- -1}}, \frac{[S(s-t)f(u(t))]^{p^+}}{a(s-t,x)^{p^+ -1}} \right\}, \end{aligned}$$

where K_Ω is the Dirichlet heat kernel on Ω and $a(s-t, x) = \int_\Omega K_\Omega(x, y; s-t)dy$. Since $K_\Omega(x, y; s-t) \leq K_{\mathbb{R}^{\mathbb{N}}}(x, y; s-t)$,[23, Lemma 7] we conclude that $a(s-t, x) \leq 1$. Thus, since $p^- \geq 1$, f is nondecreasing, inequality (9) and (19) we obtain

$$\begin{aligned} \Phi'(t) &\geq 2^{-p^+} h(t) \min\left\{ [S(s-t)f(u(t))]^{p^-}, [S(s-t)f(u(t))]^{p^+} \right\} \\ &\geq 2^{-p^+} h(t) \min\left\{ [f(S(s-t)u(t))]^{p^+}, [f(S(s-t)u(t))]^{p^-} \right\} \\ &= 2^{-p^+} h(t) \min\left\{ [f(\Phi(t))]^{p^+}, [f(\Phi(t))]^{p^-} \right\}. \end{aligned} \tag{20}$$

Set $g_2(t) = \min\{[f(t)]^{p^-}, [f(t)]^{p^+}\}$ for all $t \geq 0$. Then, by (20) we have $\Phi'(t) \geq 2^{-p^+} h(t)g_2(\Phi(t))$. Defining $G(t) = \int_t^{+\infty} \frac{d\sigma}{g_2(\sigma)}$ for $t > 0$ we obtain $[G(\Phi(t))]' = -\frac{\Phi'(t)}{g_2(\Phi(t))} \leq -2^{-p^+} h(t)$, for $0 < t < s$. Note that condition (5) guarantees that the function G is well defined.

Integrating, from 0 to s, we obtain

$$\begin{aligned} -G(S(s)u_0) &\leq \int_{G(\Phi(s))} \frac{d\sigma}{g_2(\sigma)} - \int_{G(\Phi(0))} \frac{d\sigma}{g_2(\sigma)} \\ &= G(\Phi(s)) - G(\Phi(0)) \\ &\leq -2^{-p^+} \int_0^s h(\sigma)d\sigma \end{aligned}$$

which is equivalent to $2^{-p^+} \int_0^s h(\sigma)d\sigma \leq G([S(s)u_0])$. Since G is decreasing and the left hand does not depend on x, we conclude that

$$2^{-p^+} \int_0^s h(\sigma)d\sigma \leq G(\|S(s)u_0\|_\infty),$$

which contradicts condition (10).

5 | CONCLUSIONS

We deal with the parabolic problem $u_t - \Delta u = h(t)F(x, u)$ in $\Omega \times (0, T)$, where Ω is a smooth domain (bounded or unbounded), $F(x, u) = f(u)^{p(x)}$, with $f \in C[0, \infty)$ non-decreasing, $h \in C(0, \infty)$ and $p \in C(\Omega)$ with $0 < p^- \leq p(x) \leq p^+$. We assume that $u_0 \in C_0(\Omega), u_0 \geq 0$ and consider only non-negative solutions.

Under the assumption $\int_\tau^\infty \frac{d\sigma}{\max\{f(\sigma)^{p^-}, f(\sigma)^{p^+}\}} = \infty$ we show that all the solutions non-negative are global. Moreover, we establish some conditions to get positive solutions in the case that $u_0 = 0$, extending the results of the classical case $F(x, t) = t^q$ with $0 < q < 1$. When $\int_\tau^\infty \frac{d\sigma}{\min\{f(\sigma)^{p^-}, f(\sigma)^{p^+}\}} < \infty$ we obtain blow up solutions and we use this result to improve a result established in[9].

Global existence is obtained for small initial data assuming that $\int_0^\infty h(\sigma)\mathcal{F}(\|S(\sigma)v_0\|_\infty)d\sigma < 1$ for some $v_0 \in C_0(\Omega), v_0 \neq 0$, where $\mathcal{F}(s) = \max\{f(s)^{p^+}, f(s)^{p^-}\}/s$ defined on a small interval $(0, m)$.

ACKNOWLEDGMENTS

Ricardo Castillo was supported by the ANID-FONDECYT project No. 11220152. Miguel Loayza was partially supported by CAPES-PRINT, 88881.311964/2018-01, MATHAMSUD, 88881.520205/2020-01, 21-MATH-03.

Conflict of interest

This work does not have any conflicts of interest.

References

1. R. Aboulaich, D. Meskine, and A. Souissi. New diffusion models in image processing. Comput. Math. Appl., 56 (2008), 874-882.

2. J. Aguirre and M. Escobedo. A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$. Asymptotic behaviour of solutions. Ann. Fac. Sci. Toulouse Math. 8 (1986), No. 2, 175-203.

3. S. Antontsev and J. Rodrigues. On stationary thermo-rheological viscous flows. Ann. Univ. Ferrara Sez. VII Sci. Mat., 52 (2006) (1) 19-36.

4. Bai, X.L., Zheng, S.N., Wang, W., Critical exponent for parabolic system with time-weighted sources in bounded domain. J. Funct. Anal. 265 (2013), 941-952.

5. E. Bollt, R. Chartrand, S. Esedoḡlu, P. Schultz, and K. Vixie, Graduated adaptive image denoising: local compromise between total variation and isotropic diffusion. Adv. Comput. Math., 31 (2009), 1-3, 61-85.

6. T. Cazenave, A. Haraux, An Introduction to Semilinear Evolution Equations. Transl. by Yvan Martel. Revised ed., Oxford Lecture Series in Mathematics and its Applications, vol. 13, Clarendon Press, Oxford, 1998.

7. R. Castillo, M. Loayza, On the critical exponent for some semilinear reaction-diffusion systems on general domains, J. Math. Anal. Appl. 428 (2015) 1117-1134.

8. R. Castillo, M. Loayza and C. S. Paixão, Global and nonglobal existence for a strongly coupled parabolic system on a general domain, J. Differential Equations, 261 (2016) 3344-3365.

9. R. Castillo and M. Loayza, A semilinear parabolic problem with variable reaction on a general domain, Comput. Math. Appl. 74 (2017), (3), 351-359.

10. R. Castillo, M. Loayza, Global existence and blowup for a coupled parabolic system with time-weighted sources on a general domain, Z. Angew. Math. Phys. 70 (2019) 16.

11. M. Escobedo and M. A. Herrero. A uniqueness result for a semilinear reaction-diffusion system. Proceedings of the American Mathematical Society, v. 112, (1991), 175-185.

12. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, in: Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011.

13. Li, Fengjie; Liu, Bingchen. Asymptotic analysis for blow-up solutions in parabolic equations involving variable exponents. Applicable Analysis, 92, 2013, n. 4, 651-664. .

14. R. Ferreira, A. de Pablo, M. Pérez-Llanos, J.D. Rossi, Critical exponents for a semilinear parabolic equation with variable reaction, Proc. Roy. Soc. Edinburgh Sect. A 142 (05) (2012) 1027-1042.

15. S. Kaplan. On the growth of solutions of quasi-linear parabolic equations.Comm. Pure Appl. Math., 16 (1963), (3), 305-330.

16. R. Laister, J.C. Robinson, M. Sierżega and A. Vidal-López. A complete characterization of local existence of semilinear heat equations in Lebesgue spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), 1519-1538.

17. R. Laister, J. C. Robinson, and M. Sierżega. A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations. J. Differential Equations 262 (2017), 10, 4979-4987.

18. M. Loayza and C. S. da Paixão, Existence and non-existence of global solutions for a semilinear heat equation on a general domain, Electron. J. Differential Equations, (2014), 1-9.

19. P. Meier, On the critical exponent for reaction-diffusion equations, Arch. Ration. Mech. Anal. 109 (1990), 63-71.

20. A. de Pablo and J. L. Vázquez, Traveling waves and finite propagation in a reaction-diffusion equation, J. Differential Equations, 93 (1991), 19-61.

21. P. Pinasco, Blow-up for parabolic and hyperbolic problems with variable exponents, Nonlinear Anal. 71 (2009), 1049-1058.

22. M. Růžička. Electrorheological fluids: modeling and mathematical theory, volume 1748 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000.

23. M. Van den Berg, A Gaussian lower bound for the Dirichlet heat kernel. Bulletin of the London Mathematical Society, 24 (1992), (5), 475-477.

24. F. B. Weissler, Existence and non-existence of global solutions for a semilinear heat equation, Israel J. Math., v. 38, (1981), 29-40.

$\square$