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RESEARCH ARTICLE

Delay-Adaptive Control of First-order Hyperbolic PIDEs

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Summary

We develop a delay-adaptive controller for a class of first-order hyperbolic partial integro-differential equations (PIDEs) with an unknown input delay. By employing a transport PDE to represent delayed actuator states, the system is transformed into a transport partial differential equation (PDE) with unknown propagation speed cascaded with a PIDE. A parameter update law is designed using a Lyapunov argument and the infinite-dimensional backstepping technique to establish global stability results. Furthermore, the well-posedness of the closed-loop system is analyzed. Finally, the effectiveness of the proposed method was validated through numerical simulations.

KEYWORDS:
first-order hyperbolic PIDE, delay-adaptive control, input delay, infinite-dimensional backstepping, full-state feedback

1 | INTRODUCTION

First-order hyperbolic PIDEs are widely used in various engineering applications, including traffic flow [1, 2], pipe flow [3], heat exchangers [4, 5], and oil well drilling [6, 7]. These applications often involve time delays due to the transportation of matter, energy, and information, which negatively affect the stability and performance of the system. Maintaining a stable fluid temperature is critical for the normal operation of heat exchangers, but the response speed is often limited when regulating fluid temperature, resulting in a time delay [8]. The exact value of the delay is usually hard to measure, which becomes a significant source of uncertainty within the controlled process [9]. Controlling the advection process in the presence of unknown delays is, therefore, a challenging task with practical significance. Thus, addressing the stabilization problem of first-order hyperbolic PIDEs with unknown input delays is of great practical importance.

Recently, there have been many studies on the stability of first-order hyperbolic PIDEs [10,11], and the development of infinite-dimensional backstepping techniques in [12] has provided effective methods for the PDE system control problems. [13] applies this method to the control of unstable open-loop hyperbolic PIDEs and developed a backstepping-based controller to stabilize the system. Subsequently, control problems for 2×2 first-order PDEs [14–16], $n+1$ coupled first-order hyperbolic PDEs [17], and $m+n$ anisotropic hyperbolic systems [18, 19] were investigated by employing the infinite-dimensional backstepping approach. In reference [20], a state feedback controller was designed for hyperbolic PIDEs with time-varying system parameters using this infinite-dimensional backstepping method, and the controller ensures that the system state converges to zero in the H_∞ norm within a finite time. Furthermore, a stabilizing controller and observer for hyperbolic PIDEs with Fredholm integrals were constructed in [21], and the results of [22] were extended to output regulation problems. [23] demonstrated the equivalence between finite-time stabilization and exact controllability properties for first-order hyperbolic PIDEs with Fredholm integrals. For linear anisotropic hyperbolic systems without integral terms, finite-time output regulation problems were addressed in [24], and stabilization problems for linear ODEs with linear anisotropic PDEs were solved in [25].

Infinite-dimensional backstepping has also been applied to adaptive control of hyperbolic PDEs. The pioneering work was presented in [26], where an adaptive stabilization method was developed for a one-dimensional (1-D) hyperbolic system with a single uncertain parameter. Since then, this method has been extensively applied to various types of hyperbolic PDEs with unknown parameters, as presented in the extensive literature [27–31]. The aforementioned results are built based on three traditional adaptive schemes, including the Lyapunov design, the passivity-based design, and the swapping design, which were initially proposed for nonlinear ODEs [32], and extended to the boundary adaptive control of PDEs [33–35]. Combined with backstepping design, a novel control strategy is proposed for coupled hyperbolic PDEs with multiplicative sensor faults in [36], it utilized a filter-based observer and model-based fault parameter estimation technique to achieve the tracking objective.

In recent years, studies began to pay attention to the time delays that occur in first-order PIDE systems since delays are commonly encountered in engineering practice. For instance, in [8], input delays were considered, and a backstepping boundary control was designed for first-order hyperbolic PIDEs. An observer-based output feedback control law was proposed for a class of first-order hyperbolic PIDEs with non-local coupling terms in the domain and measurement delay compensation [37]. Reference [38] addressed the output boundary regulation problem for a first-order linear hyperbolic PDE considering disturbances in the domain and on the boundary as well as state and sensor delays. Recently, the robustness of output feedback for hyperbolic PDEs with respect to small delays in actuation and measurements was discussed in [39]. Research on adaptive control for unknown arbitrary delays in PDE systems is relatively scarce. In contrast, there have been significant research achievements in the adaptive control of ODE systems with unknown delays. A notable theoretical breakthrough by developing adaptive control methods to compensate for uncertain actuator delays is achieved in [40]. Subsequently, the delay adaptive control technique has been applied to various types of unknown delays in ODE systems, including single-input delay [41, 42], multi-input delay [43] and distributed input delay [44, 45]. Inspired by these studies, recent work on parabolic systems with unknown input delays is presented in [46, 47]. However, research on hyperbolic PDE systems with delays remains relatively limited. For the first-order hyperbolic systems with uncertain transport speed, parameter estimators and adaptive controllers are designed in [48, 49] by using swapping filters. Different from these two studies, we apply a Lyapunov argument combined with the infinite-dimensional backstepping technique to design a delay-adaptive controller that achieves global stability in this paper, since the Lyapunov based adaptive methods are known to provide better transient performance [33].

In this paper, we consider a hyperbolic PIDE with an arbitrarily large unknown input delay. We extend the previous work on parabolic PDEs [46, 47] to a first-order PIDE system. We employ the infinite-dimensional backstepping method and choose the classic update law for the unknown delay, resulting in the structuring of the target system as a "cascade system", and the target transport PDE has two extra nonlinear terms which are controlled by the delay estimation error and the delay update law. The L^2 global stability of the target system is proven using appropriate Lyapunov functionals. The inverse Volterra/backstepping transformation establishes the norm equivalence relationship between the target system and the original one, thereby achieving L^2 global stability of the PDE system under the designed adaptive delay compensation controller. Furthermore, the well-posedness of the closed-loop system is analyzed.

Main contributions of this paper are:

(1) This paper develops a combined approach of the infinite-dimensional backstepping and the Lyapunov functional method for delay-adaptive control design for a class of hyperbolic PIDEs with unknown input delay. In [46], the presence of nonzero boundary conditions in the parabolic PDE target system with unknown input delay restricts us to the local stability of the closed-loop system with delay update law. However, we leverage the property first-order hyperbolic of the system to attain global stability of the closed-loop system.

(2) The well-posedness of the closed-loop system is established. Due to the presence of nonlinear terms and non-zero boundary conditions in the target system, the proof of well-posedness is not straightforward. We use the semigroup method to analyze the well-posedness of the target system, and construct Lyapunov functions to establish the system's asymptotic stability in the H^1 norm, thereby ensuring the global existence of the classical solution. Due to the invertibility of the backstepping transformation, the equivalence between the target system and the closed-loop system can be established, so that the closed-loop system is well-posed.

The structure of this paper is as follows: Section 2 briefly describes the design of a nonadaptive controller for the considered hyperbolic PIDE system. Section 3 discusses the design of the delay-adaptive control law. Section 4 is dedicated to the stability analysis of the resulting adaptive closed-loop system and the well-posedness of the closed-loop system. Section 5 provides consistent simulation results to demonstrate the feasibility of our approach. The paper ends with concluding remarks in Section 6.

Notation: Throughout the paper, we adopt the following notation to define the L^2-norm for $\chi(x) \in L^2[0,1]$:

$$\|\chi\|_{L^2}^2 = \int_{-1}^{1} |\chi(x)|^2 \mathrm{d}x, \tag{1}$$

and set $\|\chi\|^2 = \|\chi\|_{L^2}^2$.

For any given function $\psi(\cdot, \hat{D}(t))$

$$\frac{\partial \psi(\cdot, \hat{D}(t))}{\partial t} = \dot{\hat{D}}(t) \frac{\partial \psi(\cdot, \hat{D}(t))}{\partial \hat{D}(t)}. \tag{2}$$

2 | PROBLEM STATEMENT AND NON-ADAPTIVE CONTROLLER

Consider the first-order PIDE with an input delay $D > 0$,

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)\mathrm{d}y, \tag{3}$$

$$u(1,t) = U(t-D), \tag{4}$$

$$u(x,0) = u_0(x), \tag{5}$$

for $(x,t) \in (0,1)\times\mathbb{R}_+$, where $g(x), f(x,y) \in C[0,1]$ are known coefficient functions. Following [50], the delayed input $U(t-D)$ is written as a transport equation coupled with (3) as follows:

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)\mathrm{d}y, \tag{6}$$

$$u(1,t) = v(0,t), \tag{7}$$

$$u(x,0) = u_0(t), \tag{8}$$

$$Dv_t(x,t) = v_x(x,t), \quad x \in [0,1), \tag{9}$$

$$v(1,t) = U(x,t), \tag{10}$$

$$v(x,0) = v_0(x), \tag{11}$$

where the infinite-dimensional actuator state is solved as

$$v(x,t) = U(t + D(x-1)). \tag{12}$$

To design the delay-compensated controller $U(t)$, the backstepping transformation as follows can be employed:

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y, \tag{13}$$

$$z(x,t) = v(x,t) - \int_0^1 \gamma(x,y)u(y,t)\mathrm{d}y - D\int_0^x q(x-y)v(y,t)\mathrm{d}y, \tag{14}$$

where the kernel function $k(x,y)$ and $q(x-y)$ are defined on $\mathcal{T}_1 = \{(x,y) : 0 \le y \le x \le 1\}$, $\gamma(x,y)$ on $\mathcal{T}_2 = \{(x,y) : 0 \le y,\ x \le 1\}$, which gives the following target system

$$w_t(x,t) = w_x(x,t), \tag{15}$$

$$w(1,t) = z(0,t), \tag{16}$$

$$w(x,0) = w_0(x), \tag{17}$$

$$Dz_t(x,t) = z_x(x,t), \tag{18}$$

$$z(1,t) = 0, \tag{19}$$

$$z(x,0) = z_0(x), \tag{20}$$

with a mild solution for z

$$z(x,t) = \begin{cases} z_0(x + \frac{t}{D}), & 0 \le x + \frac{t}{D} \le 1, \\ 0, & x + \frac{t}{D} > 1, \end{cases} \tag{21}$$

Using the backstepping method, one can get the kernel equations

$$k_x(x,y) = -k_y(x,y) + \int_y^x f(\tau,y)k(\tau,y)\mathrm{d}\tau - f(x,y), \tag{22}$$

$$k(x,0) = \int_0^x k(x,y)g(y)\mathrm{d}y - g(x), \tag{23}$$

$$\gamma_x(x,y) = -D\gamma_y(x,y) + D\int_y^1 f(\tau,y)\gamma(x,\tau)\mathrm{d}\tau, \tag{24}$$

$$\gamma(x,0) = \int_0^1 g(y)\gamma(x,y)\mathrm{d}y, \tag{25}$$

$$\gamma(0,y) = k(1,y), \tag{26}$$

$$q(x) = \gamma(x,1). \tag{27}$$

From the boundary conditions (10) and (19), the associated control law is straightforwardly derived

$$U(t) = \int_0^1 \gamma(1,y)u(y,t)\mathrm{d}y + D\int_0^1 q(1-y)v(y,t)\mathrm{d}y. \tag{28}$$

Knowing that the transformations (13)–(14) are invertible with inverse transformation as

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)\mathrm{d}y, \tag{29}$$

$$v(x,t) = z(x,t) + \int_0^1 \eta(x,y)w(y,t)\mathrm{d}y - D\int_0^x p(x-y)z(y,t)\mathrm{d}y, \tag{30}$$

where kernels $l(x,y)$, $\eta(x,y)$ and $p(x-y)$ satisfy the following PDEs,

$$l_x(x,y) + l_y(x,y) = -\int_y^x f(\tau,y)l(\tau,y)\mathrm{d}\tau - f(x,y), \tag{31}$$

$$l(x,0) = -g(x), \tag{32}$$

$$\eta_x(x,y) + D\eta_y(x,y) = 0, \tag{33}$$

$$\eta(x,0) = 0, \tag{34}$$

$$\eta(0,y) = l(1,y), \tag{35}$$

$$p(x) = \eta(x,1). \tag{36}$$

Next, we will develop an adaptive controller with delay update law to stabilize (6)–(11) for the arbitrarily long unknown delay.

3 | DESIGN OF A DELAY-ADAPTIVE FEEDBACK CONTROL

3.1 | Adaptive control design

Considering the plant (3)–(5) with an unknown delay $D > 0$, which equivalent to the cascade system (6)–(11) with an unknown propagation speed $1/D$, we will design an adaptive boundary controller to ensure global stability result.

Assumption 1. The upper and lower bounds $\overline{D}$ and $\underline{D}$ for delay $D > 0$ are known.

Based on the certainty equivalence principle, we rewrite controller (28) by replacing D with estimated delay $\hat{D}(t)$ as the delay-adaptive controller

$$U(x,t) = \int_0^1 \gamma(1,y,\hat{D}(t))u(y,t)\mathrm{d}y + \hat{D}(t)\int_0^1 q(1-y,\hat{D}(t))v(y,t)\mathrm{d}y. \tag{37}$$

3.2 | Target system for the plant with unknown input delay

Rewriting the backstepping transformations (30) as

$$z(x,t) = v(x,t) - \int_0^1 \gamma(x,y,\hat{D}(t))u(y,t)\mathrm{d}y - \hat{D}(t)\int_0^x q(x-y,\hat{D}(t))v(y,t)\mathrm{d}y, \tag{38}$$

and its inverse (30) as:

$$v(x,t) = z(x,t) + \int_0^1 \eta(x,y,\hat{D}(t))u(y,t)\mathrm{d}y + \hat{D}(t)\int_0^x p(x-y,\hat{D}(t))z(y,t)\mathrm{d}y, \tag{39}$$

where the kernels $\gamma(x,y,\hat{D}(t))$, $q(x-y,\hat{D}(t))$, $\eta(x,y,\hat{D}(t))$, $p(x-y,\hat{D}(t))$ satisfy the same form of PDEs (22)-(27) and (31)-(36) except D replaced with $\hat{D}(t)$. Using the transformation (13) and (38), we get the following target system

$$w_t(x,t) = w_x(x,t), \tag{40}$$
$$w(1,t) = z(0,t), \tag{41}$$
$$w(x,0) = w_0(x), \tag{42}$$
$$Dz_t(x,t) = z_x(x,t) - \tilde{D}(t)P_1(x,t) - D\dot{\hat{D}}(t)P_2(x,t), \tag{43}$$
$$z(1,t) = 0, \tag{44}$$
$$z(x,0) = z_0(x), \tag{45}$$

where $\tilde{D}(t) = D - \hat{D}(t)$ is the estimation error, functions $P_i(x,t), i = 1,2$ are given below:

$$P_1(x,t) = z(0,t)M_1(x,t) + \int_0^1 w(y,t)M_2(x,y,t)\mathrm{d}y, \tag{46}$$
$$P_2(x,t) = \int_0^1 z(y,t)M_3(x,y,t)\mathrm{d}y + \int_0^1 w(y,t)M_4(x,y,t)\mathrm{d}y, \tag{47}$$

with

$$M_1(x,t) = \gamma(x,1,\hat{D}(t)), \tag{48}$$

$$\begin{aligned} M_2(x,y,t) =& \gamma(x,1,\hat{D}(t))l(1,y) - \gamma_y(x,y,\hat{D}(t)) \\ &+ \int_y^1 \left(-\gamma_y(x,\xi,\hat{D}(t))l(\xi,y) + \gamma(x,\xi,\hat{D}(t))f(\xi,y) + \int_\xi^1 \gamma(x,\tau,\hat{D}(t))f(\tau,\xi)l(\xi,y)\mathrm{d}\tau \right)\mathrm{d}\xi, \end{aligned} \tag{49}$$

$$\begin{aligned} M_3(x,y,t) =& q(x-y,\hat{D}(t)) + q_{\hat{D}(t)}(x-y,\hat{D}(t)) + \hat{D}(t)\int_y^x q(x-\xi,\hat{D}(t))p(\xi-y,\hat{D}(t))\mathrm{d}\xi \\ &+ \hat{D}(t)^2 \int_y^x q_{\hat{D}(t)}(x-\xi,\hat{D}(t))p(\xi-y,\hat{D}(t))\mathrm{d}\xi, \end{aligned} \tag{50}$$

$$\begin{aligned} M_4(x,y,t) =& \gamma_{\hat{D}(t)}(x,y,\hat{D}(t)) + \int_y^1 \gamma_{\hat{D}(t)}(x,\xi,\hat{D}(t))l(\xi,y)\mathrm{d}\xi + \int_0^x q(x-\xi,\hat{D}(t))\eta(\xi,y,\hat{D}(t))\mathrm{d}\xi \\ &+ \hat{D}(t)\int_0^x q_{\hat{D}(t)}(x-\xi,\hat{D}(t))\eta(\xi,y,\hat{D}(t))\mathrm{d}\xi. \end{aligned} \tag{51}$$

3.3 | The parameter update law

We choose the following update law

$$\dot{\hat{D}}(t) = \theta \mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau(t)\}, \quad 0 < \theta < \theta^*, \tag{52}$$

where $\tau(t)$ is given as

$$\tau(t) = \frac{-b_1 \int_0^1 (1+x)z(x,t)P_1(x,t)\mathrm{d}x}{N(t)}, \tag{53}$$

with $N(t)=\frac{1}{2}\int_0^1(1+x)w(x,t)^2\mathrm{d}x+\frac{b_1}{2}\int_0^1(1+x)z(x,t)^2\mathrm{d}x$, $b_1>2\bar{D}$ and

$$\theta^*=\frac{\min\{\underline{D},b_1-2\bar{D}\}\min\{1,b_1\}}{2b_1^2L^2}. \tag{54}$$

The standard projection operator is defined as follows

$$\mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau(t)\}=\begin{cases}0 & \hat{D}(t)=\underline{D}\text{ and }\tau(t)<0,\\ 0 & \hat{D}(t)=\overline{D}\text{ and }\tau(t)>0,\\ \tau(t) & \text{else.}\end{cases} \tag{55}$$

Remark 1. The projection is used to ensure the parameters $\hat{D}(t)$ within the known bounds $[\underline{D},\overline{D}]$ which cannot be viewed as a robust tool [33]. It prevents adaptation transients by over-limiting the size of the adaptation gain. The projection set can be taken conservatively and can be large, however, in order to ensure stability, the size needs to be inversely proportional to the adaptation gain.

4 | THE GLOBAL STABILITY OF THE CLOSED-LOOP SYSTEM UNDER THE DELAY-ADAPTIVE CONTROL

The following theorem states the global stability result of the closed-loop system (6)–(11) with update law (55) and adaptive controller (37).

Theorem 1. Consider the closed-loop system consisting of the plant (6)–(11), the control law (37), and the update law (52)–(55) under Assumption 1. There exist positive constants ρ, R such that

$$\Psi(t)\le R(e^{\rho\Psi(0)}-1),\quad \forall t\ge 0, \tag{56}$$

where

$$\Psi(t)=\int_0^1 u(x,t)^2\mathrm{d}x+\int_0^1 v(x,t)^2\mathrm{d}x+\tilde{D}(t)^2. \tag{57}$$

Furthermore,

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)|=0, \tag{58}$$

$$\lim_{t\to\infty}\max_{x\in[0,1]}|v(x,t)|=0. \tag{59}$$

The global stability of the (u,v)-system is established by the following steps:

- We establish the norm equivalence between (u,v) and (w,z).
- We introduce a Lyapunov function to prove the global stability of the (w,z)-system (40)–(45), and then get the stability of system (u,v) by using the norm equivalence.
- We arrive at the regulation of states $u(x,t)$ and $v(x,t)$.

4.1 | Global stability of the closed-loop system

First, we discuss the equivalent stability property between the plant (6)–(11) and the target system (40)–(45). Call now kernel functions $k(x,y)$, $\gamma(x,y)$, $q(x-y)$, $l(x,y)$, $\eta(x,y)$, and $p(x-y)$ are bounded by $\bar{k}$, $\bar{\gamma}$, $\bar{q}$, $\bar{l}$, $\bar{\eta}$, and $\bar{p}$ and in their respective domains. From (13), (14), (29), and (30) it is easy to find, by using Cauchy-Schwarz inequality, that

$$\|u(t)\|^2+\|v(t)\|^2\le r_1\|w(t)\|^2+r_2\|z(t)\|^2, \tag{60}$$

$$\|w(t)\|^2+\|z(t)\|^2\le s_1\|u(t)\|^2+s_2\|v(t)\|^2, \tag{61}$$

where r_i and s_i, $i = 1, 2$ are positive constants given by

$$r_1 = 2 + 2\bar{l}^2 + 3\bar{\eta}^2, \tag{62}$$

$$r_2 = 3 + 3\bar{D}^2\bar{p}^2, \tag{63}$$

$$s_1 = 2 + 2\bar{k}^2 + 3\bar{\gamma}^2, \tag{64}$$

$$s_2 = 3 + 3\bar{D}^2\bar{q}^2. \tag{65}$$

Next, we prove the global stability of the closed-loop system consisting of the (u, v)-system under the control law (37), and the update law (52)-(55). Introducing a Lyapunov-Krasovskii-type function

$$V_1(t) = D\log(1 + N(t)) + \frac{\tilde{D}(t)^2}{2\theta},$$

where $N(t) = \frac{1}{2}\int_0^1 (1 + x)w(x, t)^2\mathrm{d}x + \frac{b_1}{2}\int_0^1 (1 + x)z(x, t)^2\mathrm{d}x$, based on the target system (40)–(45) and where b_1 is a positive constant.

Taking the time derivative of (66) along (40)–(45), we get

$$\begin{aligned}
\dot{V}_1(t) =& \frac{D}{N(t)}\left(\int_0^1 (1 + x)w(x, t)w_t(x, t)\mathrm{d}x + b_1\int_0^1 (1 + x)z(x, t)z_t(x, t)\mathrm{d}x\right) - \tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\
=& \frac{1}{N(t)}\left(D\int_0^1 (1 + x)w(x, t)w_x(x, t)\mathrm{d}x + b_1\int_0^1 (1 + x)z(x, t)(z_x(x, t) - \tilde{D}(t)P_1(x, t) - D\dot{\hat{D}}(t)P_2(x, t))\mathrm{d}x\right) - \tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\
=& \frac{1}{N(t)}\Bigg(Dw(1, t)^2 - \frac{D}{2}w(0, t)^2 - \frac{D}{2}\|w\|^2 - \frac{b_1}{2}z(0, t)^2 - \frac{b_1}{2}\|z\|^2\\
&- b_1\tilde{D}(t)\int_0^1 (1 + x)z(x, t)P_1(x, t)\mathrm{d}x - b_1 D\dot{\hat{D}}(t)\int_0^1 (1 + x)z(x, t)P_2(x, t)\mathrm{d}x\Bigg) - \tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta},
\end{aligned} \tag{66}$$

where we have used integration by parts, Cauchy-Schwarz, and Young's inequalities. Using (52)–(54) and the standard properties of the projection operator leads to

$$\begin{aligned}
\dot{V}_1(t) \leq& \frac{1}{N(t)}\Bigg(- \frac{D}{2}\|w\|^2 - \frac{b_1}{2}\|z\|^2 - (\frac{b_1}{2} - D)z(0, t)^2\\
&- b_1 D\dot{\hat{D}}(t)\int_0^1 (1 + x)z(x, t)P_2(x, t)\mathrm{d}x\Bigg),
\end{aligned} \tag{67}$$

where $b_1 > 2\bar{D}$.

After a lengthy but straightforward calculation, employing the Cauchy-Schwarz and Young inequalities, along with (46) and (47), yields the following estimates

$$\int_0^1 (1 + x)z(x, t)P_1(x, t)\mathrm{d}x \leq L(\|w\|^2 + \|z\|^2 + \|z(0, t)\|^2), \tag{68}$$

$$\int_0^1 (1 + x)z(x, t)P_2(x, t)\mathrm{d}x \leq L(\|w\|^2 + \|z\|^2), \tag{69}$$

where the parameter $\bar{L}$ is defined below

$$\bar{L} = \max\left\{\overline{M}_1 + \overline{M}_2, 2\overline{M}_3 + \overline{M}_4\right\},$$

where $\overline{M}_1 = \max_{0\leq x\leq 1,\, t\geq 0}\{|M_1(x, \hat{D}(t))|\}$, $\overline{M}_i = \max_{0\leq x\leq y\leq 1,\, t\geq 0}\{|M_i(x, y, \hat{D}(t))|\}$ for $i = 2, 3, 4$.

According to the equivalent stability property between the plant (6)–(11) and the target system (40)–(45), we can get

$$\dot{V}_1 \leq -\left(\min\{\frac{D}{2}, \frac{b_1}{2} - \bar{D}\} - \frac{\theta b_1^2 L^2}{\min\{1, b_1\}}\right)\frac{\|w\|^2 + \|z\|^2 + \|z(0, t)\|^2}{N(t)}. \tag{70}$$

Choosing $\theta \in (0, \theta^\star)$, where $\theta^\star$ defined by (54), we know $\dot{V}_1(t) \leq 0$, which gives

$$V_1(t) \leq V_1(0), \tag{71}$$

for all $t \geq 0$. Hence, we get the following estimates from (66):

$$\|w\|^2 \leq 2(e^{\frac{V_1(t)}{\underline{D}}} - 1), \tag{72}$$

$$\|z\|^2 \leq \frac{2}{b_1}(e^{\frac{V_1(t)}{\underline{D}}} - 1), \tag{73}$$

$$\tilde{D}(t) \leq \frac{2\theta V_1(t)}{\underline{D}}. \tag{74}$$

Furthermore, from (66), (61) and (72)-(73), it follows that

$$\|u\|^2+\|v\|^2 \leq \left(2r_1 + \frac{2r_2}{b_1}\right)(e^{\frac{V_1(t)}{\underline{D}}} - 1), \tag{75}$$

and combining (74) and (75), we get

$$\Psi(t) \leq \left(2r_1 + \frac{2r_2}{b_1} + \frac{2\theta}{\underline{D}}\right)(e^{\frac{V_1(t)}{\underline{D}}} - 1). \tag{76}$$

So, we have bounded $\Psi(t)$ in terms of $V_1(t)$ and thus, using (71), in terms of $V_1(0)$. Now we have to bound $V_1(0)$ in terms of $\Psi(0)$. First, from (66), it follows that

$$\begin{aligned}
V_1(t) = & D\log\left(1 + \frac{1}{2}\int_0^1 (1+s)w(x,t)^2\mathrm{d}x + \frac{b_1}{2}\int_0^1 (1+s)z(x,t)^2\mathrm{d}x\right) + \frac{\tilde{D}(t)^2}{2\theta} \\
\leq & \bar{D}\|w\|^2 + b_1\bar{D}\|z\|^2 + \frac{\tilde{D}(t)^2}{2\theta} \\
\leq & \bar{D}\max\{1, b_1\}(s_1 + s_2)\left(\|u\|^2+\|v\|^2\right) + \frac{\tilde{D}(t)^2}{2\theta} \\
\leq & \left(\bar{D}\max\{1, b_1\}(s_1 + s_2) + \frac{1}{2\theta}\right)\Psi(t),
\end{aligned} \tag{77}$$

leading to the following relation

$$V_1(0) \leq \left(\bar{D}\max\{1, b_1\}(s_1 + s_2) + \frac{1}{2\theta}\right)\Psi(0). \tag{78}$$

Then, combining (71), (76) and (78), we have

$$\Psi(t) \leq R(e^{\rho\Psi(0)} - 1), \tag{79}$$

where

$$R = 2r_1 + \frac{2r_2}{b_1} + \frac{2\theta}{\underline{D}}, \tag{80}$$

$$\rho = \bar{D}\max\{1, b_1\}(s_1 + s_2) + \frac{1}{2\theta}, \tag{81}$$

so we complete the proof of the stability estimate (56).

4.2 | Pointwise boundedness and regulation of the distributed states

Now, we ensure the regulation of the distributed states. From (66) and (70), we get the boundedness of $\|w\|$, $\|z\|$ and $\hat{D}(t)$. Knowing that

$$\int_0^t \|w(\tau)\|^2\mathrm{d}\tau \leq \sup_{0\leq\tau\leq t} N(\tau)\int_0^t \frac{\|w(\tau)\|^2}{N(\tau)}\mathrm{d}\tau, \tag{82}$$

and using (71) the following inequality holds

$$N(\tau) \leq N(0)e^{\frac{\tilde{D}(0)^2}{2\theta}}. \tag{83}$$

Integrating (70) over $[0,\ t]$, we have

$$\int_0^t \frac{\|w(\tau)\|^2}{N(\tau)}\mathrm{d}\tau \le \frac{\bar{D}\log N(0) + \frac{\tilde{D}(0)^2}{2\theta}}{\min\left\{\frac{1}{2}, \frac{b_1}{2} - 1\right\} - \frac{\theta b_1^2 L^2}{\min\{1, b_1\}}}. \tag{84}$$

Substituting (83) and (84) into (82), we get $\|w\|$ is square integrable in time. One can establish that $\|z\|$ and $\|z(0,t)\|$ are square integrable in time similarly. Thus, $\|P_1\|$ and $\|P_2\|$ are bounded and integrable functions of time.

To prove the boundedness of $\|w_x\|$, we define the following Lyapunov function

$$V_2(t) = \frac{1}{2}\int_0^1 (1+x)w_x(x,t)^2\mathrm{d}x + \frac{b_2 D}{2}\int_0^1 (1+x)z_x(x,t)^2\mathrm{d}x, \tag{85}$$

where b_2 is a positive constant. Using the integration by parts, the derivative of (85) with respect to time is written as

$$\begin{aligned}
\dot{V}_2(t) =& \int_0^1 (1+x)w_x(x,t)w_{xt}(x,t)\mathrm{d}x + b_2 D\int_0^1 (1+x)z_x(x,t)z_{xt}(x,t)\mathrm{d}x \\
=& \int_0^1 (1+x)w_x(x,t)w_{xx}(x,t)\mathrm{d}x + b_2\int_0^1 (1+x)z_x(x,t)z_{xx}(x,t)\mathrm{d}x \\
&- b_2\tilde{D}(t)\int_0^1 (1+x)z_x(x,t)P_{1x}(x,t)\mathrm{d}x - b_2 D\dot{\hat{D}}(t)\int_0^1 (1+x)z_x(x,t)P_{2x}(x,t)\mathrm{d}x. \\
=& w_x(1,t)^2 - \frac{1}{2}w_x(0,t)^2 - \frac{1}{2}\|w_x\|^2 + b_2 z_x(1,t)^2 - \frac{b_2}{2}z_x(0,t)^2 - \frac{b_2}{2}\|z_x\|^2 \\
&- b_2\tilde{D}(t)\int_0^1 (1+x)z_x(x,t)P_{1x}(x,t)\mathrm{d}x - b_2 D\dot{\hat{D}}(t)\int_0^1 (1+x)z_x(x,t)P_{2x}(x,t)\mathrm{d}x.
\end{aligned} \tag{86}$$

Based on (40), (41), one can get

$$\begin{aligned}
w_x(1,t) =& w_t(1,t) = z_t(0,t) \\
=& z_x(0,t) - \tilde{D}(t)P_1(0,t) - D\dot{\hat{D}}(t)P_2(0,t),
\end{aligned} \tag{87}$$

we arrive at the following inequality

$$\begin{aligned}
\dot{V}_2(t) \le& -\frac{1}{2}\|w_x\|^2 - \frac{b_2}{2}\|z_x\|^2 - (\frac{b_2}{2} - 3)z_x(0,t)^2 + 3\tilde{D}(t)^2 P_1(0,t)^2 + 3D^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 \\
&+ 2b_2\tilde{D}(t)^2 P_1(1,t)^2 + 2b_2 D^2\dot{\hat{D}}(t)^2 P_2(1,t)^2 + 2b_2|\tilde{D}(t)|\|z_x\|\|P_{1x}(x,t)\| \\
&+ b_2 D|\dot{\hat{D}}(t)|\|z_x\|\|P_{2x}(x,t)\|.
\end{aligned} \tag{88}$$

Choosing $b_2 > 6$, we get,

$$\begin{aligned}
\dot{V}_2(t) \le& -\frac{1}{2}\|w_x\|^2 - \frac{b_2}{2}\|z_x\|^2 + 3\tilde{D}(t)^2 P_1(0,t)^2 + 3D^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 + 2b_2\tilde{D}(t)^2 P_1(1,t)^2 \\
&+ 2b_2\bar{D}^2\dot{\hat{D}}(t)^2 P_2(1,t)^2 + 2b_2 D|\dot{\hat{D}}(t)|\|z_x\|\|P_{2x}\| + 2b_2|\tilde{D}(t)|\|z_x(x,t)\|\|P_{1x}(x,t)\| \\
\le& -c_1 V_2(t) + f_1(t)V_2(t) + f_2(t),
\end{aligned} \tag{89}$$

where we use Young's and Agmon's inequalities. Here, $c_1 = \frac{1}{2}\min\{1, \frac{1}{D}\}$, and the functions $f_1(t)$ and $f_2(t)$ are given by

$$f_1(t) = b_2\overline{D}^2(|\dot{\hat{D}}(t)|^2 + 4), \tag{90}$$

$$\begin{aligned}
f_2(t) =& b_2\|P_{1x}\|^2 + b_2\|P_{2x}\|^2 + 12\bar{D}^2 P_1(0,t)^2 + 3\bar{D}^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 + 8b_2\bar{D}^2 P_1(1,t)^2 \\
&+ 2b_2 D^2\dot{\hat{D}}(t)^2 P_2(1,t)^2.
\end{aligned} \tag{91}$$

Knowing that

$$\begin{aligned}
P_1(0,t)^2 \le& 2\overline{M}_1^2 z(0,t)^2 + 2\overline{M}_2^2\|w\|^2 \\
\le& 2\overline{M}_1^2(\|z\|^2 + \|z_x\|^2) + 2\overline{M}_2^2\|w\|^2,
\end{aligned} \tag{92}$$

$$P_2(0,t)^2 \le 2\overline{M}_3^2\|z\|^2 + 2\overline{M}_4^2\|w\|^2, \tag{93}$$

with (68) and (69), we get $|\dot{\hat{D}}(t)|$, $P_1(0,t)^2$, $P_2(0,t)^2$, $P_1(1,t)^2$ and $P_2(1,t)^2$ are integrable. Then, $f_1(t)$ and $f_2(t)$ are also integrable functions of time. Using Lemma D.3 [51], we get that $\|w_x\|$ and $\|z_x\|$ are bounded, and combing the Agmon's inequality, one can deduce the boundedness of $w(x,t)$ and $z(x,t)$ for all $x \in [0,1]$.

Next we establish the boundedness of $\frac{d}{dt}(\|w\|^2)$, $\frac{d}{dt}(\|z\|^2)$ and $\frac{d}{dt}(\|z_x\|^2)$ using the following Lyapunov function

$$V_3(t) = \frac{1}{2}\int_0^1 (1+x)w_x(x,t)^2 dx + \frac{b_3 D}{2}\int_0^1 (1+x)z(x,t)^2 dx + \frac{b_3 D}{2}\int_0^1 (1+x)z_x(x,t)^2 dx, \tag{94}$$

where b_3 is a positive constant. Taking the derivative of (94) with respect to time, we obtain

$$\begin{aligned}
\dot{V}_3(t) =& \int_0^1 (1+x)w_x(x,t)w_{xt}(x,t)dx + b_3 D\int_0^1 (1+x)z(x,t)z_t(x,t)dx \\
&+ b_3 D\int_0^1 (1+x)z_x(x,t)z_{xt}(x,t)dx \\
=& w_x(1,t)^2 - \frac{1}{2}w_x(0,t)^2 - \frac{1}{2}\|w_x\|^2 - \frac{b_3}{2}z(0,t)^2 - \frac{b_3}{2}\|z\|^2 + b_3 z_x(1,t)^2 - \frac{b_3}{2}z_x(0,t)^2 \\
&- \frac{b_3}{2}\|z_x\|^2 - b_3\tilde{D}(t)\left(\int_0^1 (1+x)z(x,t)P_1(x,t)dx + \int_0^1 (1+x)z_x(x,t)P_{1x}(x,t)dx\right) \\
&- b_3 D\dot{\hat{D}}(t)\left(\int_0^1 (1+x)z(x,t)P_2(x,t)dx + \int_0^1 (1+x)z_x(x,t)P_{2x}(x,t)dx\right).
\end{aligned} \tag{95}$$

Clearly, using integrations by part and Young's inequality, the following holds,

$$\begin{aligned}
\dot{V}_3(t) \le& -\frac{1}{2}\|w_x\|^2 - \frac{b_3}{2}\|z\|^2 - (\frac{b_3}{2} - 1)z(0,t)^2 - \frac{b_3}{2}\|z_x\|^2 - (\frac{b_3}{2} - 3)z_x(0,t)^2 \\
&+ 3\tilde{D}(t)^2 P_1(0,t)^2 + 3D^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 + 2b_3\tilde{D}(t)^2 P_1(1,t)^2 + 2b_3 D^2\dot{\hat{D}}(t)^2 P_2(1,t)^2 \\
&+ 2b_3|\tilde{D}(t)|\|z\|\|P_1\| + 2b_3\overline{D}|\dot{\hat{D}}(t)|\|z\|\|P_2\| + 2b_3|\tilde{D}(t)|\|z_x\|\|P_{1x}\| \\
&+ 2b_3\overline{D}|\dot{\hat{D}}(t)|\|z_x\|\|P_{2x}\|.
\end{aligned} \tag{96}$$

Choosing $b_3 > 6$, we have

$$\begin{aligned}
\dot{V}_3(t) \le& -\frac{1}{2}\|w_x\|^2 - \frac{b_3}{2}\|z\|^2 - \frac{b_3}{2}\|z_x\|^2 + 3\tilde{D}(t)^2 P_1(0,t)^2 + 3D^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 + 2b_3\tilde{D}(t)^2 P_1(1,t)^2 \\
&+ 3D^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 + 2b_3\tilde{D}(t)^2 P_1(1,t)^2 + 2b_3 D^2\dot{\hat{D}}(t)^2 P_2(1,t)^2 + 2b_3|\tilde{D}(t)|\|z\|\|P_1\| \\
&+ 2b_3\overline{D}|\dot{\hat{D}}(t)|\|z\|\|P_2\| + 2b_3|\tilde{D}(t)|\|z_x\|\|P_{1x}\| + 2b_3\overline{D}|\dot{\hat{D}}(t)|\|z_x\|\|P_{2x}\| \\
\le& -c_1 V_3(t) + f_3(t)V_3(t) + f_4(t) < \infty,
\end{aligned} \tag{97}$$

where we use Young's and Agmon's inequalities, and

$$f_3(t) = 2b_3\overline{D}^2(|\dot{\hat{D}}(t)|^2 + 4), \tag{98}$$

$$\begin{aligned}
f_4(t) =& 3\tilde{D}(t)^2 P_1(0,t)^2 + 3\bar{D}^2\dot{\hat{D}}(t)^2 P_2(0,t)^2 + b_3(\|P_1\|^2 + \|P_2\|^2 + \|P_{1x}\|^2 + \|P_{2x}\|^2 \\
&+ 8\bar{D}^2 P_1(1,t)^2 + 2\bar{D}^2\dot{\hat{D}}(t)^2 P_2(1,t)^2),
\end{aligned} \tag{99}$$

are bounded functions. Thus, from (97), one can deduce the boundedness of $\frac{d}{dt}(\|w\|^2)$, $\frac{d}{dt}(\|z\|^2)$ and $\frac{d}{dt}(\|z_x\|^2)$. Moreover, by Lemma D.2 [51], we get $V_3(t) \to 0$, and thus $\|w_x\|$, $\|z\|$, $\|z_x\|^2 \to 0$ as $t \to \infty$. Next, from (60), we have $\|u_x\|^2$, $\|v\|^2$, $\|v_x\|^2 \to 0$ as $t \to \infty$.

From (29), we have

$$\|u_x\|^2 \le 2\|w_x\|^2 + 2\|w\|^2\bar{l}_x^2. \tag{100}$$

Since $\|w\|$, $\|w_x\|$ are bounded, $\|u_x\|$ is also bounded. By Agmon's inequality $u(x,t)^2 \le 2\|u\|\|u_x\|$, which enables one to state the regulation of $u(x,t)$ to zero uniformly in x. Similarly, one can prove the regulation of $v(x,t)$. Since $\|v\|^2$ and $\|v_x\|^2 \to 0$ as $t \to \infty$, by Agmon's inequality $v(x,t)^2 \le 2\|v\|\|v_x\|$, which enables one to state the regulation of $v(x,t)$ to zero uniformly in x and completes the proof of Theorem 1.

4.3 | Well-posedness of the closed-loop system

Following the approach in [51], we prove the well-posedness of the closed-loop system in Theorem 1. Consider the closed-loop target system $(w, z, \tilde{D}(t))$:

$$w_t(x,t) = w_x(x,t), \tag{101}$$

$$w(1,t) = z(0,t), \tag{102}$$

$$z_t(x,t) = \frac{1}{D} z_x(x,t) - \frac{\tilde{D}(t)}{D} P_1(x,t) - \theta \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\} P_2(x,t), \tag{103}$$

$$z(1,t) = 0, \tag{104}$$

$$\dot{\tilde{D}}(t) = -\theta \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\}, \tag{105}$$

we set $Z = (w, z, \tilde{D}(t))^T$, and introduce the operator

$$A = \begin{pmatrix} -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial}{D\partial x} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{106}$$

with

$$F(Z) = \begin{pmatrix} 0 \\ -\frac{\tilde{D}(t)}{D} P_1(x,t) - \theta \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\} P_2(x,t) \\ \theta \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\} \end{pmatrix}. \tag{107}$$

Then (101)-(105) can be written in abstract form as

$$Z_t = -AZ + F(Z), \tag{108}$$

$$Z(0) = Z_0. \tag{109}$$

where $Z = L^2(0,1) \times L^2(0,1) \times \mathbb{R}$, $\mathcal{B}(A) = \{f, g, l \,:\, f \in H^1(0,1), f(1) = g(0); g \in H^1(0,1), g(1) = 0; l \in \mathbb{R}\}$ and the norm $\|Z\|_H = \|w\|^2 + \|z\|^2 + \tilde{D}^2$.

Now, we establish the well-posedness of (108)–(109) with the following theorem (see as well Theorem 8.2 [51], Theorem 2.5.6 [52], for which a similar method has been employed to establish well-posedness).

Theorem 2. Consider the system (108)–(109) , where A is a maximal accretive operator from a dense subset $\mathcal{B}(A)$ in a Banach space H into H. If F is a nonlinear operator from $\mathcal{B}(A)$ to $\mathcal{B}(A)$ and satisfies the local Lipschitz condition, then for any $Z_0 \in \mathcal{B}(A)$, the problem (108)–(109) admits a unique classical solution Z such that

$$Z \in C^1([0, T_{\max}), H) \cap C([0, T_{\max}), \mathcal{B}(A)), \tag{110}$$

where
(i) either $T_{max} = +\infty$, i,e., there is a unique global classical solution
(ii) or $T_{max} < +\infty$ and $\lim_{t \to T_{max}-0} \|Z(t)\|_H = +\infty$.

Proof. Combining the proof for hyperbolic case (see, e.g., Example 2.3.1 in [52]), we obtain that A is a maximal accretive operator. Then, it is straightforward to establish that for any $Z_1,\ Z_2 \in H$,

$$\|F(Z_1) - F(Z_2)\|_H \le C\|Z_1 - Z_2\|_H \max\{\|Z_1\|_H, \|Z_2\|_H\}, \tag{111}$$

where C is a constant independent of Z_1 and Z_2. So, we get F to be locally Lipschitz on H. Hence, the system (108)–(109) has a unique classical solution.

Next, we will establish that the existence of the classical solution is global. In order to prove that $T_{\max} = +\infty$, which means there is no blowup, we need to make a priori estimates of the H^1 norm of w and z. Based on the proof of boundedness of w and z in L^2 norms, in our present work, one can obtain that w and z are bounded in H^1 by using the following new Lyapunov function

$$V_4(t) = \frac{1}{2}\int_0^1 (1+x) w_{xx}(x,t)^2 \mathrm{d}x + \frac{b_4 D}{2}\int_0^1 (1+x) z_{xx}(x,t)^2 \mathrm{d}x. \tag{112}$$

Using the integration by parts, the derivative of (112) with respect to time is written as

$$\begin{aligned}\dot{V}_4(t) =& \int_0^1 (1+x)w_{xx}(x,t)w_{xxt}(x,t)\mathrm{d}x + b_4 D\int_0^1 (1+x)z_{xx}(x,t)z_{xxt}(x,t)\mathrm{d}x\\ =& \int_0^1 (1+x)w_{xx}(x,t)w_{xxx}(x,t)\mathrm{d}x + b_4 \int_0^1 (1+x)z_{xx}(x,t)z_{xxx}(x,t)\mathrm{d}x\\ &- b_4\tilde{D}(t)\int_0^1 (1+x)z_{xx}(x,t)P_{1xx}(x,t)\mathrm{d}x - b_4 D\dot{\hat{D}}(t)\int_0^1 (1+x)z_{xx}(x,t)P_{2xx}(x,t)\mathrm{d}x\\ =& w_{xx}(1,t)^2 - \frac{1}{2}w_{xx}(0,t)^2 - \frac{1}{2}\|w_{xx}(x,t)\|^2 + b_4 z_{xx}(1,t)^2 - \frac{b_4}{2}z_{xx}(0,t)^2 - \frac{b_4}{2}\|z_{xx}(x,t)\|^2\\ &- b_4\tilde{D}(t)\int_0^1 (1+x)z_{xx}(x,t)P_{1xx}(x,t)\mathrm{d}x - b_4 D\dot{\hat{D}}(t)\int_0^1 (1+x)z_{xx}(x,t)P_{2xx}(x,t)\mathrm{d}x. \end{aligned} \tag{113}$$

Based on (40), (41), one can get

$$\begin{aligned} w_{xx}(1,t) =& w_{tx}(1,t) = w_{tt}(1,t) = z_{tt}(0,t)\\ =& \frac{1}{D^2}z_{xx}(0,t) - \frac{\tilde{D}(t)}{D^2}P_{1x}(0,t) - \frac{\dot{\hat{D}}(t)}{D}P_{2x}(0,t) + \frac{1}{D}\dot{\hat{D}}(t)P_1(0,t) - \ddot{\hat{D}}(t)P_2(0,t)\\ &- \frac{1}{D}\tilde{D}(t)P_{1t}(0,t) - \dot{\hat{D}}(t)P_{2t}(0,t), \end{aligned} \tag{114}$$

$$\begin{aligned} z_{xx}(1,t) =& \tilde{D}(t)P_{1x}(1,t) + D\dot{\hat{D}}(t)P_{2x}(1,t) - D\dot{\hat{D}}(t)P_1(1,t) + D^2\ddot{\hat{D}}(t)P_2(1,t)\\ &+ D\tilde{D}(t)P_{1t}(1,t) + D^2\dot{\hat{D}}(t)P_{2t}(1,t). \end{aligned} \tag{115}$$

Submitting (114) and (115) into (113), we arrive at the following inequality

$$\begin{aligned} \dot{V}_4(t) \leq& -\frac{1}{2}w_{xx}(0,t)^2 - \frac{1}{2}\|w_{xx}\|^2 - \frac{b_4}{2}\|z_{xx}\|^2 - \left(\frac{b_4}{2} - \frac{7}{D^4}\right)z_{xx}(0,t)^2 + 2b_4|\tilde{D}(t)|\|z_{xx}\|\|P_{1xx}\|\\ &+ 2b_4 D|\dot{\hat{D}}(t)|\|z_{xx}\|\|P_{2xx}\| + \frac{7\tilde{D}(t)^2}{D^4}P_{1x}(0,t)^2 + \frac{7\dot{\hat{D}}(t)^2}{D^2}P_{2x}(0,t)^2 + \frac{7\dot{\hat{D}}(t)^2}{D^2}P_1(0,t)^2\\ &+ 7\ddot{\hat{D}}(t)^2P_2(0,t)^2 + \frac{7}{D^2}\tilde{D}(t)^2P_{1t}(0,t)^2 + 7\dot{\hat{D}}(t)^2P_{2t}(0,t)^2 + 6b_4\tilde{D}(t)^2P_{1x}(1,t)^2\\ &+ 6b_4D^2\dot{\hat{D}}(t)^2P_{2x}(1,t)^2 + 6b_4D^2\dot{\hat{D}}(t)^2P_1(1,t)^2 + 6b_4D^4\ddot{\hat{D}}(t)^2P_2(1,t)^2 + 6b_4D^2\tilde{D}(t)^2P_{1t}(1,t)^2\\ &+ 6b_4D^4\dot{\hat{D}}(t)^2P_{2t}(1,t)^2. \end{aligned} \tag{116}$$

Choosing $b_4 > \frac{14}{\underline{D}^4}$, we get,

$$\begin{aligned} \dot{V}_4(t) \leq& -\frac{1}{2}\|w_{xx}\|^2 - \frac{b_4}{2}\|z_{xx}\|^2 + b_4\tilde{D}(t)^2\|z_{xx}\|^2 + b_4\|P_{1xx}\|^2 + b_4D^2\dot{\hat{D}}(t)^2\|z_{xx}\|^2 + b_4\|P_{2xx}\|^2\\ &+ \frac{7\tilde{D}(t)^2}{D^4}P_{1x}(0,t)^2 + \frac{7\dot{\hat{D}}(t)^2}{D^2}P_{2x}(0,t)^2 + \frac{7\dot{\hat{D}}(t)^2}{D^2}P_1(0,t)^2 + 7\ddot{\hat{D}}(t)^2P_2(0,t)^2\\ &+ \frac{7}{D^2}\tilde{D}(t)^2P_{1t}(0,t)^2 + 7\dot{\hat{D}}(t)^2P_{2t}(0,t)^2 + 6b_4\tilde{D}(t)^2P_{1x}(1,t)^2 + 6b_4D^2\dot{\hat{D}}(t)^2P_{2x}(1,t)^2\\ &+ 6b_4D^2\dot{\hat{D}}(t)^2P_1(1,t)^2 + 6b_4D^4\ddot{\hat{D}}(t)^2P_2(1,t)^2 + 6b_4D^2\tilde{D}(t)^2P_{1t}(1,t)^2 + 6b_4D^4\dot{\hat{D}}(t)^2P_{2t}(1,t)^2\\ \leq& -c_1V_4(t) + f_5(t)V_4(t) + f_6(t), \end{aligned} \tag{117}$$

where we use Young's and Agmon's inequalities. Here, $c_1 = \frac{1}{2}\min\{1, \frac{1}{\overline{D}}\}$, and the functions $f_5(t)$ and $f_6(t)$ are given by

$$f_5(t) = b_5\overline{D}^2(\dot{\hat{D}}(t)^2 + 4), \tag{118}$$

$$\begin{aligned} f_6(t) =& b_4\|P_{1xx}\|^2 + b_4\|P_{2xx}\|^2 + \frac{28\bar{D}^2}{\underline{D}^4}P_{1x}(0,t)^2 + \frac{7\dot{\hat{D}}(t)^2}{\underline{D}^2}P_{2x}(0,t)^2 + \frac{7\dot{\hat{D}}(t)^2}{\underline{D}^2}P_1(0,t)^2\\ &+ 7\ddot{\hat{D}}(t)^2P_2(0,t)^2 + \frac{28}{\underline{D}^2}\bar{D}^2P_{1t}(0,t)^2 + 7\dot{\hat{D}}(t)^2P_{2t}(0,t)^2 + 24b_4\bar{D}^2P_{1x}(1,t)^2\\ &+ 6b_4\bar{D}^2\dot{\hat{D}}(t)^2P_{2x}(1,t)^2 + 6b_4\bar{D}^2\dot{\hat{D}}(t)^2P_1(1,t)^2 + 6b_4\bar{D}^4\ddot{\hat{D}}(t)^2P_2(1,t)^2 + 24b_4\bar{D}^4P_{1t}(1,t)^2\\ &+ 6b_4\bar{D}^4\dot{\hat{D}}(t)^2P_{2t}(1,t)^2, \end{aligned} \tag{119}$$

based on all above results, we can get all terms in (118) and (119) are integrable of time. Using Lemma D.3 [51], we get that $\|w_{xx}\|$ and $\|z_{xx}\|$ are bounded. Then, from (40) and (44),

$$w_{tx}(x,t) = w_{xx}(x,t), \tag{120}$$

$$Dz_{tx}(x,t) = z_{xx}(x,t) - \tilde{D}(t)P_{1x}(x,t) - D\dot{\hat{D}}(t)P_{2x}(x,t), \tag{121}$$

we get $\|w_{tx}\|$ and $\|z_{tx}\|$ are bounded. Combing with $\|w_x\|, \|z_x\|^2 \to 0$ as $t \to \infty$ and regulation of $w(x,t)$ and $z(x,t)$, one can get $\|w_t\|, \|z_t\|^2 \to 0$ as $t \to \infty$, and then, by using the Agmon's inequality, the regulation of $w_t(x,t)$ and $z_t(x,t)$ is proven for all $x \in [0,1]$. Therefore, we have proved that $\|Z\|_H$ is bounded and global classical solution exists. □

Finally, we can get the well-posedness of the closed-loop system consisting of the plant (6)–(11), the control law (37), and the update law (52)–(55) the under Assumption 1 based on the invertibility of the backstepping transformations (29) and (39).

5 | SIMULATION

To illustrate the feasibility of the proposed adaptive controller design, we simulate the closed-loop system consisting (6)–(10), the control law (28), and the update law defined through (52)–(55). The actual delay is set to $D = 2$ assuming known upper and lower bounds defined as $\bar{D} = 4$ and $\underline{D} = 0.1$, respectively. The adaptation gain is set to $\theta = 0.021$, the plant coefficients are chosen as $g(x) = 2(1-x)$ and $f(x,y) = \cos(2\pi x) + 4\sin(2\pi y))$. The simulations are performed considering $u_0(x) = 4\sin(\pi x)$, $v_0(x) = 0$ as initial conditions with $\hat{D}_0 = 1$ and $\hat{D}_0 = 3$, respectively. Figure 1 shows the convergence of the plant's state $u(x,t)$ with and without adaptation, respectively. In the absence of adaptation, but with a "mismatch input delay" set to $\hat{D}(t) = 3$ (the true delay being $D = 2$). Figure 2 (a) shows the dynamics of the L^2-norm of the plant state $\|u(x,t)\|_{L^2}$ with and without adaptation, respectively. The control effort is displayed in Figure 2 (b) and the update law in Figure 2 (c). Finally, Figure 2 (d) reflects a good estimate of the delay with $\hat{D}(t)$ converging to the true value $D = 2$.

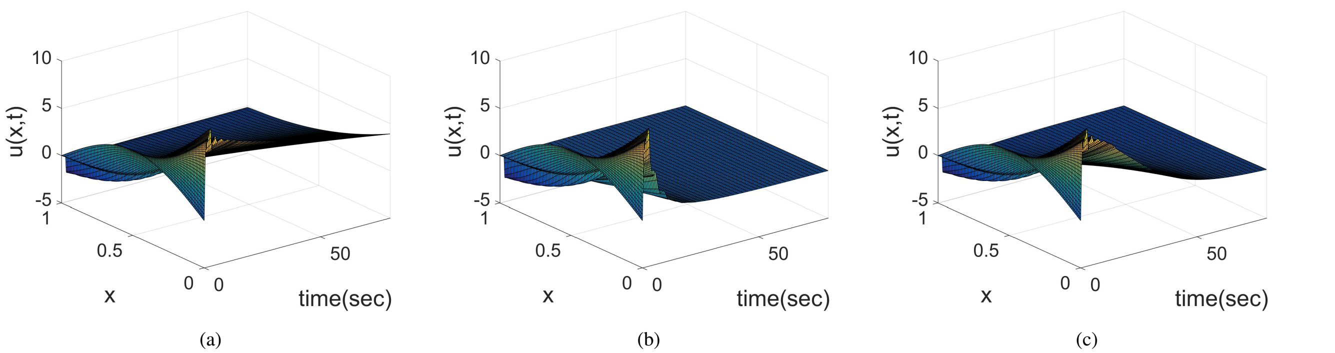

FIGURE 1 The closed-loop system dynamics with $u_0(x)$, $v_0(x)$ and $\hat{D}(0)$. (a) The distributed state $u(x,t)$ with nonadaptive control. (b) The distributed state $u(x,t)$ with $\hat{D}(0) = 1$. (c) The distributed state $u(x,t)$ with $\hat{D}(0) = 3$.

6 | CONCLUSION

We have studied a class of first-order hyperbolic PIDEs systems with an input subject to an unknown time delay. By utilizing an infinite-dimensional representation of the actuator delay, the system was transformed into a cascading structure consisting of a transport PDE and a PIDE. We successfully established global stability results by designing a parameter update law using the well-known infinite-dimensional backstepping technique and a Lyapunov argument. Furthermore, we analyzed the well-posedness of the system, taking into account the added difficulty caused by the presence of nonlinear terms. Through numerical simulations, we have demonstrated the effectiveness of the proposed method. This research contributes to the understanding

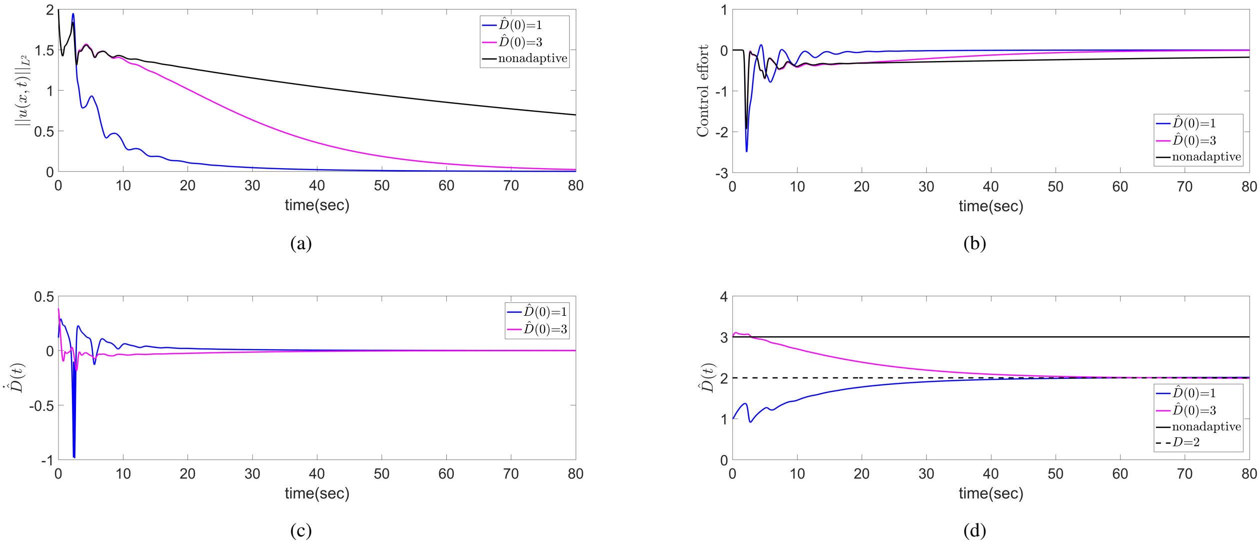

FIGURE 2 The closed-loop system dynamics with $u_0(x)$, $v_0(x)$ and to $\hat{D}(0)$ with and without adaptation. (a) L^2-norm of the plant state $u(x,t)$. (b) The time evolution of the control signal. (c) The dynamics of the update law $\dot{\hat{D}}(t)$. (d) The time-evolution of the estimate of the unknown parameter $\hat{D}(t)$.

and control of systems with unknown time delays and provides valuable insights into the stability analysis and parameter update design for such systems. Future work may involve extending these findings to more complex systems or considering additional constraints and uncertainties.

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References

[1] H. Yu and M. Krstic, "Traffic congestion control for Aw-Rascle-Zhang model," Automatica, vol. 100, pp. 38–51, 2019.

[2] N. Bekiaris-Liberis and M. Krstic, "Compensation of actuator dynamics governed by quasilinear hyperbolic PDEs," Automatica, vol. 92, pp. 29–40, 2018.

[3] O. M. Aamo, "Leak detection, size estimation and localization in pipe flows," IEEE Transactions on Automatic Control, vol. 61, no. 1, pp. 246–251, 2015.

[4] H. Sano, "Exponential stability of a mono-tubular heat exchanger equation with output feedback," Systems & control letters, vol. 50, no. 5, pp. 363–369, 2003.

[5] M. Ghousein, E. Witrant, V. Bhanot, and P. Petagna, "Adaptive boundary observer design for linear hyperbolic systems; application to estimation in heat exchangers," Automatica, vol. 114, p. 108824, 2020.

[6] R. Mlayeh, S. Toumi, and L. Beji, "Backstepping boundary observer based-control for hyperbolic PDE in rotary drilling system," Applied Mathematics and Computation, vol. 322, pp. 66–78, 2018.

[7] J. Wang and M. Krstic, "Delay-compensated control of sandwiched ODE-PDE-ODE hyperbolic systems for oil drilling and disaster relief," Automatica, vol. 120, p. 109131, 2020.

[8] H. Sano and M. Wakaiki, "Boundary stabilization of first-order hyperbolic equations with input delay," Japan Journal of Industrial and Applied Mathematics, vol. 36, no. 2, pp. 325–355, 2019.

[9] S. Gupta, R. Gupta, and S. Padhee, "Parametric system identification and robust controller design for liquid-liquid heat exchanger system," IET Control Theory & Applications, vol. 12, no. 10, pp. 1474–1482, 2018.

[10] J.-M. Coron, O. Glass, and Z. Wang, "Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed," SIAM Journal on Control and Optimization, vol. 48, no. 5, pp. 3105–3122, 2010.

[11] H. Sano, "Output tracking control of a parallel-flow heat exchange process," Systems & Control Letters, vol. 60, no. 11, pp. 917–921, 2011.

[12] M. Krstic and A. Smyshlyaev, "Boundary control of PDEs: A course on backstepping designs," in SIAM, vol. 16, 2008.

[13] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," Systems & Control Letters, vol. 57, no. 9, pp. 750–758, 2008.

[14] R. Vazquez, M. Krstic, and J.-M. Coron, "Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system," in 2011 50th IEEE conference on decision and control and european control conference, pp. 4937–4942, 2011.

[15] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, "Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping," SIAM Journal on Control and Oprimization, vol. 51, no. 3, pp. 2005–2035, 2013.

[16] L. Su, S. Chen, J.-M. Wang, and M. Krstic, "Stabilization of a 2×2 system of hyperbolic PDEs with recirculation in the unactuated channel," Automatica, vol. 120, p. 109147, 2020.

[17] F. Di Meglio, R. Vazquez, and M. Krstic, "Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input," IEEE Transactions on Automatic Control, vol. 58, no. 12, pp. 3097–3111, 2013.

[18] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic, "Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs," IEEE Transactions on Automatic Control, vol. 61, no. 11, pp. 3301–3314, 2015.

[19] L. Hu, R. Vazquez, F. Di Meglio, and M. Krstic, "Boundary exponential stabilization of 1-D inhomogeneous quasilinear hyperbolic systems," SIAM Journal on Control and Oprimization, vol. 57, no. 2, pp. 963–998, 2019.

[20] H. Anfinsen and O. M. Aamo, "Stabilization and tracking control of a time-variant linear hyperbolic PIDE using backstepping," Automatica, vol. 116, p. 108929, 2020.

[21] F. Bribiesca-Argomedo and M. Krstic, "Backstepping-forwarding control and observation for hyperbolic PDEs with fredholm integrals," IEEE Transactions on Automatic Control, vol. 60, no. 8, pp. 2145–2160, 2015.

[22] X. Xu and S. Dubljevic, "Output regulation for a class of linear boundary controlled first-order hyperbolic PIDE systems," Automatica, vol. 85, pp. 43–52, 2017.

[23] J.-M. Coron, L. Hu, and G. Olive, "Stabilization and controllability of first-order integro-differential hyperbolic equations," Journal of Functional Analysis, vol. 271, no. 12, pp. 3554–3587, 2016.

[24] J. Deutscher, "Finite-time output regulation for linear 2×2 hyperbolic systems using backstepping," Automatica, vol. 75, pp. 54–62, 2017.

[25] F. Di Meglio, F. B. Argomedo, L. Hu, and M. Krstic, "Stabilization of coupled linear hetero-directional hyperbolic PDE-ODE systems," Automatica, vol. 87, pp. 281–289, 2018.

[26] P. Bernard and M. Krstic, "Adaptive output-feedback stabilization of non-local hyperbolic PDEs," Automatica, vol. 50, no. 10, pp. 2692–2699, 2014.

[27] H. Yu, R. Vazquez, and M. Krstic, "Adaptive output feedback for hyperbolic PDE pairs with non-local coupling," in 2017 American Control Conference (ACC), pp. 487–492, 2017.

[28] H. Anfinsen and O. M. Aamo, "Adaptive stabilization of 2×2 linear hyperbolic systems with an unknown boundary parameter from collocated sensing and control," IEEE Transactions on Automatic Control, vol. 62, no. 12, pp. 6237–6249, 2017.

[29] H. Anfinsen and O. M. Aamo, Adaptive Control of Hyperbolic PDEs. Springer Cham, 2019.

[30] J. Wang, S.-X. Tang, and M. Krstic, "Adaptive output-feedback control of torsional vibration in off-shore rotary oil drilling systems," Automatica, vol. 111, p. 108640, 2020.

[31] X. Xu, Y. Yuan, S. Dubljevic, and S. Yin, "Adaptive output regulation of a class of 1-D hyperbolic PDEs with unknown boundary scaled parameters," IEEE Transactions on Automatic Control, pp. DOI: 10.1109/TAC.2023.3241830, 2023, 2023.

[32] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, Nonlinear and Adaptive Control Design. Hoboken, NJ: John Wiley and Sons, 1995.

[33] M. Krstic and A. Smyshlyaev, "Adaptive boundary control for unstable parabolic PDEs-part I: Lyapunov design," IEEE Transactions on Automatic Control, vol. 53, no. 7, pp. 1575–1591, 2008.

[34] A. Smyshlyaev and M. Krstic, "Adaptive boundary control for unstable parabolic PDEs-part II: Estimation-based designs," Automatica, vol. 43, no. 9, pp. 1543–1556, 2007.

[35] A. Smyshlyaev and M. Krstic, "Adaptive boundary control for unstable parabolic PDEs-part III: Output feedback examples with swapping identifiers," Automatica, vol. 43, no. 9, pp. 1557–1564, 2007.

[36] R. Guo, C. Qin, J. Qiu, K. Sun, and T. Wang, "Output-feedback fault-tolerant tracking control for coupled 2×2 hyperbolic PDE systems with multiplicative sensor faults.," International Journal of Robust and Nonlinear Control, vol. 116, p. DOI: 10.1002/rnc.6791, 2023.

[37] J. Qi, S. Dubljevic, and W. Kong, "Output feedback compensation to state and measurement delays for a first-order hyperbolic PIDE with recycle," Automatica, vol. 128, p. 109565, 2021.

[38] J. Zhang, J. Qi, S. Dubljevic, and B. Shen, "Output regulation for a first-order hyperbolic PIDE with state and sensor delays," European Journal of Control, vol. 65, p. 100643, 2022.

[39] J. Auriol and F. Di Meglio, "Robust output feedback stabilization for two heterodirectional linear coupled hyperbolic PDEs," Automatica, vol. 115, p. 108896, 2020.

[40] D. Bresch-Pietri and M. Krstic, "Adaptive trajectory tracking despite unknown input delay and plant parameters," Automatica, vol. 45, no. 9, pp. 2074–2081, 2009.

[41] D. Bresch-Pietri and M. Krstic, "Delay-adaptive predictor feedback for systems with unknown long actuator delay," IEEE Transactions on Automatic Control, vol. 55, no. 9, pp. 2106–2112, 2010.

[42] N. Bekiaris-Liberis and M. Krstic, "Delay-adaptive feedback for linear feedforward systems," Systems & Control Letters, vol. 59, no. 5, pp. 277–283, 2010.

[43] Y. Zhu, M. Krstic, and H. Su, "PDE boundary control of multi-input LTI systems with distinct and uncertain input delays," IEEE Transactions on Automatic Control, vol. 63, no. 12, pp. 4270–4277, 2018.

[44] Y. Zhu, M. Krstic, and H. Su, "Predictor feedback for uncertain linear systems with distributed input delays," IEEE Transactions on Automatic Control, vol. 65, no. 12, pp. 5344–5351, 2020.

[45] Y. Zhu, M. Krstic, and H. Su, "Delay-adaptive control for linear systems with distributed input delays," Automatica, vol. 116, p. 108902, 2020.

[46] S. Wang, J. Qi, and M. Diagne, "Adaptive boundary control of reaction–diffusion PDEs with unknown input delay," Automatica, vol. 134, p. 109909, 2021.

[47] S. Wang, M. Diagne, and J. Qi, "Delay-adaptive predictor feedback control of reaction-advection-diffusion PDEs with a delayed distributed input," IEEE Transactions on Automatic Control, vol. 67, no. 7, pp. 3762–3769, 2022.

[48] H. Anfinsen and O. M. Aamo, "Estimation of parameters in a class of hyperbolic systems with uncertain transport speeds," in 2017 25th Mediterranean Conference on Control and Automation (MED), pp. 80–87, 2017.

[49] H. Anfinsen, H. Holta, and O. M. Aamo, "Adaptive control of a linear hyperbolic PDE with uncertain transport speed and a spatially varying coefficient," in 2020 28th Mediterranean Conference on Control and Automation (MED), pp. 945–951, 2020.

[50] M. Krstic and D. Bresch-Pietri, "Delay-adaptive full-state predictor feedback for systems with unknown long actuator delay," in 2009 American Control Conference, pp. 4500–4505, 2009.

[51] A. Smyshlyaev and M. Krstic, Adaptive Control of Parabolic PDEs. Princeton University Press, 2010.

[52] S. Zheng, Nonlinear evolution equations. Chapman & Hall/CRC, Boca Raton, Fla., 2004.

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