

ARTICLE TYPE

Stability analysis of nonlinear fuzzy hybrid control systems subject to saturation and delays via step-function method

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Abstract

Under the framework of the step-function method, the stability of a nonlinear fuzzy hybrid control system combining an impulsive controller and a continuous state feedback controller is investigated. Both the two controllers are assumed to be subject to both actuator saturation and time-varying delays, which has received little attention if any, in the existing studies. A new assumption is established enabling the use of generalized sector conditions to tackle the double saturation, and the conservatism of the stability results is remarkably reduced thanks to the improved step-function method. The stability theorem proposed in this paper removes restriction on the time delays of both controllers, which can be also applied to wider scopes of systems, including hybrid control systems with both stabilizing and instabilizing impulses, systems with varying impulsive gain, and systems with Zeno behavior. Numerical simulations of stabilization for different systems by delayed saturated hybrid control have been conducted, which demonstrate the validity of proposed theorems.

KEYWORDS:

impulsive systems, stability analysis, step-function method, actuator saturation

1 | INTRODUCTION

Various phenomena and processes in real life such as biological population management and spacecraft control can be modeled by impulsive systems which involve both continuous dynamics and discrete state changes occurring at specific instants. Milman and Myshkis conducted initial research on impulsive systems in 1960.¹ Since then, considerable studies on the stability of systems with impulses have been carried out^{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17} and several methods have been established to investigate impulsive systems including the impulse time window method^{2,4} and B-equivalence method.^{11,12,13} More recently, T-S fuzzy systems with impulse effects, namely, T-S fuzzy impulsive systems, have also been explored and their stability has been studied.^{13,14,15,16,17} The use of fuzzy expressions simplifies the analysis process, allowing the application of analysis methods for linear systems to impulsive systems with nonlinear continuous dynamics. Currently, impulsive control systems, an important branch of impulsive systems that utilizes a series of impulses as the controller, have garnered significant research interest. It is noteworthy that, unlike other classes of impulsive systems, the dynamics of impulses in impulsive control systems are inherently stable. For its accessibility, robustness, and relatively low cost, impulsive control has attracted considerable interest among researchers and engineers. Moreover, aiming to further enhance control performance and tackle some more complex nonlinear systems, hybrid control involving impulsive controller has captured the attention of scientists in recent years.^{2,3,4,18,19,20,21,22,23,24,25,26,27,28,29}

Actually, the combination of an impulsive controller and another controller, such as a continuous state-feedback controller, in nonlinear systems has demonstrated the ability to achieve superior control performance. The impulsive controller introduces intermittent control inputs at specific time instants, which allows for rapid adjustments and responses to sudden changes or disturbances in the system. On the other hand, the state-feedback controller operates in real-time, continuously monitoring and adjusting the control inputs to maintain stability and optimize performance. When combined, these two controllers complement each other's capabilities. This hybrid control approach may leverage the strengths of both controllers and has proven particularly effective in diverse fields, such as neural networks^{18,19,20,21,22,23,24}, multi-agent control^{2,3,4}, and integrated pest management.²⁵

Unfortunately, it is worth noting that the phenomenon of saturation is common and, in many circumstances, inevitable in practical control systems. It often comes from the output limitations of the actuator, which is a device stimulated by the controller and amplifies the control signals to directly drive the controlled object in a closed-loop control system. In real control systems, the output of the actuator (e.g., an impulse thruster) is always constrained as the output amplitude and power of a certain device are bounded. When the actuator reaches its saturation threshold, a discrepancy arises between the expected and actual output, which can degrade the control performance, such as reduced speed of response, severe oscillation, and even instability. Given their importance for practical engineering, control systems with actuator saturation constraints have gained much attention recently. Several methods addressing saturated nonlinearity have been developed. The convex hull analysis method^{18,26,27,28} and the sector nonlinearity method^{23,24,29,30} are two approaches with the widest application. Via the former approach, a convex polyhedron is used to indicate the saturation term, while the latter method aims to transform a system with actuator saturation into a non-saturation system with an additional nonlinear feedback loop, with the help of a decentralized dead-zone term. The dead zone function is proven to satisfy some inequalities (called sector conditions) which can help to simplify the system when certain set relations are fulfilled. In addition to actuator saturation, the time delay is another unavoidable phenomenon that may damage the performance and stability of a specific control system, occurring during signal transmission, computation, actuation, or measurement processes. Considering the challenges it may pose, time-varying delays should also be considered when designing a nonlinear impulsive control system.^{8,18,30,31,32,33,34} Specially, state feedback control of neural networks systems with impulsive inputs has obtained extensive attention considering time delays in either impulse inputs or continuous feedback control.^{18,19,20,21,22,23,24} Very recently, taking the actuator saturation and/or time delay into account, some interesting and inspiring results in the field of hybrid control involving impulse effect have been reported. For example, studies conducted by Li²⁹ establish a framework of composite control involving both sampled-data and constrained impulsive controllers and reveal its effectiveness in stabilizing a class of nonlinear dynamic systems. Recent research conducted by Yu focuses on the stabilization of nonlinear systems using impulsive control and continuous control simultaneously, where both controllers are assumed to have saturated outputs.²⁶ The saturation nonlinearity in the two controllers is converted into convex hulls utilizing the poly-topic representation approach, and a new set relation is given to address the problem of double saturation.

Nevertheless, to the best of our knowledge, double saturation in composite control systems has been investigated by few researchers^{26,27}, and, unfortunately, none of these works considers both saturation and delays in both controllers simultaneously, despite their theoretical and practical significance. In fact, each controller in practical hybrid control systems may experience saturation and time-varying delays, and the combination of double saturation and double delays will result in significant additional complexity. This is an open question that warrants further investigation.

There are some other shortcomings or limitations that call for improvement in current studies. For example, despite extensive studies reported considering the stability of systems with impulses, room for improvement remains. When analyzing the stability of hybrid control systems, the Lyapunov-like function approach is generally applied. However, many present works require the monotonic decreasing with time of the Lyapunov function in each interval between two adjacent impulse actions, which causes significant conservatism in stability conditions. To tackle this problem and derive less conservative stability conditions, a novel analysis framework called the step-function method has been developed recently¹⁷, involving constructing a step auxiliary function that is always greater than or equal to the Lyapunov-like function. This method only requires the Lyapunov function to decrease every step of the step function (equivalently, every m impulsive intervals, where m can be a bit large number), and imposing fewer restrictions on the dynamics during the m impulsive intervals. Both theoretical and simulation evidence has revealed that results derived via the step-function method exhibit less conservatism. In this paper, this approach is expected to be applied to systems that are subject to saturation and time delays. Apart from that, the impulsive gain matrices in many related studies are assumed to be fixed, or, time-invariant, for the sake of analysis simplicity.^{18,22,23,24,26,29} This assumption seriously restricts the applicability of the current findings and results. In addition, set inclusion relations proposed in some literature are difficult to verify²⁶, which highlights the need for improved expression of these conditions.

Motivated by the preceding discussions on existing research, in this paper, a nonlinear hybrid control system that incorporates saturated delayed impulsive and state feedback controllers is presented using the T-S fuzzy model framework. The saturation in both controllers is tackled through local sector conditions, while the step-function method is then employed to obtain relaxed conditions for local asymptotic stability of the hybrid control system. The following summarizes the advantages and innovations of the methodologies and results presented in this paper compared to existing studies:

- (i). Incorporating double saturation and double time-varying delays in the framework of fuzzy hybrid control systems, which is previously unexplored because of its complexity;
- (ii). Reducing the conservatism of stability results by relaxing the limitations of the Lyapunov-like function through the utilization of a novel step function-based method;
- (iii). The impulsive gain matrix of the impulsive controller in the hybrid control system proposed in this paper can be variable at each impulsive instance, which is a rare feature in current studies;
- (iv). The proposed method can handle systems with varying time intervals between impulses and can also address Zeno behavior, which is not effectively handled by popular approaches for analysis of delayed impulsive systems such as the average impulsive interval-based method;^{31,32}
- (v). the proposed stability theorem removes the common assumption in most existing research that the delays of controllers must be shorter than the impulsive intervals.

In the subsequent parts of this paper, Section 2 introduces the problem and provides necessary background information regarding fuzzy hybrid control systems. The stabilization analysis of a simplified case, where only the impulsive controller is involved, is presented in Section 3. Furthermore, applying the step-function and sector nonlinearity method, the main findings of this paper are presented in Section 4, where sufficient stability conditions for hybrid control systems considering double saturation and time-varying delays are derived. To verify the effectiveness of these results, Section 5 provides two illustrative simulation examples. The paper concludes with a summary of the key points in Section 6.

2 | PROBLEM STATEMENT AND PRELIMINARIES

Consider a general nonlinear impulsive system which is described as

$$\begin{cases} \dot{z}(t) = y(t, z(t)), t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k) = \partial(\tau_k, z(\tau_k)) \\ z(\tau_0^+) = z_0, \end{cases} \quad k = 1, 2, 3, \dots \quad (1)$$

where $z(t) \in \mathfrak{R}^n$ denotes the state vector; $y : \mathfrak{R}^+ \times \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is a continuous nonlinear function with respect to both time variable t and state vector z , which denotes the continuous behavior of the system; τ_k ($k = 1, 2, 3, \dots$) represents time instant when there is an impulse, satisfying $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_k < \dots < \tau_\infty$. Suppose that $z(\tau_k^+) = \lim_{\Delta t \rightarrow 0^+} z(\tau_k + \Delta t)$, and $z(\tau_k^-) = \lim_{\Delta t \rightarrow 0^-} z(\tau_k + \Delta t)$, and $\Delta z(\tau_k) = z(\tau_k^+) - z(\tau_k^-)$ represents the instantaneous changes of states at impulsive instants. As a common assumption in previous studies, we suppose that $z(t)$ is left-continuous at each time instant of impulse, that is to say, $z(\tau_k^-) = z(\tau_k)$, $k = 1, 2, 3, \dots$

Then, system (1) is assumed to be in the form of T-S fuzzy impulsive system with q fuzzy rules, and the i th subsystem is given as follows:

Module Rule i : IF $e_1(t)$ is M_{i1} , $e_2(t)$ is M_{i2} , ..., $e_p(t)$ is M_{ip} , THEN

$$\begin{cases} \dot{z}(t) = A_i z(t), t \in (\tau_{k-1}, \tau_k] \\ z(\tau_k^+) = [I + G_{ik}] z(\tau_k), k = 1, 2, \dots \end{cases} \quad (2)$$

where $i = 1, 2, \dots, q$, $e(t) = [e_1(t), e_2(t), \dots, e_p(t)]^T$ is a vector composed of p premise variables, M_{ij} is the fuzzy set of module rule i corresponding to the j th premise variable, $A_i \in \mathfrak{R}^{n \times n}$ is the system matrix of the linear subsystem, matrix $G_{ik} \in \mathfrak{R}^{n \times n}$ denotes the impulsive control gain at τ_k and $I \in \mathfrak{R}^{n \times n}$ is the identity matrix.

By singleton fuzzifier, product fuzzy inference and center average defuzzifier, we can construct the overall T-S fuzzy impulsive system as

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^q d_i(e(t)) A_i z(t), t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k^+) = \sum_{i=1}^q d_i(e(\tau_k)) G_{ik} z(\tau_k) \end{cases} \quad (3)$$

where $d_i(e(t)) = \phi_i(e(t)) / \sum_{i=1}^q \phi_i(e(t))$, $\phi_i(e(t)) = \prod_{j=1}^p \mathcal{M}_{ij}(e_j(t))$, \mathcal{M}_{ij} represents the membership function. Obviously, $d_i(e(t)) \geq 0$, $\sum_{i=1}^q d_i(e(t)) = 1$ for $t \geq \tau_0$. Consider the saturation function, which can be defined as

$$Sat(z) = \begin{cases} \Gamma_{\max}, z > \Gamma_{\max} \\ z, \Gamma_{\min} \leq z \leq \Gamma_{\max} \\ \Gamma_{\min}, z < \Gamma_{\min} \end{cases} \quad (4)$$

where Γ_{\min} and Γ_{\max} are the lower and upper saturation thresholds, respectively. For simplicity, the saturation function is assumed to be symmetrical in this paper:

$$Sat(z) = \begin{cases} u_0, z > u_0 \\ z, -u_0 \leq z \leq u_0 \\ -u_0, z < -u_0 \end{cases} \quad (5)$$

Then the fuzzy saturated impulsive system with nonlinear continuous dynamics can be formulated by

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^q d_i(e(t)) A_i z(t), t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k) = Sat \left\{ \sum_{i=1}^q d_i(e(\tau_k)) G_{ik} z(\tau_k) \right\} \end{cases} \quad (6)$$

For brevity, we assume that matrix $Q_k = \sum_{i=1}^q d_i(e(t)) G_{ik} \in \mathbb{R}^{n \times n}$. And then, a hybrid control system composing an impulsive controller and a continuous state feedback controller can be constructed as

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^q d_i(e(t)) [A_i - K_i] z(t), t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k) = \sum_{i=1}^q d_i(e(\tau_k)) G_{ik} z(\tau_k), t = \tau_k \end{cases} \quad (7)$$

If the saturation and the time-varying delay in each controller are considered, the composite control system can be further modeled by fuzzy model as

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^q d_i(e(t)) A_i z(t) - Sat_1 \left\{ \sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right\}, t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k) = Sat_2 \left\{ \sum_{i=1}^q d_i(e(\tau_k)) G_{ik} z(\tau_k - J_2(\tau_k)) \right\}, t = \tau_k \\ z(\tau_0 + \xi) = \theta(\tau_0 + \xi), \xi \in [-\kappa, 0] \end{cases} \quad (8)$$

where time delays in the two controllers $J_1(t) \geq 0$, $J_2(\tau_k) \geq 0$, $\forall t \geq \tau_0$, and $\kappa = \max[\sup[J_1(t)], \sup[J_2(\tau_k)]] \geq 0$, $k = 1, 2, \dots$. For brevity, we suppose that $D = \sum_{i=1}^q d_i(e(t)) K_i$. To ensure self-containment of this paper, we now present some definitions, lemmas, and other preliminaries.

Definition 1. Three classes of functions are defined as follows:

- i. If a continuous function $f_1 : [0, a) \rightarrow [0, \infty)$ is strictly increasing with $f_1(0) = 0$, then we say $f_1 \in K$;
- ii. If a function $f_2 \in K$ with $a = +\infty$ and $m \rightarrow +\infty$, $f_2(m) \rightarrow +\infty$, then we say $f_2 \in K_\infty$;
- iii. If a continuous function $f_3 : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ satisfies $f_3(o_1, o_2) \in K$ for each fixed o_2 , and $f_3(o_1, o_2)$ is decreasing with respect to o_2 , and $f_3(o_1, o_2) \rightarrow 0$ as $o_2 \rightarrow \infty$, then we say $f_3 \in KL$.

The Lyapunov stability of non-autonomous nonlinear systems are defined using these classes of functions in most existing works.

Definition 2.³³ The equilibrium point of (1), or, the origin is uniformly attractively stable if there exists a function $\vartheta \in KL$, such that, for any $t \geq \tau_0$, there is

$$\|z(t, \tau_0, z_0)\| \leq \vartheta(\|z_0\|, t - \tau_0) \quad (9)$$

The uniform attractive stability of system (6) and (8) can be defined similarly. To investigate the stability of impulsive systems, a so-called step-function method is proposed¹⁷, which has shown a great advantage in reducing the conservatism of the stability conditions, with the ability to address various kinds of impulsive systems. In this way, the stability of the system can be derived when the step auxiliary function exhibits a decreasing trend and is convergent to the equilibrium point.

Lemma 1.¹⁷ Let $z(t)$ be a solution of (6) with $z_0 \in \Omega \subseteq \mathbb{R}^n$, $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a positive definite scalar function. If there exists an integer $m \geq 2$ such that the step function

$$\mathcal{W}(t) = \begin{cases} \sup_{t \in [\tau_0, \tau_m]} \mathcal{V}(z(t)), & t \in [\tau_0, \tau_m] \\ \sup_{t \in [\tau_{mk}, \tau_{m(k+1)}]} \mathcal{V}(z(t)), & t \in (\tau_{mk}, \tau_{m(k+1)}], k = 1, 2, 3, \dots \\ 0, & t > \tau_\infty \end{cases} \quad (10)$$

can fulfil the following conditions:

- i. $\mathcal{W}(\tau_m) \leq \alpha(\mathcal{V}(z_0))$, where $\alpha \in K$;
- ii. $\mathcal{W}(t)$ decreases with t and $\mathcal{W}(t) \rightarrow 0, t \rightarrow \tau_\infty$;

then the origin of (6) is locally uniformly attractively stable for any $z_0 \in \Omega$.

Moreover, considering the case when each impulse is trying to stabilizing the system to the origin, in other words, $\sum_{i=1}^q d_i(e(t)) \mathcal{G}_{ik} < 0$ holds for every $k = 1, 2, 3, \dots$, Lemma 1 can be reduced to the following lemma.

Lemma 2. Let $z(t)$ be a solution of (6) with $z_0 \in \Omega \subseteq \mathbb{R}^n$, $\mathcal{V} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a positive definite scalar function. If there exists an integer $m \geq 2$ such that the step function

$$\mathcal{W}(t) = \begin{cases} \sup_{t \in [\tau_0, \tau_m]} \mathcal{V}(z(t)), & t \in [\tau_0, \tau_m] \\ \mathcal{V}(z(\tau_{mk})), & t \in (\tau_{mk}, \tau_{m(k+1)}], k = 1, 2, 3, \dots \\ 0, & t > \tau_\infty \end{cases} \quad (11)$$

can fulfil the following conditions:

- i. $\mathcal{W}(\tau_m) \leq \alpha(\mathcal{V}(z_0))$, where $\alpha \in K$;
- ii. $\mathcal{W}(t)$ decreases with t and $\mathcal{W}(t) \rightarrow 0, t \rightarrow \tau_\infty$;
- iii. $\mathcal{W}(t) \geq \mathcal{V}(z(t)), t \in (\tau_{m(k-1)}, \tau_{mk}]$;

then the origin of (6) is locally uniformly attractively stable for any $z_0 \in \Omega$.

In this paper, the method utilizing the sector conditions in order to address the saturation term in the impulsive system. Consider a general nonlinear control system with actuator saturation

$$\begin{cases} \dot{z}(t) = y(z(t)) + h(t) \\ h(z(t)) = \text{Sat}(B(z(t))) \end{cases} \quad (12)$$

We define a decentralized dead-zone function

$$\varphi(B) = [\varphi(B_{(1)}), \varphi(B_{(2)}), \dots, \varphi(B_{(n)})]^T \quad (13)$$

where $\varphi(B_{(i)})$ satisfies

$$\varphi(B_{(i)}) = \text{Sat}(B_{(i)}) - B_{(i)} = \begin{cases} u_{0(i)} - B_{(i)}, & B_{(i)} > u_{0(i)} \\ 0, & -u_{0(i)} \leq B_{(i)} \leq u_{0(i)} \\ -u_{0(i)} - B_{(i)}, & B_{(i)} < -u_{0(i)} \end{cases} \quad (14)$$

where $i = 1, 2, 3, \dots, n$. Substituting (14) into (12), we can remove the saturation nonlinearity in the system as

$$\dot{z}(t) = y(z(t)) + B(z(t)) + \varphi(B(z(t))) \quad (15)$$

The dead-zone function has been proven to fulfill some sector conditions, which is helpful in the following deduction. However, before that, we need to define some sets.

Definition 3. ^{35,36} Define a set S as

$$S(v - \delta, \Gamma_{\min}, \Gamma_{\max}) = \{v \in \mathfrak{R}^n, \delta \in \mathfrak{R}^n : -\Gamma_{\min(i)} \leq (v - \delta)_{(i)} \leq \Gamma_{\max(i)}\} \quad (16)$$

where, $v_{(i)}, \delta_{(i)}, \Gamma_{\max(i)}, \Gamma_{\min(i)}$ denote the i th row of $v, \delta, \Gamma_{\max}, \Gamma_{\min}$, respectively. Define a polyhedron set as

$$\bar{S}(\mathcal{Q}_k - \mathcal{N}, \Gamma_{\min}, \Gamma_{\max}) = \{z \in \mathfrak{R}^n : -\Gamma_{\min(i)} \leq (\mathcal{Q}_{k(i)} - \mathcal{N}_{(i)})z \leq \Gamma_{\max(i)}\} \quad (17)$$

In particular, when $\Gamma_{\max} = \Gamma_{\min} = u_0$, we have

$$S(|\mathcal{Q}_k - \mathcal{N}|, u_0) = \{z \in \mathfrak{R}^n : |(\mathcal{Q}_{k(i)} - \mathcal{N}_{(i)})z| \leq u_{0(i)}\} \quad (18)$$

According to the results presented in recent researches^{35,36}, the so-called extended local sector conditions can be stated as follows:

Lemma 3. For any $(v, \delta) \in S(v - \delta, \Gamma_{\min}, \Gamma_{\max})$ and any diagonal positive definite matrix $H \in \mathfrak{R}^{m \times m}$, the dead-zone function $\varphi(v)$ always fulfills that

$$\varphi^T(v) H [\varphi(v) + \delta] \leq 0 \quad (19)$$

Another problem emerges due to the fact that the equilibrium point of systems with strong nonlinearity (for example, saturation nonlinearity) is generally not globally stable. To end this section, we shall define the region of attraction and the region of asymptotic stability (RAS) of the saturated impulsive system (6).

Definition 4. The region of attraction R_a is defined as a set of all $z^* \in \mathfrak{R}^n$ satisfying that $z(t, z(0))$ will converge asymptotically to the origin when $z(\tau_0) = z(0) = z^*$.

In practice, we tend to investigate some subsets with some regular forms of the domain of attraction, especially, ellipsoidal sets and polyhedral sets, as it is both difficult and unnecessary to derive the accurate region of attraction.

Definition 5. R_s is called the RAS if $R_s \subseteq R_a$ and $0 \in R_s$.

3 | STEP-FUNCTION APPROACH FOR NONLINEAR FUZZY IMPULSIVE SYSTEM WITH SATURATED IMPULSES

The stability of the nonlinear system with saturated impulses has been studied by several scholars very recently.^{37,38,39} However, existing studies and results are still very few, bearing some serious defects. Specifically, the continuous subsystems of the

saturated impulsive systems are restricted to be stable^{37,38}, which is too strict for most impulsive control systems. Although the research conducted by Li has removed this requirement³⁹, it further assumes that all the impulses must be stabilizing, ignoring the case that stabilizing impulses may be coupled with impulsive disturbances at some instants which will exert an instabilizing effect.

In the following, we shall present a theorem providing relaxed sufficient conditions for the stability of a general nonlinear fuzzy impulsive system. The proof is developed subsequently utilizing the step-function method and extended local sector condition.

Theorem 1. Given a column vector $u_0 \in \mathfrak{R}^n$ and a positive integer $m \geq 1$, if there exist a positive definite symmetric matrix $\mathcal{P} \in \mathfrak{R}^{n \times n}$, a positive definite diagonal matrix $\mathcal{T} \in \mathfrak{R}^{n \times n}$, a matrix $\mathcal{N} \in \mathfrak{R}^{n \times n}$, scalars $\alpha, \beta_k > 0, \beta_1 = 1, \eta > 0$ and positive integer $k \geq 1, i = 1, 2, \dots, q$, assuming that $\tau_0 = 0, z(\tau_0) = z_0$, such that the following conditions are satisfied:

- i. $\mathcal{P}A_i + A_i^T \mathcal{P} - \alpha \mathcal{P} \leq 0$;
- ii. $\Pi_k = \begin{bmatrix} (I + \mathcal{Q}_k)^T \mathcal{P} (I + \mathcal{Q}_k) - \beta_k \mathcal{P} (I + \mathcal{Q}_k)^T \mathcal{P} - \mathcal{N}^T \mathcal{T} & \\ \mathcal{P} (I + \mathcal{Q}_k) - \mathcal{T} \mathcal{N} & \mathcal{P} - 2\mathcal{T} \end{bmatrix} \leq 0$;
- iii. $\begin{bmatrix} \mathcal{P} & [\mathcal{Q}_{k(i)} - \mathcal{N}_{(i)}]^T \\ [\mathcal{Q}_{k(i)} - \mathcal{N}_{(i)}] & u_{0(i)}^2 \end{bmatrix} \geq 0$;
- iv. $\max_{i=m(k-1)+1, \dots, mk} \left\{ \sup_{t \in (\tau_{i-1}, \tau_i]} \left[\left(\prod_{j=m(k-1)}^{i-1} \beta_j \right) e^{\alpha(t - \tau_{m(k-1)})} \right] \right\} \leq \rho_k < 1$;

then the high-dimensional ellipsoid set $\varepsilon(\mathcal{P}, \eta) = \{z \in \mathfrak{R}^n : z^T \mathcal{P} z \leq \eta\}$ is a RAS of the origin of nonlinear fuzzy systems with saturated impulses (6), where $\eta = \rho_1^{-1}$.

Proof. The theoretical deduction of Theorem 1 is based on Lemma 2 and we are going to verify all three conditions in Lemma 2. To begin with, consider a quadratic Lyapunov function candidate $\mathcal{V}(z(t)) = z^T(t) \mathcal{P} z(t)$, and let $\mathcal{V}_0 = z_0^T \mathcal{P} z_0$. When $t \in (\tau_{k-1}, \tau_k]$, the time derivative of $\mathcal{V}(z(t))$ satisfies

$$\begin{aligned} D^+ \mathcal{V}(z(t)) &= \left[\sum_{i=1}^q d_i(e(t)) A_i z(t) \right]^T \mathcal{P} z(t) + z^T(t) \mathcal{P} \left[\sum_{i=1}^q d_i(e(t)) A_i z(t) \right] \\ &= \sum_{i=1}^q d_i(e(t)) z^T(t) [A_i^T \mathcal{P} + \mathcal{P} A_i - \alpha \mathcal{P}] z(t) + \alpha z^T(t) \mathcal{P} z(t) \\ &\leq \alpha \mathcal{V}(z(t)) \end{aligned} \quad (20)$$

In this way, obviously we can derive $\mathcal{V}(z(t)) \leq e^{\alpha(t - \tau_{k-1})} \mathcal{V}(z(\tau_{k-1}^+))$ for any $t \in (\tau_{k-1}, \tau_k]$, and $\mathcal{V}(z(\tau_k)) \leq e^{\alpha(\tau_k - \tau_{k-1})} \mathcal{V}(z(\tau_{k-1}^+))$. Specially, when $t \in (\tau_0, \tau_1]$, supposing that $\mathcal{V}(z_0) = \mathcal{V}_0$, then we have $\mathcal{V}(z(t)) \leq e^{\alpha t} \mathcal{V}_0$.

Now, we will investigate the changes of Lyapunov candidate function at impulsive instants. Noting that if $z(\tau_1) \in \varepsilon(\mathcal{P}, 1) \subset S(|\mathcal{Q}_k - \mathcal{N}|, u_0)$ holds, we have

$$\begin{aligned} \mathcal{V}(z(\tau_1^+)) &= z^T(\tau_1^+) \mathcal{P} z(\tau_1^+) \\ &= [(I + \mathcal{Q}_1) z(\tau_1) + \varphi(\mathcal{Q}_1 z(\tau_1))]^T \mathcal{P} [(I + \mathcal{Q}_1) z(\tau_1) + \varphi(\mathcal{Q}_1 z(\tau_1))] \\ &= z^T(\tau_1) (I + \mathcal{Q}_1)^T \mathcal{P} (I + \mathcal{Q}_1) z(\tau_1) + z^T(\tau_1) (I + \mathcal{Q}_1)^T \mathcal{P} \varphi(\mathcal{Q}_1 z(\tau_1)) \\ &\quad + \varphi^T(\mathcal{Q}_1 z(\tau_1)) \mathcal{P} (I + \mathcal{Q}_1) z(\tau_1) + \varphi^T(\mathcal{Q}_1 z(\tau_1)) \mathcal{P} \varphi(\mathcal{Q}_1 z(\tau_1)) \end{aligned} \quad (21)$$

Reminding of the sector condition presented in Lemma 3, apparently, we have

$$\begin{aligned} \mathcal{V}(z(\tau_1^+)) &\leq z^T(\tau_1) (I + \mathcal{Q}_1)^T \mathcal{P} (I + \mathcal{Q}_1) z(\tau_1) + z^T(\tau_1) (I + \mathcal{Q}_1)^T \mathcal{P} \varphi(\mathcal{Q}_1 z(\tau_1)) \\ &\quad + \varphi^T(\mathcal{Q}_1 z(\tau_1)) \mathcal{P} (I + \mathcal{Q}_1) z(\tau_1) + \varphi^T(\mathcal{Q}_1 z(\tau_1)) \mathcal{P} \varphi(\mathcal{Q}_1 z(\tau_1)) \\ &\quad - 2\varphi^T(\mathcal{Q}_1 z(\tau_1)) \mathcal{T} \varphi(\mathcal{Q}_1 z(\tau_1)) - \varphi^T(\mathcal{Q}_1 z(\tau_1)) \mathcal{T} \mathcal{N} z(\tau_1) - z^T(\tau_1) \mathcal{N}^T \mathcal{T} \varphi(\mathcal{Q}_1 z(\tau_1)) \end{aligned} \quad (22)$$

It follows that

$$\begin{aligned}
\mathcal{V}(z(t_1^+)) &\leq [z^T(t_1) \ \varphi^T(Q_1 z(t_1))] \begin{bmatrix} (I + Q_1)^T \mathcal{P} (I + Q_1) - \beta_1 \mathcal{P} (I + Q_1)^T \mathcal{P} - \mathcal{N}^T \mathcal{T} \\ \mathcal{P} (I + Q_1) - \mathcal{T} \mathcal{N} & \mathcal{P} - 2\mathcal{T} \end{bmatrix} \begin{bmatrix} z(t_1) \\ \varphi(Q_1 z(t_1)) \end{bmatrix} \\
&+ \beta_1 \mathcal{V}(z(t_1)) \\
&= [z^T(t_1) \ \varphi^T(Q_1 z(t_1))] \Pi_1 \begin{bmatrix} z(t_1) \\ \varphi(Q_1 z(t_1)) \end{bmatrix} + \beta_1 \mathcal{V}(z(t_1)) \\
&\leq \beta_1 \mathcal{V}(z(t_1))
\end{aligned} \tag{23}$$

Thus, for $t \in (\tau_1, \tau_2]$, $\mathcal{V}(z(t)) \leq e^{a(t-\tau_1)} \mathcal{V}(z(t_1^+)) \leq e^{at} \beta_1 \mathcal{V}_0$ and $\mathcal{V}(z(t_2)) \leq e^{a\tau_2} \beta_1 \mathcal{V}_0$ under the condition that $z(t_1) \in \varepsilon(\mathcal{P}, 1) \subset S(|Q_1 - \mathcal{N}|, u_0)$. Similarly, when $z(t_2) \in \varepsilon(\mathcal{P}, 1) \subset S(|Q_2 - \mathcal{N}|, u_0)$, we have

$$\begin{aligned}
\mathcal{V}(z(t_2^+)) &\leq [z^T(t_2) \ \varphi^T(Q_2 z(t_2))] \Pi_2 \begin{bmatrix} z(t_2) \\ \varphi(Q_2 z(t_2)) \end{bmatrix} + \beta_2 \mathcal{V}(z(t_2)) \\
&\leq \beta_2 \mathcal{V}(z(t_2))
\end{aligned} \tag{24}$$

Therefore, for $t \in (\tau_2, \tau_3]$,

$$\mathcal{V}(z(t)) \leq \beta_2 e^{a(t-\tau_2)} \mathcal{V}(z(t_2)) \leq \beta_1 \beta_2 e^{at} \mathcal{V}_0 \tag{25}$$

$$\mathcal{V}(z(t_3)) \leq e^{a\tau_3} \beta_1 \beta_2 \mathcal{V}_0 \tag{26}$$

Generally, if $z(t_i) \in \varepsilon(\mathcal{P}, 1) \subset S(|Q_i - \mathcal{N}|, u_0)$, $i = 1, 2, \dots, k-1$, we can derive

$$\mathcal{V}(z(t_i^+)) \leq \beta_i \mathcal{V}(z(t_i)), i = 1, 2, \dots, k-1 \tag{27}$$

And for any $t \in (\tau_{k-1}, \tau_k]$, $k = 1, 2, \dots$, we have

$$\mathcal{V}(z(t)) \leq \beta_{k-1} e^{a(t-\tau_{k-1})} \mathcal{V}(z(t_{k-1})) \leq \dots \leq \left(\prod_{i=1}^{k-1} \beta_i \right) e^{at} \mathcal{V}_0 \tag{28}$$

Next, we consider the conditions given by Lemma 2. For this, an m -span step function is constructed as follows:

$$\mathcal{W}(t) = \begin{cases} \sup_{t \in [\tau_0, \tau_m]} \mathcal{V}(z(t)), & t \in [\tau_0, \tau_m] \\ \sup_{t \in [\tau_{mk}, \tau_{m(k+1)})} \mathcal{V}(z(t)), & t \in (\tau_{mk}, \tau_{m(k+1)}) \\ 0, & t > \tau_\infty \end{cases} \tag{29}$$

When $t \in [\tau_0, \tau_m]$, we have

$$\begin{aligned}
\mathcal{W}(t) &= \sup_{t \in [\tau_0, \tau_m]} \mathcal{V}(z(t)) \\
&\leq \max_{i=1,2,\dots,m} \left[\sup_{t \in [\tau_{i-1}, \tau_i]} \mathcal{V}(z(t)) \right] \\
&\leq \max_{i=1,2,\dots,m} \left\{ \sup_{t \in (\tau_{i-1}, \tau_i]} \left[\left(\prod_{j=1}^{i-1} \beta_j \right) e^{at} \right] \right\} \mathcal{V}_0 \\
&= \rho_1 \mathcal{V}_0
\end{aligned} \tag{30}$$

Note that function $\mathcal{F}(\mathcal{V}_0) = \rho_1 \mathcal{V}_0$ belongs to class K where ρ_1 is a constant, so the condition i in Lemma 2 is satisfied. As for the k th step, when $t \in (\tau_{m(k-1)}, \tau_{mk}]$, $k = 1, 2, \dots$

$$\begin{aligned}
\mathcal{V}(z(t)) &\leq \max_{i=m(k-1)+1, \dots, mk} \left[\sup_{t \in [\tau_{i-1}, \tau_i]} \mathcal{V}(z(t)) \right] \\
&\leq \max_{i=m(k-1)+1, \dots, mk} \left\{ \sup_{t \in [\tau_{i-1}, \tau_i]} \left[\left(\prod_{j=m(k-1)}^{i-1} \beta_j \right) e^{\alpha(t-\tau_{m(k-1)})} \right] \mathcal{V}(z(\tau_{m(k-1)})) \right\} \\
&\leq \rho_k \mathcal{V}(z(\tau_{m(k-1)}))
\end{aligned} \tag{31}$$

Then it fulfills that

$$\mathcal{W}(t) = \sup_{t \in (\tau_{m(k-1)}, \tau_{mk}]} \mathcal{V}(z(t)) \leq \rho_k \mathcal{V}(z(\tau_{m(k-1)})) \tag{32}$$

The third condition has thus been proved. We then consider the monotonicity of step function during the interval $t \in (\tau_{m(k-1)}, \tau_{mk}]$,

$$\begin{aligned}
\mathcal{W}(\tau_{mk}) - \mathcal{W}(\tau_{m(k-1)}) &= \sup_{t \in (\tau_{m(k-1)}, \tau_{mk}]} \mathcal{V}(z(t)) - \sup_{t \in (\tau_{m(k-2)}, \tau_{m(k-1)})} \mathcal{V}(z(t)) \\
&\leq \rho_k \mathcal{V}(z(\tau_{m(k-1)})) - \sup_{t \in (\tau_{m(k-2)}, \tau_{m(k-1)})} \mathcal{V}(z(t)) \\
&\leq -(1 - \rho_k) \mathcal{V}(z(\tau_{m(k-1)})) \\
&\leq 0
\end{aligned} \tag{33}$$

Obviously, it shows that $\mathcal{W}(t)$ decreases with time t , and we have

$$\mathcal{V}(z(\tau_{m(k-1)})) \leq \rho_{k-1} \mathcal{V}(z(\tau_{m(k-2)})) \leq \dots \leq \left(\prod_{i=1}^{k-1} \rho_i \right) \mathcal{V}_0 \tag{34}$$

It follows that

$$\lim_{t \rightarrow \tau_\infty} \mathcal{W}(t) \leq \lim_{k \rightarrow \infty} \left[\left(\prod_{i=1}^k \rho_i \right) \mathcal{V}_0 \right] = 0 \tag{35}$$

Consider the fact that $\lim_{t \rightarrow \tau_\infty} \mathcal{W}(t) \geq 0$, according to the Squeeze Theorem, we have

$$\lim_{t \rightarrow \infty} \mathcal{W}(t) = 0 \tag{36}$$

which means condition ii has been satisfied. In this way, on the bases of Lemma 2, we can say that the origin of (6) is locally asymptotically stable.

Remark 1. It is noteworthy that condition iv in Theorem 1 is less conservative than the conditions given in the present works. Employing the step-function approach, we divide the entire time axis into segments with a length of m . For any moment within each segment, the value of the quadratic Lyapunov function must be less than or equal to the function value at the beginning of that segment after the impulse. As a comparison, take the latest study³⁹ as an example, condition $\ln \mu + \tau \lambda \leq 0$ was proposed in the stability theorem requiring the quadratic Lyapunov function to decrease at every impulsive interval. In this way, the step function-based approach reduces the conservatism of the stability conditions. Moreover, the theorem also indicates that $\mu \in (0, 1)^{39}$, which implies that the theorem can only handle the case that the impulses are stabilizing.

Now, we consider the last step of the proof of Theorem 1, confirming the set relations. Note that during the process of proof above, we have made an assumption:

$$z(\tau_k) \in \varepsilon(\mathcal{P}, 1) \subset \mathcal{S}(|\mathcal{Q}_k - \mathcal{N}|, u_0) = \left\{ z \in \mathfrak{R}^n : |(Q_{k(i)} - \mathcal{N}_{(i)})z| \leq u_{0(i)} \right\}, i = 1, 2, \dots, k \tag{37}$$

Now we will verify this assumption in two steps: Firstly, we prove $\varepsilon(\mathcal{P}, 1) \subset \mathcal{S}(|\mathcal{Q}_k - \mathcal{N}|, u_0)$; Then, we derive $z(\tau_k) \in \varepsilon(\mathcal{P}, 1)$ when $z_0 \in \varepsilon(\mathcal{P}, \eta)$. Consider the condition iv in Theorem 1. According to Schur Complement Theorem, we have

$$\mathcal{P} - \frac{1}{u_{0(i)}^2} [\mathcal{Q}_{k(i)} - \mathcal{N}_{(i)}]^T [\mathcal{Q}_{k(i)} - \mathcal{N}_{(i)}] \geq 0 \tag{38}$$

It follows

$$z^T(t_k) \mathcal{P} z(t_k) - \frac{1}{u_{0(i)}^2} \left| (Q_{k(i)} - \mathcal{N}_{(i)}) z(t_k) \right|^2 \geq 0 \quad (39)$$

With $z^T(t_k) \mathcal{P} z(t_k) \leq 1$, one observes

$$\left| (Q_{k(i)} - \mathcal{N}_{(i)}) z(t_k) \right| \leq u_{0(i)} \quad (40)$$

which means $\varepsilon(\mathcal{P}, 1) \subset S(|Q_k - \mathcal{N}|, u_0)$.

Subsequently, we prove that $z_0 \in \varepsilon(\mathcal{P}, \eta)$ implies $z(\tau_k)^T \mathcal{P} z(\tau_k) \leq \eta = \rho_1$. Suppose that k is a positive integer smaller than m , when $t \in (\tau_{k-1}, \tau_k]$, we have

$$\mathcal{V}(z(t)) \leq \left(\prod_{i=1}^{k-1} \beta_i \right) e^{\alpha t} \mathcal{V}_0 \leq \rho_1 \mathcal{V}_0 \leq \rho_1 \eta = 1 \quad (41)$$

Furthermore, suppose that there exists an integer $a \geq 1$, such that

$$\tau_0 < \tau_1 < \dots < \tau_{a(m-1)} \leq \dots \leq \tau_{k-1} < \tau_k \leq \dots \leq \tau_{am} < \dots < \tau_\infty \quad (42)$$

When $t = \tau_k$,

$$\begin{aligned} \mathcal{V}(z(\tau_k)) &\leq \left(\prod_{i=1}^{k-1} \beta_i \right) e^{\alpha \tau_k} \mathcal{V}_0 \\ &\leq \left(\prod_{j=1}^{a-1} \rho_j \right) \cdot \left(\prod_{i=(a-1)m}^{k-1} \beta_i \right) e^{\alpha(\tau_k - \tau_{(a-1)m})} \mathcal{V}_0 \\ &\leq \left(\prod_{j=1}^a \rho_j \right) \mathcal{V}_0 \end{aligned} \quad (43)$$

Consider that $\mathcal{V}_0 \leq \eta = \rho_1^{-1}$, we have

$$\mathcal{V}(z(\tau_k)) \leq \rho_1 \mathcal{V}_0 \leq \rho_1 \eta = \rho_1 \rho_1^{-1} = 1 \quad (44)$$

Combine the two parts above, we verify the assumption in the process of proof of stability

$$z(\tau_k) \in \varepsilon(\mathcal{P}, 1) \subset S(|Q_k - \mathcal{N}|, u_0) \quad (45)$$

Thus, we can say that the ellipsoid set $\varepsilon(\mathcal{P}, \eta) = \{z \in \mathfrak{R}^n : z^T \mathcal{P} z \leq \eta\}$ is the RAS of the origin of nonlinear fuzzy systems with saturated impulses (6), which means when the trajectory of the impulsive system starts in the ellipsoid $\varepsilon(\mathcal{P}, \eta) = \{z_0 \in \mathfrak{R}^n : z_0^T \mathcal{P} z_0 \leq \rho_1^{-1}\}$, it will converge to the equilibrium point $z^* = 0$ asymptotically. \square

Regarding the situation where the impulses of the system(6) are always unstable, when $m = 2$, Theorem 1 yields the following corollary.

Corollary 1. For nonlinear fuzzy impulsive system (6), given a column vector $u_0 \in \mathfrak{R}^n$, if there exist a positive definite symmetric matrix $\mathcal{P} \in \mathfrak{R}^{n \times n}$, a positive definite diagonal matrix $\mathcal{T} \in \mathfrak{R}^{n \times n}$, a matrix $\mathcal{N} \in \mathfrak{R}^{n \times n}$, scalars $\alpha > 0$, $0 < \beta_k \leq 1$, $\beta_1 = 1$, $\eta > 0$ and positive integer $k \geq 1$, $i = 1, 2, \dots, q$, assuming that $\tau_0 = 0$, $z(\tau_0) = z_0$, such that the following conditions are satisfied:

- i. $\mathcal{P} A_i + A_i^T \mathcal{P} - \alpha \mathcal{P} \leq 0$;
- ii. $\Pi_k = \begin{bmatrix} (I + Q_k)^T \mathcal{P} (I + Q_k) - \beta_k \mathcal{P} (I + Q_k)^T \mathcal{P} - \mathcal{N}^T \mathcal{T} & \\ \mathcal{P} (I + Q_k) - \mathcal{T} \mathcal{N} & \mathcal{P} - 2\mathcal{T} \end{bmatrix} \leq 0$;
- iii. $\begin{bmatrix} \mathcal{P} & [Q_{k(i)} - \mathcal{N}_{(i)}]^T \\ [Q_{k(i)} - \mathcal{N}_{(i)}] & u_{0(i)}^2 \end{bmatrix} \geq 0$;
- iv. $\beta_{2k-2} e^{\alpha(\tau_{2k-1} - \tau_{2k-2})} \leq 1$, $\beta_{2k-2} \beta_{2k-1} e^{\alpha(\tau_{2k} - \tau_{2k-2})} \leq 1$, $\beta_{2k-1} \beta_{2k} e^{\alpha(\tau_{2k} - \tau_{2k-2})} \leq 1$;

then the high-dimensional ellipsoid set $\varepsilon(\mathcal{P}, \eta) = \{z \in \mathfrak{R}^n : z^T \mathcal{P} z \leq \eta\}$ is a RAS of the origin of T-S fuzzy systems with saturated impulses (6), where $\eta = \min [e^{-\alpha\tau_1}, e^{-\alpha\tau_2 - \ln \beta_1}]$.

Remark 2. As far as we know, there are few works considering fuzzy impulsive systems with saturation, and Theorem 1 is also the first time using the step-function method to investigate saturated systems, which leads to related stability criterion. It should be noted that Theorem 1 can address some more challenging cases, such as the situation when the impulsive gain matrix is varying, and the saturated impulsive systems with Zeno behavior, which can not be tackled by existing methods^{37,38,39}.

4 | STABILITY ANALYSIS OF HYBRID CONTROL SYSTEMS WITH ACTUATOR SATURATION AND TIME DELAY IN BOTH IMPULSIVE CONTROLLER AND STATE FEEDBACK CONTROLLER

In recent years, hybrid control systems have gained massive interest because of their ability to control and regulate complex behaviors in engineering and natural systems. However, designing a reliable hybrid control system with actuator saturation and time delays is a challenging task, as these factors can significantly affect the stability of the system, and few studies have been conducted.

In this section, we focus on the stability analysis of hybrid control systems where both impulsive controller and state feedback controller are employed, and they face actuator saturation and time delay respectively. Utilizing novel methods (step function-based approach) for the analysis of such systems, we present stability results that reduce the conservatism of the stability conditions. Several applications and simulation experiments will be provided to show the efficacy of the proposed methods in the next section.

Similar to condition iii in Theorem 1, we present the following assumption in order to apply the generalized local sector condition to the two types of saturation in system (8).

Assumption 1. Given matrices $\mathcal{D}, \mathcal{Q}_k \in \mathfrak{R}^{n \times n}$, column vector $u_{01}, u_{02} \in \mathfrak{R}^n$, when there are matrices $\mathcal{N}_1, \mathcal{N}_2 \in \mathfrak{R}^{n \times n}$ and a positive definite matrix $\mathcal{P} \in \mathfrak{R}^{n \times n}$, such that

$$\begin{bmatrix} \mathcal{P} & [\mathcal{D}_{(\varpi)} - \mathcal{N}_{1(\varpi)}]^T \\ [\mathcal{D}_{(\varpi)} - \mathcal{N}_{1(\varpi)}] & u_{01(\varpi)}^2 \end{bmatrix} \geq 0, \varpi = 1, 2, \dots, n \quad (46)$$

And for some $i, j \in \{\varpi \in \mathbb{Z}^+ : 1 \leq \varpi \leq n\}$, $k \in \mathbb{Z}^+$, if it holds that $\Omega_k^{(1)}(i, j) \leq \Omega_k^{(2)}(i, j)$, then

$$\begin{bmatrix} \mathcal{P} & [\mathcal{Q}_{k(i)} - \mathcal{N}_{2(i)}]^T \\ [\mathcal{Q}_{k(i)} - \mathcal{N}_{2(i)}] & u_{02(i)}^2 \end{bmatrix} \geq 0 \quad (47)$$

where,

$$\Omega_k^{(1)}(i, j) = \frac{1}{u_{01(i)}} \left| (\mathcal{D}_{(ij)} - \mathcal{N}_{1(ij)}) \cdot z_{(j)}(\tau_k) \right| \quad (48)$$

$$\Omega_k^{(2)}(i, j) = \frac{1}{u_{02(i)}} \left| (\mathcal{Q}_{k(ij)} - \mathcal{N}_{2(ij)}) \cdot z_{(j)}(\tau_k) \right| \quad (49)$$

Note that the stability Lemma 2 proposed in the previous section can not cope with the impulsive systems with time delays. Therefore, a generalized m -span step function is then constructed as follows:

$$\mathcal{W}(t) = \begin{cases} \sup_{t \in [\tau_0 - \kappa, \tau_m]} \mathcal{V}(z(t)), & t \in [\tau_0 - \kappa, \tau_0] \\ \sup_{t \in [\tau_0, \tau_m]} \mathcal{V}(z(t)), & t \in (\tau_0, \tau_m] \\ \mathcal{V}(z(\tau_{mk})), & t \in (\tau_{mk}, \tau_{m(k+1)}] \\ 0, & t > \tau_\infty \end{cases} \quad (50)$$

Based on which, a stability lemma ensues according to the idea of step-function method:

Lemma 4. Let $z(t)$ be a solution of (8) with $z_0 \in \Omega \subseteq \mathfrak{R}^n$, $\mathcal{V} : \mathfrak{R}^n \rightarrow \mathfrak{R}^+$ be a positive definite scalar function. If there exists an integer $m \geq 1$, such that the step function defined by (50) satisfies

- i. $\mathcal{W}(t_0 - \kappa) \leq \vartheta(\mathcal{V}(z_0))$, where $\vartheta \in K$;
- ii. $\mathcal{W}(t)$ decreases with t and fulfills $\mathcal{W}(t) \rightarrow 0, t \rightarrow \tau_\infty$;
- iii. $\mathcal{W}(t) \geq \mathcal{V}(z(t))$ on each impulsive interval $t \in (\tau_{m(k-1)}, \tau_{mk}]$;

then the origin of (8) is locally uniformly attractively stable for any $z_0 \in \Omega$.

Proof. As $\mathcal{V}(z)$ is a positive definite function with respect to z , then there exist functions $\chi_1, \chi_2 \in K_\infty$ fulfill that

$$\chi_1(\|z\|) \leq \mathcal{V}(z) \leq \chi_2(\|z\|) \quad (51)$$

Apparently,

$$\mathcal{W}(\tau_0 - \kappa) = \mathcal{W}(\tau_0) \leq \vartheta(\mathcal{V}(z_0)) \leq \vartheta(\chi_2(\|z_0\|)) \quad (52)$$

Then, we can construct the following comparison function

$$F(\mathcal{W}(\tau_0 - \kappa), t - \tau_0) = \begin{cases} \mathcal{W}(\tau_0 - \kappa), & \tau_0 - \varepsilon \leq t \leq \tau_0 \\ \frac{\mathcal{W}(\tau_0) - \mathcal{W}(\tau_0 - \kappa)}{\tau_0 - \tau_0 + \kappa} (t - \tau_0 + \kappa) + \mathcal{W}(\tau_0), & \tau_0 \leq t < \tau_m \\ \frac{\mathcal{W}(\tau_{mk}) - \mathcal{W}(\tau_{m(k-1)})}{\tau_{mk} - \tau_{m(k-1)}} (t - \tau_{m(k-1)}) + \mathcal{W}(\tau_{m(k-1)}), & \tau_{m(k-1)} \leq t \leq \tau_{mk} \\ 0, & t \geq \tau_\infty \end{cases} \quad (53)$$

Suppose that

$$\tilde{F}(\mathcal{W}(\tau_0 - \kappa), t) = F(\mathcal{W}(\tau_0 - \kappa), t) + e^{-t + \tau_0 - \kappa} \mathcal{W}(\tau_0 - \kappa) \quad (54)$$

Apparently, $\tilde{F}(\mathcal{W}(\tau_0 - \kappa), t) \in KL$. Then, for $t \geq \tau_0 - \kappa$, it follows that

$$\chi_1(\|z\|) \leq \mathcal{V}(z(t)) \leq \mathcal{W}(t) \leq \tilde{F}(\mathcal{W}(\tau_0 - \kappa), t) \quad (55)$$

which leads to

$$\|z(t)\| \leq \chi_1^{-1}(\tilde{F}(\vartheta(\chi_2(\|z_0\|)), t)) \in KL \quad (56)$$

The proof of Lemma 4 is then finished. \square

Thus, we can now present the stability theorem of the hybrid control system suffering both saturation and time delay in both controllers, following the m -span step-function approach with the help of sector conditions.

Theorem 2. Given column vectors $u_{01}, u_{02} \in \mathfrak{R}^n$ and a positive integer $m \geq 1$ and a constant $\kappa > 0$, if there exist a positive definite symmetric matrix $\mathcal{P} \in \mathfrak{R}^{n \times n}$, positive definite diagonal matrices $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{R}^{n \times n}$, matrices $\mathcal{N}_1, \mathcal{N}_2 \in \mathfrak{R}^{n \times n}$, scalars $g > 0, \alpha > 0, 0 < \beta_k \leq 1, \beta_1 = 1$, and positive integers $k \geq 1, i = 1, 2, \dots, q$, assuming that $\tau_0 = 0, z(\tau_0) = z_0$, suppose that the Assumption 1 is fulfilled, such that the following conditions are fulfilled:

- i. $\mathfrak{K} = \begin{bmatrix} A_i^T \mathcal{P} + \mathcal{P} A_i - \alpha \mathcal{P} & -\mathcal{P} K_i & -\mathcal{P} \\ -K_i^T \mathcal{P} & 0 & 0 \\ -\mathcal{P} & -\mathcal{N}_1 \mathcal{T}_1 & -\mathcal{T}_1 \end{bmatrix} \leq 0$;
- ii. $\Pi_k = \begin{bmatrix} \mathcal{P} - \beta_k \mathcal{P} & \mathcal{P} Q_k & \mathcal{P} \\ Q_k^T \mathcal{P} & Q_k^T \mathcal{P} Q_k & Q_k^T \mathcal{P} \\ \mathcal{P} & \mathcal{P} Q_k - \mathcal{T}_2 \mathcal{N}_2 & \mathcal{P} - \mathcal{T}_2 \end{bmatrix} \leq 0$;
- iii. $\theta^T(t - \kappa) \mathcal{P} \theta(t - \kappa) \leq g \cdot z^T(t) \mathcal{P} z(t), t \in [\tau_0, \tau_0 + \kappa], 0 < g \leq \rho_1^{-1}$;
- iv. $\max_{i=m(k-1)+1, \dots, mk} \left\{ \left(\prod_{j=m(k-1)}^{i-1} \beta_j \right) e^{\alpha(\tau_i - \tau_{m(k-1)})} \right\} \leq \rho_k < 1$;

$$\text{v. } \left(\prod_{i=m(k-2)}^{m(k-1)-1} \beta_i \right) e^{\alpha(\tau_{m(k-1)} - \tau_{m(k-2)})} \leq \varsigma < 1, k \geq 2;$$

then the high-dimensional ellipsoid set $\varepsilon(\mathcal{P}, 1) = \{z \in \mathfrak{R}^n : z^T \mathcal{P} z \leq 1\}$ is a RAS of the origin of nonlinear fuzzy hybrid control systems with saturation and time delay (8).

Proof. Consider the quadratic Lyapunov function $\mathcal{V}(z(t)) = z^T(t) \mathcal{P} z(t)$, $\forall t \geq \tau_0 - \kappa$, and we let $\mathcal{V}_0 = \sup_{s \in [t_0 - \kappa, t_0]} \mathcal{V}(z(s))$. The deductive process can be categorized into three steps as follows:

- To deduce the two fundamental conditions $D^+ \mathcal{V}(z(t)) \leq \alpha \mathcal{V}$ and $\mathcal{V}(z(\tau_k^+)) \leq \beta_k \mathcal{V}(z(\tau_k))$ in the subsequent part, utilizing sector inequalities;
- The stability criterion of the hybrid control system (8) will be established within the framework of step-function approach;
- To confirm the requirement for the generalized local sector condition.

Firstly, when $t \in (\tau_{k-1}, \tau_k]$,

$$\begin{aligned} D^+ \mathcal{V}(z(t)) &= \dot{z}^T(t) \mathcal{P} z(t) + z^T(t) \mathcal{P} \dot{z}(t) \\ &= \sum_{i=1}^q d_i(e(t)) z^T(t) [A_i^T \mathcal{P} + \mathcal{P} A_i] z(t) - \sum_{i=1}^q d_i(e(t)) z^T(t - J_1(t)) K_i^T \mathcal{P} z(t) \\ &\quad - \sum_{i=1}^q d_i(e(t)) z^T(t) \mathcal{P} K_i z(t - J_1(t)) - \varphi_1^T \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \mathcal{P} z(t) \\ &\quad - z^T(t) \mathcal{P} \varphi_1 \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \end{aligned} \quad (57)$$

We assume that $z(t - J_1(t)) \in \varepsilon(\mathcal{P}, 1) \subset \mathcal{S}(|D - \mathcal{N}_1|, u_{01})$ holds for all $t \geq \tau_0$, according to the generalized local sector conditions, we have

$$\varphi_1^T \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \mathcal{T}_1 \left[\varphi_1 \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] + \mathcal{N}_1 z(t - J_1(t)) \right] \leq 0 \quad (58)$$

Therefore, we obtain

$$\begin{aligned} D^+ \mathcal{V}(z(t)) &= \sum_{i=1}^q d_i(e(t)) \{ z^T(t) [A_i^T \mathcal{P} + \mathcal{P} A_i - \alpha \mathcal{P}] z(t) - z^T(t - J_1(t)) K_i^T \mathcal{P} z(t) - z^T(t) \mathcal{P} K_i z(t - J_1(t)) \\ &\quad - \varphi_1^T \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \mathcal{P} z(t) - z^T(t) \mathcal{P} \varphi_1 \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \\ &\quad - \varphi_1^T \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \mathcal{T}_1 \mathcal{N}_1 z(t - J_1(t)) \\ &\quad - \varphi_1^T \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \mathcal{T}_1 \varphi_1 \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] + \alpha z^T(t) \mathcal{P} z(t) \} \end{aligned} \quad (59)$$

It follows that,

$$\begin{aligned} D^+ \mathcal{V}(z(t)) &\leq \sum_{i=1}^q d_i(e(t)) \left[z^T(t) z^T(t - J_1(t)) \varphi_1^T \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \right] \cdot \mathfrak{N} \\ &\quad \left[\begin{array}{c} z(t) \\ z(t - J_1(t)) \\ \varphi_1 \left[\sum_{i=1}^q d_i(e(t)) K_i z(t - J_1(t)) \right] \end{array} \right] + \alpha z^T(t) \mathcal{P} z(t) \\ &\leq \alpha \mathcal{V}(z(t)) \end{aligned} \quad (60)$$

Then, as for the impulsive instants, similar to the demonstration of Theorem 1, we can obtain

$$\begin{aligned}\mathcal{V}(z(\tau_k^+)) &= z^T(\tau_k^+) \mathcal{P} z(\tau_k^+) \\ &= [z^T(\tau_k) + z^T(\tau_k - J_2(\tau_k)) \mathcal{Q}_k^T + \varphi_2^T [\mathcal{Q}_k z(\tau_k - J_2(\tau_k))]] \mathcal{P} \\ &\quad [z(\tau_k) + \mathcal{Q}_k z(\tau_k - J_2(\tau_k)) + \varphi_2 [\mathcal{Q}_k z(\tau_k - J_2(\tau_k))]]\end{aligned}\quad (61)$$

In the case of $z(\tau_k - J_2(\tau_k)) \in \varepsilon(\mathcal{P}, 1) \subset S(|\mathcal{Q}_k - \mathcal{N}_2|, u_{02})$, it yields

$$\begin{aligned}\mathcal{V}(z(\tau_k^+)) &\leq [z^T(\tau_k) \quad z^T(\tau_k - J_2(\tau_k)) \quad \varphi_2^T [\mathcal{Q}_k z(\tau_k - J_2(\tau_k))]] \Pi_k \begin{bmatrix} z(\tau_k) \\ z(\tau_k - J_2(\tau_k)) \\ \varphi_2 [\mathcal{Q}_k z(\tau_k - J_2(\tau_k))] \end{bmatrix} \\ &\quad + \beta_k z^T(\tau_k) \mathcal{P} z(\tau_k) \\ &\leq \beta_k \mathcal{V}(z(\tau_k))\end{aligned}\quad (62)$$

Combining (60) and (62), we can find that $\mathcal{V}(z(t)) \leq e^{\alpha(t-\tau_{k-1})} \mathcal{V}(z(\tau_{k-1}^+))$ for any $t \in (\tau_{k-1}, \tau_k]$. Hence, we can derive

$$\mathcal{V}(z(t)) \leq \left(\prod_{i=1}^{k-1} \beta_i \right) e^{\alpha(t-\tau_0)} \mathcal{V}(z_0) \leq \left(\prod_{i=1}^{k-1} \beta_i \right) e^{\alpha t} \mathcal{V}_0 \quad (63)$$

Now, consider the Step b, which aims to verify the stability of the hybrid control system. We are going to check the three conditions required by Lemma 4 under the assumptions made in Theorem 2. Consider the first step of the step function, that is to say, when $t \in (\tau_0, \tau_m)$, we have

$$\begin{aligned}\mathcal{W}(t) &= \sup_{t \in [\tau_0, \tau_m]} \mathcal{V}(z(t)) \\ &\leq \max_{i=1,2,\dots,m} \left\{ \sup_{t \in [\tau_{i-1}, \tau_i]} \mathcal{V}(z(t)) \right\} \\ &\leq \max_{i=1,2,\dots,m} \left\{ \left[\left(\prod_{j=0}^{i-1} \beta_j \right) e^{\alpha(\tau_i-\tau_0)} \right] \cdot \mathcal{V}(z_0) \right\} \\ &= \rho_1 \mathcal{V}(z_0)\end{aligned}\quad (64)$$

As ρ_1 is a constant, the first condition in Lemma 4 is fulfilled. Then, as for the $(k-1)$ th step, when $t \in (\tau_{m(k-1)}, \tau_{mk}]$, $k \geq 1$,

$$\begin{aligned}\mathcal{V}(z(t)) &\leq \max_{i=m(k-1)+1,\dots,mk} \left\{ \sup_{t \in [\tau_{i-1}, \tau_i]} \mathcal{V}(z(t)) \right\} \\ &\leq \max_{i=m(k-1)+1,\dots,mk} \left\{ \left[\left(\prod_{j=m(k-1)}^{i-1} \beta_j \right) e^{\alpha(\tau_i-\tau_{m(k-1)})} \right] \cdot \mathcal{V}(z(\tau_{m(k-1)})) \right\} \\ &\leq \rho_k \mathcal{V}(z(\tau_{m(k-1)})) \\ &\leq \mathcal{W}(t)\end{aligned}\quad (65)$$

which indicates that condition iii is satisfied. Furthermore, when $t \in (\tau_{m(k-1)}, \tau_{mk}]$, $k \geq 1$,

$$\begin{aligned}\mathcal{W}(\tau_{mk}) - \mathcal{W}(\tau_{m(k-1)}) &= \mathcal{V}(z(\tau_{m(k-1)})) - \mathcal{V}(z(\tau_{m(k-2)})) \\ &\leq - \left[1 - \beta_{m(k-1)-1} \cdots \beta_{m(k-2)} e^{\alpha(\tau_{m(k-1)} - \tau_{m(k-2)})} \right] \mathcal{V}(z(\tau_{m(k-2)})) \\ &\leq - (1 - \varsigma) \mathcal{V}(z(\tau_{m(k-2)})) \\ &= - (1 - \varsigma) \mathcal{W}(\tau_{m(k-1)}) \\ &< 0\end{aligned}\quad (66)$$

Because the inequity (66) holds for every $k \geq 1$, we can write

$$\mathcal{W}(t) = \mathcal{W}(\tau_{mk}) \leq \varsigma \mathcal{W}(\tau_{m(k-1)}) \leq \varsigma^{k-1} \mathcal{W}(\tau_m) \leq \varsigma^{k-1} \rho_1 \mathcal{V}(z_0) \quad (67)$$

Therefore,

$$0 \leq \lim_{t \rightarrow \tau_\infty} \mathcal{W}(t) \leq \lim_{k \rightarrow \infty} [\varsigma^{k-1} \rho_1 \mathcal{V}(z_0)] = 0 \quad (68)$$

According to the Squeeze Theorem, we finally arrive at

$$\lim_{t \rightarrow \tau_\infty} \mathcal{W}(t) = 0 \quad (69)$$

Now all the conditions in Theorem 2 is confirmed. Next, however, we need to verify the follow relations:

$$\begin{cases} z(t - J_1(t)) \in \varepsilon(\mathcal{P}, 1) \subset S(|\mathcal{D} - \mathcal{N}_1|, u_{01}), \forall t \geq \tau_0 \\ z(\tau_k - J_2(\tau_k)) \in \varepsilon(\mathcal{P}, 1) \subset S(|\mathcal{Q}_k - \mathcal{N}_2|, u_{02}), \forall k \geq 1 \end{cases} \quad (70)$$

We shall first demonstrate that $z(t) \in \varepsilon(\mathcal{P}, 1)$, $\forall t \geq \tau_0 - \kappa$. When $t \in (\tau_{k-1}, \tau_k]$, and $t - J_1(t) \in (\tau_{k-1}, \tau_k]$, we suppose that

$$\tau_0 < \dots < \tau_{(a-1)m} < \tau_{k-1} < t - J_1(t) \leq t \leq \tau_k < \tau_{am} < \dots < \tau_\infty \quad (71)$$

then we get

$$\begin{aligned} \mathcal{V}(z(t - J_1(t))) &\leq \left(\prod_{i=(a-1)m}^{k-1} \beta_i \right) e^{\alpha(t - J_1(t) - \tau_{(a-1)m})} \mathcal{V}(z(\tau_{(a-1)m})) \\ &\leq \left(\prod_{j=1}^{a-1} \rho_j \right) \cdot \left(\prod_{i=(a-1)m}^{k-1} \beta_i \right) e^{\alpha(t - J_1(t) - \tau_{(a-1)m})} \mathcal{V}(z_0) \end{aligned} \quad (72)$$

As $\alpha > 0$, it yields that

$$\mathcal{V}(z(t)) \leq \left(\prod_{j=1}^{a-1} \rho_j \right) \cdot \left(\prod_{i=(a-1)m}^{k-1} \beta_i \right) e^{\alpha(\tau_k - \tau_{(a-1)m})} \mathcal{V}(z_0) \leq \left(\prod_{j=1}^a \rho_j \right) \mathcal{V}(z_0) \quad (73)$$

Because $\mathcal{V}(z(t_0)) \leq 1$,

$$\mathcal{V}(z(t)) \leq \left(\prod_{j=1}^a \rho_j \right) \leq 1 \quad (74)$$

That is to say, $z(t - J_1(t)) \in \varepsilon(\mathcal{P}, 1)$. When $t \in (\tau_{k-1}, \tau_k]$, $t - J_1(t) \in (\tau_{i-1}, \tau_i]$, $1 \leq i < k$. We suppose that

$$\tau_0 < \dots < \tau_{(b-1)m} < \tau_{i-1} < t - J_1(t) \leq \tau_i < \tau_{bm} < \dots < \tau_\infty \quad (75)$$

One can find that

$$\mathcal{V}(z(t - J_1(t))) \leq \left(\prod_{j=1}^{b-1} \rho_j \right) \cdot \left(\prod_{i=(b-1)m}^{k-1} \beta_i \right) e^{\alpha(t - J_1(t) - \tau_{(b-1)m})} \mathcal{V}(z_0) \leq \left(\prod_{j=1}^b \rho_j \right) \mathcal{V}(z_0) \leq 1 \quad (76)$$

Moreover, when $t \in (\tau_{k-1}, \tau_k]$, $t - J_1(t) \in (\tau_0 - \kappa, \tau_0]$,

$$\mathcal{V}(z(t - J_1(t))) = \theta^T(t - J_1(t)) \mathcal{P} \theta(t - J_1(t)) \leq g \cdot z^T(t^*) \mathcal{P} z(t^*) \leq g \mathcal{V}(z(t^*)) \quad (77)$$

where $t^* = t - J_1(t) + \kappa \geq \tau_0$. Hence, we obtain

$$\mathcal{V}(z(t - J_1(t))) \leq g \mathcal{V}(z(t^*)) \leq g \left(\prod_{i=1}^n \rho_i \right), n \geq 1 \quad (78)$$

Since $g \leq \rho_1^{-1}$, we can get

$$\mathcal{V}(z(t - J_1(t))) \leq 1 \quad (79)$$

Combing (74), (76) as well as (79), it yields that

$$z(t - J_1(t)) \in \varepsilon(\mathcal{P}, 1), \forall t \geq \tau_0 - \kappa \quad (80)$$

Moreover, note that the scope of $t - J_1(t)$ covers the entire region of $t - J_2(\tau_k)$ can possibly cover, except $[\tau_0 - \kappa, \tau_0]$. As a consequence, we only need to consider the case when $t - J_2(\tau_k) < \tau_0$. Specifically, we derive

$$\mathcal{V}(z(t - J_2(\tau_k))) = \theta^T(t - J_2(\tau_k)) \mathcal{P} \theta(t - J_2(\tau_k)) \leq g \mathcal{V}(z(\bar{t})) \quad (81)$$

where, $\hat{t} = t - J_2(\tau_k) + \kappa \geq \tau_0$. With the third condition in Theorem 2, it follows that

$$\mathcal{V}(z(t - J_2(t_k))) \leq g\mathcal{V}(z(\hat{t})) \leq 1 \quad (82)$$

Combing (80) and (82), we therefore arrive at

$$z(t) \in \varepsilon(\mathcal{P}, 1), \forall t \geq \tau_0 - \kappa \quad (83)$$

Finally, in the Step c, we need to prove

$$\varepsilon(\mathcal{P}, 1) \subset S(|D - \mathcal{N}_1|, u_{01}) \quad (84)$$

$$\varepsilon(\mathcal{P}, 1) \subset S(|Q_k - \mathcal{N}_2|, u_{02}) \quad (85)$$

which is actually very similar to the proof of Theorem 1 considering Assumption 1. By applying Schur Complement Theorem, from (46) it can be shown that

$$\mathcal{P} - \frac{1}{u_{01(i)}^2} [D_{(i)} - \mathcal{N}_{1(i)}]^T [D_{(i)} - \mathcal{N}_{1(i)}] \geq 0 \quad (86)$$

Substituting $z(t) \in \varepsilon(\mathcal{P}, 1)$, into (86), we find that

$$|(D_{(i)} - \mathcal{N}_{1(i)}) z(t)| \leq u_{01(i)} \quad (87)$$

which means that $\varepsilon(\mathcal{P}, 1) \subset S(|D - \mathcal{N}_1|, u_{01})$. When it fulfills that $\Omega_k^{(1)}(i, j) \leq \Omega_k^{(2)}(i, j)$, which indicates that the set $S(|D - \mathcal{N}_1|, u_{01})$ is not fully included in the set $S(|Q_k - \mathcal{N}_2|, u_{02})$. In this case, as a consequence of (47), we derive

$$\mathcal{P} - \frac{1}{u_{02(i)}^2} [Q_{k(i)} - \mathcal{N}_{2(i)}]^T [Q_{k(i)} - \mathcal{N}_{2(i)}] \geq 0 \quad (88)$$

which follows by

$$\varepsilon(\mathcal{P}, 1) \subset S(|Q_k - \mathcal{N}_2|, u_{02}) \quad (89)$$

Now, we have verified all the assumptions made in the process of proof of stability. Eventually, the proof of Theorem 2 is finished, and we can claim that the $\varepsilon(\mathcal{P}, 1)$ is a ellipsoid RAS of the delayed saturated hybrid control system (8). \square

Remark 3. The condition iv and v derived under the framework of step-function method, gain an advantage over most current studies in reducing the conservatism of the sufficient conditions. Moreover, Theorem 2 in this paper takes some cases into account which have attracted little attention in the previous research. For example, the impulsive gain matrix can be varying with the increase of the number of impulses, and the hybrid control system with impulsive controller exhibiting Zeno behavior, which is further demonstrated using the numerical examples in the next section. The theorem also eliminates the restriction that the delays have to be smaller than the impulsive internal.

Overall, this section contributes to the development of new methodologies for designing hybrid control systems that can effectively cope with the challenges posed by actuator saturation and time delay, while ensuring local stability.

5 | NUMERICAL SIMULATION

Two numerical examples are presented in this section based on the theoretical results. We first consider a two-dimensional fuzzy hybrid control system. The stabilization of the system is analyzed and the RAS is estimated applying Theorem 2. In the second example, the stability of Chua's circuit system is investigated under saturated delayed hybrid control, which further verifies the validity of the proposed theorem.

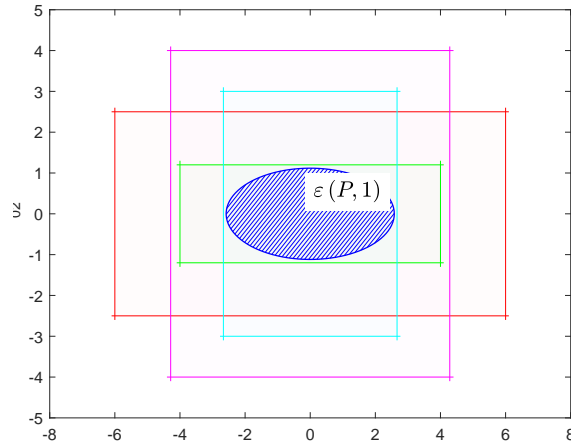


Figure 1 The RAS of delayed saturated hybrid control system (21).

5.1 | Hybrid control for a class of nonlinear system

Consider a fuzzy state feedback system

$$\dot{z}(t) = \sum_{i=1}^q d_i(e(t)) (A_i - K_i) z(t) \quad (90)$$

where, $A_i = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -0.3 \end{bmatrix}$, $K_i = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.2 \end{bmatrix}$, $i = 1, 2, \dots, q$. Note that the system (90) is asymptotic stable itself. However, when considering the time delay and actuator saturation which are generally inevitable in state feedback controller, the system may be not stable. To stabilize the system, an impulsive controller is then designed, which is also assumed to be subject to saturation and delay. The overall delayed saturated hybrid control system is presented as follows:

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^q d_i(e(t)) A_i z(t) - Sat_1 \left\{ \sum_{i=1}^q d_i(e(t)) K_i z(t - J_1) \right\}, t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k) = Sat_2 \left\{ \sum_{i=1}^q d_i(e(\tau_k)) G_{ik} z(\tau_k - J_2) \right\}, t = \tau_k \\ z(\tau_0 + \xi) = \theta(\tau_0 + \xi), \xi \in [-\kappa, 0] \end{cases} \quad (91)$$

We let the thresholds of saturation $u_{01} = [0.03 \ 0.02]^T$, $u_{02} = [0.08 \ 0.06]^T$, fixed time delays $J_1 = 0.6$, $J_2 = 0.5$. Suppose that function $\theta_{(j)}(t) = \frac{z_j(\tau_0)}{\kappa} (t - \tau_0 + \kappa)$, $j = 1, 2$, $t \in [\tau_0 - \kappa, \tau_0]$, and the impulsive interval is fixed, satisfying $\tau_k - \tau_{k-1} \equiv \delta = 0.2$, $k = 1, 2, \dots$

Let positive definite matrix $P = diag[0.15, 0.8]$, We can further compute that the impulsive gain matrix $G_{ik} = diag[-0.5, -0.5]$ satisfying the conditions given in Theorem 2, which also exhibits a stabilizing effect on the system. Therefore, we derive the estimation of region of attraction $\varepsilon(P, 1)$ in Figure 1, where the rectangles denotes the constraints presented by the conditions in Theorem 2.

According to Theorem 2, system (91) is asymptotic stable when the trajectory starts from any point within ellipsoid $\varepsilon(P, 1)$. Let the initial states fulfil $z_0^{(1)} = [0.5 \ 0.2]^T \in \varepsilon(P, 1)$ and $z_0^{(2)} = [5 \ 0.2]^T \notin \varepsilon(P, 1)$, then the time response curves, with or without impulsive controller, are shown in Figure 2 and Figure 3, respectively. Apparently, the impulses play an indispensable role in stabilizing the system (91). Take the trajectory starts from $z_0^{(1)}$ as example, the simulations show that the number of times that saturation of impulsive controller occurs is 10. We further suppose that the impulsive intervals varies with the number of impulses and satisfies that $\tau_k = 4.6 - \frac{3.8}{k}$, $k \geq 1$. It is obvious that the system will exhibit Zeno behavior, as shown in Figure 4.

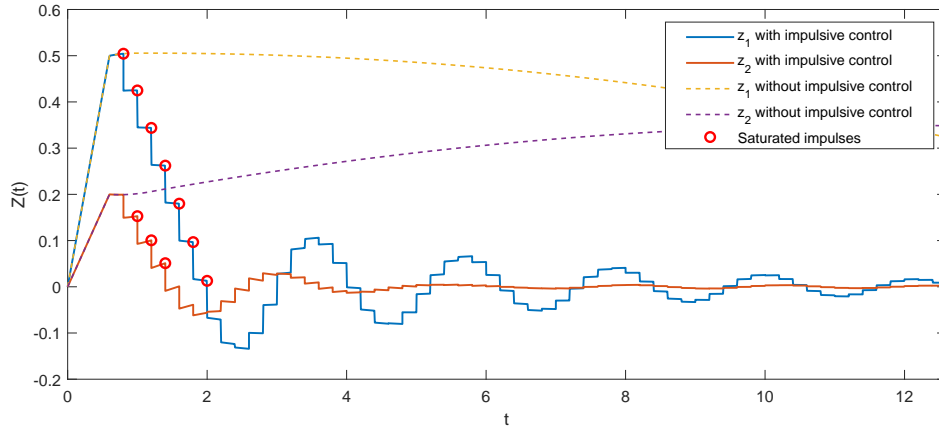


Figure 2 The time response curve of system (91) starts from $z_0^{(1)}$.

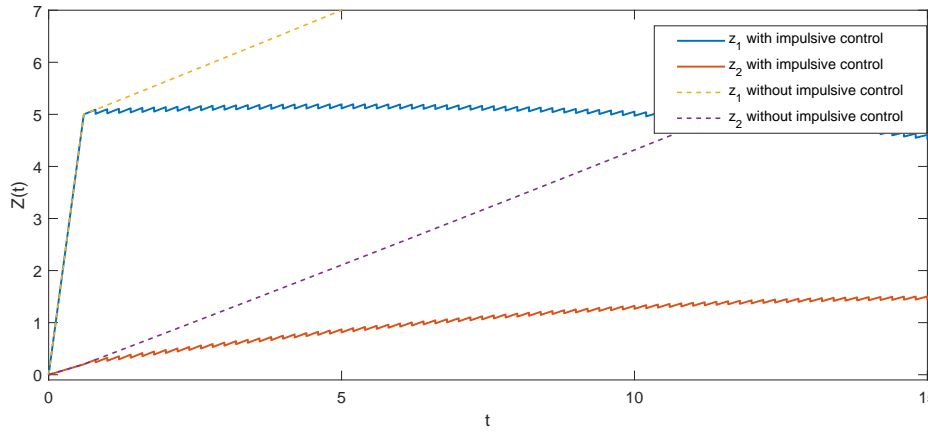


Figure 3 The time response curve of system (91) starts from $z_0^{(2)}$.

5.2 | Stabilization of nonlinear Chua's circuit system

Consider a typical nonlinear circuit system, Chua's circuit, which can exhibit chaotic behavior with appropriate conditions. The Chua's oscillator is expected to be described as¹⁷

$$\begin{cases} \dot{z}_1(t) = -\tilde{\alpha} [z_1(t) - z_2(t) + g(z_1(t))] \\ \dot{z}_2(t) = z_1(t) - z_2(t) + z_3(t) \\ \dot{z}_3(t) = -\tilde{\beta} z_2(t) \end{cases} \quad (92)$$

where, $\tilde{\alpha} = 10$, $\tilde{\beta} = 14.86$, and the piece-wise function $g(\cdot)$ is assumed as

$$g(z_1) = bz_1 + \frac{1}{2}(a-b)(|z_1+1| - |z_1-1|), a = -1.27, b = -0.68 \quad (93)$$

For simplifying the results, we construct the T-S fuzzy model for Chua's oscillator. Suppose that $z = [z_1, z_2, z_3]^T$ represents the state vector, the fuzzy model rules considering hybrid control combining a continuous state feedback controller and an impulsive controller are presented as below:

Module Rule 1: IF $z_1(t)$ is M_1 , THEN

$$\begin{cases} \dot{z}(t) = A_1 z(t) - K_1 z(t), t \in (\tau_{k-1}, \tau_k] \\ z(\tau_k^+) = [1 + G_{1k}] z(\tau_k) \end{cases}, k = 1, 2, \dots \quad (94)$$

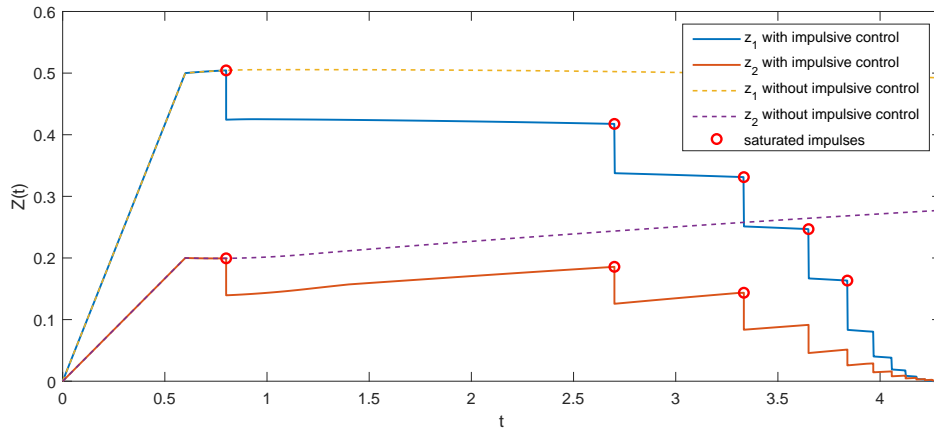


Figure 4 The time response curve of system (91) starts from $z_0^{(1)}$ exhibiting Zeno behavior.

Module Rule 2: IF $z_1(t)$ is M_2 , THEN

$$\begin{cases} \dot{z}(t) = A_2 z(t) - K_2 z(t), t \in (\tau_{k-1}, \tau_k], k = 1, 2, \dots \\ z(\tau_k^+) = [1 + G_{2k}] z(\tau_k) \end{cases} \quad (95)$$

where, M_i denotes the fuzzy sets. The matrices in fuzzy linear subsystems are given as follows:

$$A_1 = \begin{bmatrix} (\omega - 1)\tilde{\alpha} & \tilde{\alpha} & 0 \\ 1 & -1 & 1 \\ 0 & -\tilde{\beta} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -(\omega + 1)\tilde{\alpha} & \tilde{\alpha} & 0 \\ 1 & -1 & 1 \\ 0 & -\tilde{\beta} & 0 \end{bmatrix}, K_1 = K_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \quad (96)$$

where, $\omega = 1.8$. The membership functions can be described as¹⁴

$$\begin{aligned} \mathcal{M}_1(z_1(t)) &= 0.5 - \frac{1}{2\omega} \psi(z_1(t)) \\ \mathcal{M}_2(z_1(t)) &= 0.5 + \frac{1}{2\omega} \psi(z_1(t)) \end{aligned} \quad (97)$$

$$\text{where } \psi(z_1(t)) = \begin{cases} g(z_1(t))/z_1(t), & z_1(t) \neq 0 \\ a, & z_1(t) = 0 \end{cases}.$$

Taking the actuator saturation and time-varying delays into consideration, the nonlinear fuzzy hybrid control system is thus derived as follows:

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^2 d_i(e(t)) A_i z(t) - Sat_1 \left\{ \sum_{i=1}^2 d_i(e(t)) K_i z(t - J_1(t)) \right\}, t \in (\tau_{k-1}, \tau_k] \\ \Delta z(\tau_k) = Sat_2 \left\{ \sum_{i=1}^2 d_i(e(\tau_k)) G_{ik} z(\tau_k - J_2(\tau_k)) \right\}, t = \tau_k \\ z(\tau_0 + \xi) = \theta(\tau_0 + \xi), \xi \in [-\kappa, 0] \end{cases} \quad (98)$$

where the saturation thresholds are set as $u_{01} = [2 \ 1.5 \ 2]^T$, $u_{02} = [0.3 \ 0.1 \ 0.2]^T$, the impulsive interval $\delta = 0.1$, function $\theta(\cdot)$ is same as it in the first example.

Now, we consider a relatively complicated case in which the impulsive gain matrix and time delays in (98) are both time-variant, which, as far as we know, has rarely been studied before. Suppose that

$$\begin{aligned} G_{ik} &= \text{diag} [-0.5 \times (k+1)^{-0.1}, -0.2 \times (k+1)^{-0.2}, -0.5 \times (k+1)^{-0.1}] \\ J_1(t) &= 0.05, J_2(t) = 0.05 \times (k+1)^{-1.1} \end{aligned} \quad (99)$$

According to the conditions given by Theorem 2, the initial state of system (98) is chosen as

$$z_0 = z(\tau_0) = [0.5 \ 0.2 \ 0.3]^T \quad (100)$$

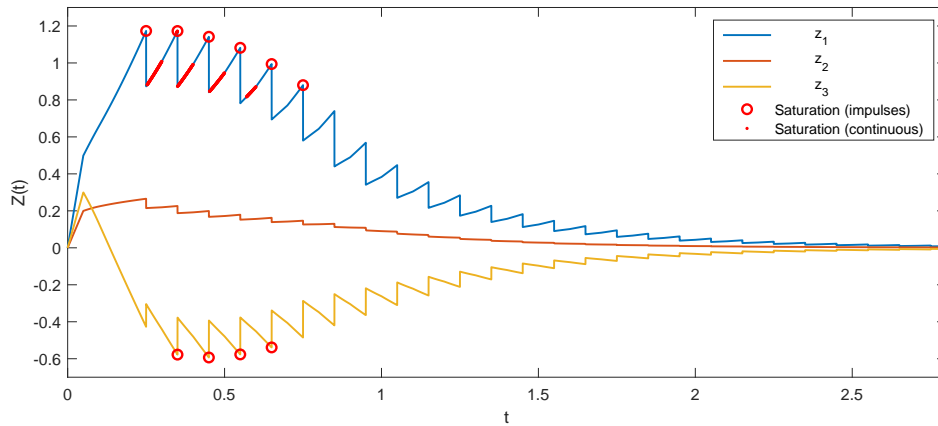


Figure 5 The time response curve of Chua's circuit system under delayed saturated hybrid control.

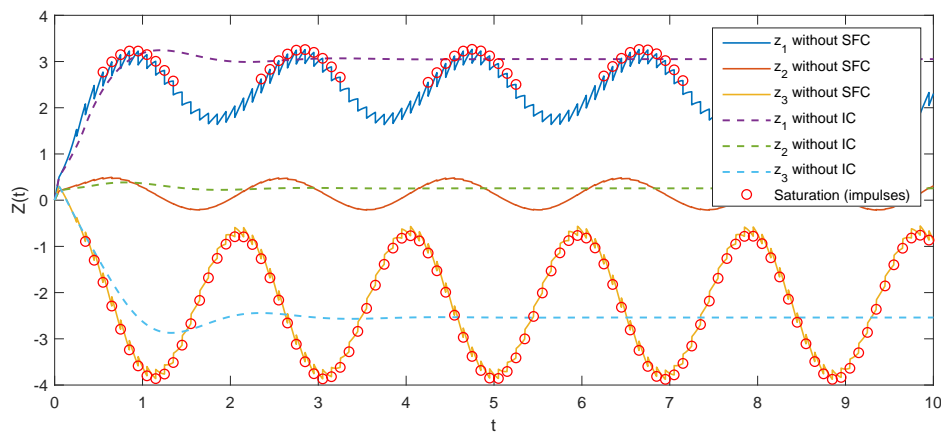


Figure 6 The time response curve of Chua's circuit system with only one controller.

In this situation, the time response curve of (98) with hybrid control is shown in Figure 5, and the curves with only one controller are given in Figure 6, where SFC represents the continuous state feedback controller, and IC denotes the impulsive controller. The numerical results show that both SFC and IC have the effect of stabilizing Chua's circuit system, and the system can not achieve asymptotic stability without any of the two controllers.

6 | CONCLUSION

In this paper, the local stability of a nonlinear fuzzy hybrid control system considering both saturation and time-varying delays in both impulsive controller and continuous controller has been studied for the first time. The stability conditions in the form of LMI has been derived using the step-function method and extended local sector condition, and improved sufficient conditions with less conservatism have been provided which guarantee the stability when the system is subject to double saturation and time delays. The stability theorem proposed in this paper relaxes some common requirements of lengths of time delays and can be applied to systems with variable impulsive gain and impulsive intervals, as well as the systems exhibiting Zeno behavior. Simulation examples have been proposed for the demonstration of theoretical findings.

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Upon reasonable request, the corresponding author can provide the data that back this study.

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