

Blowing-up solutions for the Moore-Gibson-Thompson equation with a visco-elastic memory and a non-linear external force

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Abstract

The Moore-Gibson-Thompson equation with a viscoelastic memory and a forcing is considered. The existence and uniqueness of a local solution is obtained via the Faedo-Galerkin's method. Furthermore, blowing-up solutions with or without a positive initial energy exist due to the nonlinear forcing.

Keywords: Moore-Gibson-Thompson equation; Viscoelastic memory; Faedo-Galerkin's method; Blow-up.

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1. Introduction

We present the existence of local solutions, and the occurrence of blowing-up solutions of the Moore-Gibson-Thompson (MGT in short) equation with a

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viscoelastic memory and a nonlinear forcing

$$v_{ttt} + \alpha v_{tt} - c^2 \Delta v - \beta \Delta v_t + \int_0^t \mathcal{K}(t - \sigma) \Delta v(\sigma) d\sigma = |v|^{p-2} v, \quad (1)$$

for $P := (x, t) \in \Omega_T := \Omega \times (0, T)$, with initial position and velocity

$$v(x, 0) = \varphi(x), \quad v_t(x, 0) = \omega(x), \quad v_{tt}(x, 0) = \Upsilon(x), \quad x \in \Omega, \quad (2)$$

and supplemented with the boundary condition

$$v(P) = 0, \quad P \in \Gamma \times (0, T), \quad T < \infty, \quad (3)$$

where the bounded domain $\Omega \subset \mathbb{R}^N$ is smooth and with boundary Γ ; the kernel $\mathcal{K} : [0, +\infty) \rightarrow [0, +\infty)$, $\varphi(x)$, $\omega(x)$ and $\Upsilon(x)$ are given functions; $\alpha > 0$ and $\beta > 0$ are real numbers. In acoustics, the variable v denotes a scalar acoustic velocity potential, c^2 the speed of sound, β describes damping effect associated with an acoustic environment, see Lebon and Cloot [14] for further details; the memory effects of an acoustic environment is described by $\int_0^t \mathcal{K}(t - s) \Delta v(\sigma) d\sigma$.

Before we present our results, let us dwell on some existing literature. In [17], Caixeta et al. studied global solutions of the equation

$$\tau v_{ttt} + \alpha v_{tt} + c^2 A v + \beta A v_t = f(v, v_t, v_{tt}), \quad P \in \Omega_T,$$

where A is a self-adjoint, positive operator densely defined in a Hilbert space H and f is nonlinear. In [13],

$$\tau v_{ttt} + \alpha v_{tt} + c^2 A v + \beta A v_t - \int_0^t \mathcal{K}(t - \sigma) A w(\sigma) d\sigma = 0, \quad P \in \Omega_T, \quad (4)$$

has been studied; it is shown that the memory causes energy decay whenever \mathcal{K} satisfies some conditions.

The blow-up for the MGT equation with viscoelastic memory and nonlinear forcing is considered as a new problem. The works [2]-[17], examine blow-up results of (1) – (3); No blow-up result when $\Xi(0) \geq 0$ or $\Xi(0) < 0$ ($\Xi(t)$ is the

energy of the system) is mentioned. We will then study blowing-up solutions of problem (1) – (3) when $\Xi(0) \geq 0$ and when $\Xi(0) < 0$.

The outline of the paper is as follows. In Section 2, we recall some spaces, and we define a weak solution of problem (1) – (3) and establish some useful inequalities which will be used in the sequel. Section 3 is devoted to the study of local existence and uniqueness of the weak solution of (1) – (3) via Galerkin's method. Finally, we present the main results on blowing-up solutions and estimates of the blow-up time.

2. Assumptions and main results

The norm of $L^p(\Omega)$ is denoted by $\|v\|_p := \left(\int_{\Omega} |v|^p dx \right)^{\frac{1}{p}}$.

The scalar product in $L^2(\Omega)$ is defined by

$$(v(x), u(x)) = \int_{\Omega} v(x)u(x) dx.$$

Set

$$\mathcal{V}(\Omega_T) := \left\{ v : v \in H_0^1(\Omega_T), v_t \in H^1(\Omega_T) \right\},$$

and

$$\mathcal{W}(\Omega_T) := \left\{ u : u \in V(\Omega_T), u(x, T) = 0 \right\}.$$

Definition 2.1. *A function $v \in \mathcal{V}(\Omega_T)$ is called a generalized solution of problem (1) – (3) if it satisfies, for each $u \in \mathcal{W}(\Omega_T)$,*

$$\begin{aligned} & - (v_{tt}(t), u_t(t))_{2, \Omega_T} - \alpha (v_t(t), u_t(t))_{2, \Omega_T} \\ & + c^2 (\nabla v(t), \nabla u(t))_{2, \Omega_T} + \beta (\nabla v_t(t), \nabla u(t))_{2, \Omega_T} \\ & - \left(\int_0^t \mathcal{K}(t-s) \nabla v(s) ds, \nabla u(t) \right)_{2, \Omega_T} \\ & = \alpha (\omega(x), u(0))_2 + (\Upsilon(x), u(0))_2 + (|v|^{p-2} v(t), u(t))_{2, \Omega_T}. \end{aligned} \quad (5)$$

Lemma 2.2. *[1] If $1 \leq q \leq \frac{2N}{N-2}$, $N \geq 3$, then the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is continuous.*

Lemma 2.3. [16] Let $4 < p \leq \frac{4(N-1)}{N-2}$, $N \geq 3$. Then there exists a constant $C_p = C_p(p, N, \Omega) > 0$ such that

$$\begin{aligned} \left\| |v_1|^{p-2} v_1 - |v_2|^{p-2} v_2 \right\|_2^2 &\leq C_p \left(1 + \left(\|v_1\|_{H^1(\Omega)} + \|v_2\|_{H^1(\Omega)} \right)^{\frac{1}{N}} \right. \\ &\quad \left. + \left(\|v_1\|_{H^1(\Omega)} + \|v_2\|_{H^1(\Omega)} \right)^{p-2} \right)^2 \|v_1 - v_2\|_{H^1(\Omega)}^2, \end{aligned}$$

for $v_1, v_2 \in H^1(\Omega)$.

Lemma 2.4. [15] Let $\varrho > 0$ and let $0 \leq B(t) \in C^2(0, \infty)$ be such that

$$B''(t) - 4(\varrho + 1)B'(t) + 4(\varrho + 1)B(t) \geq 0.$$

If $B'(0) > r_* B(0) + k_0$, then

$$B'(t) > k_0,$$

for $t > 0$, where k_0 is a constant, $r_* := 2(\varrho + 1) - 2\sqrt{(\varrho + 1)\varrho}$ is the smallest root of the equation

$$r^2 - 4(\varrho + 1)r + 4(\varrho + 1) = 0.$$

Lemma 2.5. [15] If $M(t)$ is a non-increasing function on $[t_*, \infty)$, $t_* \geq 0$, and satisfies

$$M'(t)^2 \geq \nu + RM(t)^{2+\frac{1}{\varrho}}, \quad \text{for } t \geq t_*, \nu > 0,$$

where R is a real number, then

$$\lim_{t \rightarrow T_{max}-} M(t) = 0,$$

for some $T_{max} < \infty$. An upper bound of T_{max} is given in the following cases:

(i) When $R < 0$ and $M(t_*) < \min \left\{ 1, \left(\frac{\nu}{-R} \right)^{1/2} \right\}$, then

$$T_{max} \leq t_* + \frac{1}{\sqrt{-R}} \ln \frac{\left(\frac{\nu}{-R} \right)^{1/2}}{\left(\frac{\nu}{-R} \right)^{1/2} - M(t_*)}.$$

(ii) When $R = 0$, then

$$T_{max} \leq t_* + \frac{M(t_*)}{\sqrt{\nu}}.$$

(iii) When $R > 0$, then

$$T_{max} \leq \frac{M(t_*)}{\sqrt{\nu}},$$

or

$$T_{max} \leq t_* + 2 \frac{3\varrho + 1}{2\varrho} \frac{\varrho\gamma}{\sqrt{\nu}} \left(1 - (1 + \gamma M(t_*))^{\frac{1}{2\varrho}} \right), \quad \gamma := \left(\frac{R}{\nu} \right)^{\varrho/(2+\varrho)}.$$

Our results need some assumptions:

(H₁) We assume that $\mathcal{K} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\mathcal{K}'(t) \leq 0, \quad \mathcal{K}''(t) \geq 0, \quad \bar{\mathcal{K}} := \int_0^\infty \mathcal{K}(\sigma) d\sigma < \gamma^2, \quad t \geq 0.$$

(H₂) The exponent p satisfies

$$4 < p, \text{ if } N = 1, 2; \quad 4 < p \leq \frac{4(N-1)}{N-2}, \text{ if } N \geq 3.$$

3. Existence and uniqueness

The existence and uniqueness of the weak solution of problem (1) – (3) is obtained using the Faedo–Galerkin method.

3.1. Existence

Theorem 3.1. *Let $\varphi(x) \in H_0^1(\Omega)$, $\omega(x) \in H^1(\Omega)$ and $\Upsilon(x) \in L^2(\Omega)$. Under (H₁) – (H₂), there is at least one generalized solution of (1) – (3) in $\mathcal{V}(\Omega_T)$.*

Proof. Let $\{\psi_l(x)\}_{l \geq 1}$ be an orthonormal basis of $H_0^1(\Omega)$. We look for an approximating solution

$$v^{(\kappa)}(x, t) = \sum_{l=1}^{l=\kappa} G_l(t) \psi_l(x), \tag{6}$$

with

$$G_l(t) = (\psi_l(x), v^{(\kappa)}(P))_2, \quad l = 1, \dots, \kappa.$$

They can be found, for $k = 1, \dots, \kappa$, via the relations

$$\begin{aligned} & \left(v_{ttt}^{(\kappa)}, \psi_k(x) \right)_2 + \alpha \left(v_{tt}^{(\kappa)}, \psi_k(x) \right)_2 + c^2 \left(\nabla v^{(\kappa)}, \nabla \psi_k(x) \right)_2 + b \left(\nabla v_t^{(\kappa)}, \nabla \psi_k(x) \right)_2 \\ & - \left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}(s) ds, \nabla \psi_k(x) \right)_2 = \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), \psi_k(x) \right)_2. \end{aligned} \quad (7)$$

System (7) is supplemented with initial conditions

$$G_l(0) = (\psi_l, \varphi(x))_2, \quad G'_l(0) = (\psi_l, \omega(x))_2, \quad G''_l(t) = (\psi_l, \Upsilon(x))_2.$$

Carathéodory theorem [2] ensures the existence of solutions $G_l(t)$, $l = 1, \dots, \kappa$, for $t \in [0, t_\kappa)$. In order to prolong the solution on $[0, T]$ some bounds for $v^{(\kappa)}$ independent of κ are needed.

Set

$$\Sigma^{(\kappa)}(t) := \|v^{(\kappa)}(t)\|_{H_0^1(\Omega)}^2 + \|v_t^{(\kappa)}(t)\|_{H^1(\Omega)}^2 + \|v_{tt}^{(\kappa)}(t)\|_2^2. \quad (8)$$

For, we have

$$\begin{aligned} & \left(v_{ttt}^{(\kappa)}(t), v_t^{(\kappa)}(t) \right)_{2, \Omega_\tau} + \alpha \left(v_{tt}^{(\kappa)}(t), v_t^{(\kappa)}(t) \right)_{2, \Omega_\tau} \\ & + c^2 \left(\nabla v^{(\kappa)}(t), \nabla v_t^{(\kappa)}(t) \right)_{2, \Omega_\tau} + b \left(\nabla v_t^{(\kappa)}(t), \nabla v_t^{(\kappa)}(t) \right)_{2, \Omega_\tau} \\ & - \left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}(s) ds, \nabla v_t^{(\kappa)}(t) \right)_{2, \Omega_\tau} \\ & = \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), v_t^{(\kappa)}(t) \right)_{2, \Omega_\tau}. \end{aligned} \quad (9)$$

Now, we evaluate each term of (9). integrating by parts, we get

$$\begin{aligned}
& \frac{1}{2} \left(c^2 - \int_0^\tau \mathcal{K}(s) ds \right) \|\nabla v^{(\kappa)}(\tau)\|_2^2 + \frac{\alpha}{2} \|v_\tau^{(\kappa)}(\tau)\|_2^2 \\
& + (v_{\tau\tau}^{(\kappa)}(\tau), v_\tau^{(\kappa)}(\tau))_2 + \frac{1}{2} (\mathcal{K} \circ \nabla v^{(\kappa)}) (\tau) \\
& = \frac{c^2}{2} \|\nabla v^{(\kappa)}(0)\|_2^2 + \frac{\alpha}{2} \|v_t^{(\kappa)}(0)\|_2^2 + (v_{tt}^{(\kappa)}(0), v_t^{(\kappa)}(0))_2 \\
& - \beta \int_0^\tau \|\nabla v_t^{(\kappa)}(t)\|_2^2 dt + \int_0^\tau \|v_{tt}^{(\kappa)}(t)\|_2^2 dt \\
& + \frac{1}{2} \int_0^\tau (\mathcal{K}' \circ \nabla v^{(\kappa)})(t) dt - \frac{1}{2} \int_0^\tau \mathcal{K}(t) \|\nabla v^{(\kappa)}(t)\|_2^2 dt \\
& + \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), v_t^{(\kappa)}(t) \right)_{L^2(\Omega_\tau)}, \tag{10}
\end{aligned}$$

where

$$(\mathcal{K} \circ \nabla v^{(\kappa)})(t) := \int_\Omega \int_0^t \mathcal{K}(t-s) |\nabla v^{(\kappa)} P - \nabla v^{(\kappa)}(x, s)|^2 ds dx.$$

Using ε -Young's inequality with $\varepsilon = \vartheta_1$, we obtain

$$-\frac{\vartheta_1}{2} \|v_{\tau\tau}^{(\kappa)}(\tau)\|_2^2 - \frac{1}{2\vartheta_1} \|v_\tau^{(\kappa)}(\tau)\|_2^2 \leq (v_{\tau\tau}^{(\kappa)}(\tau), v_\tau^{(\kappa)}(\tau))_2. \tag{11}$$

It holds

$$(v_{tt}^{(\kappa)}(0), v_t^{(\kappa)}(0))_2 \leq \frac{1}{2} \|v_{tt}^{(\kappa)}(0)\|_2^2 + \frac{1}{2} \|v_t^{(\kappa)}(0)\|_2^2. \tag{12}$$

Using Cauchy-Schwarz inequality, Young's inequality, the embedding $H^1(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$, and (8), we obtain

$$\begin{aligned}
& \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), v_t^{(\kappa)}(t) \right)_{L^2(\Omega_\tau)} \\
& \leq C_T \|v^{(\kappa)}(0)\|_2^{2(p-1)} + C_T \int_0^\tau (\Sigma^{(\kappa)}(t))^{p-1} dt + \int_0^\tau \Sigma^{(\kappa)}(t) dt. \tag{13}
\end{aligned}$$

Using inequalities (11)-(13) into (10), we obtain

$$\begin{aligned}
& \frac{(c^2 - \bar{\mathcal{K}})}{2} \|\nabla v^{(\kappa)}(\tau)\|_2^2 + \left(\frac{\alpha}{2} - \frac{1}{2\vartheta_1} \right) \|v_\tau^{(\kappa)}(\tau)\|_2^2 \\
& - \frac{\vartheta_1}{2} \|v_{\tau\tau}^{(\kappa)}(\tau)\|_2^2 + \frac{1}{2} (\mathcal{K} \circ \nabla v^{(\kappa)}) (\tau) \\
& \leq C_T \|v^{(\kappa)}(0)\|_2^{2(p-1)} + \frac{c^2}{2} \|\nabla v^{(\kappa)}(0)\|_2^2 \\
& + \frac{(\alpha + 1)}{2} \|v_t^{(\kappa)}(0)\|_2^2 + \frac{1}{2} \|v_{tt}^{(\kappa)}(0)\|_2^2 \\
& + C_T \int_0^\tau (\Sigma^{(\kappa)}(t))^{p-1} dt + 2 \int_0^\tau \Sigma^{(\kappa)}(t) dt. \tag{14}
\end{aligned}$$

For, multiplying each equation of (7) by $G_k''(t)$, add them up from 1 to m and then integrate on $(0, \tau)$, $\tau \leq T$, we obtain

$$\begin{aligned}
& \left(v_{ttt}^{(\kappa)}(t), v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau} + \alpha \left(v_{tt}^{(\kappa)}(t), v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau} \\
& + c^2 \left(\nabla v^{(\kappa)}(t), \nabla v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau} + b \left(\nabla v_t^{(\kappa)}(t), \nabla v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau} \\
& - \left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}(s) ds, \nabla v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau} \\
& = \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau}. \tag{15}
\end{aligned}$$

An integration by parts leads to

$$\begin{aligned}
& c^2 (\nabla v^{(\kappa)}(\tau), \nabla v_\tau^m(\tau))_2 + \frac{\beta}{2} \|\nabla v_\tau^{(\kappa)}(\tau)\|_2^2 \\
& + \frac{\mathcal{K}\tau}{2} \|\nabla v^{(\kappa)}(\tau)\|_2^2 + \frac{1}{2} (-\mathcal{K}' \circ \nabla v^{(\kappa)}) (\tau) + \frac{1}{2} \|v_{\tau\tau}^{(\kappa)}(\tau)\|_2^2 \\
& - \int_0^\tau \mathcal{K}(\tau-s) (\nabla v^{(\kappa)}(s), \nabla v_\tau^{(\kappa)}(\tau))_2 ds \\
& = \frac{\mathcal{K}(0)}{2} \|\nabla v^{(\kappa)}(0)\|_2^2 + c^2 (\nabla v^{(\kappa)}(0), \nabla v_t^{(\kappa)}(0))_2 + \frac{\beta}{2} \|\nabla v_t^{(\kappa)}(0)\|_2^2 \\
& + \frac{1}{2} \|v_{tt}^{(\kappa)}(0)\|_2^2 + \frac{1}{2} \int_0^\tau \mathcal{K}'(t) \|\nabla v^{(\kappa)}(t)\|_2^2 dt \\
& + c^2 \int_0^\tau \|\nabla v_t^{(\kappa)}(t)\|_2^2 dt - \alpha \int_0^\tau \|v_{tt}^{(\kappa)}(t)\|_2^2 dt \\
& - \frac{1}{2} \int_0^\tau (\mathcal{K}'' \circ \nabla v^{(\kappa)}) (t) dt + \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), v_{tt}^{(\kappa)}(t) \right)_{2, \Omega_\tau}. \quad (16)
\end{aligned}$$

Young's inequality with $\varepsilon = \vartheta_2$ allows to get

$$-\frac{c^2 \vartheta_2}{2} \|\nabla v^{(\kappa)}(\tau)\|_2^2 - \frac{c^2}{2\vartheta_2} \|\nabla v_\tau^{(\kappa)}(\tau)\|_2^2 \leq c^2 (\nabla v^{(\kappa)}(\tau), \nabla v_\tau^{(\kappa)}(\tau))_2. \quad (17)$$

Using Young's inequality with $\varepsilon = \vartheta_2$, and $-\int_0^\tau \mathcal{K}(\sigma) d\sigma \geq -\bar{\mathcal{K}}$, we obtain

$$\begin{aligned}
& -\frac{\vartheta_2 \bar{\mathcal{K}}}{2} \|\nabla v^{(\kappa)}(\tau)\|_2^2 - \frac{\vartheta_2}{2} (\mathcal{K} \circ \nabla v^{(\kappa)}) (\tau) - \frac{\bar{\mathcal{K}}}{\vartheta_2} \|\nabla v_\tau^{(\kappa)}(\tau)\|_2^2 \\
& \leq -\int_0^\tau \mathcal{K}(\tau-s) (\nabla v^{(\kappa)}(s), \nabla v_\tau^{(\kappa)}(\tau))_2 ds. \quad (18)
\end{aligned}$$

It holds

$$c^2 (\nabla v^{(\kappa)}(0), \nabla v_t^m(0))_2 \leq \frac{c^2}{2} \|\nabla v^{(\kappa)}(0)\|_2^2 + \frac{c^2}{2} \|\nabla v_t^{(\kappa)}(0)\|_2^2. \quad (19)$$

Using Cauchy-Schwarz inequality, Young's inequality, $H^1(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$, and (8), we obtain

$$\begin{aligned}
& \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), v_{tt}^{(\kappa)}(t) \right)_{2, D_\tau} \\
& \leq C_T \|v^{(\kappa)}(0)\|_2^{2(p-1)} + C_T \int_0^\tau (\Sigma^{(\kappa)}(t))^{p-1} dt + \int_0^\tau \Sigma^{(\kappa)}(t) dt. \quad (20)
\end{aligned}$$

Combining inequalities (17) – (20) with (16), we obtain

$$\begin{aligned}
& -\frac{\vartheta_2(c^2 + \bar{\mathcal{K}})}{2} \left\| \nabla v^{(\kappa)}(\tau) \right\|_2 + \left(\frac{\beta}{2} - \frac{c^2}{2\vartheta_2} - \frac{\bar{\mathcal{K}}}{\vartheta_2} \right) \left\| \nabla v^{(\kappa)}_\tau(\tau) \right\|_2^2 \\
& + \frac{1}{2} \left\| v^{(\kappa)}_{\tau\tau}(\tau) \right\|_2^2 - \frac{\vartheta_2}{2} (\mathcal{K} \circ \nabla v^{(\kappa)}) (\tau) \\
& \leq C_T \left\| v^{(\kappa)}(0) \right\|_2^{2(p-1)} + \frac{\mathcal{K}(0) + c^2}{2} \left\| \nabla v^{(\kappa)}(0) \right\|_2^2 \\
& + \frac{(c^2 + b)}{2} \left\| \nabla v^{(\kappa)}_t(0) \right\|_2^2 + \frac{1}{2} \left\| v^{(\kappa)}_{tt}(0) \right\|_2^2 \\
& + C_T \int_0^\tau (\Sigma^{(\kappa)}(t))^{p-1} dt + (c^2 + 1) \int_0^\tau \Sigma^{(\kappa)}(t) dt. \tag{21}
\end{aligned}$$

Multiplying (14) by λ_1 , (21) by λ_2 and making use of the following inequalities

$$\left\| v^{(\kappa)}(\tau) \right\|_2^2 \leq \int_0^\tau \Sigma^{(\kappa)}(t) dt + \left\| v^{(\kappa)}(0) \right\|_2^2,$$

$$\alpha_1 \left\| \nabla v^{(\kappa)}(\tau) \right\|_2^2 \leq \alpha_1 \int_0^\tau \Sigma^{(\kappa)}(t) dt + \alpha_1 \left\| \nabla v^{(\kappa)}(0) \right\|_2^2,$$

and

$$\alpha_2 \left\| v^{(\kappa)}_\tau(\tau) \right\|_2^2 \leq \alpha_2 \int_0^\tau \Sigma^{(\kappa)}(t) dt + \alpha_2 \left\| v^{(\kappa)}_t(0) \right\|_2^2,$$

where

$$\alpha_1 = \frac{\lambda_2 \vartheta_2 (c^2 + \bar{\mathcal{K}})}{2} > 0, \quad \vartheta_2 = \frac{2(c^2 + 2\bar{\mathcal{K}})}{\beta} > 0,$$

$$\alpha_2 = \frac{\lambda_1}{2\vartheta_2} > 0, \quad \vartheta_1 = \frac{1}{8\vartheta_2} > 0,$$

we obtain

$$\begin{aligned}
& \|v^{(\kappa)}(\tau)\|_2^2 + \frac{\lambda_1(c^2 - \bar{\mathcal{K}})}{2} \|\nabla v^{(\kappa)}(\tau)\|_2^2 + \frac{\lambda_1\alpha}{2} \|v_\tau^{(\kappa)}(\tau)\|_2^2 \\
& + \frac{\lambda_2 b}{4} \|\nabla v_\tau^{(\kappa)}(\tau)\|_2^2 + \frac{\lambda_2}{4} \|v_{\tau\tau}^{(\kappa)}(\tau)\|_2^2 \\
& \leq \|v^{(\kappa)}(0)\|_2^2 + C_1 \|\nabla v^{(\kappa)}(0)\|_2^2 + C_2 \|v_t^{(\kappa)}(0)\|_2^2 \\
& + \frac{\lambda_2(c^2 + b)}{2} \|\nabla v_t^{(\kappa)}(0)\|_2^2 + \frac{(\lambda_1 + \lambda_2)}{2} \|v_{tt}^{(\kappa)}(0)\|_2^2 \\
& + (\lambda_1 + \lambda_2)C_T \|v^{(\kappa)}(0)\|_2^{2(p-1)} \\
& + C_3 \int_0^\tau \Sigma^{(\kappa)}(t) dt + (\lambda_1 + \lambda_2)C_T \int_0^\tau (\Sigma^{(\kappa)}(t))^{p-1} dt,
\end{aligned}$$

where

$$\begin{aligned}
C_1 & : = \frac{\lambda_1 c^2 + \lambda_2(\mathcal{K}(0) + c^2) + 2\alpha_1}{2} > 0, \\
C_2 & : = \frac{\lambda_1(\alpha + 1) + 2\alpha_2}{2} > 0, \\
C_3 & : = 2\lambda_1 + \lambda_2(c^2 + 1) + 1 + \alpha_1 + \alpha_2 > 0.
\end{aligned}$$

We have

$$\begin{aligned}
\Sigma^{(\kappa)}(\tau) & \leq \omega \left(\|v^{(\kappa)}(0)\|_{H_0^1(\Omega)}^2 + \|v_t^{(\kappa)}(0)\|_{H^1(\Omega)}^2 + \|v_{tt}^{(\kappa)}(0)\|_2^2 + \|v^{(\kappa)}(0)\|_2^{2(p-1)} \right) \\
& + \omega \int_0^\tau \Sigma^{(\kappa)}(t) dt + \omega \int_0^\tau (\Sigma^{(\kappa)}(t))^{p-1} dt, \tag{22}
\end{aligned}$$

where

$$\omega := \frac{\max\{1, C_1, C_2, \lambda_2(c^2 + b)/2, (\lambda_1 + \lambda_2)/2, (\lambda_1 + \lambda_2)C_T, C_3\}}{\min\{1, \lambda_1(c^2 - \bar{\mathcal{K}})/2, \lambda_1\alpha/2, \lambda_2 b/4, \lambda_2/4\}} > 0.$$

By solving (22) and then integrating on $(0, \tau)$, we obtain

$$\int_0^\tau \Sigma^{(\kappa)}(t) dt \leq \omega T \left(\|v(0)\|_{H_0^1(\Omega)}^2 + \|v_t(0)\|_{H^1(\Omega)}^2 + \|v_{tt}^{(\kappa)}(0)\|_2^2 + \|v(0)\|_2^{2(p-1)} \right).$$

We deduce from (8) that

$$\|v^{(\kappa)}(t)\|_{H_0^1(\Omega_T)}^2 + \|v_t^{(\kappa)}(t)\|_{H^1(\Omega_T)}^2 + \|v_{tt}^{(\kappa)}(t)\|_{2, \Omega_T}^2 \leq C.$$

Whereupon, the sequence $\{v^{(\kappa)}\}_{\kappa \geq 1} \subset \mathcal{V}(\Omega_T)$ is bounded; so, a subsequence (with the same notation) can be extracted that converges weakly in $\mathcal{V}(\Omega_T)$ to $v(P)$. Let us show that $v(P)$ is a generalized solution of (1) – (3). Since $\lim_{\kappa \rightarrow \infty} \|v^{(\kappa)}(P) - v(P)\|_{2, \Omega_T} = 0$ and $\lim_{\kappa \rightarrow \infty} \|v^{(\kappa)}(x, 0) - \varphi(x)\|_{2, \Omega} = 0$, then $v(x, 0) = \varphi(x)$. Now to prove (5), we multiply each of the relations (7) by a function $\Psi_l(t) \in W_2^1(0, T)$, $\Psi_l(T) = 0$, then add up the obtained equalities ranging from $l = 1$ to $l = \kappa$, and integrate on $(0, T)$. If we set $\phi^{(\kappa)} := \sum_{l=1}^{\kappa} \Psi_l(t) \psi_l(x)$, then we have

$$\begin{aligned} & - \left(v_{tt}^{(\kappa)}(t), \phi_t^{(\kappa)}(t) \right)_{2, \Omega_T} - \alpha \left(v_t^{(\kappa)}(t), \phi_t^{(\kappa)}(t) \right)_{2, \Omega_T} \\ & + c^2 \left(\nabla v^{(\kappa)}, \nabla \phi^{(\kappa)}(t) \right)_{2, \Omega_T} + \beta \left(\nabla v_t^m, \nabla \phi^{(\kappa)}(t) \right)_{2, \Omega_T} \\ & - \left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}(s) ds, \nabla \phi^{(\kappa)}(t) \right)_{2, \Omega_T} \\ & = \alpha \left(v_t^{(\kappa)}(0), \phi^{(\kappa)}(0) \right)_2 + \left(v_{tt}^{(\kappa)}(0), \phi^{(\kappa)}(0) \right)_2 \\ & + \left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), \phi^{(\kappa)}(t) \right)_{2, \Omega_T}. \end{aligned}$$

Since

$$\begin{aligned} & - \left(\int_0^t \mathcal{K}(t-s) (\nabla v^{(\kappa)}(s) - \nabla v(s)) ds, \nabla \phi^{(\kappa)}(t) \right)_{2, \Omega_T} \\ & \leq \frac{\sup(\mathcal{K}(t)) T}{\sqrt{2}} \|\nabla \phi^{(\kappa)}(t)\|_{2, \Omega_T} \|\nabla v^{(\kappa)}(t) - \nabla v(t)\|_{2, \Omega_T}, \end{aligned}$$

and

$$\lim_{\kappa \rightarrow \infty} \|v^{(\kappa)}(t) - v(t)\|_{H^1(\Omega_T)} = 0,$$

then,

$$\lim_{\kappa \rightarrow \infty} \left[\left(\int_0^t \mathcal{K}(t-s) \nabla v^{(\kappa)}(s) ds, \nabla \phi^{(\kappa)}(t) \right)_{2, \Omega_T} - \left(\int_0^t \mathcal{K}(t-s) \nabla v(s) ds, \nabla \phi(t) \right)_{2, \Omega_T} \right] = 0.$$

As the function $z \mapsto |z|^{p-2} z$ is continuous,

$$|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t) \longrightarrow |v|^{p-2} v(t), \quad \text{and a.e. in } \Omega_T. \quad (23)$$

Also,

$$\left\| |v^{(\kappa)}|^{p-2} v^{(\kappa)}(t) \right\|_{2, \Omega_T}^2 \leq C, \quad (24)$$

for some constant C . Using [[16], Lemma 1.3], it follows from (23) and (24) that

$$\left(|v^{(\kappa)}|^{p-2} v^{(\kappa)}(t), \phi^{(\kappa)}(t) \right)_{2, \Omega_T} \longrightarrow \left(|v|^{p-2} v(t), \phi(t) \right)_{2, \Omega_T}, \text{ as } \kappa \longrightarrow \infty.$$

Hence, the limit v satisfies (5) for every $\phi^{(\kappa)}(P) := \sum_{l=1}^{l=\kappa} \Psi_l(t) \psi_l(x)$. We denote by \mathbb{Q}_κ all functions of the form $\phi^{(\kappa)}(P) := \sum_{l=1}^{l=\kappa} \Psi_l(t) \psi_l(x)$, with $\Psi_l(t) \in W_2^1(0, T)$, $\Psi_l(T) = 0$. As $\bigcup_{\kappa=1}^\infty \mathbb{Q}_\kappa \subset \mathcal{W}(\Omega_T)$ is dense, then (5) holds for all $u \in \mathcal{W}(\Omega_T)$. Hence, $v(P)$ is a generalized solution, in $\mathcal{V}(\Omega_T)$, of problem (1) – (3). \square

3.2. Uniqueness

Theorem 3.2. *Under $(\mathbf{H}_1) - (\mathbf{H}_2)$, problem (1) – (3) admits, in $\mathcal{V}(\Omega_T)$, a unique generalized solution.*

Proof. Assume that $v_1, v_2 \in \mathcal{V}(\Omega_T)$ are two different solutions of (1) – (3). Then $v := v_1 - v_2$ solves

$$v_{ttt} + \alpha v_{tt} - c^2 \varrho v - b \Delta v_t + \int_0^t \mathcal{K}(t-s) \Delta v(s) ds = |v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, \quad (25)$$

$$v(x, 0) = v_t(x, 0) = v_{tt}(x, 0) = 0,$$

$$v(P) = 0, \quad P \in \partial\Omega \times (0, T).$$

So,

$$\begin{aligned} & - (v_{tt}(t), u_t(t))_{L^2(\Omega_T)} - \alpha (v_t(t), u_t(t))_{2, \Omega_T} \\ & + c^2 (\nabla v(t), \nabla u(t))_{2, \Omega_T} + b (\nabla v_t(t), \nabla u(t))_{2, \Omega_T} \\ & - \left(\int_0^t \mathcal{K}(t-s) \nabla v(s) ds, \nabla u(t) \right)_{2, \Omega_T} \\ & = (|v_1|^{p-2} v_1(t) - |v_2|^{p-2} v_2(t), u)_{2, \Omega_T}. \end{aligned} \quad (26)$$

Define the function $u(P)$ by

$$u(P) := \begin{cases} \int_t^\tau v(x, s) ds, & 0 \leq t \leq \tau, \\ 0, & \tau \leq t \leq T. \end{cases} \quad (27)$$

It holds $u \in \mathcal{W}(\Omega_T)$ and $u_t(P) = -v(P)$, $0 \leq t \leq \tau$. Integration by parts allows to write

$$\begin{aligned} & \frac{\alpha}{2} \|v(\tau)\|_2^2 + (v_\tau(\tau), v(\tau))_2 + \frac{c^2}{2} \|\nabla u(0)\|_2^2 \\ &= \left(\int_0^t \mathcal{K}(t-s) \nabla v(s) ds, \nabla u(t) \right)_{2, \Omega_T} \\ &+ \int_0^\tau \|v_t(t)\|_2^2 dt - b \int_0^\tau \|\nabla u_t(t)\|_2^2 dt \\ &+ (|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, u)_{2, \Omega_T}. \end{aligned} \quad (28)$$

We set

$$\Sigma(t) := \|v(t)\|_{H_0^1(\Omega)}^2 + \|v_t(t)\|_{H^1(\Omega)}^2 + \|v_{tt}(t)\|_2^2 + \|\nabla \vartheta(t)\|_2^2, \quad (29)$$

where

$$\vartheta(P) := \int_0^t v(x, s) ds. \quad (30)$$

Using (27) and (30), we get

$$u(P) = \vartheta(x, \tau) - \vartheta(P), \quad \nabla u(x, 0) = \nabla \vartheta(x, \tau),$$

and

$$\int_0^\tau \|\nabla u(t)\|_2^2 dt \leq 2\tau \|\nabla \vartheta(\tau)\|_2^2 + 2 \int_0^\tau \|\nabla \vartheta(t)\|_2^2 dt. \quad (31)$$

Using (27) and (29), we get

$$\int_0^\tau \|u(t)\|_2^2 dt \leq \frac{T^2}{2} \int_0^\tau \Sigma(t) dt. \quad (32)$$

It holds

$$-\frac{1}{2} \|v_\tau(\tau)\|_2^2 - \frac{1}{2} \|v(\tau)\|_2^2 \leq (v_\tau(\tau), v(\tau))_2. \quad (33)$$

Cauchy-Schwarz inequality, (27) and (31), lead to

$$\begin{aligned} & \left(\int_0^t \mathcal{K}(t-s) \nabla v(s) ds, \nabla u(t) \right)_{2, \Omega_T} \\ & \leq \frac{\sup(\mathcal{K}^2(t))T^2 + 4}{4} \int_0^\tau \Sigma(t) dt + \tau \|\nabla \vartheta(\tau)\|_2^2. \end{aligned} \quad (34)$$

Using Cauchy-Schwarz inequality, Lemma 2.3, (27), and (29), we obtain

$$\begin{aligned} & (|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, u)_{2, \Omega_T} \\ & \leq C'_T \int_0^\tau (\Sigma(t))^{p-1} dt + \frac{(2+T^2)}{2} \int_0^\tau \Sigma(t) dt. \end{aligned} \quad (35)$$

Combining (33)-(35), and (28), results in

$$\begin{aligned} & \frac{(\alpha-1)}{2} \|v(\tau)\|_2^2 - \frac{1}{2} \|v_\tau(\tau)\|_2^2 + \left(\frac{c^2}{2} - \tau \right) \|\nabla \vartheta(\tau)\|_2^2 \\ & \leq C'_T \int_0^\tau (\Sigma(t))^{p-1} dt + \frac{\sup(\mathcal{K}^2(t))T^2 + 12 + 2T^2}{4} \int_0^\tau \Sigma(s) ds. \end{aligned} \quad (36)$$

Now, multiplying (25) by v_t , then integrating over Ω_τ , we get

$$\begin{aligned} & \frac{1}{2} \left(c^2 - \int_0^\tau \mathcal{K}(s) ds \right) \|\nabla v(\tau)\|_2^2 + \frac{\alpha}{2} \|v_\tau(\tau)\|_2^2 \\ & + (v_{\tau\tau}(\tau), v_\tau(\tau))_2 + \frac{1}{2} (\mathcal{K} \circ \nabla v)(\tau) \\ & = -\beta \int_0^\tau \|\nabla v_t(t)\|_2^2 dt + \int_0^\tau \|v_{tt}(t)\|_2^2 dt \\ & + \frac{1}{2} \int_0^\tau (\mathcal{K}' \circ \nabla v)(s) ds - \frac{1}{2} \int_0^\tau \mathcal{K}(s) \|\nabla v(s)\|_2^2 ds \\ & + (|v_1|^{p-2} v_1(t) - |v_2|^{p-2} v_2(t), v_t(t))_{2, \Omega_T}. \end{aligned} \quad (37)$$

It holds

$$-\frac{\vartheta_3}{2} \|v_{\tau\tau}(\tau)\|_2^2 - \frac{1}{2\vartheta_3} \|v_\tau(\tau)\|_2^2 \leq (v_{\tau\tau}(\tau), v_\tau(\tau))_2. \quad (38)$$

Using Cauchy-Schwarz inequality, Lemma 2.3 and (29), we obtain

$$(|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, v_t)_{2, \Omega_T} \leq C'_T \int_0^\tau (\Sigma(s))^{p-1} ds + \frac{3}{2} \int_0^\tau \Sigma(s) ds. \quad (39)$$

Inequalities (38) and (39) in (37) allow to write

$$\begin{aligned}
& \frac{(c^2 - \bar{\mathcal{K}})}{2} \|\nabla v(\tau)\|_2^2 + \left(\frac{\alpha}{2} - \frac{1}{2\vartheta_3} \right) \|v_\tau(\tau)\|_2^2 \\
& - \frac{\vartheta_3}{2} \|v_{\tau\tau}(\tau)\|_2^2 + \frac{1}{2} (\mathcal{K} \circ \nabla v)(\tau) \\
& \leq C'_T \int_0^\tau (\Sigma(s))^{p-1} ds + \frac{5}{2} \int_0^\tau \Sigma(s) ds.
\end{aligned} \tag{40}$$

Clearly, we have

$$\begin{aligned}
& c^2 (\nabla v(\tau), \nabla v_\tau(\tau))_2 + \frac{\beta}{2} \|\nabla v_\tau(\tau)\|_2^2 \\
& + \frac{\mathcal{K}(\tau)}{2} \|\nabla v(\tau)\|_2^2 + \frac{1}{2} (-\mathcal{K}' \circ \nabla v)(\tau) + \frac{1}{2} \|v_{\tau\tau}(\tau)\|_2^2 \\
& - \int_0^\tau \mathcal{K}(\tau - \sigma) (\nabla v(\sigma), \nabla v_\tau(\tau))_2 d\sigma \\
& = \frac{1}{2} \int_0^\tau \mathcal{K}'(s) \|\nabla v(s)\|_2^2 ds + c^2 \int_0^\tau \|\nabla v_t(t)\|_2^2 dt \\
& - \alpha \int_0^\tau \|v_{tt}(t)\|_2^2 dt - \frac{1}{2} \int_0^\tau (\mathcal{K}'' \circ \nabla v)(s) ds \\
& + (|v_1|^{p-2} v_1(t) - |v_2|^{p-2} v_2(t), v_{tt}(t))_{2, \Omega_T}.
\end{aligned} \tag{41}$$

It holds,

$$-\frac{\vartheta_4 c^2}{2} \|\nabla v(\tau)\|_2^2 - \frac{c^2}{2\vartheta_4} \|\nabla v_\tau(\tau)\|_2^2 \leq c^2 (\nabla v(\tau), \nabla v_\tau(\tau))_2. \tag{42}$$

Appealing to Young's inequality with $\varepsilon = \vartheta_5$ and using $-\int_0^\tau h(s) ds \geq -\bar{\mathcal{K}}$, we get

$$\begin{aligned}
& -\frac{\vartheta_5 \bar{\mathcal{K}}}{2} \|\nabla v(\tau)\|_2^2 - \frac{\vartheta_5}{2} (\mathcal{K} \circ \nabla v)(\tau) - \frac{\bar{\mathcal{K}}}{\vartheta_5} \|\nabla v_\tau(\tau)\|_2^2 \\
& \leq -\int_0^\tau h(\tau - s) (\nabla v(s), \nabla v_\tau(\tau))_2 ds.
\end{aligned} \tag{43}$$

Using Cauchy-Schwarz inequality, Lemma 2.3, and (29), we obtain

$$(|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2, v_{tt})_{2, \Omega_T} \leq C'_T \int_0^\tau (\Sigma(s))^{p-1} ds + \frac{3}{2} \int_0^\tau \Sigma(s) ds. \tag{44}$$

Combining inequalities (42) – (44) in (41), we get

$$\begin{aligned}
& - \left(\frac{\vartheta_4 c^2}{2} + \frac{\vartheta_5 \bar{\mathcal{K}}}{2} \right) \|\nabla v(\tau)\|_2^2 + \left(\frac{\beta}{2} - \frac{c^2}{2\vartheta_4} - \frac{\bar{\mathcal{K}}}{\vartheta_5} \right) \|\nabla v_\tau(\tau)\|_2^2 \\
& + \frac{1}{2} \|v_{\tau\tau}(\tau)\|_2^2 - \frac{\vartheta_5}{2} (\mathcal{K} \circ \nabla v)(\tau) \\
& \leq C'_T \int_0^\tau (\Sigma(s))^{p-1} ds + \frac{(2c^2 + 3)}{2} \int_0^\tau \Sigma(s) ds. \tag{45}
\end{aligned}$$

Multiplying (36) by λ_3 , (40) by λ_4 , (45) by λ_5 , and using the following inequalities

$$\alpha_4 \|v(\tau)\|_2^2 \leq \alpha_4 \int_0^\tau \Sigma(s) ds,$$

$$\alpha_5 \|\nabla v(\tau)\|_2^2 \leq \alpha_5 \int_0^\tau \Sigma(s) ds,$$

and

$$\alpha_6 \|v_\tau(\tau)\|_2^2 \leq \alpha_6 \int_0^\tau \Sigma(s) ds,$$

where

$$\alpha_4 := \frac{\lambda_3}{2} > 0, \alpha_5 := \frac{\lambda_5(\vartheta_4 c^2 + \vartheta_5 \bar{\mathcal{K}})}{2} > 0, \quad \vartheta_4 = \frac{3c^2}{\beta} > 0,$$

$$\alpha_6 := \frac{\lambda_3}{2} + \frac{\lambda_4}{2\vartheta_3} > 0, \quad \vartheta_3 = \vartheta_5 = \frac{\lambda_5}{2\lambda_4} = \frac{6\bar{\mathcal{K}}}{\beta} > 0,$$

we get

$$\begin{aligned}
& \frac{\lambda_3 \alpha}{2} \|v(\tau)\|_2^2 + \frac{\lambda_4 (c^2 - \bar{\mathcal{K}})}{2} \|\nabla v(\tau)\|_2^2 + \frac{\lambda_4 \alpha}{2} \|v_\tau(\tau)\|_2^2 + \frac{\lambda_5 b}{6} \|\nabla v_\tau(\tau)\|_2^2 \\
& + \frac{\lambda_5}{4} \|v_{\tau\tau}(\tau)\|_2^2 + \frac{\lambda_4}{4} (h \circ \nabla v)(\tau) + \frac{\lambda_3 (c^2 - 2\tau)}{2} \|\nabla v(\tau)\|_2^2 \\
& \leq (\lambda_3 + \lambda_4 + \lambda_5) C'_T \int_0^\tau (\Sigma(s))^{p-1} ds + C_5 \int_0^\tau \Sigma(s) ds, \tag{46}
\end{aligned}$$

where

$$C_5 := C_4 + \alpha_4 + \alpha_5 + \alpha_6 > 0.$$

Since τ is arbitrary, we assume that $c^2 - 2\tau > 0$; thus inequality (46) takes the form

$$\Sigma(\tau) \leq \omega' \int_0^\tau (\Sigma(s))^{p-1} ds + \omega' \int_0^\tau \Sigma(s) ds, \quad (47)$$

where

$$\omega' := \frac{\max \{(\lambda_3 + \lambda_4 + \lambda_5) C'_T, C_5\}}{\min \left\{ \frac{\lambda_3 \alpha}{2}, \frac{\lambda_4 (c^2 - \bar{K})}{2}, \frac{\lambda_4 \alpha}{2}, \frac{\lambda_5 b}{6}, \frac{\lambda_5}{4}, \frac{\lambda_3 (c^2 - \tau)}{2} \right\}} > 0.$$

Then, solving (47) (see [18]), we obtain

$$\|v(\tau)\|_{H_0^1(\Omega)}^2 + \|v_\tau(\tau)\|_{H^1(\Omega)}^2 + \|v_{\tau\tau}(\tau)\|_2^2 + \|\nabla \vartheta(\tau)\|_2^2 \leq 0, \quad \tau \in \left[0, \frac{c^2}{2}\right].$$

Similarly, for $\tau \in \left[\frac{(N-1)c^2}{2}, \frac{Nc^2}{2}\right]$ to cover $[0, T]$, one shows that $v = 0$ on $[0, T]$. \square

4. Blowing-up Solutions

Definition 4.1. A blowing-up solution u of problem (1) – (3) is a solution that exists till a finite time T_{max} such that

$$\lim_{t \rightarrow T_{max}^-} \|\nabla v(t)\|_2 = \infty. \quad (48)$$

The energy of problem (1) – (3) is

$$\begin{aligned} \Xi(t) : &= \frac{k}{2} \|v_{tt}\|_2^2 + \frac{\alpha}{2} \|v_t\|_2^2 + (v_{tt}, v_t)_2 + kc^2 (\nabla v, \nabla v_t)_2 \\ &+ \frac{k\beta}{2} \|\nabla v_t\|_2^2 + \frac{k}{2} (-\mathcal{K}' \circ \nabla v)(t) + \frac{k}{2} \mathcal{K}(t) \|\nabla v\|_2^2 \\ &+ \frac{1}{2} \left(c^2 - \int_0^t \mathcal{K}(\sigma) d\sigma \right) \|\nabla v\|_2^2 - k \int_0^t \mathcal{K}(t-s) (\nabla v(\sigma), \nabla v_t(t))_2 d\sigma \\ &+ \frac{1}{2} (\mathcal{K} \circ \nabla v)(t) - \frac{1}{p} \|v\|_p^p, \end{aligned} \quad (49)$$

where the constant k satisfies

$$\frac{1}{\alpha} < 1 < k < \frac{\beta}{2c^2}. \quad (50)$$

Lemma 4.2. *Let u be a solution of problem (1) – (3), then on $[0, t]$*

$$\begin{aligned}
\Xi'(t) &= (1 - k\alpha) \|v_{tt}\|_2^2 + (kc^2 - \beta) \|\nabla v_t\|_2^2 \\
&\quad - \frac{k}{2} (\mathcal{K}'' \circ \nabla v)(t) + \frac{1}{2} (\mathcal{K}' \circ \nabla v)(t) \\
&\quad + \frac{k}{2} \mathcal{K}'(t) \|\nabla v\|_2^2 - \frac{1}{2} \mathcal{K}(t) \|\nabla v\|_2^2 \\
&\leq 0.
\end{aligned} \tag{51}$$

Proof. Multiplying equation of (1) by $v_t + kv_{tt}$ and integrating by parts, we obtain (51). \square

Remark 4.3. *Integrating (51) over $(0, t)$, it comes*

$$\begin{aligned}
\Xi(t) &= \Xi(0) + (1 - k\alpha) \int_0^t \|v_{\sigma\sigma}\|_2^2 d\sigma + (kc^2 - \beta) \int_0^t \|\nabla v_\sigma\|_2^2 d\sigma \\
&\quad - \frac{k}{2} \int_0^t (\mathcal{K}'' \circ \nabla v)(\sigma) d\sigma + \frac{1}{2} \int_0^t (\mathcal{K}' \circ \nabla v)(\sigma) d\sigma \\
&\quad + \frac{k}{2} \int_0^t \mathcal{K}'(\sigma) \|\nabla v\|_2^2 d\sigma - \frac{1}{2} \int_0^t \mathcal{K}(\sigma) \|\nabla v\|_2^2 d\sigma.
\end{aligned} \tag{52}$$

For a solution v of (1) – (3), we let

$$\Psi(t) := \Psi_1(t) + \Psi_2(t), \tag{53}$$

where

$$\begin{aligned}
\Psi_1(t) &:= (v_t, v)_2 + \int_0^T \left| \frac{d}{ds} (v_s, v)_2 \right| ds \\
&\quad + |(v_t(0), v(0))_2| + (T - t) \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \\
&\quad + \frac{\alpha}{2} \|v\|_2^2 + \alpha (T - t) \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds \\
&\quad + \frac{\beta}{2} \int_0^t \|\nabla v\|_2^2 ds - \frac{3}{2} \int_0^t \|v_s\|_2^2 ds + \frac{3}{2} \int_0^T \|v_s\|_2^2 ds,
\end{aligned}$$

and

$$\begin{aligned}
\Psi_2(t) &:= k_1 \int_0^t \int_0^s \|\nabla v\|_2^2 d\mu ds + k_1 (T - t) \int_0^T \|\nabla v\|_2^2 ds \\
&\quad + k_2 \int_0^t \int_0^s \|\nabla v_\mu\|_2^2 d\mu ds + k_2 (T - t) \int_0^T \|\nabla v_s\|_2^2 ds,
\end{aligned}$$

$k_1 > 0, k_2 > 0$ are constants.

Lemma 4.4. *Suppose that the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_2)$ hold and*

$$\bar{\mathcal{K}} < \frac{p(p-2)}{(p-1)^2 + 1} c^2 < c^2, \quad (54)$$

then we have

$$\frac{p\beta}{2} \|\nabla v\|_2^2 + \frac{p\beta}{2} \int_0^t \|\nabla v_s\|_2^2 ds - p\Xi(0) \leq \Psi''(t). \quad (55)$$

Proof. Form (53) and using the solution v of problem (1) – (3), we get

$$\begin{aligned} \Psi''(t) &= (k_1 - c^2) \|\nabla v\|_2^2 + \alpha \|v_t\|_2^2 + k_2 \|\nabla v_t\|_2^2 \\ &\quad + \|v\|_p^p + \int_0^t \mathcal{K}(t-s) (\nabla v(s), \nabla v(t))_2 ds. \end{aligned} \quad (56)$$

Multiplying Eq. (52) by p and summing up in (56), we obtain

$$\begin{aligned} \Psi''(t) &= -p\Xi(0) + \frac{pk}{2} \|v_{tt}\|_2^2 + p(k\alpha - 1) \int_0^t \|v_{ss}\|_2^2 ds \\ &\quad + p(\beta - kc^2) \int_0^t \|\nabla v_s\|_2^2 ds + \left(\alpha + \frac{p\alpha}{2}\right) \|v_t\|_2^2 \\ &\quad + \left(k_1 + \frac{(p-2)c^2}{2} - \frac{p}{2} \left(\int_0^t \mathcal{K}(s) ds\right)\right) \|\nabla v\|_2^2 \\ &\quad + \left(k_2 + \frac{pk\beta}{2}\right) \|\nabla v_t\|_2^2 + p(v_{tt}, v_t)_2 \\ &\quad + pkc^2 (\nabla v, \nabla v_t)_2 + \frac{p}{2} (\mathcal{K} \circ \nabla v)(t) \\ &\quad + \int_0^t \mathcal{K}(t-s) (\nabla v(s), \nabla v(t))_2 ds \\ &\quad - pk \int_0^t \mathcal{K}(t-s) (\nabla v(s), \nabla v_t(t))_2 ds \\ &\quad + \frac{pk}{2} \mathcal{K}(t) \|\nabla v\|_2^2 + \frac{pk}{2} (-\mathcal{K}' \circ \nabla v)(t) \\ &\quad + \frac{p}{2} \int_0^t \mathcal{K}(s) \|\nabla v\|_2^2 ds - \frac{pk}{2} \int_0^t h'(s) \|\nabla v\|_2^2 ds \\ &\quad - \frac{p}{2} \int_0^t (\mathcal{K}' \circ \nabla v)(s) ds + \frac{pk}{2} \int_0^t (\mathcal{K}'' \circ \nabla v)(s) ds. \end{aligned} \quad (57)$$

Young's inequality with $\varepsilon = k$, allows to write get

$$p(v_{tt}, v_t)_2 \geq -\frac{pk}{2} \|v_{tt}\|_2^2 - \frac{p}{2k} \|v_t\|_2^2. \quad (58)$$

It holds

$$pkc^2 (\nabla v, \nabla v_t)_2 \geq -\frac{pkc^2}{2} \|\nabla v\|_2^2 - \frac{pkc^2}{2} \|\nabla v_t\|_2^2. \quad (59)$$

Young's inequality, with $\varepsilon = \frac{p}{2}$, allows to get

$$\begin{aligned} & \int_0^t \mathcal{K}(t-s) (\nabla v(s), \nabla v(t))_2 ds \\ & \geq \left(1 - \frac{1}{p}\right) \left(\int_0^t \mathcal{K}(s) ds\right) \|\nabla v\|_2^2 - \frac{p}{4} (\mathcal{K} \circ \nabla v)(t). \end{aligned} \quad (60)$$

Using $-\int_0^t \mathcal{K}(s) ds \geq -c^2$, we obtain

$$\begin{aligned} & -pk \int_0^t \mathcal{K}(t-s) (\nabla u(s), \nabla u_t(t))_2 ds \\ & \geq -\frac{p}{4} (\mathcal{K} \circ \nabla v)(t) - pk \left(k + \frac{1}{2}\right) c^2 \|\nabla v_t\|_2^2 - \frac{pkc^2}{2} \|\nabla v\|_2^2. \end{aligned} \quad (61)$$

Combining inequalities (58) – (61) into (57) and using (50), (54), we get

$$\begin{aligned} & \Psi''(t) - (k_1 - pkc^2) \|\nabla v\|_2^2 \\ & \geq -p\Xi(0) + p(\beta - kc^2) \int_0^t \|\nabla v_s\|_2^2 ds + (k_2 - pk^2c^2) \|\nabla v_t\|_2^2. \end{aligned}$$

Let $k_1 := pkc^2 + \frac{p\beta}{2}$ and $k_2 := pk^2c^2$, then, for $p > 4$, (55) is obtained. \square

Lemma 4.5. *Suppose $(\mathbf{H}_1) - (\mathbf{H}_2)$ and (54) hold and that either one of the following condition is satisfied*

- (i) $\Xi(0) < 0$,
- (ii) $\Xi(0) = 0$, and

$$\Psi'(0) > \frac{\beta}{2} \|\nabla v(0)\|_2^2 + |(v_{tt}(0), v(0))_2| + \alpha |(v_t(0), v(0))_2|. \quad (62)$$

(iii) $\Xi(0) > 0$ and

$$\begin{aligned} \Psi'(0) &> r_* [\Psi(0) + k_0] \\ &+ \frac{\beta}{2} \|\nabla v(0)\|_2^2 + |(v_{tt}(0), v(0))_{L^2(\Omega)}| + \alpha |(v_t(0), v(0))_2|, \end{aligned} \quad (63)$$

where

$$r_* := \frac{p - \sqrt{p(p-4)}}{2},$$

and

$$k_0 := \frac{\beta}{2} \|\nabla v(t)\|_2^2 + |(v_{tt}(t), v(t))_2| + \alpha |(v_t(t), v(t))_2| + \Xi(t). \quad (64)$$

Then

$$\Psi'(t) \geq \frac{\beta}{2} \|\nabla v(t)\|_2^2 + |(v_{tt}(t), v(t))_2| + \alpha |(v_t(t), v(0))_2|, \quad (65)$$

for $t > t_*$, and

$$t^* := \max \left\{ 0, \frac{\Lambda}{p\Xi(0)} \right\}, \quad (66)$$

where we have set

$$\Lambda := \Psi'(0) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2|,$$

and $t_* = t^*$ in case (i) and $t_* = 0$ in case (ii) and (iii).

Proof. (i) If $\Xi(0) < 0$, then $-p\Xi(t) > 0$.

Using (55), we get

$$\Psi''(t) \geq -p\Xi(t); \quad (67)$$

whereupon

$$\Psi'(t) - \Psi'(0) \geq -p\Xi(0)t. \quad (68)$$

Then (68) is written

$$\begin{aligned} \Psi'(t) &\geq \frac{\beta}{2} \|\nabla v(0)\|_2^2 + |(v_{tt}(0), v(0))_2| + \alpha |(v_t(0), v(0))_2| \\ &+ \left(-p\Xi(0)t + \Psi'(0) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right). \end{aligned} \quad (69)$$

Let

$$t \geq \frac{\Psi'(0) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2|}{p\Xi(0)}. \quad (70)$$

Substituting (70) into (69), using the definition of t^* given in (66), we obtain directly (65).

(ii) If $\Xi(0) = 0$, then using (55), we obtain

$$\Psi''(t) \geq 0, \quad t \geq 0; \quad (71)$$

whereupon

$$\Psi'(t) - \Psi'(0) \geq 0. \quad (72)$$

Then (72) is written in form

$$\begin{aligned} \Psi'(t) &\geq \frac{\beta}{2} \|\nabla v(0)\|_2^2 + |(v_{tt}(0), v(0))_2| + \alpha |(v_t(0), v(0))_2| \\ &+ \left\{ \Psi'(0) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right\}. \end{aligned} \quad (73)$$

Furthermore, if (62) holds, hence we obtain (65) for $t \geq 0$.

(iii) When $\Xi(0) > 0$, making use of (53), we get

$$\begin{aligned} \Psi'(t) &- \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\ &\leq (v_{tt}, v)_2 - |(v_{tt}(0), v(0))_2| - \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \\ &+ \frac{\beta}{2} \|\nabla v\|_2^2 - \frac{\beta}{2} \|\nabla v(0)\|_2^2 + \alpha (v_t, v)_2 \\ &- \alpha |(v_t(0), v(0))_2| - \alpha \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds. \end{aligned} \quad (74)$$

Using

$$(v_{tt}, v)_2 - |(v_{tt}(0), v(0))_2| - \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \leq 0 \quad (75)$$

and

$$\alpha (v_t, v)_2 - \alpha |(v_t(0), v(0))_2| - \alpha \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds \leq 0 \quad (76)$$

into (74), we obtain

$$\begin{aligned}\Psi'(t) &= \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\ &\leq \Psi(t) + \frac{\beta}{2} \|\nabla v\|_2^2.\end{aligned}$$

Then

$$\begin{aligned}p(\Psi'(t) - \Psi(t)) &= \frac{p\beta}{2} \|\nabla v(0)\|_2^2 - p|(v_{tt}(0), v(0))_2| - p\alpha |(v_t(0), v(0))_2| \\ &\leq \frac{p\beta}{2} \|\nabla v\|_2^2.\end{aligned}\tag{77}$$

Using (55), we get

$$\frac{p\beta}{2} \|\nabla v\|_2^2 \leq \Psi''(t) + p\Xi(0).\tag{78}$$

Using (78) into (77), we get

$$\begin{aligned}p(\Psi'(t) - \Psi(t)) &= \frac{p\beta}{2} \|\nabla v(0)\|_2^2 - p|(v_{tt}(0), v(0))_2| - p\alpha |(v_t(0), v(0))_2| \\ &\leq \Psi''(t) + p\Xi(0).\end{aligned}$$

Then

$$\Psi''(t) - p\Psi'(t) + p\Psi(t) + pk_0 \geq 0,$$

where k_0 is defined in (64).

Let

$$B(t) := \Psi(t) + k_0,$$

then $B'(t) = \Psi'(t)$, $B''(t) = \Psi''(t)$, and $B(t)$ satisfies

$$B''(t) - pB'(t) + pB(t) \geq 0.\tag{79}$$

Therefore, using Lemma 2.4 in (79) and (63), (65) is obtained. \square

Theorem 4.6. *Suppose that the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_2)$ and (54) hold and that either one of the following conditions is satisfied*

- (i) $\Xi(0) < 0$,
- (ii) $\Xi(0) = 0$ and (62) holds,

$$(iii) \quad 0 < \Xi(0) < \frac{\vartheta}{\left(\frac{(p-2)^2}{2} - 2\right) \left(\frac{1}{2} - \frac{1}{p-2}\right)},$$

where

$$\vartheta := \left(\frac{p-4}{4}\right)^2 \left(\Psi'(t_*) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2|\right)^2 \Lambda(t_*)^{1/\xi},$$

and (63) holds. Then the solution v blows-up at finite time T^* .

In case (i),

$$T^* \leq t_* - \frac{\Lambda(t_*)}{\Lambda'(t_*)}.$$

Furthermore, when $\Lambda(t_*) < \min \left\{ 1, \left(\frac{\nu}{-\beta}\right)^{1/2} \right\}$, then

$$T^* \leq t_* + \frac{1}{\sqrt{-\beta}} \ln \frac{\left(\frac{\nu}{-\beta}\right)^{1/2}}{\left(\frac{\nu}{-\beta}\right)^{1/2} - \Lambda(t_*)}.$$

In case (ii)

$$T^* \leq t_* - \frac{\Lambda(t_*)}{\Lambda'(t_*)},$$

or

$$T^* \leq t_* + \frac{\Lambda(t_*)}{\Lambda'(t_*)}.$$

In case (iii)

$$T^* \leq \frac{\Lambda(t_0)}{\sqrt{\mu}},$$

or

$$T_{max} \leq t_* + \frac{\xi c}{\sqrt{\mu}} 3^{(3\xi+1)/2\xi} \left(1 - (1 + c\Lambda(t_*))^{1/2\xi}\right),$$

where $c := (\beta/\mu)^{\xi/(2+\xi)}$, $\xi := (p-4)/4$, and $\Lambda(t)$, μ and β are given here after in (80), (94) and (95), respectively.

Note that in **case (i)**, $t_* = t^*$ is given in (66) and $t_* = 0$ in **case (ii)** and **(iii)**.

Proof. Let

$$\begin{aligned} \Lambda(t) \quad : \quad &= \left(\Psi(t) + (T-t) \frac{\beta}{2} \|\nabla v(0)\|_2^2 + (T-t) |(v_{tt}(0), v(0))_2| \right. \\ &\quad \left. + (T-t) |(v_t(0), v(0))_2| \right)^{-\xi}. \end{aligned} \quad (80)$$

It holds

$$\Lambda'(t) = -\xi \Lambda(t)^{1+\frac{1}{\xi}} \left(\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - |(v_t(0), v(0))_2| \right).$$

Then

$$\Lambda''(t) = -\xi \Lambda(t)^{1+\frac{2}{\xi}} Q(t), \quad (81)$$

for

$$\begin{aligned} Q(t) \quad : \quad &= \Psi''(t) \left(\Psi(t) + (T-t) \frac{\beta}{2} \|\nabla v(0)\|_2^2 \right. \\ &\quad \left. + (T-t) |(v_{tt}(0), v(0))_2| + (T-t) |(v_t(0), v(0))_2| \right) \\ &\quad - (1+\xi) \left(\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| \right. \\ &\quad \left. - |(v_t(0), v(0))_2| \right)^2. \end{aligned} \quad (82)$$

From (53), we have

$$\begin{aligned} \Psi(t) \quad &+ \quad (T-t) \frac{\beta}{2} \|\nabla v(0)\|_2^2 + (T-t) |(v_{tt}(0), v(0))_2| \\ &+ \quad (T-t) |(v_t(0), v(0))_2| \\ &\geq \quad \frac{\beta}{2} \int_0^t \|\nabla v\|_2^2 \, ds, \end{aligned} \quad (83)$$

and from (55), we have

$$\Psi''(t) \geq -p\Xi(0) + \frac{p\beta}{2} \int_0^t \|\nabla v_s\|_2^2 \, ds. \quad (84)$$

From (53), we get

$$\begin{aligned}
& \Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\
& \leq (v_{tt}, v)_2 - |(v_{tt}(0), v(0))_2| - \int_0^T \left| \frac{d}{ds} |(v_{ss}, v)_2| \right| ds \\
& + \frac{\beta}{2} \|\nabla v\|_2^2 - \frac{\beta}{2} \|\nabla v(0)\|_2^2 + \alpha (v_t, v)_2 \\
& - \alpha |(v_t(0), v(0))_2| - \alpha \int_0^T \left| \frac{d}{ds} |(v_t, v)_2| \right| ds.
\end{aligned}$$

Using (75) and (76), and

$$\frac{\beta}{2} \|\nabla v(t)\|_2^2 - \frac{\beta}{2} \|\nabla v(0)\|_2^2 = \beta \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds,$$

we get

$$\begin{aligned}
\Psi'(t) & - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \\
& \leq \beta \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds.
\end{aligned}$$

Then

$$\begin{aligned}
& \left(\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right)^2 \quad (85) \\
& \leq 4 \left(\frac{\beta}{2} \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds \right)^2.
\end{aligned}$$

Multiplying (85) by $-(1 + \xi)$, we obtain

$$\begin{aligned}
& - (1 + \xi) \left(\Psi'(t) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 - |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right)^2 \\
& \geq -4(1 + \xi) \left(\frac{\beta}{2} \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds \right)^2. \quad (86)
\end{aligned}$$

Substituting (83), (84) and (86) into (82), we get

$$\begin{aligned}
Q(t) \geq & -p\Xi(0) \left(\Psi(t) + (T-t) \frac{\beta}{2} \|\nabla v(0)\|_2^2 \right. \\
& \left. + (T-t) |(v_{tt}(0), v(0))_2| + (T-t) |(v_t(0), v(0))_2| \right) \\
& + p \left(\frac{\beta}{2} \int_0^t \|\nabla v_s\|_2^2 ds \right) \left(\frac{\beta}{2} \int_0^t \|\nabla v\|_2^2 ds \right) \\
& - 4(1+\xi) \left(\frac{\beta}{2} \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds \right)^2.
\end{aligned}$$

Let us set

$$\begin{aligned}
\mathcal{A} & : = \frac{\beta}{2} \int_0^t \|\nabla v\|_2^2 ds, \\
\mathcal{B} & : = \frac{\beta}{2} \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds, \\
\mathcal{C} & : = \frac{\beta}{2} \int_0^t \|\nabla v_s\|_2^2 ds.
\end{aligned}$$

Then, we have

$$Q(t) \geq -p\Xi(0)\Lambda(t)^{-\frac{1}{\xi}} + p(\mathcal{A}\mathcal{C} - \mathcal{B}^2). \quad (87)$$

Observe that, for all $\rho, \eta \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned}
\mathcal{A}\rho^2 + 2\mathcal{B}\rho\eta + \mathcal{C}\eta^2 & = \rho^2 \frac{\beta}{2} \int_0^t \|\nabla v\|_2^2 ds + \beta\rho\eta \int_0^t (\nabla v(s), \nabla v_s(s))_2 ds \\
& + \frac{\beta}{2} \eta^2 \int_0^t \|\nabla v_s\|_2^2 ds.
\end{aligned}$$

Whereupon

$$\mathcal{A}\rho^2 + 2\mathcal{B}\rho\eta + \mathcal{C}\eta^2 = \frac{\beta}{2} \int_0^t \|\rho\nabla v + \eta\nabla v_s\|_2^2 ds \geq 0.$$

Consequently,

$$\mathcal{B}^2 - \mathcal{A}\mathcal{C} \leq 0.$$

Hence, thanks to (87),

$$Q(t) \geq -p\Xi(0)\Lambda(t)^{-\frac{1}{\xi}}, \quad t \geq t_*. \quad (88)$$

Therefore, using (88) and (81), we get

$$\Lambda''(t) \leq \left(\frac{(p-2)^2}{4} - 1 \right) \Xi(0) \Lambda(t)^{1+\frac{1}{\xi}}, \quad t \geq t_*. \quad (89)$$

Note that by Lemma 4.5, $\Lambda'(t) < 0$ for $t \geq t_*$. We have

$$\Lambda''(t) \Lambda'(t) \geq \left(\frac{(p-2)^2}{4} - 1 \right) \Xi(0) \Lambda'(t) M(t)^{1+\frac{1}{\xi}}, \quad t \geq t_*. \quad (90)$$

Integrating (90) on (t_*, t) , it comes

$$\int_{t_*}^t \Lambda''(s) \Lambda'(s) ds \geq \left(\frac{(p-2)^2}{4} - 1 \right) \Xi(0) \int_{t_*}^t \Lambda'(s) \Lambda(s)^{1+\frac{1}{\xi}} ds. \quad (91)$$

By using (81) in (80), we obtain

$$\begin{aligned} \int_{t_*}^t \Lambda'' \Lambda'(s) ds &= \frac{1}{2} \Lambda'^2(t) - \frac{1}{2} \left(\frac{p-4}{4} \right)^2 \left[\Psi'(t_*) - \frac{\beta}{2} \|\nabla v(0)\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. - \left| (v_{tt}(0), v(0))_{L^2(\Omega)} \right| - \alpha \left| (v_t(0), v(0))_{L^2(\Omega)} \right| \right]^2 \Lambda(t_*)^{2+\frac{2}{\xi}}. \end{aligned} \quad (92)$$

We also have

$$\begin{aligned} \int_{t_*}^t \Lambda'(s) \Lambda(s)^{1+\frac{1}{\xi}} ds &= \int_{t_*}^t \frac{\xi}{2\xi+1} \frac{d}{ds} \left[\Lambda(s)^{\frac{2\xi+1}{\xi}} \right] ds \\ &= \frac{\xi}{2\xi+1} M(t)^{\frac{2\xi+1}{\xi}} - \frac{\xi}{2\xi+1} \Lambda(t_*)^{\frac{2\xi+1}{\xi}} \\ &= \left(\frac{1}{2} - \frac{1}{p-2} \right) M(t)^{\frac{2\xi+1}{\xi}} - \left(\frac{1}{2} - \frac{1}{p-2} \right) \Lambda(t_*)^{\frac{2\xi+1}{\xi}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(\frac{(p-2)^2}{4} - 1 \right) \Xi(0) \int_{t_*}^t \Lambda'(s) \Lambda(s)^{1+\frac{1}{\xi}} ds \\ &= \left(\frac{(p-2)^2}{4} - 1 \right) \left(\frac{1}{2} - \frac{1}{p-2} \right) \Xi(0) \Lambda(t)^{2+\frac{1}{\xi}} \\ &\quad - \left(\frac{(p-2)^2}{4} - 1 \right) \left(\frac{1}{2} - \frac{1}{p-2} \right) \Xi(0) \Lambda(t_*)^{2+\frac{1}{\xi}}. \end{aligned} \quad (93)$$

By substituting (92) and (93) in (91), it comes

$$\begin{aligned}
& \frac{1}{2} \Lambda'(t)^2 - \frac{1}{2} \left(\frac{p-4}{4} \right)^2 \left[\Psi'(t_*) - \frac{\beta}{2} \|\nabla v(t)\|_{L^2(\Omega)}^2 \right. \\
& - \left. |(v_{tt}(0), v(0))_{L^2(\Omega)}| - \alpha |(v_t(t), v(t))_{L^2(\Omega)}| \right]^2 \Lambda(t_*)^{2+\frac{2}{\xi}} \\
& \geq \left(\frac{(p-2)^2}{4} - 1 \right) \left(\frac{1}{2} - \frac{1}{p-2} \right) \Xi(t) \Lambda(t)^{2+\frac{1}{\xi}} \\
& - \left(\frac{(p-2)^2}{4} - 1 \right) \left(\frac{1}{2} - \frac{1}{p-2} \right) \Xi(0) \Lambda(t_*)^{2+\frac{1}{\xi}}.
\end{aligned}$$

Thus

$$\Lambda'(t)^2 \geq \mu + \beta \Lambda(t)^{2+\frac{1}{\gamma}},$$

where

$$\begin{aligned}
\mu & : = \left(\left(\frac{p-4}{4} \right)^2 \left(\Psi'(t_*) - \frac{\beta}{2} \|\nabla v(0)\|_2^2 \right. \right. \\
& - \left. |(v_{tt}(0), v(0))_2| - \alpha |(v_t(0), v(0))_2| \right)^2 \\
& - \left(\frac{(p-2)^2}{2} - 2 \right) \left(\frac{1}{2} - \frac{1}{p-2} \right) E(0) \Lambda(t_*)^{-\frac{1}{\gamma}} \right) \Lambda(t_*)^{2+\frac{2}{\gamma}} \\
& > 0,
\end{aligned} \tag{94}$$

and

$$\beta := \left(\frac{(p-2)^2}{2} - 2 \right) \left(\frac{1}{2} - \frac{1}{p-2} \right) \Xi(0). \tag{95}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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