WELL-POSEDNESS OF REYNOLDS AVERAGED EQUATIONS FOR COMPRESSIBLE FLUIDS WITH A VANISHING PRESSURE

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Abstract. We show that the Reynolds averaged equations for compressible fluids (neglecting third order correlations) are well-posed in H^s when the pressure vanishes in dimensions $d = 2$ and 3. In order to do this, we show that the system is Friedrichs-symmetrizable. This model belongs to the class of non-conservative hyperbolic systems. Hence the usual symmetrisation method for conservation laws can not be used here.

1. Introduction and main results

We study the Reynolds averaged equations for compressible fluids, where third-order correlations are neglected. This system can be written in Eulerian coordinates as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & \text{(1.1a)} \\ \partial_t v + (v \cdot \nabla v) + \dfrac{1}{\rho}\left(\nabla p + \operatorname{div}(\rho P)^T\right) = 0, & \text{(1.1b)} \\ \partial_t P + (v \cdot \nabla)P + \dfrac{\partial v}{\partial x} P + P \left(\dfrac{\partial v}{\partial x}\right)^T = 0. & \text{(1.1c)} \end{cases}$$

The variables are the averaged density $\rho > 0$, the averaged velocity $v \in \mathbb{R}^d$ ($d = 2$ or 3), and P is the Reynolds stress tensor, $P \in S_d^{++}(\mathbb{R})$. The function p is the pressure of the fluid and is a function of the density ρ, through an equation of state $p = p(\rho)$. The map $\rho \mapsto p(\rho)$ is supposed to be of class C^1 and non-decreasing. Typical pressure laws are of the form $p(\rho) = a\rho^\gamma$, with $a > 0$ and $\gamma > 0$ two constants.

The tensor P is a classical Reynolds tensor appearing in the Reynolds averaging of turbulent flows for barotropic compressible fluids (see [14], [18], [12]). It also appears in the description of free surface shear flows, where the averaging operator is the depth averaging (cf. Annexe C, A7 in [15] for a derivation of the model). In that latter case, the density ρ must be replaced by the water depth, often denoted h. The pressure is then given by $p(h) = gh^2/2$ (cf. [15] for instance).

System (1.1) is hyperbolic whenever $p'(\rho) \geq 0$ and P is definite positive, as it has been proved in [15]. Equation (1.1a) shows that the density ρ is conserved. The conservation of momentum ρv also holds: (1.1b) rewrites as

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + pI_d + \rho P)^T = 0. \tag{1.2}$$

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One can also deduce from (1.1) the conservation of energy:

$$\partial_t e + \operatorname{div}(ev + pv + \rho P v) = 0, \text{ with } e := \frac{1}{2}\rho|v|^2 + \rho\mathcal{E}(\rho) + \frac{1}{2}\operatorname{Tr}(\rho P). \tag{1.3}$$

The map $\rho \mapsto \rho\mathcal{E}(\rho)$ is called the volumic internal energy, and is linked to the pressure via the relation $p(\rho) = \rho^2\mathcal{E}'(\rho)$. The term $\rho|v|^2/2$ is the volumic kinetic energy of the fluid, and the term $\operatorname{Tr}(\rho P)/2$ is the energy associated to the tensor P.

In [6], it was shown that system (1.1) admits a variational formulation, as it is often the case in physics when the energy of a system is conserved. Define the Lagrangian density

$$\mathcal{L}(\rho, v, P) := \frac{1}{2}\rho|v|^2 - \rho\mathcal{E}(\rho) - \frac{1}{2}\operatorname{Tr}(\rho P), \tag{1.4}$$

and the corresponding action

$$\mathcal{A} := \int_{t_0}^{t_1}\int_{\mathbb{R}^d} \mathcal{L}\mathrm{d}x\mathrm{d}t. \tag{1.5}$$

Then one can show that (1.1b) is the Euler-Lagrange equation given by the stationnary action principle applied to the action (1.5), under the two constraints (1.1a) and (1.1c).

The tensor P admits an additional conservation law, sometimes called conservation of enstrophy, that is a consequence of (1.1c) and can be written

$$\partial_t\left(\frac{\det P}{\rho^2}\right) + v\cdot\nabla\left(\frac{\det P}{\rho^2}\right) = 0. \tag{1.6}$$

Note that equation (1.1c) implies that the symmetry of P is conserved by the evolution. Equation (1.6) then implies that, if $P(t) \in S_d^{++}(\mathbb{R})$ for some instant t, then this property is true for all times.

It has been proved in [7] that system (1.1) does not admit any further conservation law. Thus, in dimension $d = 2$ or 3, system (1.1) is not conservative. Hence the usual symmetrisation method of Godunov (cf. [8]) and Lax and Friedrichs (cf. [5]) for hyperbolic systems of conservation laws (see for instance [17], pages 83-84) can not be used here.

However, one can show that system (1.1) is Friedrichs-symmetrizable when the pressure vanishes. More precisely, we state the following proposition:

Proposition 1. *Let $d = 2$ or 3. Suppose that ρ takes values in $\mathbb{R}_+^*$ and P takes values in $S_d^{++}(\mathbb{R})$. Then the two following properties are equivalent:*

1. *System (1.1) is Friedrichs-symmetrizable*
2. *The pressure p is constant: $p'(\rho) = 0$, or the tensor P is a scalar matrix, i.e. there exists $\lambda = \lambda(t, x) \in \mathbb{R}$ such that $P = \lambda I_d$.*

Let us make few comments about this proposition:

- The tensor $P = \lambda I_d$ is a solution of system (1.1) only for trivial velocities; hence, for applications of this result, the case $p' = 0$ seems more interesting.
- This property holds in the variables (ρ, v, P). It could be possible that system (1.1), written in different variables, appears to be symmetrizable even when $p' \neq 0$.

- When $d = 1$, system (1.1) is symmetrizable, even when $p' \neq 0$. In fact, one can prove that one-dimensional hyperbolic systems are always Friedrichs-symmetrizable (cf. [13]). In higher dimension $d \geq 2$, this does not hold anymore.

As a consequence of Proposition 1, we have the following result regarding the well-posedness of system (1.1):

Theorem 1. *Let $d = 2$ or 3, $s > 1 + d/2$ and $\mathcal{U} := \mathbb{R}_+^* \times \mathbb{R}^d \times S_d^{++}(\mathbb{R})$. Let $\bar{Y} := (\bar{\rho}, \bar{u}, \bar{P}) \in \mathcal{U}$ and Y_0 taking values in $\mathcal{U}$ such that $Y_0 - \bar{Y} \in H^s(\mathbb{R}^d)$. We consider the Cauchy problem associated to (1.1) with initial data Y_0. There exists $T > 0$ such that (1.1) with $p' = 0$ has a unique classical solution $Y(t)$ in $\mathcal{C}^1([0,T] \times \mathbb{R}^d)$ with values in $\mathcal{U}$ achieving the initial data $Y(0) = Y_0$. Furthermore, $Y - \bar{Y}$ belongs to $\mathcal{C}([0,T], H^s(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T], H^{s-1}(\mathbb{R}^d))$.*

Theorem 1 is a consequence of Proposition 1 and of the theory of well-posedness of Kato for quasilinear evolution equations (cf. [11]). For a detailed proof of this result, see for instance Sect. 10 in [1].

System (1.1) has been used over the last years in the modeling of turbulent flows, including for numerical simulations (see [16], [10], [7], [9]). However, the well-posedness of (1.1) in dimension $d \geq 2$ is still uncertain today. Theorem 1 states an answer to this question in the case of a vanishing pressure. One could also obtain system (1.1) with $p' = 0$ when modeling a fluid for which the pressure gradient ∇p is negligible compared to the turbulence of the fluid $\mathrm{div}(\rho P)$ in (1.1b).

The hypothesis $p' = 0$ can also be found in the litterature, in a model called "pressureless gas dynamics". The pressureless Euler equations are obtained from the Euler equations with a pressure assumed to be 0. The pressureless gas dynamics are used to model astrophysical systems (see [19] for instance) and is only weakly hyperbolic. As a consequence, phenomena like creation of vacuum or high concentrations (delta shocks) can occur (see for instance [3], [4], [2]). System (1.1) with $p' = 0$ can thus be seen as a hyperbolic version of the pressureless gas model. In this sense, the Reynolds tensor P brings more regularity to the model.

2. Proof of the proposition

Notations. We denote $\partial_i := \partial/\partial x_i$ the partial derivative of the variable x_i, for $1 \leq i \leq d$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a scalar function, we denote by $\nabla f \in \mathbb{R}^d$ the gradient of f, i.e. the vector field of components $\partial_i f$, $1 \leq i \leq d$.

If $Z = (Z_1, \dots, Z_n) : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, the divergence of Z is the scalar function defined by

$$\mathrm{div}(Z) := \partial_1 Z_1 + \cdots + \partial_n Z_n.$$

We also denote by $\partial Z/\partial x$ the Jacobian matrix of Z, i.e. the matrix of coefficients $(\partial Z/\partial x)_{i,j} = \partial Z_i/\partial x_j$, for $1 \leq i, j \leq d$.

If $Z = (Z_i)_{1 \leq i \leq d}$ and $Z' = (Z_i')_{1 \leq i \leq d}$ are two vector fields, we denote $Z \otimes Z'$ the second order tensor defined by $Z \otimes Z' := Z(Z')^T$, i.e. the matrix of coefficients $(Z \otimes Z')_{i,j} = Z_i Z_j'$, for $1 \leq i, j \leq d$.

If $A : \mathbb{R}^d \to M_d(\mathbb{R})$ is a second order tensor, we defined the divergence of A as the line vector of $\mathbb{R}^d$ whose i-th component is given by the divergence of the i-th column of A.

For any positive integer d, we denote $I_d \in M_d(\mathbb{R})$ the identity matrix of size d. We denote $S_d^{++}(\mathbb{R})$ the set of symmetric definite positive matrices, i.e. the symmetric matrices of size d with a positive spectrum.

We now give the proof of Proposition 1.

Proof. We write system (1.1) in matricial form:

$$\partial_t Y + A(Y,\nabla)Y = 0, \text{ with } Y := \begin{pmatrix} \rho \\ v \\ \tilde{P} \end{pmatrix} \in \mathbb{R}^{1+d+d(d+1)/2}$$

and, if $\xi = (\xi_i)_{1\le i\le d} \in \mathbb{R}^d$,

$$A(Y,\xi) := \begin{pmatrix} v\cdot\xi & \rho\xi^T & 0 \\ \frac{1}{\rho}(p'(\rho)I_d + P)\xi & (v\cdot\xi)I_d & C(\xi) \\ 0 & D(\xi) & (v\cdot\xi)I_{d(d+1)/2} \end{pmatrix}. \tag{2.1}$$

When $d=2$, the symmetric matrix $P = (P_{ij})_{1\le i,j\le 2}$ can be identified as a vector $\tilde{P} \in \mathbb{R}^3$ given by

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix}.$$

The matrices $C(\xi)$ and $D(\xi)$ are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \text{ and } D(\xi) := \begin{pmatrix} 2P_{11}\xi_1 + 2P_{12}\xi_2 & 0 \\ P_{21}\xi_1 + P_{22}\xi_2 & P_{11}\xi_1 + P_{12}\xi_2 \\ 0 & 2P_{12}\xi_1 + 2P_{22}\xi_2 \end{pmatrix}.$$

When $d=3$, the symmetric matrix P can be identified as a vector $\tilde{P} \in \mathbb{R}^6$:

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix}.$$

The matrices $C(\xi)$ and $D(\xi)$ are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}$$

and

$$D(\xi) := \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix},$$

where $(P\xi)_i$ denotes the i-th component of the vector $P\xi$.

We first show the implication (2) $\Rightarrow$ (1)**.** Namely, if $p'=0$ or $P=\lambda I_d$, then system (1.1) is Friedrichs-symmetrizable.

We thus suppose that $p'=0$ or $P=\lambda I_d$.

Consider $S=S(Y)\in S_n^{++}(\mathbb{R})$ (with $n=(d+1)(d+2)/2$) defined as a block matrix, compatible with A:

$$S=\begin{pmatrix}1 & 0 & 0\\ 0 & S_2 & 0\\ 0 & 0 & S_3\end{pmatrix},\ \text{ with }\ S_2\in M_d(\mathbb{R})\ \text{ and }\ S_3\in M_{d(d+1)/2}(\mathbb{R}). \tag{2.2}$$

Note that S is symmetric definite positive if and only if the matrices S_2 and S_3 are also symmetric definite positive.

We can now compute the product SA by block matrix mutiplication. We obtain that

$$SA=\begin{pmatrix} v\cdot\xi & \rho\xi^T & 0\\ \frac{1}{\rho}S_2(p'(\rho)Id+P)\xi & (v\cdot\xi)S_2 & S_2C(\xi)\\ 0 & S_3D(\xi) & (v\cdot\xi)S_3\end{pmatrix} \tag{2.3}$$

Case $d=2$. Let us choose

$$S_2:=\mu P^{-1}\ \text{ and }\ S_3:=\mu\begin{pmatrix}\frac12 q_{11}^2 & q_{12}q_{11} & \frac12 q_{12}^2\\ q_{12}q_{11} & q_{11}q_{22}+q_{12}^2 & q_{12}q_{22}\\ \frac12 q_{12}^2 & q_{12}q_{22} & \frac12 q_{22}^2\end{pmatrix},$$

where we denoted

$$P^{-1}=\begin{pmatrix}q_{11} & q_{12}\\ q_{12} & q_{22}\end{pmatrix}\in S_2^{++}(\mathbb{R}), \tag{2.4}$$

and

$$\mu:=\begin{cases}\rho^2 & \text{when }\ p'(\rho)=0,\\ \rho^2\dfrac{\lambda}{p'(\rho)+\lambda} & \text{when }\ P=\lambda Id.\end{cases} \tag{2.5}$$

The matrix S_2 is symmetric definite positive.

We see that the principal minors of $\mu^{-1}S_3$ are given by $M_1=\frac12 q_{11}^2>0$, $M_2=\frac12 q_{11}^2(q_{11}q_{22}-q_{12}^2)>0$, and $M_3=\frac14(q_{11}q_{22}-q_{12}^2)^3>0$ (recall that P is positive definite). Hence by Sylvester's criterion, S_3 is definite positive, and S, defined by (2.2), is a positive definite matrix.

We compute that

$$\frac{1}{\rho}S_2\left(p'(\rho)Id+P\right)\xi=\rho\xi, \tag{2.6}$$

$$S_2C(\xi)=\mu\begin{pmatrix}q_{11}\xi_1 & q_{11}\xi_2+q_{12}\xi_1 & q_{12}\xi_2\\ q_{12}\xi_1 & q_{12}\xi_2+q_{22}\xi_1 & q_{22}\xi_2\end{pmatrix},$$

and, since $q_{i1}P_{1j}+q_{i2}P_{2j}=\delta_{ij}$ by (2.4),

$$S_3D(\xi)=\mu\begin{pmatrix}q_{11}\xi_1 & q_{12}\xi_1\\ q_{12}\xi_1+q_{11}\xi_2 & q_{22}\xi_1+q_{12}\xi_2\\ q_{12}\xi_2 & q_{22}\xi_2\end{pmatrix}=[S_2C(\xi)]^T.$$

Hence for any $Y\in\mathcal{U}$, and for any $\xi\in\mathbb{R}^2$, the matrix $S(Y)$ is symmetric definite positive and (2.3) shows that the matrix $S(Y)A(Y,\xi)$ is symmetric. As a consequence, (1.1) is Friedrichs-symmetrizable when $d=2$.

Case $d=3$. Define again S by equation (2.2). Choose $S_2=\mu P^{-1}$ with μ as in (2.5), such that S_2 is symmetric definite positive and (2.6) holds again. Define S_3 by

$$S_3 := \mu \begin{pmatrix} \frac{1}{2}q_{11}^2 & q_{11}q_{12} & q_{11}q_{13} & \frac{1}{2}q_{12}^2 & q_{12}q_{13} & \frac{1}{2}q_{13}^2 \\ q_{11}q_{12} & q_{11}q_{22}+q_{12}^2 & q_{11}q_{23}+q_{12}q_{13} & q_{12}q_{22} & q_{12}q_{23}+q_{22}q_{13} & q_{13}q_{23} \\ q_{11}q_{13} & q_{11}q_{23}+q_{13}q_{12} & q_{11}q_{33}+q_{13}^2 & q_{12}q_{23} & q_{12}q_{33}+q_{23}q_{13} & q_{13}q_{33} \\ \frac{1}{2}q_{12}^2 & q_{12}q_{22} & q_{12}q_{23} & \frac{1}{2}q_{22}^2 & q_{22}q_{23} & \frac{1}{2}q_{23}^2 \\ q_{12}q_{13} & q_{12}q_{23}+q_{13}q_{22} & q_{12}q_{33}+q_{13}q_{23} & q_{22}q_{23} & q_{22}q_{33}+q_{23}^2 & q_{23}q_{33} \\ \frac{1}{2}q_{13}^2 & q_{13}q_{23} & q_{13}q_{33} & \frac{1}{2}q_{23}^2 & q_{23}q_{33} & \frac{1}{2}q_{33}^2 \end{pmatrix},$$

where we denoted again

$$P^{-1} := \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}.$$

We see that S_3 is symmetric. Furthermore, the principal minors of $\mu^{-1}S_3$ are given by $M_1=q_{11}^2/2>0$, $M_2=q_{11}^2(q_{11}q_{22}-q_{12}^2)/2>0$, $M_3=q_{11}^3\det(P^{-1})/2>0$, $M_4=q_{11}(q_{11}q_{22}-q_{12}^2)^2\det(P^{-1})/4>0$, $M_5=(q_{11}q_{22}-q_{12}^2)^2\det(P^{-1})^2/4>0$ and $M_6=\det(P^{-1})^4/8>0$. Hence, by Sylvester's criterion, S_3 is definite positive and S is symmetric definite positive.

We also check that

$$S_3D(\xi)=[S_2C(\xi)]^T.$$

Hence equation (2.3) shows that SA is symmetric, and, consequently, (1.1) is Friedrichs-symmetrizable when $d=3$.

We now show that (1) $\Rightarrow$ (2)**.** Suppose that system (1.1) is Friedrichs-symmetrizable, i.e. there is a matrix S such that the product SA is symmetric.

We write S as a block matrix, compatible with A:

$$S := \begin{pmatrix} S_1 & \alpha & \beta \\ \alpha^T & S_2 & \gamma \\ \beta^T & \gamma^T & S_3 \end{pmatrix}.$$

We compute the product SA by block multiplication. Since SA is symmetric, for any block of SA, denoted $(SA)_{i,j}$ $(1\le i,j\le 3)$, we must have $(SA)_{i,j}=(SA)_{j,i}^T$. For $i=j=3$, we obtain the constraint

$$\gamma^TC(\xi)+(v\cdot\xi)S_3=\left[\gamma^TC(\xi)+(v\cdot\xi)S_3\right]^T.$$

Since S_3 is symmetric, we deduce that the product $\gamma^TC(\xi)$ is symmetric, for any $\xi\in\mathbb{R}^d$. By computing explicitely the product, we obtain that the only possibility is that $\gamma=0$.

For $i=1$ and $j=3$, we obtain the constraint

$$[\alpha C(\xi)+(v\cdot\xi)\beta]^T=(v\cdot\xi)\beta^T+\frac{1}{\rho}\gamma^T(p'(\rho)Id+P)\xi.$$

Since $\gamma=0$, we obtain that $\alpha C(\xi)=0$, for any $\xi\in\mathbb{R}^d$. By computing explicitely the product, we also obtain that $\alpha=0$. Hence S has to be of the form

$$S=\begin{pmatrix} S_1 & 0 & \beta \\ 0 & S_2 & 0 \\ \beta^T & 0 & S_3 \end{pmatrix}.$$

For $i = 2$ and $j = 3$, we obtain the constraint

$$\rho\beta^T\xi^T + S_3D(\xi) = [S_2C(\xi)]^T \tag{2.7}$$

Solving the linear system (2.7) for S_2, S_3 and β gives after some computations that there exists two constants λ_1, λ_2 such that

$$S_2 = \lambda_1 P^{-1} \text{ and } \beta = \begin{cases} \lambda_2(q_{11}, 2q_{12}, q_{22}) \text{ when } d = 2, \\ \lambda_2(q_{11}, 2q_{12}, 2q_{13}, q_{22}, 2q_{23}, q_{33}) \text{ when } d = 3. \end{cases} \tag{2.8}$$

Note that it follows from these computations that $\beta D(\xi) = 2\lambda_2\xi^T$.

For $i = 1$ and $j = 2$ we obtain the constraint

$$\frac{1}{\rho}S_2(p'(\rho)Id + P)\xi = \left[S_1\rho\xi^T + \beta D(\xi)\right]^T = (S_1\rho + 2\lambda_2)\xi. \tag{2.9}$$

Equation (2.9) implies that S_2 is proportional to $(p'(\rho)Id + P)^{-1}$. Since S_2 is also proportional to P^{-1} by (2.8), we obtain that the two matrices P and $P + p'(\rho)Id$ are proportional (recall that S_2 is invertible). Hence $p' = 0$ or $P = \lambda Id$.

Declarations

- Conflict Of Interest statement: there is no competing interest.
- Data availability statement: there is no associated data.

□

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