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Some properties for the fifth-order Camassa-Holm type equation

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Abstract

In this paper, we study several problems of the fifth-order Camassa-Holm type (FOCHT) equation. The local well-posedness and blow-up scenario are established at first. Then we prove the global existence under some conditions and analyze the large-time behavior of the support of momentum density. Finally, we discuss the persistence property in Sobolev space.

KEYWORDS:
The fifth-order Camassa-Holm type equation; blow-up criterion; global existence; large time behavior; persistence property

1 | INTRODUCTION

The well-known Camassa-Holm (CH) equation was introduced by Camassa and Holm[1] to model the shallow water waves. Later, the Degasperis-Procesi (DP) equation was discovered by Degasperis and Procesi[2] when they were searching for integrable systems in similar forms as the CH equation. These equations possess many common properties such as integrability and the existence of Lax pair and explicit solutions, including the classical soliton, cuspon, and peakon solutions.

It is well-known that the CH equation is completely integrable and has many useful properties, such as conservation laws[3,4]. About the physical relevance of the CH and DP equations, we suggest the reference book written by Constantin and Lannes[5]. For the CH equation, the local well-posedness in H^s space with $s > \frac{3}{2}$ was proved[6,7] and the blow-up scenario was obtained[6,7,8,9,10]. The global existence of solution was proved[11,12,13], orbital stability of peakon solution was proved in Constanin et al[14]. The persistence and unique continuity of the solution were obtained in[15,16]. The large-time behavior of the support of momentum density was studied in the same paper. Meanwhile, for the DP equation, there are a large number of studies on the well-posedness, global existence, and blow-up phenomena, see for example[17,18,19,20,21,22,23].

Finding integrable models is an important task in the theory of integrable systems and solitons. There are several ways to generalize the peakon models and obtain new integrable systems. One way to do that is by increasing the order of nonlinearity. For example, the CH and DP equations are typical peakon models with quadratic nonlinearities and the Fokas-Olver-Rosenau-Qiao (FORQ) equation[24,25,26] with cubic nonlinearities. Another way is by introducing new potential functions to form the so-called multi-component CH systems with quadratic or cubic nonlinearities[27].

The standard CH models were generalized to fifth-order equations:

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0, x \in \mathbb{R}, \\ m = 4(1-\partial_x^2)(1-\frac{1}{4}\partial_x^2)u, & t > 0, x \in \mathbb{R}, \end{cases}$$

by Holm and Hone[28] . They obtained a conservation law: $(m^{\frac{1}{b}})_t = -(m^{\frac{1}{b}}u)_x$. For the same model, the infinite propagation speed and asymptotic behavior were obtained in Han rt al[29]. Liu and Qiao[30] studied the peakon system with fifth-order derivatives

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0, x \in \mathbb{R}, \\ m = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u, & t > 0, x \in \mathbb{R}. \end{cases} \tag{1.1}$$

They obtained some interesting solutions of (1.1) including single pseudo-peakon solutions, two-peakon, and N-peakon interactional solutions. There are extensive studies on high-order CH type equations ([31,32,33,34]). Zhu, Cao, Jiang et al[35] established the local well-posedness and blow-up scenario for equation (1.1), then they proved global existence under different conditions and studied large time behavior of the support of momentum density. For another fifth-order CH equation

$$\begin{cases} m_t + um_x + 2u_x m = 0, & t > 0, x \in \mathbb{R}, \\ m = u - \alpha u_{xx} + u_{xxxx}, & t > 0, x \in \mathbb{R}, \end{cases}$$

the local well-posedness for $\alpha = 1$ was proved in Sobolev space H^s with $s > \frac{9}{2}$ by Kato's theory in Tian et al[34]. The stationary solution and general mild traveling solution for $\alpha = 1$ were considered in Ding et al[31]. The global existence and convergence of conservative solutions were studied[34,36], respectively. With $\alpha = 2$, Tang and Liu[37] proved the local well-posedness in the critical Besov space $B^{7/2}_{2,1}$, as well as the existence of peakon-like solution and ill-posedness in $B^{7/2}_{2,\infty}$.

In this paper, we consider the fifth-order Camassa-Holm type (FOCHT) equation with high-order nonlinearities:

$$\begin{cases} m_t + m_x u^k + bmu^{k-1}u_x = 0, & t > 0, x \in \mathbb{R}, \\ m = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0, m_0 := (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0, & t > 0, x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $b \in \mathbb{R}$, $k \in \mathbb{Z}^+$ $\alpha, \beta > 0$ are constants. Without loss of generality, we always assume $\alpha \geq \beta > 0$. To our knowledge, this paper is the first work that considers the fifth-order CH equation of degree k.

The organization of this paper is as follows. In section 2, the local well-posedness (Theorem 2.6), blow-up scenario (Theorem 3.1), and the global existence under different conditions (Theorem 3.4) are established. In section 3, we analyze the large-time behavior of the support of momentum density (Theorem 4.3, Theorem 4.4). Persistence property in Sobolev spaces (Theorem 5.5) is presented in section 4.

2 | LOCAL WELL-POSEDNESS

In this section, we present the local well-posedness of problem (1.2). In order to apply Kato's theory[38] to our problem, we prove some lemmas, which ensure that the conditions in Kato's theorem are satisfied.

Consider the abstract quasi-linear evolution equation:

$$\begin{cases} \frac{dv}{dt} + A(v)v = f(v), & t > 0, x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Let X and Y be two Hilbert spaces such that Y is continuously and densely embedded in X. Suppose $Q : X \to Y$ be a topological isomorphism. We use $L(Y,X)$ to denote the space of all bounded linear operators from Y to X and let $L(X) = L(X,X)$ be the space of linear operators from X to itself. We introduce the following assumptions.

(i) Suppose $A(y) \in L(Y,X)$ for all $y \in X$ and

$$\|\big(A(y) - A(z)\big)\omega\|_X \leq \mu_1 \|y - z\|_X \cdot \|\omega\|_Y, \quad \text{for any } y, z, \omega \in Y.$$

We further assume that $A(y) \in G\big(X, 1, \beta\big)$, where $G\big(X, 1, \beta\big)$, $\beta \in \mathbb{R}$, denotes the set of all linear operators A in X such that $-A$ generates a C_0-semigroup e^{-tA} satisfying $||e^{-tA}|| \leq Me^{\beta t}$ for some constant M and $t \geq 0$.

(ii) Let $B(y) = QA(y)Q^{-1} - A(y)$. Suppose that $B(y) \in L(X)$ is uniformly bounded for y belongs to any bounded sets in Y, and

$$\|\big(B(y) - B(z)\big)\omega\|_X \leq \mu_2 \|y - z\|_Y \|\omega\|_X, \quad y, z \in Y, \omega \in X.$$

(iii) Suppose f is X-Lipschitz continuous as an operator from X to X, and Y-Lipschitz continuous as an operator from Y to itself, i.e.

$$\|f(y) - f(z)\|_Y \leq \mu_3 \|y - z\|_Y, \quad y, z \in Y,$$
$$\|f(y) - f(z)\|_X \leq \mu_4 \|y - z\|_X, \quad y, z \in X.$$

Here μ_1, μ_2, μ_3 and μ_4 are constants depending only on $\max\{||y||_Y, ||z||_Y\}$.

Theorem 2.1 (Kato[38]). Assume that (i), (ii) and (iii) hold. For any given $v_0 \in Y$, there exists a unique solution $v(\cdot, v_0) \in C\big([0,T); Y\big) \cap C^1\big([0,T); X\big)$ to (2.1) for some $T > 0$, which depends only on $||v_0||_Y$. Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is continuous from Y to $C\big([0,T); Y\big) \cap C^1\big([0,T); X\big)$.

Problem (1.2) can be transformed into

$$\begin{cases} u_t + u^k u_x = -\left((1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)\right)^{-1} \\ \qquad \left(m_x u^k + bmu^{k-1}u_x - (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)(u^k u_x)\right), & t>0, x\in\mathbb{R}, \\ u(x,0)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{2.2}$$

Let $A(u) := u^k\partial_x$, $Q := \left((1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)\right)^{\frac{1}{4}}$. The operator $Q^{-4} = ((1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2))^{-1}$ can be expressed by its associated Green's function where

$$G(x) := \begin{cases} \dfrac{\alpha^2}{\alpha^2-\beta^2} g_1 - \dfrac{\beta^2}{\alpha^2-\beta^2} g_2, & \alpha\neq\beta, \\ \dfrac{1}{4\alpha} e^{-\frac{|x|}{\alpha}}(1+\dfrac{|x|}{\alpha}), & \alpha=\beta, \end{cases} \tag{2.3}$$

with $g_1 := \frac{1}{2\alpha}e^{-\frac{|x|}{\alpha}}$, $g_2 := \frac{1}{2\beta}e^{-\frac{|x|}{\beta}}$.

Then the right hand side of (2.2) can be reformulated as

$$\begin{aligned} f(u) :=& -G * \left(m_x u^k + bmu^{k-1}u_x - (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)(u^k u_x)\right) \\ =& -G * f_1(u) - \partial_x G * f_2(u) - \partial_x^2 G * f_3(u), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} f_1(u) =& bu^k u_x + (3k-b)(\alpha^2+\beta^2)u^{k-1}u_x u_{xx} + k(k-1)(\alpha^2+\beta^2)u^{k-2}u_x^3 \\ & - k(k-1)(k-2)(k-3)\alpha^2\beta^2 u^{k-4}u_x^5 + (b-5k)(k-1)(k-2)\alpha^2\beta^2 u^{k-3}u_x^3 u_{xx} \\ & + \frac{5}{2}(k-1)(b-k)\alpha^2\beta^2 u^{k-2}u_x u_{xx}^2, \\ f_2(u) =& -\frac{b-5k}{k}\alpha^2\beta^2 (u^k)_{xx}u_{xx} - (b+5k)\alpha^2\beta^2(u^{k-1})_x u_x u_{xx} - \frac{b+5k}{2}\alpha^2\beta^2 u^{k-1}u_{xx}^2, \\ f_3(u) =& \frac{b-5k}{k}\alpha^2\beta^2(u^k)_x u_{xx}. \end{aligned}$$

Note that f_1, f_2 and f_3 have at most second-order derivatives of u. Let $Y = H^s$, $X = H^{s-1}$, and $Q = [(1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)]^{1/4}$. Obviously, Q is an isomorphism from H^s onto H^{s-1}. In order to apply Theorem 2.1 to obtain local well-posedness of (2.2), we only need to verify that $A(u)$ and $f(u)$ satisfy conditions (i)-(iii). The following four lemmas aims to verify these conditions.

Lemma 2.2. The operator $A(u) = u^k\partial_x$ with $u\in H^s, > 3/2$, belongs to $G(H^{s-1}, 1, \beta)$ for some $\beta > 0$.

Proof. Note that H^s is a Banach algebra for any $s > 1/2$. So $u^k \in H^s$ for any $u \in H^s$, $s > 1/2$, $k\in\mathbb{N}^+$. This lemma is a direct consequence of Lemma 2.7 in Li et al[35]. □

Lemma 2.3. Let $A(u) = u^k\partial_x$, $u\in H^s$, $s > 3/2$ be given. Then $A(u)\in L(H^s, H^{s-1})$ and for any $u, y, \omega\in H^s$, we have

$$\|(A(u)-A(y))\omega\|_{H^{s-1}} \le C\|u-y\|_{H^{s-1}}\|\omega\|_{H^s}.$$

Proof. Note that H^{s-1} is a Banach algebra for $s > 3/2$ and

$$(A(u)-A(y))\omega = (u^k - y^k)\partial_x\omega.$$

Then we have

$$\|(A(u)-A(y))\omega\|_{H^{s-1}} = \|(u^k-y^k)\partial_x\omega\|_{H^{s-1}} \le C\|u^k-y^k\|_{H^{s-1}}\|\omega\|_{H^s}.$$

Due to the fact that for any $k_1, k_2\in\mathbb{N}$, $\|u^{k_1}y^{k_2}\|_{H^{s-1}} \le \|u^{k_1}\|_{H^{s-1}}\|y^{k_2}\|_{H^{s-1}} \le \|u\|_{H^{s-1}}^{k_1}\|y\|_{H^{s-1}}^{k_2}$, we can get

$$\begin{aligned} \|u^k-y^k\|_{H^{s-1}} =& \|(u-y)(u^{k-1}+u^{k-2}y+\cdots+y^{k-1})\|_{H^{s-1}} \\ \le& C\|u-y\|_{H^{s-1}}\left(\|u^{k-1}\|_{H^{s-1}}\| + \|u^{k-2}y\|_{H^{s-1}} + \cdots + \|y^{k-1}\|_{H^{s-1}}\right) \le C\|u-y\|_{H^{s-1}}. \end{aligned}$$

So we have

$$\|(A(u)-A(y))\omega\|_{H^{s-1}} \le C\|u-y\|_{H^{s-1}}\|\omega\|_{H^s}.$$

Taking $y = 0$ in the above inequality, we obtain that $A(u)\in L(H^s, H^{s-1})$. This completes the proof of this lemma. □

Lemma 2.4. Let $B(u) := QA(u)Q^{-1} - A(u)$ where $A(u) = u^k\partial_x$, $Q = \left((1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)\right)^{1/4}$, $u \in H^s$, $s > 3/2$. Then $B(u) \in L(H^{s-1})$, and for any $u, y \in H^s, \omega \in H^{s-1}$, we have

$$\|(B(u)-B(y))\omega\|_{H^{s-1}} \le C\|u-y\|_{H^s}\|\omega\|_{H^{s-1}}.$$

Proof. By definition of B, we know

$$(B(u)-B(y))\omega = Q(u^k-y^k)Q^{-1}\partial_x\omega - (u^k-y^k)\partial_x\omega = [Q, u^k-y^k]Q^{-1}\partial_x\omega.$$

Then we have

$$\begin{aligned}\|(B(u)-B(y))\omega\|_{H^{s-1}} =&\|[Q,u^k-y^k]Q^{-1}\partial_x\omega\|_{H^{s-1}} = \|Q^{s-1}[Q,u^k-y^k]Q^{1-s}Q^{s-2}\partial_x\omega\|_{L^2}\\ \le& C\|Q^{s-1}[Q,u^k-y^k]Q^{1-s}\|_{L(L^2)}\|Q^{s-1}\omega\|_{L^2} \le C\|u^k-y^k\|_{H^s}\|\omega\|_{H^{s-1}}\\ \le& C\|u-y\|_{H^s}\|\omega\|_{H^{s-1}},\end{aligned}$$

where we applied Lemma 2.2 in Yin[39]. Taking $z = 0$ in the above inequality, we obtain $B(u) \in L(H^{s-1})$. This completes the proof of Lemma 2.4. □

Lemma 2.5. Let $f(u)$ be given by (2.4), $u \in H^s, s > 7/2$, then we have
(i) $\|f(u)-f(v)\|_{H^{s-1}} \le C\|u-v\|_{H^{s-1}}$,
(ii) $\|f(u)-f(v)\|_{H^s} \le C\|u-v\|_{H^s}$.

Proof. From the expression of f, we have

$$f(u)-f(v) = -G*(f_1(u)-f_1(v)) - \partial_x G*(f_2(u)-f_2(v)) - \partial_x^2 G*(f_3(u)-f_3(v))$$

We only prove (i), since the method to obtain (ii) is similar. We only estimate the last term $\partial_x^2 G * (f_3(u)-f_3(v))$ since other estimates can be obtained similarly.

$$\begin{aligned}&\|\partial_x^2 G*(f_3(u)-f_3(v))\|_{H^{s-1}}\\ \le& C\|\partial_x^2 G*\left((u^k)_x u_{xx} - (v^k)_x v_{xx}\right)\|_{H^{s-1}} \le C\|(u^k)_x u_{xx} - (v^k)_x v_{xx}\|_{H^{s-3}}\\ \le& C\|(u^k)_x(u_{xx}-v_{xx})\|_{H^{s-3}} + C\|v_{xx}\left((u^k)_x-(v^k)_x\right)\|_{H^{s-3}}\\ \le& C\|u\|^k_{H^{s-2}}\|u-v\|_{H^{s-1}} + C\|v\|_{H^{s-1}}\|u^k-v^k\|_{H^{s-2}}\\ \le& C\|u-v\|_{H^{s-1}}.\end{aligned}$$

Here we have used the fact that H^{s-3} is a Banach algebra for $s > 7/2$. This completes the proof of this lemma. □

By Kato's theory, we obtain the following local well-posedness results.

Theorem 2.6. Let $u_0 \in H^s(\mathbb{R})$ with $s > 7/2$. Then there exists a constant $T > 0$ depending only on $\|u_0\|_{H^s}$, such that the FOCHT model (1.2) has a unique solution

$$u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Moreover, the map $u_0 \in H^s \mapsto u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.

3 | BLOW-UP SCENARIO AND GLOBAL EXISTENCE

Now we prove the blow-up scenario for solutions of (1.2).

Theorem 3.1. Let u be a solution of equation (1.2) with initial data $u_0 \in H^4(\mathbb{R})$. Suppose T be the maximal existence time of u.
(i) When $k < 2b$, then solution u blows up in finite time if and only if

$$\liminf_{t\to T^-}\inf_{x\in\mathbb{R}}(u^{k-1}u_x) = -\infty.$$

(ii) When $k > 2b$, then solution u blows up in finite time if and only if

$$\limsup_{t\to T^-}\sup_{x\in\mathbb{R}}(u^{k-1}u_x) = +\infty.$$

(iii) When $k = 2b$, then $T = +\infty$. Namely, solution u does not blow up within finite time.

Proof. From the second equation of (1.2),

$$\int_{\mathbb{R}} m^2 dx = \int_{\mathbb{R}} u^2 + 2(\alpha^2+\beta^2)u_x^2 + ((\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2)u_{xx}^2 + 2(\alpha^2+\beta^2)\alpha^2\beta^2 u_{xxx}^2 + (\alpha^2\beta^2)^2 u_{xxxx}^2 dx.$$

There exist constants c_1 and c_2, depending only on α and β, such that

$$c_1\|u\|_{H^4}^2 \le \|m\|_{L^2}^2 \le c_2\|u\|_{H^4}^2.$$

Since $u_0 \in H^4(\mathbb{R})$, we know $m_0 \in L^2(\mathbb{R})$. Multiply (1.2) by m, and integrate over $\mathbb{R}$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = (k-2b)\int_{\mathbb{R}} m^2 u^{k-1} u_x dx. \tag{3.1}$$

In the case $k < 2b$, we use contradiction argument to prove result (i).

On one hand, suppose for any $t \in (0,T]$.

$$\inf_{x\in\mathbb{R}} (u^{k-1}u_x) \ge -M,$$

for some $M > 0$. Then we have

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx \le (k-2b)\inf_{x\in\mathbb{R}}(u^{k-1}u_x)\int_{\mathbb{R}} m^2 dx \le -(k-2b)M\int_{\mathbb{R}} m^2 dx.$$

By Grönwall's inequality, we have

$$\|m\|_{L^2}^2 \le e^{-(k-2b)Mt}\|m_0\|_{L^2}^2.$$

Therefore, the L^2 norm of m, as well as H^4 norm of u, is bounded for finite T and $t \in (0,T]$. This contradicts the fact that T is the maximal time of existence.

On the other hand, the solution u does not blow up, that is $\|u\|_{H^4}$ is bounded, by Morrey's inequality, we have

$$\|u^{k-1}u_x\|_{L^\infty} \le \|u\|_{L^\infty}^{k-1}\|u_x\|_{L^\infty} \le C\|u\|_{H^4}^k < +\infty.$$

The result for $k > 2b$ can be proved by similar argument.

In the case $k = 2b$, $\|m\|_{L^2}$ is conserved by (3.1). Then $\|u\|_{H^4}$ and $\|u^{k-1}u_x\|_{L^\infty}$ are uniformly bounded for any $t \ge 0$. Hence $T = +\infty$. □

Before presenting global existence, we first show some conservation laws.

Lemma 3.2. Assume that $u_0 \in H^4(\mathbb{R})$ and u is a solution of equation (1.2) in its lifespan. Then for any nonzero b, it holds that

$$\int_{\mathbb{R}} m^{k/b} dx = \int_{\mathbb{R}} m_0^{k/b} dx, \qquad \int_{\mathbb{R}} |m|^{k/b} dx = \int_{\mathbb{R}} |m_0|^{k/b} dx. \tag{3.2}$$

Moreover, when $k = b-1$, we have

$$\int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)u_x^2 + \alpha^2\beta^2 u_{xx}^2 dx = \int_{\mathbb{R}} u_0^2 + (\alpha^2+\beta^2)u_{0x}^2 + \alpha^2\beta^2 u_{0xx}^2 dx. \tag{3.3}$$

Proof. We first prove (3.2). Let q be the particle trajectory satisfying

$$\begin{cases} q_t = u^k(q,t), & 0 < t < T, x \in \mathbb{R}, \\ q(x,0) = x, & x \in \mathbb{R}, \end{cases} \tag{3.4}$$

where T is the lifespan of solution u. Take derivative of (3.4) with respect to x, we obtain

$$\frac{dq_t}{dx} = q_{xt} = ku^{k-1}(q,t)u_x(q,t)q_x, \qquad t \in (0,T).$$

Therefore,

$$\begin{cases} q_x = \exp\left(\int_0^t ku^{k-1}(q,s)u_x(q,s)ds\right), & 0 < t < T, \quad x \in \mathbb{R}, \\ q_x(x,0) = 1, & x \in \mathbb{R}. \end{cases}$$

Since q_x is always positive before blow-up, $q(x,t)$ is increasing with respect to x and trajectories never coincide before blow-up. In fact, direct calculation yields

$$\frac{d}{dt}\Big(m(q(x,t),t)q_x^{b/k}(x,t)\Big) = \Big(m_t(q,t) + u^k(q,t)m_x(q,t) + bu^{k-1}u_x(q,t)m(q,t)\Big)q_x^{b/k} = 0.$$

Hence,

$$m(q(x,t),t)q_x^{k/b}(x,t) = m_0(x), \qquad 0 < t < T, x \in \mathbb{R}. \tag{3.5}$$

It follows for any nonzero b that

$$\int_{\mathbb{R}} m_0^{k/b} dx = \int_{\mathbb{R}} m^{k/b}((q(x,t),t)q_x(x,t)dx = \int_{\mathbb{R}} m^{k/b} dx,$$

$$\int_{\mathbb{R}} |m_0|^{k/b} dx = \int_{\mathbb{R}} |m|^{k/b}((q(x,t),t)q_x(x,t)dx = \int_{\mathbb{R}} |m|^{k/b} dx.$$

Hence equation (3.2) holds.

Now we prove (3.3) for $k = b-1$. Take derivative of the left hand side of (3.3) with respect to t, use integration by parts twice, then we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)u_x^2 + \alpha^2\beta^2 u_{xx}^2 dx \\
=&2\int_{\mathbb{R}} u\Big(u_t - (\alpha^2+\beta^2)u_{xxt} + \alpha^2\beta^2 u_{xxxxt}\Big)dx \\
=&2\int_{\mathbb{R}} m_t u dx = -2\int_{\mathbb{R}} \Big(m_x u^{k+1} + bmu^k u_x\Big)dx \\
=&2(k+1-b)\int_{\mathbb{R}} mu^k u_x dx = 0.
\end{aligned}$$

This completes the proof of Lemma 3.2. ☐

Remark 3.3. The proof of conservation law (3.2) can also be achieved through direct computation:

$$\frac{d}{dt}\Big(\int_{\mathbb{R}} m^{k/b} dx\Big) = \frac{d}{dt}\Big(\int_{\mathbb{R}} |m|^{k/b} dx\Big) = 0.$$

Our proof of (3.2) in the lemma illustrates pointwise relations along trajectories.

Since $u(x,t) = G * m$, G is given in (2.3), u and u_x can be presented as

$$u(x,t) = \begin{cases} \displaystyle\int_{\mathbb{R}} \Big(\frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{|x-\xi|}{\alpha}} - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{|x-\xi|}{\beta}}\Big)m(\xi,t)d\xi, & \alpha \neq \beta, \\ \displaystyle\frac{1}{4\alpha}\int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}}\Big(1 + \frac{|x-\xi|}{\alpha}\Big)m(\xi,t)d\xi, & \alpha = \beta, \end{cases}$$

$$= \begin{cases} \displaystyle\frac{\alpha}{2(\alpha^2-\beta^2)}\Big(e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{\frac{x}{\alpha}}\int_{x}^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi\Big) & \\ \displaystyle\qquad - \frac{\beta}{2(\alpha^2-\beta^2)}\Big(e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi + e^{\frac{x}{\beta}}\int_{x}^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\Big), & \alpha \neq \beta, \\ \displaystyle\frac{1}{4\alpha}\int_{-\infty}^{x} e^{-\frac{x-\xi}{\alpha}}\Big(1 + \frac{x-\xi}{\alpha}\Big)m(\xi,t)d\xi + \frac{1}{4\alpha}\int_{x}^{+\infty} e^{-\frac{\xi-x}{\alpha}}\Big(1 + \frac{\xi-x}{\alpha}\Big)m(\xi,t)d\xi, & \alpha = \beta, \end{cases} \tag{3.6}$$

and

$$
u_x(x,t)=\begin{cases}
\frac{1}{2(\alpha^2-\beta^2)}\Big(e^{\frac{x}{\alpha}}\int\limits_x^{+\infty} e^{-\frac{\xi}{\alpha}}m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int\limits_{-\infty}^{x} e^{\frac{\xi}{\alpha}}m(\xi,t)d\xi\Big) \\
\qquad + \frac{1}{2(\alpha^2-\beta^2)}\Big(e^{-\frac{x}{\beta}}\int\limits_{-\infty}^{x} e^{\frac{\xi}{\beta}}m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int\limits_x^{+\infty} e^{-\frac{\xi}{\beta}}m(\xi,t)d\xi\Big), & \alpha\neq\beta, \\
-\frac{1}{4\alpha^2}\int\limits_{-\infty}^{x} e^{-\frac{x-\xi}{\alpha}}\frac{x-\xi}{\alpha}m(\xi,t)d\xi + \frac{1}{4\alpha^2}\int\limits_x^{+\infty} e^{-\frac{\xi-x}{\alpha}}\frac{\xi-x}{\alpha}m(\xi,t)d\xi, & \alpha=\beta.
\end{cases}
\tag{3.7}
$$

Theorem 3.4. Assume that $u_0(x)\in H^4(\mathbb{R})$, b, k and the initial momentum density satisfy one of the following three conditions:
(i) $k=2b$,
(ii) $k=b-1$,
(iii) $0<b\le k$ and $m_0\in L^{k/b}$.
Then equation (1.2) possesses at least one global in time solution.

Proof. In order to prove global existence, we only need to establish the boundedness of $u^{k-1}u_x$.

(i) When $k=2b$, global existence is a direct consequence of local existence and the blow-up scenario (iii) of Theorem 3.1.

(ii) Suppose $k=b-1$. By conservation law (3.3) and Sobolev embedding, we have

$$
\begin{aligned}
\|u^{k-1}u_x\|_{L^\infty}\le\|u\|_{H^2}^k &\le C(\alpha,\beta)\Big(\int\limits_{\mathbb{R}} u^2+(\alpha^2+\beta^2)u_x^2+\alpha^2\beta^2u_{xx}^2dx\Big)^{k/2}\\
&= C(\alpha,\beta)\Big(\int\limits_{\mathbb{R}} u_0^2+(\alpha^2+\beta^2)u_{0x}^2+\alpha^2\beta^2u_{0xx}^2dx\Big)^{k/2}.
\end{aligned}
$$

(iii) Suppose $0<b\le k$ and $m_0\in L^{k/b}$. The proof will be divided into two parts: $\alpha>\beta>0$ and $\alpha=\beta>0$.

a) We first consider the case $\alpha>\beta>0$. When $b=k$, we have from (3.2) that

$$
\int\limits_{\mathbb{R}}|m|dx=\int\limits_{\mathbb{R}}|m_0|dx. \tag{3.8}
$$

From (3.6) and (3.8), it is easy to see that

$$
\begin{aligned}
|u|\le&\frac{\alpha}{2(\alpha^2-\beta^2)}\Big(\int\limits_{-\infty}^{x}e^{\frac{\xi-x}{\alpha}}|m|d\xi+\int\limits_x^{+\infty}e^{\frac{x-\xi}{\alpha}}|m|d\xi\Big)+\frac{\beta}{2(\alpha^2-\beta^2)}\Big(\int\limits_{-\infty}^{x}e^{\frac{\xi-x}{\beta}}|m|d\xi+\int\limits_x^{+\infty}e^{\frac{x-\xi}{\beta}}|m|d\xi\Big)\\
\le&\Big(\frac{\alpha}{2(\alpha^2-\beta^2)}+\frac{\beta}{2(\alpha^2-\beta^2)}\Big)\int\limits_{\mathbb{R}}|m|d\xi\\
\le&\frac{1}{2(\alpha-\beta)}\int\limits_{\mathbb{R}}|m_0|dx.
\end{aligned}
$$

Similarly, we have by (3.7) and (3.8) that

$$
|u_x|\le\frac{1}{\alpha^2-\beta^2}\int\limits_{\mathbb{R}}|m_0|dx.
$$

When $0<b<k$, we first notice that

$$
\int\limits_{-\infty}^{x}e^{\frac{\xi-x}{\alpha}\cdot\frac{k}{k-b}}d\xi=\frac{\alpha(k-b)}{k}=\int\limits_x^{+\infty}e^{\frac{x-\xi}{\alpha}\cdot\frac{k}{k-b}}d\xi.
$$

Hence by (3.2), (3.6) and Hölder's inequality, we know that

$$
\begin{aligned}
|u| &\le \frac{\alpha}{2(\alpha^2-\beta^2)}\Big(\int_{-\infty}^{x} e^{\frac{\xi-x}{\alpha}}|m|d\xi+\int_{x}^{+\infty} e^{\frac{x-\xi}{\alpha}}|m|d\xi\Big)+\frac{\beta}{2(\alpha^2-\beta^2)}\Big(\int_{-\infty}^{x} e^{\frac{\xi-x}{\beta}}|m|d\xi+\int_{x}^{+\infty} e^{\frac{x-\xi}{\beta}}|m|d\xi\Big)\\
&\le \frac{\alpha}{2(\alpha^2-\beta^2)}\Big(\frac{\alpha(k-b)}{k}\Big)^{\frac{k-b}{k}}\Big(\int_{\mathbb{R}}|m|^{k/b}dx\Big)^{b/k}+\frac{\beta}{2(\alpha^2-\beta^2)}\Big(\frac{\beta(k-b)}{k}\Big)^{\frac{k-b}{k}}\Big(\int_{\mathbb{R}}|m|^{k/b}dx\Big)^{b/k}\\
&\le \frac{1}{2(\alpha-\beta)}\Big(\alpha\frac{k-b}{k}\Big)^{\frac{k-b}{k}}\Big(\int_{\mathbb{R}}|m_0|^{k/b}dx\Big)^{b/k}.
\end{aligned}
\tag{3.9}
$$

Similarly, it can be proved by (3.2), (3.7) and Hölder's inequality that

$$
|u_x|\le \frac{1}{(\alpha^2-\beta^2)}\Big(\alpha\frac{k-b}{k}\Big)^{k-b/k}\Big(\int_{\mathbb{R}}|m_0|^{k/b}dx\Big)^{b/k}.
$$

Local existence result together with boundedness of u and u_x implies that the global solution exists.

b) Now we consider the case $\alpha=\beta>0$. When $b=k$, note that $\sup_{x\in\mathbb{R}} e^{-|x|}|x|=\frac{1}{e}$. We have by (3.6)-(3.8) that

$$
|u|\le \frac{1}{4\alpha}\big(1+\frac{1}{e}\big)\int_{\mathbb{R}}|m_0|dy,\qquad |u_x|\le\frac{1}{4e\alpha^2}\int_{\mathbb{R}}|m_0|dy.
$$

When $0<b<k$, we have by (3.6) that

$$
\begin{aligned}
u=&\frac{1}{4\alpha}\int_{-\infty}^{x} e^{-\frac{x-y}{\alpha}}m(y)dy+\frac{1}{4\alpha}\int_{-\infty}^{x} e^{-\frac{x-y}{\alpha}}\frac{x-y}{\alpha}m(y)dy+\frac{1}{4\alpha}\int_{x}^{+\infty} e^{-\frac{y-x}{\alpha}}m(y)dy\\
&+\frac{1}{4\alpha}\int_{x}^{+\infty} e^{-\frac{y-x}{\alpha}}\frac{y-x}{\alpha}m(y)dy\\
=:&\ I_1+I_2+I_3+I_4.
\end{aligned}
$$

For I_1 and I_3, by similar argument as in (3.9), it is easy to derive that

$$
|I_1|+|I_3|\le\frac{1}{4\alpha}\big(\frac{\alpha(k-b)}{k}\big)^{k-b/k}\Big(\int_{\mathbb{R}}|m_0|^{k/b}dx\Big)^{b/k}.
$$

It remains to prove the boundedness of I_2 and I_4. We first obtain the following equality by changing of variables. Let $s=\frac{x-y}{\alpha}p$ for any $1<p<+\infty$. Then

$$
\int_{-\infty}^{x}\big(e^{-\frac{x-y}{\alpha}}\frac{x-y}{\alpha}\big)^p dy=\frac{\alpha}{p}\int_0^{+\infty}e^{-s}\big(\frac{s}{p}\big)^p ds=\frac{\alpha}{p^{p+1}}\int_0^{+\infty}e^{-s}s^p ds=\frac{\alpha}{p^{p+1}}\Gamma(p+1).
$$

Let $s=\frac{y-x}{\alpha}p$, $1<p<+\infty$. Then we have

$$
\int_{x}^{+\infty}\big(e^{-\frac{y-x}{\alpha}}\frac{y-x}{\alpha}\big)^p dy=\frac{\alpha}{p}\int_0^{+\infty}e^{-s}\big(\frac{s}{p}\big)^p ds=\frac{\alpha}{p^{p+1}}\int_0^{+\infty}e^{-s}s^p ds=\frac{\alpha}{p^{p+1}}\Gamma(p+1).
$$

Note that $\Gamma(p+1)$ is bounded for any fixed $p\in(1,+\infty)$. Hence we obtain by Hölder's inequality that

$$
|I_2|+|I_4|\le\frac{1}{4\alpha}\Big(\alpha\big(\frac{k-b}{k}\big)^{\frac{2k-b}{k-b}}\Gamma\big(\frac{2k-b}{k-b}\big)\Big)^{\frac{k-b}{k}}\big(\int_{\mathbb{R}}|m_0|^{k/b}dx\big)^{b/k}.
$$

Combining the above estimates on $I_i, i=1,2,3,4$, we obtain

$$
|u|\le C(\alpha,k,b)\Big(\int_{\mathbb{R}}|m_0|^{k/b}\Big)^{b/k},
$$

where C is a constant depending only on k, b and α. By similar argument as above, we can also obtain

$$|u_x| \leq C(\alpha, k, b)\Big(\int_{\mathbb{R}} |m_0|^{k/b} dy\Big)^{b/k}.$$

So we obtain the boundedness of u and u_x, which yields the global existence result. □

4 | LARGE TIME BEHAVIOR FOR THE SUPPORT OF THE MOMENTUM DENSITY

Let

$$E(t) := \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} |m(\xi, t)| d\xi, \qquad F(t) := \int_{\mathbb{R}} e^{-\frac{\xi}{\alpha}} |m(\xi, t)| d\xi, \tag{4.1}$$

$$E_\varepsilon(t) := \int_{\mathbb{R}} e^{\frac{(1-\varepsilon)\xi}{\alpha}} |m(\xi, t)| d\xi, \qquad F_\varepsilon(t) := \int_{\mathbb{R}} e^{-\frac{(1-\varepsilon)\xi}{\alpha}} |m(\xi, t)| d\xi. \tag{4.2}$$

Lemma 4.1. Assume (u, m) is a solution of (1.2), and q are trajectories given by (3.4). Suppose initial data $m_0 \not\equiv 0$ has compact support in $[a, c]$, and m_0 does not change sign on $\mathbb{R}$.

(1) If $\alpha > \beta > 0$, then u satisfies the following properties for any $t > 0$ in its lifespan:

$$\frac{1}{2(\alpha+\beta)} e^{-x/\alpha} E(t) < |u(x, t)| < \frac{\alpha}{2(\alpha^2 - \beta^2)} e^{-x/\alpha} E(t), \quad \text{for} \quad x > q(c, t), \tag{4.3}$$

$$\frac{1}{2(\alpha+\beta)} e^{x/\alpha} F(t) < |u(x, t)| < \frac{\alpha}{2(\alpha^2 - \beta^2)} e^{x/\alpha} F(t), \quad \text{for} \quad x < q(a, t). \tag{4.4}$$

(2) If $\alpha = \beta > 0$, then u satisfies the following properties for any $0 < \varepsilon < 1$ and $t > 0$ in its lifespan:

$$\frac{1}{4\alpha} e^{-x/\alpha} E(t) \leq |u(x, t)| \leq \frac{C(\varepsilon)}{4\alpha} e^{-(1-\varepsilon)x/\alpha} E_\varepsilon(t), \quad \text{for} \quad x > q(c, t), \tag{4.5}$$

$$\frac{1}{4\alpha} e^{x/\alpha} F(t) \leq |u(x, t)| \leq \frac{C(\varepsilon)}{4\alpha} e^{(1-\varepsilon)x/\alpha} F_\varepsilon(t), \quad \text{for} \quad x < q(a, t), \tag{4.6}$$

where $E(t)$, $F(t)$, $E_\varepsilon(t)$, $F_\varepsilon(t)$ given by (4.1) and (4.2) denote continuous non-vanishing functions, and $C(\epsilon)$ is a positive constant depending only on ϵ.

Proof. Since $m_0 \not\equiv 0$ has compact support set in $[a, c]$, we know from (3.5) that $m \not\equiv 0$, m does not change sign, and supp $m(x, t) \subset [q(a, t), q(c, t)]$ for any fixed $t > 0$.

(1) We first consider the case $\alpha > \beta > 0$. By (3.6), we have

$$u(x, t) = \int_{q(a,t)}^{q(c,t)} \Big(\frac{\alpha}{2(\alpha^2 - \beta^2)} e^{-\frac{|x-\xi|}{\alpha}} - \frac{\beta}{2(\alpha^2 - \beta^2)} e^{-\frac{|x-\xi|}{\beta}}\Big) m(\xi) d\xi.$$

It is easy to see that

$$0 < \frac{1}{2(\alpha+\beta)} e^{-\frac{|x-\xi|}{\alpha}} < \frac{\alpha}{2(\alpha^2 - \beta^2)} e^{-\frac{|x-\xi|}{\alpha}} - \frac{\beta}{2(\alpha^2 - \beta^2)} e^{-\frac{|x-\xi|}{\beta}} < \frac{\alpha}{2(\alpha^2 - \beta^2)} e^{-\frac{|x-\xi|}{\alpha}}.$$

Since m does not change sign, we have

$$\frac{1}{2(\alpha+\beta)} \int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}} |m(\xi)| d\xi < |u(x, t)| < \frac{\alpha}{2(\alpha^2 - \beta^2)} \int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}} |m(\xi)| d\xi.$$

Hence inequalities (4.3) and (4.4) holds.

(2) In the case $\alpha = \beta > 0$, we have by (3.6)

$$u = G * m = \frac{1}{4\alpha} \int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}} \big(1 + \frac{|x - \xi|}{\alpha}\big) m(\xi, t) d\xi,$$

Note that for any $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that

$$1 + \frac{|x-\xi|}{\alpha} < C(\varepsilon)e^{\varepsilon\frac{|x-\xi|}{\alpha}}.$$

Hence

$$e^{-\frac{|x-\xi|}{\alpha}} < e^{-\frac{|x-\xi|}{\alpha}}\left(1 + \frac{|x-\xi|}{\alpha}\right) < C(\varepsilon)e^{-(1-\varepsilon)\frac{|x-\xi|}{\alpha}}$$

Since m does not change sign, we have

$$\frac{1}{4\alpha}\int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}}|m(\xi,t)|d\xi < |u(x,t)| < \frac{C(\varepsilon)}{4\alpha}\int_{q(a,t)}^{q(c,t)} e^{-\frac{(1-\varepsilon)|x-\xi|}{\alpha}}|m(\xi,t)|d\xi.$$

Therefore (4.5) and (4.6) holds. □

Then we discuss the large time behavior for the support of momentum density of equation (1.2). The main idea comes from Jiang et al.[13] which solved the same problem for the Camassa-Holm equation.

Lemma 4.2. Let (u,m) be a solution of (1.2), and q be trajectories given by (3.4). Suppose that the initial data $m_0 \not\equiv 0$, supp $u_0 \subset [a,c]$ and m_0 does not change sign.

(1) If $m_0 \geq 0$, then

$$\lim_{t\to+\infty} F(t) = 0.$$

(2) Suppose $m_0 \leq 0$. Then $\lim_{t\to+\infty} E(t) = 0$ for k odd, $\lim_{t\to+\infty} F(t) = 0$ for k even.

Proof. Easy to see that $m_0 \not\equiv 0$, supp $m_0 \subset [a,c]$. We first consider the case $\alpha > \beta > 0$.

When $m_0(x) \geq 0$, we have from (3.5) that $m(x,t) \geq 0$. Hence $u = G * m \geq 0$. Use contradiction argument, we assume that

$$\lim_{t\to+\infty} F(t) \neq 0.$$

Since $F(t) > 0$, there exists a constant $\epsilon_0 > 0$, such that for any $T > 0$, there exists $t > T$, satisfying $F(t) \geq \epsilon_0$.

For $x < a$, from (3.4) and the first inequality in (4.4) we have

$$\frac{d}{dt}q(x,t) = u^k(q(x,t),t) \geq \frac{1}{2^k(\alpha+\beta)^k}e^{\frac{kq}{\alpha}}F^k(t) \geq \frac{1}{2^k(\alpha+\beta)^k}e^{\frac{kq}{\alpha}}\epsilon_0^k.$$

It follows that

$$e^{-\frac{kq}{\alpha}} \leq -\frac{k}{\alpha}\cdot\frac{\epsilon_0^k}{2^k(\alpha+\beta)^k}t + e^{-\frac{kx}{\alpha}}.$$

It is obvious that the right hand side becomes negative for sufficiently large t. This leads to a contradiction. Therefore $\lim_{t\to+\infty} F(t) = 0$ when $m_0 \geq 0$.

Suppose $m_0 \leq 0$, we have from (3.5) that $m(x,t) \leq 0$. Hence $u = G * m \leq 0$ for any $t \geq 0$. Assume k is odd and use contradiction argument, we assume that

$$\lim_{t\to+\infty} E(t) \neq 0.$$

Since $E(t) > 0$, there exists a constant $\epsilon_0 > 0$, such that for any $T > 0$, there exists $t > T$, satisfying $E(t) \geq \epsilon_0$.

For $x > c$, from (3.4) and the first inequality in (4.3) we have

$$\frac{d}{dt}q(x,t) = u^k(q(x,t),t) \leq \frac{-1}{2^k(\alpha+\beta)^k}e^{-\frac{kq}{\alpha}}E^k(t) \leq \frac{-1}{2^k(\alpha+\beta)^k}e^{-\frac{kq}{\alpha}}\epsilon_0^k.$$

It follows that

$$e^{\frac{kq}{\alpha}} \leq \frac{k}{\alpha}\cdot\frac{-1}{2^k(\alpha+\beta)^k}\epsilon_0^k t + e^{\frac{kx}{\alpha}}.$$

It is obvious that the right hand side becomes negative for sufficiently large t. This leads to a contradiction. Therefore $\lim_{t\to+\infty} E(t) = 0$ when $m_0 \leq 0$ and k is odd.

When k is even, assume that $\lim_{t\to+\infty} F(t) \neq 0$. There exists a constant $\epsilon_0 > 0$, such that for any $T > 0$, there exists $t > T$, satisfying $F(t) \geq \epsilon_0$. For $x > c$, from (3.4) and the first inequality in (4.4) we have

$$\frac{d}{dt}q(x,t) = u^k(q(x,t),t) \geq \frac{1}{2^k(\alpha+\beta)^k}e^{\frac{kq}{\alpha}}F^k(t) \geq \frac{1}{2^k(\alpha+\beta)^k}e^{\frac{kq}{\alpha}}\epsilon_0^k.$$

Similar argument will leads to a contradiction. Hence $\lim_{t\to+\infty} F(t) = 0$ when $m_0 \leq 0$ and k is even.

The case $\alpha = \beta > 0$ can be proved by similar argument as above, and we only make use of the first inequalities of (4.5) and (4.6). □

Theorem 4.3. Assume that (u, m) is a solution of (1.2), and q are trajectories given by (3.4). Suppose $k \le b$, m_0 has compact support in $[a, c]$, m_0 does not change sign and belongs to $L^{k/b}$.

(1) If $m_0(x) \ge 0$ or $m_0(x) \le 0$ and k is even, then

$$\lim_{t\to+\infty} q(c,t) = +\infty.$$

(2) If $m_0(x) \le 0$ and k is odd, then

$$\lim_{t\to+\infty} q(a,t) = -\infty.$$

Proof. (1) We first consider the case $k < b$. By conservation law (3.2), we have

$$\begin{aligned}\int_a^c |m_0|^{k/b} dx &= \int_{q(a,t)}^{q(c,t)} |m(\xi,t)|^{k/b} d\xi \\ &\le \int_{q(a,t)}^{q(c,t)} |m| e^{-\xi/\alpha} d\xi \cdot \Big(\int_{q(a,t)}^{q(c,t)} e^{\frac{kx}{\alpha(b-k)}} dx \Big)^{b-k/b} \\ &= F(t) \cdot \Big(\frac{\alpha(b-k)}{k} (e^{\frac{k}{\alpha(b-k)} q(c,t)} - e^{\frac{k}{\alpha(b-k)} q(a,t)}) \Big)^{b-k/b}.\end{aligned}$$

By Lemma 4.2, we know that $\lim_{t\to+\infty} F(t) = 0$ when $m_0 \ge 0$, or $m_0 \le 0$ and k is even. Hence

$$\lim_{t\to+\infty} e^{\frac{k}{\alpha(b-k)} q(c,t)} - e^{\frac{k}{\alpha(b-k)} q(a,t)} = +\infty.$$

Therefore, $\lim_{t\to+\infty} q(c,t) = +\infty$ when $m_0 \ge 0$, or $m_0 \le 0$ and k is even.

Similarly, when $m_0 \le 0$ and k is odd, we have

$$\begin{aligned}\int_a^c |m_0|^{k/b} dx &= \int_{q(a,t)}^{q(c,t)} |m(\xi,t)|^{k/b} d\xi \\ &\le \int_{q(a,t)}^{q(c,t)} |m| e^{\xi/\alpha} d\xi \cdot \Big(\int_{q(a,t)}^{q(c,t)} e^{-\frac{kx}{\alpha(b-k)}} dx \Big)^{b-k/k} \\ &= E(t) \cdot \Big(\frac{\alpha(b-k)}{k} (e^{-\frac{k}{\alpha(b-k)} q(a,t)} - e^{-\frac{k}{\alpha(b-k)} q(c,t)}) \Big)^{b-k/k}.\end{aligned}$$

We know from Lemma 4.2 that $\lim_{t\to+\infty} E(t) = 0$ when $m_0 \le 0$ and k is odd, hence

$$\lim_{t\to+\infty} e^{-\frac{k}{\alpha(b-k)} q(a,t)} - e^{-\frac{k}{\alpha(b-k)} q(c,t)} = +\infty.$$

Therefore, $\lim_{t\to+\infty} q(a,t) = -\infty$ when $m_0 \le 0$ and k is odd. Theorem holds for $k < b$.

(2) When $k = b$, by conservation law (3.2), we have

$$\int_a^c |m_0| dx = \int_{q(a,t)}^{q(c,t)} |m(\xi,t)| d\xi \le e^{\frac{q(c,t)}{\alpha}} \cdot \int_{q(a,t)}^{q(c,t)} |m| e^{-\frac{\xi}{\alpha}} d\xi = e^{\frac{q(c,t)}{\alpha}} \cdot F(t).$$

By Lemma 4.2, we know $\lim_{t\to+\infty} F(t) = 0$ when $m_0 \ge 0$, or $m_0 \le 0$ and k is even. Hence $\lim_{t\to+\infty} q(c,t) = +\infty$ when $m_0 \ge 0$, or $m_0 \le 0$ and k is even.

On the other hand, when $m_0 \le 0$ and k is odd, we have

$$\int_a^c |m_0| dx = \int_{q(a,t)}^{q(c,t)} |m(\xi,t)| d\xi \le e^{\frac{q(a,t)}{\alpha}} \cdot \int_{q(a,t)}^{q(c,t)} |m| e^{\frac{\xi}{\alpha}} d\xi = e^{\frac{q(a,t)}{\alpha}} \cdot E(t).$$

We know from Lemma 4.2 that $\lim_{t\to+\infty} E(t) = 0$ when $m_0 \le 0$ and k is odd, hence $\lim_{t\to+\infty} q(a,t) = -\infty$ in present case. Theorem holds for $k = b$. □

Theorem 4.4. Assume (u, m) is a solution of (1.2), and q are trajectories given by (3.4). Suppose $0 < b < k$, m_0 has compact support in $[a, c]$, m_0 does not change sign and belongs to $L^{k/b}$.

(1) If $m_0 \geq 0$ or $m_0 \leq 0$ and k is even, then

$$\lim_{t\to+\infty} q(c,t) - 2\alpha(k-b)\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds = +\infty.$$

(2) If $m_0 \leq 0$ and k is odd, then

$$\lim_{t\to+\infty} -q(a,t) - 2\alpha(k-b)\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds = +\infty.$$

Proof. From the proof of Theorem 3.4, we know that u and u_x are bounded for $0 < b < k$. Multiply sign(m) to the first line of (1.2), integrate with respect to x over $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |m(x,t)|dx &= -\int_{\mathbb{R}} \Big((|m|)_x u^k + b|m|u^{k-1}u_x\Big)dx \\ &= (k-b)\int_{\mathbb{R}} |m|u^{k-1}u_x dx \\ &\geq (k-b)\inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x \cdot \int_{\mathbb{R}} |m(x,t)|dx.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}} |m(x,t)|dx \geq e^{(k-b)\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds} \cdot \int_{\mathbb{R}} |m_0|dx. \tag{4.7}$$

By Hölder's inequality and conservation law (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}} |m(x,t)|dx &= \int_{\mathbb{R}} |m(x,t)|^{1/2}|m(x,t)|^{1/2}e^{-x/2\alpha}e^{x/2\alpha}dx \\ &\leq \Big(\int_{\mathbb{R}} |m(x,t)|^{k/b}dx\Big)^{b/2k} \cdot \Big(\int_{\mathbb{R}} |m(x,t)|e^{-x/2\alpha}dx\Big)^{1/2} \cdot \Big(\int_{\mathbb{R}} e^{\frac{kx}{\alpha(k-b)}}dx\Big)^{k-b/2k} \\ &= \Big(\int_{\mathbb{R}} |m_0|^{k/b}dx\Big)^{b/2k} \cdot F(t)^{1/2} \cdot \Big(\frac{\alpha(k-b)}{k}\big(e^{\frac{k}{\alpha(k-b)}q(c,t)} - e^{\frac{k}{\alpha(k-b)}q(a,t)}\big)\Big)^{k-b/2k}.\end{aligned} \tag{4.8}$$

Meanwhile, similar argument leads to

$$\begin{aligned}\int_{\mathbb{R}} |m(x,t)|dx &= \int_{\mathbb{R}} |m(x,t)|^{1/2}|m(x,t)|^{1/2}e^{x/2\alpha}e^{-x/2\alpha}dx \\ &\leq \Big(\int_{\mathbb{R}} |m(x,t)|^{k/b}dx\Big)^{b/2k} \cdot \Big(\int_{\mathbb{R}} |m(x,t)|e^{x/\alpha}dx\Big)^{1/2} \cdot \Big(\int_{\mathbb{R}} e^{-\frac{kx}{\alpha(k-b)}}dx\Big)^{k-b/2k} \\ &= \Big(\int_{\mathbb{R}} |m_0|^{k/b}dx\Big)^{b/2k} \cdot E(t)^{1/2} \cdot \Big(\frac{\alpha(k-b)}{k}\big(e^{-\frac{k}{\alpha(k-b)}q(a,t)} - e^{-\frac{k}{\alpha(k-b)}q(c,t)}\big)\Big)^{k-b/2k}.\end{aligned} \tag{4.9}$$

When $m_0 \geq 0$ or $m_0 \leq 0$ and k is even, we know $F(t)$ converges to zero as t goes to infinity from Lemma 4.2. Therefore, from (4.7) and (4.8), we obtain

$$\big(e^{\frac{k}{\alpha(k-b)}q(c,t)} - e^{\frac{k}{\alpha(k-b)}q(a,t)}\big) \cdot e^{-2k\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds} \to +\infty, \qquad \text{as} \quad t\to+\infty.$$

Hence,

$$q(c,t) - 2\alpha(k-b)\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds \to +\infty, \qquad \text{as} \quad t\to+\infty.$$

When $m_0 \leq 0$ and k is odd, we know $E(t)$ converges to zero as t goes to infinity from Lemma 4.2. By similar argument as above, we obtain from (4.7) and (4.9) that

$$\big(e^{-\frac{k}{\alpha(k-b)}q(a,t)} - e^{-\frac{k}{\alpha(k-b)}q(c,t)}\big) \cdot e^{-2k\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds} \to +\infty, \qquad \text{as} \quad t\to+\infty.$$

Hence,

$$-q(a,t) - 2\alpha(k-b)\int_0^t \inf_{x\in[q(a,t),q(c,t)]} u^{k-1}u_x ds \to +\infty, \qquad \text{as} \quad t \to +\infty.$$

The proof of theorem is finished. □

5 | PERSISTENCE PROPERTY

In this section, we build the persistence property for the solutions of (1.2) in weighted Sobolev spaces.

Definition 5.1. A non-negative function $\upsilon : \mathbb{R}^n \to \mathbb{R}$ is called *sub-multiplicative* if $\upsilon(x+y) \le \upsilon(x)\upsilon(y)$ holds for all $x, y \in \mathbb{R}^n$.

Definition 5.2. Given a sub-multiplicative function υ. A positive function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called *υ-moderate* if there exists a constant $C_0 > 0$ such that $\phi(x+y) \le C_0\upsilon(x)\phi(y)$ holds for all $x, y \in \mathbb{R}^n$.

It is proved in Brandolese[40] that ϕ is υ-moderate if and only if the weighted Young's inequality

$$\|(f_1 * f_2)\phi\|_{L^p} \le C_0\|f_1\upsilon\|_{L^1}\|f_2\phi\|_{L^p} \tag{5.1}$$

holds for any two measurable functions f_1, f_2 and $1 \le p \le \infty$.

Definition 5.3. We say that $\phi : \mathbb{R} \to (0, +\infty)$ is an *admissible weight* for (1.2) if the following properties hold:
i) ϕ is locally absolutely continuous,
ii) there exists a constant A such that $|\phi'(x)| \le A|\phi(x)|$ for almost all $x \in \mathbb{R}$,
iii) ϕ is v-moderate for a sub-multiplicative function v, which satisfies $\inf_{\mathbb{R}} \upsilon \ge \delta_0 > 0$ and

$$\int_{\mathbb{R}} \upsilon(x)e^{-\frac{|x|}{\max\{\alpha,\beta\}}}dx < M_0 \tag{5.2}$$

for some constants δ_0 and M_0.

Remark 5.4. The examples for admissible weight functions can be found in Tian et al.[34], such as

$$\phi(x) = \phi_{\alpha,\beta,\gamma,\delta}(x) = e^{\alpha|x|^\beta}(1+|x|)^\gamma \log(e+|x|^\delta),$$

where we require that $\alpha \ge 0$, $0 \le \beta \le 1$, $\alpha\beta < 1$.

Now we state the main result of this section.

Theorem 5.5. Let $u_0 \in H^s(\mathbb{R})$ with $s \ge 4$, and $u \in C([0,T); H^s(\mathbb{R}))\cap C^1([0,T); H^{s-1}(\mathbb{R}))$ be a strong solution to (1.2) starting from u_0. Suppose that ϕu_0, $\phi u_{0x} \in L^\infty(\mathbb{R})$ for an admissible weight function ϕ. Then the following estimate holds

$$\|\phi u(\cdot,t)\|_{L^\infty} + \|\phi u_x(\cdot,t)\|_{L^\infty} \le e^{CM^K t}\left(\|\phi u_0(\cdot)\|_{L^\infty} + \|\phi u_{0x}(\cdot)\|_{L^\infty}\right), \qquad t \in [0,T),$$

where constant C depends on α, β, b, k, functions υ, ϕ, and $M := \sup_{t\in[0,T)} \|u\|_{W^{4,\infty}}$.

This theorem asserts that if the initial data possesses some exponential decay as $|x|$ goes to infinity, then for any fixed $t \in [0,T)$ the solution u also possesses an exponential decay at infinity.

Proof. Rewrite (1.2) as

$$u_t + u^k u_x + G * \mathcal{F}(u) = 0. \tag{5.3}$$

where G is given by (2.3) and

$$\mathcal{F}(u) := m_x u^k + bmu^{k-1}u_x - (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)(u^k u_x).$$

has $(k+1)$ degree of nonlinearities on u, and up to fourth-order derivatives of u with respect to x. The coefficients of $\mathcal{F}(u)$ depend only on α, β, k and b.

For any $N \in \mathbb{R}^+$, we define the N-truncation of ϕ as $\phi_N(x) := \min\{\phi(x), N\}$. It is easy to check that $\phi_N : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function satisfying $\|\phi_N\|_{L^\infty} \le N$ and $|\phi_N'| \le A|\phi_N|$ for almost every $x \in \mathbb{R}$. Since ϕ is

v-moderate with $\inf_{\mathbb{R}} v > 0$, there exists a constant $C_0 > 0$ such that

$$\phi(x+y) \le C_0 v(x)\phi(y), \qquad x, y \in \mathbb{R}.$$

Hence it yields by choosing $\tilde{C_0} := \max\{C_0, \delta_0^{-1}\}$ that

$$\begin{aligned}\phi_N(x+y) = \min\{\phi(x+y), N\} &\le \min\{C_0 v(x)\phi(y), N\} \\ &\le \max\{C_0, \frac{1}{\inf_{\mathbb{R}} v}\} v(x) \min\{\phi(y), N\} \\ &\le \tilde{C_0} v(x)\phi_N(y), \qquad x, y \in \mathbb{R}.\end{aligned}$$

The N-truncation function ϕ_N is also v-moderate. Therefore, ϕ_N is an admissible weight.

From the definition of $\mathcal{F}(u)$, it is easy to check for any $p \in [1, +\infty]$ that

$$\begin{aligned}\|\phi_N \mathcal{F}(u)\|_{L^p} &\le C_1(\alpha, \beta, k, b)\Big(\sum_{i=0}^{4} \|\partial_x^i u\|_{L^\infty}^k\Big) \cdot \Big(\|\phi_N u\|_{L^p} + \|\phi_N u_x\|_{L^p}\Big) \\ &\le C_1(\alpha, \beta, k, b) M^k \Big(\|\phi_N u\|_{L^p} + \|\phi_N u_x\|_{L^p}\Big). \end{aligned} \tag{5.4}$$

Now we derive differential inequalities for $\phi_N u$ and $\phi_N u_x$ respectively. Multiplying (5.3) by $|\phi_N u|^{p-2}\phi_N^2 u$, $2 \le p < +\infty$, integrating over $\mathbb{R}$, we have

$$\|\phi_N u\|_{L^p}^{p-1} \frac{d}{dt}\|\phi_N u\|_{L^p} = -\int_{\mathbb{R}} u^{k-1}|\phi_N u|^p u_x dx - \int_{\mathbb{R}} \phi_N\big(G * \mathcal{F}(u)\big)|\phi_N u|^{p-2}\phi_N u dx =: J_1 + J_2. \tag{5.5}$$

It is easy to check that

$$|J_1| \le \|u^{k-1}u_x\|_{L^\infty} \cdot \|\phi_N u\|_{L^p}^p \le C_1(k)\Big(\|u\|_{L^\infty}^k + \|u_x\|_{L^\infty}^k\Big) \cdot \|\phi_N u\|_{L^p}^p. \tag{5.6}$$

By Hölder's inequality, we know

$$|J_2| \le \|\phi_N\big(G * \mathcal{F}(u)\big)\|_{L^p} \cdot \|\phi_N u\|_{L^p}^{p-1}.$$

Since ϕ_N is an admissible weight, we have by using (5.1) and (5.2) that

$$\|\phi_N(G * \mathcal{F}(u))\|_{L^p} \le \tilde{C_0}\|Gv\|_{L^1} \cdot \|\phi_N \mathcal{F}(u)\|_{L^p} \le C_2(C_0, \delta_0, M_0, \alpha, \beta) \cdot \|\phi_N \mathcal{F}(u)\|_{L^p}.$$

Thus, we have

$$|J_2| \le C_2(C_0, \delta_0, M_0, \alpha, \beta) \cdot \|\phi_N \mathcal{F}(u)\|_{L^p} \cdot \|\phi_N u\|_{L^p}^{p-1}. \tag{5.7}$$

Put (5.6) and (5.7) into (5.5), we obtain for any $2 \le p < +\infty$ that

$$\frac{d}{dt}\|\phi_N u\|_{L^p} \le C_1(k)\Big(\|u\|_{L^\infty}^k + \|u_x\|_{L^\infty}^k\Big)\|\phi_N u\|_{L^p} + C_2(C_0, \delta_0, M_0, \alpha, \beta)\|\phi_N \mathcal{F}(u)\|_{L^p}. \tag{5.8}$$

In order to derive a differential inequality for $\phi_N u_x$, we first take derivatives of (5.3) with respect to x. It is derived that

$$u_{xt} + ku^{k-1}u_x^2 + u^k u_{xx} + \partial_x(G * \mathcal{F}(u)) = 0.$$

Multiplying the above equation by $|\phi_N u_x|^{p-2}\phi_N^2 u_x$, $p \in [2, +\infty)$, and integrating over the real line, one has

$$\begin{aligned}&\|\phi_N u_x\|_{L^p}^{p-1} \frac{d}{dt}\|\phi_N u_x\|_{L^p} \\ &= -k\int_{\mathbb{R}} u^{k-1}|\phi_N u_x|^{p-2}\phi_N^2 u_x^3 dx - \int_{\mathbb{R}} u^k u_{xx}|\phi_N u_x|^{p-2}\phi_N^2 u_x dx - \int_{\mathbb{R}} \partial_x\big(G * \mathcal{F}(u)\big) \cdot |\phi_N u_x|^{p-2}\phi_N^2 u_x dx \\ &=: J_3 + J_4 + J_5. \end{aligned} \tag{5.9}$$

Note that $|\phi_N'| \leq A|\phi_N|$ almost everywhere over $\mathbb{R}$. Direct computation gives

$$
\begin{aligned}
|J_4| &= \Big|\int_{\mathbb{R}} \Big((\phi_N u_x)_x - (\phi_N)_x u_x\Big) u^k |\phi_N u_x|^{p-2}\phi_N u_x dx\Big| \\
&= \Big|\int_{\mathbb{R}} u^k \partial_x\Big(\frac{|\phi_N u_x|^p}{p}\Big)dx - \int_{\mathbb{R}} u^k |\phi_N u_x|^{p-2}\phi_N u_x^2 (\phi_N)_x dx\Big| \\
&\leq \frac{k}{p}\|u^{k-1}u_x\|_{L^\infty}\|\phi_N u_x\|_{L^p}^p + A\|u\|_{L^\infty}^k\|\phi_N u_x\|_{L^p}^p \\
&\leq C_3(k,A)\Big(\|u\|_{L^\infty}^k + \|u_x\|_{L^\infty}^k\Big)\|\phi_N u_x\|_{L^p}^p,
\end{aligned} \tag{5.10}
$$

$$
|J_3| \leq k\|u^{k-1}u_x\|_{L^\infty}\cdot\|\phi_N u_x\|_{L^p}^p \leq C_4(k)\Big(\|u\|_{L^\infty}^k + \|u_x\|_{L^\infty}^k\Big)\cdot\|\phi_N u_x\|_{L^p}^p. \tag{5.11}
$$

By Hölder's inequality, we have

$$
|J_5| \leq \|\phi_N \partial_x\big(G * \mathcal{F}(u)\big)\|_{L^p}\cdot\|\phi_N u_x\|_{L^p}^{p-1}.
$$

By (5.1), (5.2) and the fact $\partial_x G = \frac{-\alpha}{\alpha^2-\beta^2}\mathrm{sign}(x)g_1 + \frac{\beta}{\alpha^2-\beta^2}\mathrm{sign}(x)g_2$ in weak sense, we have

$$
\|\phi_N \partial_x\big(G * \mathcal{F}(u)\big)\|_{L^p} \leq \tilde{C}_0\|(\partial_x G)v\|_{L^1}\|\phi_N\mathcal{F}(u)\|_{L^p} \leq C_5(C_0,\delta_0,M_0,\alpha,\beta)\cdot\|\phi_N\mathcal{F}(u)\|_{L^p}.
$$

Hence

$$
|J_5| \leq C_5(C_0,\delta_0,M_0,\alpha,\beta)\cdot\|\phi_N\mathcal{F}(u)\|_{L^p}\cdot\|\phi_N u_x\|_{L^p}^{p-1}. \tag{5.12}
$$

Put (5.10), (5.11) and (5.12) into (5.9), we obtain for any $2 \leq p < +\infty$ that

$$
\frac{d}{dt}\|\phi_N u_x\|_{L^p} \leq C_6(k,A)\Big(\|u\|_{L^\infty}^k + \|u_x\|_{L^\infty}^k\Big)\cdot\|\phi_N u_x\|_{L^p} + C_5(C_0,\delta_0,M_0,\alpha,\beta)\|\phi_N\mathcal{F}(u)\|_{L^p}. \tag{5.13}
$$

Add (5.8) and (5.13) together. By making use of inequality (5.4), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\phi_N u\|_{L^p} + \|\phi_N u_x\|_{L^p}\Big) \\
\leq& C_7(k,A)\Big(\|u\|_{L^\infty}^k + \|u_x\|_{L^\infty}^k\Big)\cdot\Big(\|\phi_N u\|_{L^p} + \|\phi_N u_x\|_{L^p}\Big) + C_8(C_0,\delta_0,M_0,\alpha,\beta)\|\phi_N\mathcal{F}(u)\|_{L^p} \\
\leq& C(\alpha,\beta,k,b,A,C_0,\delta_0,M_0)M^k\cdot\Big(\|\phi_N u\|_{L^p} + \|\phi_N u_x\|_{L^p}\Big).
\end{aligned}
$$

By Grönwall's inequality, we have

$$
\|\phi_N u\|_{L^p} + \|\phi_N u_x\|_{L^p} \leq e^{CM^k t}(\|\phi_N u_0\|_{L^p} + \|\phi_N u_{0x}\|_{L^p}).
$$

Note that C and M are independent of $p \in [2,\infty)$ and $N \in \mathbb{R}^+$. Letting $p \to +\infty$, it implies that

$$
\|\phi_N u\|_{L^\infty} + \|\phi_N u_x\|_{L^\infty} \leq e^{CM^k t}(\|\phi_N u_0\|_{L^\infty} + \|\phi_N u_{0x}\|_{L^\infty}).
$$

Finally, letting $N \to +\infty$ completes the proof of this theorem. □

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Conflict of interest

The authors declare no potential conflict of interests.

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