# Positive solutions for one Schrodinger-Possion system involving subcritical or critical exponent

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**Abstract:** In this paper, we study one Schrödinger-Possion syetem involving subcritical or critical exponent. By Nehari manifold and Variational methods, we prove that the system has at least one positive solution under different cases.

**Keywords:** critical Sobolev exponent; Schrödinger-Possion system; Nehari manifold; Mountain pass lemma; positive solution

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## 1 Introduction and Definition

We consider the following Schrödinger-Possion system

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)\left|u\right|^{p-2}u + \lambda\left|u\right|^{4}u, \ x \in \mathbf{R}^3, \\ -\Delta\phi = K(x)u^2, \ x \in \mathbf{R}^3, \end{cases} \tag{1.1}$$

where $4 \le p < 6,\ \lambda > 0$ is a positive parameter. $K(x), a(x) : \mathbf{R}^3 \to \mathbf{R}$ are positive functions. we make the following assumption.

(A) $\lim_{|x|\to\infty} a(x) = a_\infty > 0,\ \alpha(x) := a(x) - a_\infty \in L^{\frac{6}{6-p}}(\mathbf{R}^3)$.

(B) $a(x) \ge a_\infty > 0$ and $\alpha(x) > 0$ in a positive measure set.

(C) $K(x) \in L^2(\mathbf{R}^3),\ \lim_{|x|\to\infty} K(x) = 0,\ K(x) \ge 0$ and $K(x) \not\equiv 0$ , $x \in \mathbf{R}^3$.

Schrödinger-Possion system have a strong physical meaning because they appear in quantum mechanics models (see [1]). In recent years, many people have studied the following Schrödinger-Possion system

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = f(x,u), \\ -\Delta\phi = u^2,\ x \in \mathbf{R}^3. \end{cases}$$

Ambrosetti, Azzollini and Wang obtained the existence of ground states in [2-6]. By using Lusternik-Schnirelmann category theory, Marco and He [7,8] proved the existence of many

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critical points for the case $f(x,u) = f(u)$ in a bounded domain. Cerami and Vaira [9] discussed the existence result of solutions for (1.1) in the case $\lambda = 0$. For the other subcritical cases, we refer the reader to [10-12] and the references therein.

However, to the best our knowledge, there is few information on the existence of solution to the Schrödinger-Poisson system in the critical case. In this paper, we shall prove that the problem (1.1) has at least one positive solution for the subcritical case $\lambda = 0$ and the critical case $\lambda \neq 0$.

Let $E = H^1(\mathbf{R}^3)$ be the usual Sobolev space equipped with the norm

$$\|u\| = \left( \int_{\mathbf{R}^3} (|\nabla u|^2 \, + u^2) \mathrm{d}x \right)^{1/2}.$$

Let $D^{1,2}(\mathbf{R}^3)$ be the completion of $C_0^\infty(\mathbf{R}^3)$:

$$D^{1,2}(\mathbf{R}^3) = \{u \in L^6(\mathbf{R}^3) : \nabla u \in L^2(\mathbf{R}^3)\}$$

with respect to the norm

$$\|u\|_D = \left( \int_{\mathbf{R}^3} |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}.$$

The norm of $L^s(\mathbf{R}^3)(2 \leq s \leq 6)$ is denoted by

$$\|u\|_s = \left( \int_{\mathbf{R}^3} |u|^s \, \mathrm{d}x \right)^{1/s}.$$

For $u \neq 0$, suppose that

$$S = \inf \frac{\int_{\mathbf{R}^3} |\nabla u|^2 \, \mathrm{d}x}{\left( \int_{\mathbf{R}^3} |u|^6 \, \mathrm{d}x \right)^{1/3}},$$

then

$$\|u\|_6^6 \leq \|u\|^6 \, S^{-3}. \tag{1.2}$$

$S$ is achieved by the family of functions

$$U_\varepsilon(x) = \frac{(C\varepsilon)^{1/4}}{(\varepsilon + |x|^2)^{1/2}},$$

and $U_\varepsilon(x)$ satisfies the equation:

$$-\Delta u = u \, |u|^4 \, , u \in D^{1,2}(\mathbf{R}^3).$$

Hence we have

$$\|U_\varepsilon\|^2 = \|U_\varepsilon\|_6^6 = S^{3/2}.$$

Since that $K(x) \in L^2(\mathbf{R}^3)$, then using Hölder inequality and (1.2) we have

$$\begin{aligned} \int_\Omega K(x)u^2\psi \mathrm{d}x &\leq \|K(x)\|_2 \left\|u^2\right\|_3 \|\psi\|_6 \\ &\leq S^{-\frac{1}{2}} \|K(x)\|_2 \|u\|_6^2 \|\psi\|, \quad \forall \psi \in D^{1,2}(\mathbf{R}^3). \end{aligned} \tag{1.3}$$

By Lax-Milgram theorem there exists a unique solution $\phi_u \in D^{1,2}(\mathbf{R}^3)$ such that

$$\int_\Omega \nabla\phi_u \nabla v \mathrm{d}x = \int_\Omega K(x)u^2 v \mathrm{d}x, \quad \forall v \in D^{1,2}(\mathbf{R}^3),$$

and $\phi_u$ has the following properties

(1) $\phi_u : H^1(\mathbf{R}^3) \to D^{1,2}(\mathbf{R}^3)$ is continuous and $\phi$ maps bounded sets into bounded sets;

(2) If $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbf{R}^3)$;

(3) $\phi_u \geq 0,\ \|\phi_u\| \leq C\|u\|^2,\ \int_{\mathbf{R}^3} \phi_u u^2 \mathrm{d}x \leq C\|u\|_{12/5}^4 \leq C\|u\|^4;$

(4) $\phi_{tu}(x) = t^2\phi_u$ for all $t \in \mathbf{R}$.

See [9] for the details.

We now introduce the main results in this paper.

**Theorem 1.2** Assume that (A)(B)(C) with $4 \leq p < 6, \lambda = 0$, then the problem (1.1) has at least one positive solution.

**Theorem 1.3** Assume that (A)(B)(C) with $4 < p < 6, \lambda \neq 0$, then the problem (1.1) has at least one positive solution for any $\lambda > 0$.

## 2 The proof of Theorem 1.2

For $\lambda = 0$, we define the energy functional associated with (1.1)

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbf{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbf{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbf{R}^3} a(x)|u|^p \mathrm{d}x \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}P(u) - \frac{1}{p}A(u), \end{aligned}$$

where

$$P(u) = \int_{\mathbf{R}^3} K(x)\phi_u u^2 \mathrm{d}x, \ \ A(u) = \int_{\mathbf{R}^3} a(x)|u|^p \mathrm{d}x.$$

Suppose that $S_0$ is the Sobolev constant for the embedding of $E = H^1(\mathbf{R}^3)$ in $L^p(\mathbf{R}^3)(2 < p < 6)$, then Hölder inequality and the condition (A)(B) imply that

$$\begin{aligned} A(u) &= \int_{\mathbf{R}^3} a(x)|u|^p \mathrm{d}x = \int_{\mathbf{R}^3} (\alpha(x) + a_\infty)|u|^p \mathrm{d}x \\ &\leq \|\alpha(x) + a_\infty\|_{\frac{6}{6-p}} \|u\|_p \leq C_0 S_0^p \|u\|^p, \end{aligned} \tag{2.1}$$

where

$$C_0 = \|\alpha(x) + a_\infty\|_{\frac{6}{6-p}}.$$

By (1.3) and (2.1), we get $J \in C^1(E, R)$.

Consider the Nehari manifold

$$N = \{u \in E \backslash \{0\} \mid \langle J'(u), u \rangle = 0\} .$$

It is easy to see that $u \in N$ if and only if

$$\langle J'(u), u \rangle = \|u\|^2 + P(u) - A(u) = 0.$$

Define

$$M(u) = \langle J'(u), u \rangle .$$

It is clear that $M(u)$ is of $C^1$ class. For $u \in N$, we have

$$\begin{aligned} \langle M'(u), u \rangle &= 2\|u\|^2 + 4P(u) - pA(u) \\ &= (4-p)A(u) - 2\|u\|^2 < 0. \end{aligned} \tag{2.2}$$

which implies that $N$ is a $C^1$ manifold.

**Lemma 2.1** Assume that $u_0$ is a local minimizer for $J$ on $N$ and $\langle M'(u), u \rangle \neq 0$, then $J'(u_0) = 0$.

**proof** Our proof is almost the same as that in Brown and Zhang (see [13]), we omit it.

Let $u \in N$, we have for $4 \leq p < 6$

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}P(u) - \frac{1}{p}A(u) \\ &= \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{p})A(u) > 0. \end{aligned}$$

Then $J(u)$ is bounded below on $N$. Since that $J(u)$ is bounded below on $N$. we may define

$$\xi^- = \inf_{u \in N} J(u).$$

**Lemma 2.2** There exists $C_1 = C_1(C_0, p, S_0) > 0$ such that $\xi^- > C_1$.

**Proof** Let $u \in N$, then

$$A(u) = \|u\|^2 + P(u) \geq \|u\|^2 . \tag{2.3}$$

From (2.1)(2.3) we get

$$1 \leq C_0 S_0^p \|u\|^{p-2} , \tag{2.4}$$

by (2.4) we deduce that there exists $C_1 = C_1(C_0, p, S_0) > 0$ such that $\xi^- > C_1$.

**Lemma 2.3** For each $u \in E$, $u \neq 0$, then there exists a unique $t_0$ such that $t_0 u \in N$ and

$$J(t_0 u) = \sup_{t \geq 0} J(tu).$$

**Proof** Suppose that $u \in E,\ u \neq 0$. Set

$$h(t) = J(tu) = \frac{t^2}{2}\left\|u\right\|^2 + \frac{t^4}{4}P(u) - \frac{t^p}{p}A(u),$$

then

$$h'(t) = t(\left\|u\right\|^2 + t^2 P(u) - t^{p-2}A(u).$$

From $h'(t) = 0$ we have $t = t_0 > 0$. Moreover, we have $h(0) = 0$ and $h(t) \to -\infty$ as $t \to +\infty$. Observe that $h'(t) > 0$ for $t \in [0, t_0)$ and $h'(t) < 0$ for $t \in [t_0, \infty)$, it follows that $h(t)$ achieves its maximum at $t = t_0$. By computation we have

$$\langle J'(t_0 u), t_0 u\rangle = t_0^2\left\|u\right\|^2 + t_0^4 P(u) - t_0^p A(u) = 0,$$

$$\langle M'(t_0 u), t_0 u\rangle = (4-p)t_0^p A(u) - 2t_0^2\left\|u\right\|^2 < 0.$$

Then $t_0 u \in N$ and $J(t_0 u) = \sup\limits_{t\geq 0} J(tu)$.

**Lemma 2.4** Assume that (A)(B)(C) with $4 \leq p < 6$, then $J(u)$ has a minimizer $u_0 \in N$, and it satisfies

(1) $J(u_0) = \xi^-$,

(2) $u_0$ is a nontrivial negative solution to (1.1).

**Proof** Let $u_n \in N$ be a minimizing sequence for $J(u)$, that is

$$\lim_{n\to\infty} J(u_n) = \inf_{u\in N} J(u).$$

Then we get

$$J(u_n) = \frac{1}{4}\left\|u_n\right\|^2 + (\frac{1}{4} - \frac{1}{p})A(u_n) > 0.$$

Which implies that $\{u_n\}$ is bounded in $E$. We can extract a subsequence(still denoted $\{u_n\}$) and $u_0 \in E$ such that

$$u_n \rightharpoonup u_0 \text{ in } E,$$

$$u_n \rightharpoonup u_0 \text{ in } L^i(\mathbf{R}^3)(2 \leq i < 2^*),$$

$$u_n \to u_0 \ a.e. \text{ in } \mathbf{R}^3.$$

Define the operator $T:\ L^{6/p}(\mathbf{R}^3) \to \mathbf{R}$ by

$$\langle T, w\rangle = \int_{\mathbf{R}^N} a(x) w \mathrm{d}x.$$

From (2.1) we get that $T$ is linear and continuous, then

$$A(u_n) = \int_{\mathbf{R}^3} a(x)\left|u_n\right|^p \mathrm{d}x \to \int_{\mathbf{R}^3} a(x)\left|u_0\right|^p \mathrm{d}x = A(u_0), \tag{2.5}$$

With the same arguments as [9] we can show that

$$\int_{\mathbf{R}^3} K(x)\phi_u u_n^2 \mathrm{d}x \to \int_{\mathbf{R}^3} K(x)\phi_u u_0^2 \mathrm{d}x. \tag{2.6}$$

Therefore

$$\begin{aligned} 0 < C_0 < \xi^- &\le J(u_n) \\ &= (\frac{1}{2} - \frac{1}{p})A(u_n) - \frac{1}{4}P(u_n) \\ &\to (\frac{1}{2} - \frac{1}{p})A(u_0) - \frac{1}{4}P(u_0). \end{aligned}$$

then $u_0 \neq 0$. Now we prove that: $u_n \to u_0$ in $E$. Supposing the contrary, by Fatou Lemma,

$$\|u_0\| < \lim_{n\to\infty} \inf \|u_n\|, \tag{2.7}$$

By Lemma 2.3, there exists a unique $t_0^-$ such that $t_0^- u_0 \in N$. Note that $u_n \in N, J(tu_n)$ achieves its maximum at $t = 1$. Hence we have $J(tu_n) \le J(u_n)$ for $t \ge 0$. From (2.7) we get that

$$\begin{aligned} J(t_0^- u_0) &= \frac{(t_0^-)^2}{4} \|u_0\|^2 + (\frac{1}{4} - \frac{1}{p})(t_0^-)^2 A(u_0) \\ &< \lim_{n\to\infty} \inf J(t_0^- u_n) \le \lim_{n\to\infty} J(u_n) = \xi^-, \end{aligned}$$

which is a contradiction. Hence $u_n \to u_0$ in $E$. If $n \to \infty, J(u_n) \to J(u_0) = \xi^-$. Since $u_0 \neq 0$. then Lemma 2.1 implies that $u_0$ is a nontrivial negative solution to (1.1).

let $u_0^+ = \max\{u_0, 0\}$. Replacing $u_0$ in $J(u)$ by $u_0^+$, we get one nontrivial negative solution to (1.1). From the Harnack inequality [14] we deduce that $u_0$ is a positive solution to (1.1). Then we complete the proof of Theorem 1.1.

## 3 Proof of Theorem 1.2

Now we turn to the case $\lambda > 0$.

Definition the energy functional

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbf{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbf{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbf{R}^3} a(x)|u|^p \mathrm{d}x \\ &- \frac{\lambda}{6}\int_{\mathbf{R}^3} |u|^6 \mathrm{d}x = \frac{1}{2}\|u\|^2 + \frac{1}{4}P(u) - \frac{1}{p}A(u) - \frac{1}{6}B(u), \end{aligned}$$

where

$$B(u) = \lambda \int_{\mathbf{R}^3} |u|^6 \mathrm{d}x.$$

**Definition 3.1** We say that a sequence $\{u_n\} \subset E$ is a $(PS)_c$ sequence, if there exists $c \in \mathbf{R}$ such that

$$I(u_n) \to c,\ I'(u_n) \to 0, n \to \infty.$$

**Lemma 3.2** Assume $\{u_n\} \subset E$ is a $(PS)_c$ sequence for $J$, then there exists a subsequence(still denoted $\{u_n\}$) and $u \in E$ such that $u_n \rightharpoonup u \in E$ and $u$ is a solution to (1.1). That is, $\langle J'(u), u \rangle = 0$.

**Proof** From the definition of $(PS)_c$ sequence, there exists a sequence $\{u_n\}$ in E such that for some $c \in \mathbf{R}$ we have

$$J(u_n) \to c, J'(u_n) \to 0, n \to \infty.$$

$$J(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}P(u_n) - \frac{1}{p}A(u_n) - \frac{1}{6}B(u_n) = c + o_n(1),$$

$$\langle J'(u_n), u_n \rangle = \|u_n\|^2 + P(u_n) - A(u_n) - B(u_n) = o_n(1),$$

hence

$$\begin{aligned} &c + o_n(1) = J(u_n) - \frac{1}{p}\langle J'(u_n), u_n \rangle \\ =&(\frac{1}{2} - \frac{1}{p})\|u_n\|^2 + (\frac{1}{4} - \frac{1}{p})P(u_n) + (\frac{1}{p} - \frac{1}{6})B(u_n) \geq (\frac{1}{2} - \frac{1}{p})\|u_n\|^2. \end{aligned}$$

Hence $\{u_n\}$ is bounded.

Since $\{u_n\}$ is bounded, we can extract a subsequence in $E$(still denoted $\{u_n\}$)and $u \in E$ such that

$$u_n \rightharpoonup u_0 \text{ in } E,$$

$$u_n \rightharpoonup u_0 \text{ in } L^i(\mathbf{R}^3)(2 \leq i < 2^*),$$

$$u_n \to u_0 \ a.e. \text{ in } \mathbf{R}^3.$$

By using (2.5)(2.6) we have $P(u_n) \to P(u),\ A(u_n) \to A(u)$. Therefore $u$ is a solution to (1.1) and $J'(u) = 0$.

**Lemma 3.3** Let $c \in \mathbf{R}$, if $\{u_n\} \subset E$ is a $(PS)_c$ sequence for $J$ and $u_n \rightharpoonup u$ with

$$c < \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}$$

then $u_n \to u$.

**Proof** From the definition of $(PS)_c$ sequence we have

$$J(u_n) \to c,\ J'(u_n) \to 0$$

as $n \to \infty$. From Lemma 3.2, there exists a subsequence(still denoted $\{u_n\}$) and $u \in E$ such that $u_n \rightharpoonup u \in E$ and $u$ is a solution to (1.1). Set $v_n = u_n - u$, then we deduce that $v_n \rightharpoonup 0$. Brezis-lemma Lemma [15] implies that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1),$$

$$B(u_n) = B(v_n) + B(u) + o_n(1).$$

Since $P(u_n) \to P(u),\ A(u_n) \to A(u)$, then we get

$$P(u_n) = P(u) + o_n(1),$$

$$A(u_n) = A(u) + o_n(1).$$

Therefore

$$\langle J'(u_n), u_n\rangle = \|v_n\|^2 - B(v_n) + o_n(1).$$

We may therefore assume that

$$\lim_{n\to\infty} \|v_n\|^2 = \lim_{n\to\infty} B(v_n) = l,$$

By the definition of $S$ we have

$$\|v_n\|^2 \geq \|\nabla v_n\|_2^2 \geq S\,\|v_n\|_6^2\,.$$

Then $\lambda^{\frac{1}{3}} l \geq S l^{\frac{1}{3}}$, it follows that either $l = 0$ or $l \geq \lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$, If $l \geq \lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$,

$$J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle = J(u) + \frac{1}{3}B(v_n) + o_n(1).$$

Let $n \to \infty$,

$$J(u) = c - \frac{1}{3}l \leq c - \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}} < 0,$$

we deduce that $J(u) < 0$.

On the other hand, since $u$ is a solution to (1.1) then $J'(u) = 0$.

$$\begin{aligned} J(u) &= J(u) - \frac{1}{4}\langle J'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{p})A(u) + \frac{1}{12}B(u) \geq 0, \end{aligned}$$

which is a contradiction. Hence $l = 0, u_n \to u$.

**Lemma 3.4** The following results hold

(1) There exist $\delta, \rho > 0$ such that $J(u) \geq \delta > 0$ for any $u \in E$ with $\|u\| = \rho$;

(2) There exists $\phi \in E$ such that $\lim_{t\to\infty} J(t\phi) = -\infty$;

**proof** (1)From (2.1)(1.2) we have

$$\begin{aligned} &J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}P(u) - \frac{1}{p}A(u) - \frac{1}{6}B(u) \\ \geq &\frac{1}{2}\|u\|^2 - \frac{1}{p}A(u) - \frac{1}{6}B(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{6}C_0 S_0^p \|u\|^p - S^{-3}\|u\|^6 \end{aligned}$$

We may choose $\|u\| = \rho$ small enough such that $\delta > 0, J(u) \geq \delta > 0$, then the first conclusion of Lemma 3.4 holds true.

(2) Let $\phi \in E, \phi \geq 0, \phi \neq 0$, by the properties of $\phi_u$ and the condition (B)(C), we have for $t \to \infty$,

$$J(t\phi) = \frac{t^2}{2}\|\phi\|^2 + \frac{t^4}{4}P(\phi) - \frac{t^p}{p}A(\phi) - \frac{t^6}{6}B(\phi) \to -\infty.$$

We now complete the proof of Theorem 1.2.

**The proof of Theorem 1.2** By using (1) (2) of Lemma 3.4 and Mountain Pass Theorem, there exists a $(PS)_c$ sequence $\{u_n\}$ in $E$ such that

$$J(u_n) \to c_0,\ J'(u_n) \to 0,\ n \to \infty,$$

where

$$c_0 = \inf_{\gamma\in\Gamma, t\in[0,1]} \max J(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], E), \gamma(0) = 0, \gamma(1) = (t\phi)\}.$$

From Lemma 3.2 we obtain $u_n \rightharpoonup u$ and $J'(u) = 0$.

Let $\xi(x) \in C^\infty(\mathbf{R}^3)$ is a cut-off function such that $\xi(x) = 1$ for $|x| \leq R$ and $\xi(x) = 0$ for $|x| \geq 2R$, where $|\nabla\xi(x)| \leq C, B_{2R}(0) \subset \mathbf{R}^3$. We define the function $u_\varepsilon(x)$ as follows

$$u_\varepsilon(x) = \phi U_\varepsilon \|\phi U_\varepsilon\|_6^{-1},$$

then we have the properties of the function $u_\varepsilon$:

$$\int_\Omega |\nabla u_\varepsilon|^2 \mathrm{d}x = S + O(\varepsilon^{1/2}), \tag{3.1}$$

and

$$\int_\Omega |u_\varepsilon|^q \mathrm{d}x = \begin{cases} O(\varepsilon^{t/4}), & q \in [2,3), \\ O(\varepsilon^{3/4}|\ln\varepsilon|), & q = 3, \\ O(\varepsilon^{(6-t)/4}), & q \in (3,6). \end{cases} \tag{3.2}$$

see [16-18] for the details.

Consider the functions

$$g(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{t^4}{4}P(u_\varepsilon) - \frac{t^p}{p}a_\infty \int_\Omega |u_\varepsilon|^p \mathrm{d}x - \lambda\frac{t^6}{6},$$

Note that $\lim\limits_{t\to+\infty} g(t) = -\infty$ and $\lim\limits_{t\to 0^+} g(t) > 0$, then $\sup g(t)$ is attained for some $t_\varepsilon > 0$. By computation we get

$$0 = g'(t_\varepsilon) = t_\varepsilon(\|u_\varepsilon\|^2 + t_\varepsilon^2 P(u_\varepsilon) - t_\varepsilon^{p-2}a_\infty \int_\Omega |u_\varepsilon|^p \mathrm{d}x - \lambda t_\varepsilon^4),$$

$$\|u_\varepsilon\|^2 + t_\varepsilon^2 P(u_\varepsilon) = t_\varepsilon^{p-2}a_\infty \int_\Omega |u_\varepsilon|^p \mathrm{d}x + \lambda t_\varepsilon^4 \geq \lambda t_\varepsilon^4. \tag{3.3}$$

Then there exists $t_1 > 0$ such that $t_\varepsilon \leq t_1$. On the other hand from (3.3) we deduce that

$$\|\nabla u_\varepsilon\|_2^2 \leq \|u_\varepsilon\|^2 \leq t_\varepsilon^{p-2} a_\infty \int_\Omega |u_\varepsilon|^p \mathrm{d}x + \lambda t_\varepsilon^4.$$

Using(3.1)(3.2) we deduce that there exists $M > 0$ such that

$$t_\varepsilon \geq M.$$

Set

$$h(t) = \frac{1}{2} t^2 \|\nabla u_\varepsilon\|_2^2 - \lambda \frac{t^6}{6},$$

then sup $h(t)$ is attained for some $t_2 = \lambda^{\frac{-1}{4}} \|\nabla u_\varepsilon\|_2^{1/2}$. If $4 < p < 6$, we have

$$\frac{6-p}{4} < \frac{1}{2} < 1,$$

then (3.1)(3.2) and the properties of $\phi_u$ imply that

$$g(t) \leq h(t_2) + \frac{1}{2} t_\varepsilon^2 \|u_\varepsilon\|_2^2 + \frac{t_\varepsilon^4}{4} C \|u_\varepsilon\|_{12/5}^4 - \frac{t_\varepsilon^p}{p} a_\infty \|u_\varepsilon\|_p^p$$
$$\leq \frac{1}{3} \lambda^{-\frac{1}{2}} S^{\frac{3}{2}} + C_1 \varepsilon^{\frac{1}{2}} + C_2 \varepsilon - C_3 \varepsilon^{\frac{6-p}{4}} < \frac{1}{3} \lambda^{-\frac{1}{2}} S^{\frac{3}{2}},$$

where $C_i (i = 2, 3, 4)$ are constants independent of $\varepsilon$. Thus we get that

$$\sup J(t u_\varepsilon) \leq g(t) < \frac{1}{3} \lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$$

$$c_0 = \inf_{\gamma \in \Gamma, t \in [0,1]} \max J(\gamma(t)) < \frac{1}{3} \lambda^{-\frac{1}{2}} S^{\frac{3}{2}},$$

By Lemma 3.3 we have that $u_n \to u, J(u) = c_0 > 0$. From the above steps, we obtain that the problem (1.1) has one nontrivial solution. Using the same argument as that in the proof of theorem 1.1, we deduce that the solution is positive. This completes the proof of Theorem 1.2.

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