

GENERALIZED INTEGRAL TYPE HILBERT OPERATOR ACTING BETWEEN WEIGHTED BLOCH SPACE

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ABSTRACT. Let μ be a finite Borel measure on $[0, 1)$. In this paper, we consider the generalized integral type Hilbert operator

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}} d\mu(t) \quad (\alpha > -1).$$

The operator \mathcal{I}_{μ_1} has been extensively studied recently. The aim of this paper is to study the boundedness (resp. compactness) of $\mathcal{I}_{\mu_{\alpha+1}}$ acting from the normal weight Bloch space into another of the same kind. As consequences of our study, we get completely results for the boundedness of $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch type spaces, logarithmic Bloch spaces among others.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane \mathbb{C} and $H(\mathbb{D})$ denote the space of all analytic functions in \mathbb{D} .

A positive continuous function ν on $[0, 1)$ is called normal if there exist $0 < a \leq b < \infty$ and $0 \leq s_0 < 1$ such that $\frac{\nu(s)}{(1-s^2)^a}$ almost decreasing on $[s_0, 1)$ and $\frac{\nu(s)}{(1-s^2)^b}$ almost increasing on $[s_0, 1)$.

A function g is almost increasing if there exists $C > 0$ such that $r_1 < r_2$ implies $g(r_1) \leq Cg(r_2)$. Almost decreasing functions are defined in an analogous manner.

Functions such as

$$\nu(s) = (1-s^2)^t \log^\delta \frac{e}{1-s^2} \quad (t > 0, \delta \in \mathbb{R}) \quad \text{and} \quad \nu(s) = \left(\sum_{k=1}^{\infty} \frac{ks^{2k-2}}{\log^3(k+1)} \right)^{-1}$$

are normal functions.

In this paper, we use \mathcal{N} to denote the set of all normal functions on $[0, 1)$ and let $s_0 = 0$. The letters a and b always represent the parameters in the definition of normal function.

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Let $\nu \in \mathcal{N}$, the normal weight Bloch space \mathcal{B}_ν consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}_\nu} = |f(0)| + \sup_{z \in \mathbb{D}} \nu(|z|) |f'(z)| < \infty.$$

In particular, if $\nu(|z|) = (1 - |z|^2)^\gamma (\gamma > 0)$, then \mathcal{B}_ν is the Bloch type space \mathcal{B}^γ . If $\nu(|z|) = (1 - |z|^2) \log^{-\beta} \frac{e}{1-|z|^2} (\beta \in \mathbb{R})$, then \mathcal{B}_ν is just the logarithmic Bloch space \mathcal{B}_{\log^β} .

Let μ be a positive Borel measure on $[0, 1)$, $0 \leq \gamma < \infty$ and $0 < s < \infty$. Then μ is a γ -logarithmic s -Carleson measure if there exists a positive constant C , such that

$$\mu([t, 1)) \log^\gamma \frac{e}{1-t} \leq C(1-t)^s, \quad \text{for all } 0 \leq t < 1.$$

In particular, μ is an s -Carleson measure if $\gamma = 0$. See [1] for more about logarithmic Carleson measure.

Let μ be a finite Borel measure on $[0, 1)$ and $n \in \mathbb{N}$. We use μ_n to denote the sequence of order n of μ , that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$. Let \mathcal{H}_μ be the Hankel matrix $(\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix \mathcal{H}_μ induces an operator on $H(\mathbb{D})$ by its action on the Taylor coefficients: $a_n \rightarrow \sum_{k=0}^{\infty} \mu_{n,k} a_k$, $n = 0, 1, 2, \dots$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, the generalized Hilbert operator defined as follows:

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

It's known that the generalized Hilbert operator \mathcal{H}_μ is closely related to the integral operator

$$\mathcal{I}_\mu(f)(z) = \int_0^1 \frac{f(t)}{1-tz} d\mu(t)$$

If μ is the Lebesgue measure on $[0, 1)$, then \mathcal{H}_μ and \mathcal{I}_μ reduce to the classic Hilbert operator \mathcal{H} and \mathcal{I} .

The action of the operators \mathcal{I}_μ and \mathcal{H}_μ on distinct spaces of analytic functions have been studied in a number of articles (see, e.g., [2–8]). In this paper, we consider the generalized integral type Hilbert operator

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}} d\mu(t), \quad (\alpha > -1).$$

If $\alpha = 0$, the operator $\mathcal{I}_{\mu_{\alpha+1}}$ is just \mathcal{I}_μ . The integral type operator $\mathcal{I}_{\mu_{\alpha+1}}$ is closely related to the Hilbert type operator

$$\mathcal{H}_\mu^\alpha(f)(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad (\alpha > -1),$$

whenever the right hand side makes sense and defines an analytic function in \mathbb{D} . The operator \mathcal{H}_μ^α can be regarded as the fractional derivative of \mathcal{H}_μ . If $\alpha = 1$, then \mathcal{H}_μ^α called the Derivative-Hilbert operator which has been studied in [9, 10].

The connection between \mathcal{I}_μ (or \mathcal{H}_μ) and $\mathcal{I}_{\mu_{\alpha+1}}$ (or \mathcal{H}_μ^α) motivates us to consider the operator $\mathcal{I}_{\mu_{\alpha+1}}$ in a unified manner. In [11] (see also [5]), the authors have studied the boundedness of \mathcal{I}_μ acting on \mathcal{B} . Li and Zhou studied the operator \mathcal{H}_μ from Bloch type spaces to the BMOA and the Bloch space in [12]. Ye and Zhou investigated \mathcal{I}_{μ_2} acting on \mathcal{B} in [9] and $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch-type space in [13]. But only partial results were obtained for the boundedness of $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch-type spaces. The aim of this article is to deal with the operator $\mathcal{I}_{\mu_{\alpha+1}}$ acting from normal weight Bloch space into another of the same kind. As consequences of our study, we obtain complete results for the boundedness of $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch type spaces, logarithmic Bloch spaces among others.

Throughout the paper, the letter C will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation “ $P \lesssim Q$ ” if there exists a constant $C = C(\cdot)$ such that “ $P \leq CQ$ ”, and “ $P \gtrsim Q$ ” is understood in an analogous manner. In particular, if “ $P \lesssim Q$ ” and “ $P \gtrsim Q$ ”, then we will write “ $P \asymp Q$ ”.

2. Preliminary Results

In [14], a sequence $\{V_n\}$ was constructed in the following way: Let ψ be a C^∞ -function on \mathbb{R} such that (1) $\psi(s) = 1$ for $s \leq 1$, (2) $\psi(s) = 0$ for $s \geq 2$, (3) ψ is decreasing and positive on the interval $(1, 2)$.

Let $\varphi(s) = \psi(\frac{s}{2}) - \psi(s)$, and let $v_0 = 1 + z$, for $n \geq 1$,

$$V_n(z) = \sum_{k=0}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right) z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi\left(\frac{k}{2^{n-1}}\right) z^k.$$

The polynomials V_n have the properties:

- (1) $f(z) = \sum_{n=0}^{\infty} V_n * g(z)$, for $f \in H(\mathbb{D})$;
- (2) $\|V_n * f\|_p \lesssim \|f\|_p$, for $f \in H^p$, $p > 0$;
- (3) $\|V_n\|_p \asymp 2^{n(1-\frac{1}{p})}$, for all $p > 0$, where $*$ denotes the Hadamard product and $\|\cdot\|_p$ denotes the norm of Hardy space H^p .

Lemma 2.1. *Let $\nu \in \mathcal{N}$ and $f \in H(\mathbb{D})$, then $f \in \mathcal{B}_\nu$ if and only if*

$$\sup_{n \geq 0} \nu(1 - 2^{-n}) 2^n \|V_n * f\|_\infty < \infty.$$

Moreover,

$$\|f\|_{\mathcal{B}_\nu} \asymp \sup_{n \geq 0} \nu(1 - 2^{-n}) 2^n \|V_n * f\|_\infty.$$

The proof of this Lemma is similar to that Theorem 3.1 in [15], we leave it to the interested readers.

Lemma 2.2. *Let $\nu \in \mathcal{N}$ and*

$$g(\zeta) = 1 + \sum_{s=1}^{\infty} 2^s \zeta^{n_s} \quad (\zeta \in \mathbb{D}),$$

where n_s is the integer part of $(1 - r_s)^{-1}$, $r_0 = 0$, $\nu(r_s) = 2^{-s}$ ($s = 1, 2, \dots$). Then $g(r)$ is strictly increasing on $[0, 1)$ and there exist two positive constants N_1 and N_2 such that

$$\inf_{[0,1)} \nu(r)g(r) = N_1 > 0, \quad \sup_{\zeta \in \mathbb{D}} \nu(|\zeta|)|g(\zeta)| = N_2 < +\infty.$$

This result is originated from Theorem 1 in [16].

Lemma 2.3. *If $\nu \in \mathcal{N}$, then*

$$\frac{\nu(|z|)}{\nu(|w|)} \lesssim \left(\frac{1 - |z|^2}{1 - |w|^2} \right)^a + \left(\frac{1 - |z|^2}{1 - |w|^2} \right)^b \quad \text{for all } z, w \in \mathbb{D}.$$

This result comes from Lemma 2.2 in [17].

Lemma 2.4. *Let $\nu \in \mathcal{N}$, $0 < \delta < \frac{1}{e^2}$, then*

$$\int_e^\infty \frac{e^{-\delta t} dt}{t\nu(1 - \frac{1}{t})} \lesssim \frac{1}{\nu(1 - \delta)}.$$

Proof.

$$\int_e^\infty \frac{e^{-\delta t} dt}{t\nu(1 - \frac{1}{t})} = \int_e^{\frac{1}{\delta}} \frac{e^{-\delta t} dt}{t\nu(1 - \frac{1}{t})} + \int_{\frac{1}{\delta}}^\infty \frac{e^{-\delta t} dt}{t\nu(1 - \frac{1}{t})} = I_1 + I_2.$$

By the definition of normal function, we have

$$I_1 \leq \int_e^{\frac{1}{\delta}} \frac{dt}{t\nu(1 - \frac{1}{t})} \lesssim \frac{\delta^a}{\nu(1 - \delta)} \int_e^{\frac{1}{\delta}} t^{a-1} dt \lesssim \frac{1}{\nu(1 - \delta)}.$$

If $t > \frac{1}{\delta}$, then $1 - \frac{1}{t} > 1 - \delta$. The definition of normal function shows that

$$\frac{\nu(1 - \delta)}{[1 - (1 - \delta)]^b} \lesssim \frac{\nu(1 - \frac{1}{t})}{[1 - (1 - \frac{1}{t})]^b}.$$

Hence, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{\delta}}^\infty \frac{\nu(1 - \delta)}{\nu(1 - \frac{1}{t})} \frac{e^{-\delta t} dt}{t\nu(1 - \delta)} \\ &\lesssim \int_{\frac{1}{\delta}}^\infty \frac{\delta^b t^{b-1} e^{-\delta t}}{\nu(1 - \delta)} dt = \frac{1}{\nu(1 - \delta)} \int_1^\infty e^{-s} s^{b-1} ds \\ &\lesssim \frac{1}{\nu(1 - \delta)}. \end{aligned}$$

The proof is complete. □

Lemma 2.5. *Let μ be a positive Borel measure on $[0, 1)$, $\beta > 0$, $\gamma > 0$. Let τ be the Borel measure on $[0, 1)$ defined by*

$$d\tau(t) = \frac{d\mu(t)}{(1-t)^\gamma}.$$

Then, the following two conditions are equivalent.

- (a) τ is a β -Carleson measure.*
- (b) μ is a $\beta + \gamma$ -Carleson measure.*

Proof. (a) \Rightarrow (b). Assume (a). Then there exists a positive constant $C > 0$ such that

$$\int_t^1 \frac{d\mu(r)}{(1-r)^\gamma} \leq C(1-t)^\beta, \quad t \in [0, 1).$$

Using this and the fact that the function $x \rightarrow \frac{1}{(1-x)^\gamma}$ is increasing in $[0, 1)$, we obtain

$$\frac{\mu([t, 1))}{(1-t)^\gamma} \leq \int_t^1 \frac{d\mu(r)}{(1-r)^\gamma} \leq C(1-t)^\beta, \quad t \in [0, 1).$$

This shows that μ is a $\beta + \gamma$ -Carleson measure.

(b) \Rightarrow (a). Assume (b). Then there exists a positive constant $C > 0$ such that

$$\mu(t) \leq C(1-t)^{\beta+\gamma}, \quad t \in [0, 1).$$

For $0 < x < 1$, let $h(x) = \mu([0, x)) - \mu([0, 1)) = -\mu([x, 1))$. Integrating by parts and using the inequality above, we obtain

$$\begin{aligned} \tau([t, 1)) &= \int_t^1 \frac{d\mu(x)}{(1-x)^\gamma} \\ &= \frac{1}{(1-t)^\gamma} \mu([t, 1)) - \lim_{x \rightarrow 1} \frac{1}{(1-x)^\gamma} \mu([x, 1)) + \gamma \int_t^1 \frac{\mu([x, 1))}{(1-x)^{\gamma+1}} dx \\ &= \frac{1}{(1-t)^\gamma} \mu([t, 1)) + \gamma \int_t^1 \frac{\mu([x, 1))}{(1-x)^{\gamma+1}} dx \\ &\lesssim (1-t)^\beta + \int_t^1 (1-x)^{\beta-1} dx \lesssim (1-t)^\beta. \end{aligned}$$

Thus, τ is an β -Carleson measure. □

Lemma 2.6. *Let $\omega, \nu \in \mathcal{N}$. If T is a bounded operator from \mathcal{B}_ω into \mathcal{B}_ν , then T is compact operator from \mathcal{B}_ω into \mathcal{B}_ν if and only if for any bounded sequence $\{h_n\}$ in \mathcal{B}_ω which converges to 0 uniformly on every compact subset of \mathbb{D} , we have $\lim_{n \rightarrow \infty} \|T(h_n)\|_{\mathcal{B}_\nu} = 0$.*

The proof is similar to that of Proposition 3.11 in [18], we omit the details.

3. Nonnegative Coefficients of normal weight Bloch functions

First, we give a characterization of the functions $f \in H(\mathbb{D})$ whose sequence of Taylor coefficients is non-negative which belongs to \mathcal{B}_ν .

Theorem 3.1. *Let $\nu \in \mathcal{N}$ and $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \geq 0$ for all $n \geq 0$. Then $f \in \mathcal{B}_\nu$ if and only if*

$$S(f) := \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{k=1}^n k a_k < \infty.$$

Moreover,

$$\|f\|_{\mathcal{B}_\nu} \asymp S(f) + a_0.$$

Proof. If $f \in \mathcal{B}_\nu$, then for each $n \in \mathbb{N}$,

$$\begin{aligned} \|f\|_{\mathcal{B}_\nu} &\geq \sup_{z=1-\frac{1}{n}} \nu(|z|) |f'(z)| \\ &\geq \nu(1 - \frac{1}{n}) \left| \sum_{k=1}^{\infty} k a_k (1 - \frac{1}{n})^{k-1} \right| \\ &\gtrsim \nu(1 - \frac{1}{n}) \sum_{k=1}^n k a_k, \end{aligned}$$

and hence $S(f) \lesssim \|f\|_{\mathcal{B}_\nu}$. Since $a_0 = |f(0)| \leq \|f\|_{\mathcal{B}_\nu}$, we may obtain

$$S(f) + a_0 \lesssim \|f\|_{\mathcal{B}_\nu}.$$

On the other hand, if $S(f) < \infty$, then

$$\nu(1 - 2^{-j}) \sum_{k=2^j}^{2^{j+1}-1} k a_k \lesssim S(f), \quad j \in \mathbb{N}.$$

For each $z \in \mathbb{D}$ with $\frac{1}{2} \leq |z| < 1$, we have

$$\begin{aligned} |f'(z)| &= \left| \sum_{j=0}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} k a_k z^{k-1} \right| \leq \sum_{j=0}^{\infty} \left(\sum_{k=2^j}^{2^{j+1}-1} k a_k |z|^{k-1} \right) \\ &\lesssim S(f) \sum_{j=0}^{\infty} \frac{|z|^{2^j}}{\nu(1 - 2^{-j})}. \end{aligned}$$

To finish the proof, it suffices to prove that

$$\sum_{j=0}^{\infty} \frac{|z|^{2^j}}{\nu(1 - 2^{-j})} \lesssim \frac{1}{\nu(|z|)} \quad \text{for all } \frac{1}{2} \leq |z| < 1. \quad (3.1)$$

For each $\frac{1}{2} \leq |z| = r < 1$, by choosing $m \geq 2$ such that $r_{m-1} \leq r \leq r_m$, where $r_m = 1 - 2^{-m}$. Then

$$\sum_{j=0}^{\infty} \nu^{-1}(1 - 2^{-j})r^{2^j} \leq \sum_{j=0}^m \nu^{-1}(1 - 2^{-j}) + \sum_{j=m+1}^{\infty} \nu^{-1}(1 - 2^{-j})r^{2^j} = S_1 + S_2.$$

Using Lemma 2.3 we have

$$\begin{aligned} S_1 &\lesssim \nu^{-1}(1 - 2^{-m}) \sum_{j=0}^m \left(\left(\frac{1}{2} \right)^{(m-j)a} + \left(\frac{1}{2} \right)^{(m-j)b} \right) \\ &\lesssim \nu^{-1}(1 - 2^{-m}). \end{aligned}$$

On the other hand,

$$\begin{aligned} S_2 &= \sum_{j=m+1}^{\infty} \nu^{-1}(1 - 2^{-j})r^{2^j} \leq \sum_{j=m+1}^{\infty} \nu^{-1}(1 - 2^{-j})r_m^{2^m \cdot 2^{j-m}} \\ &\leq \sum_{j=m+1}^{\infty} \nu^{-1}(1 - 2^{-j})e^{-2^{j-m}} = \sum_{l=1}^{\infty} \nu^{-1}(1 - 2^{-(l+m)})e^{-2^l} \\ &\lesssim \nu^{-1}(1 - 2^{-m}) \sum_{l=1}^{\infty} e^{-2^l} 2^{lb} \lesssim \nu^{-1}(1 - 2^{-m}). \end{aligned}$$

Since $\nu^{-1}(1 - 2^{-m}) \asymp \nu^{-1}(r)$, it follows that (3.1) is valid for all $\frac{1}{2} \leq |z| < 1$.

Therefore,

$$|f(0)| + \sup_{z \in \mathbb{D}} \nu(|z|)|f'(z)| \lesssim a_0 + S(f).$$

The proof is complete. \square

Corollary 3.2. *Let $\gamma > 0$ and $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \geq 0$ for all $n \geq 0$. Then $f \in \mathcal{B}^\gamma$ if and only if*

$$\sup_{n \geq 1} n^{-\gamma} \sum_{k=1}^n k a_k < \infty.$$

If $f \in \mathcal{B}_\nu$ has nonnegative and non-increasing coefficients, then the result of Theorem 3.1 can be state as follows.

Theorem 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ with a_n nonnegative and non-increasing. Then $f \in \mathcal{B}_\nu$ if and only if*

$$\sup_{n \geq 1} n^2 \nu \left(1 - \frac{1}{n} \right) a_n < \infty.$$

Moreover,

$$\|f\|_{\mathcal{B}_\nu} \asymp a_0 + \sup_{n \geq 1} n^2 \nu \left(1 - \frac{1}{n} \right) a_n.$$

Proof. If a_n nonnegative and non-increasing, then $\sum_{k=1}^n ka_k \gtrsim n^2 a_n$. The proof of the necessity follows from Theorem 3.1 immediately.

On the other hand, if $M := \sup_{n \geq 1} n^2 \nu(1 - \frac{1}{n}) a_n < \infty$, then

$$a_n \lesssim \frac{M}{n^2 \nu(1 - \frac{1}{n})} \quad \text{for all } n \geq 1.$$

For every $z \in \mathbb{D}$ and $\frac{1}{2} < |z| < 1$,

$$|f'(z)| \leq \sum_{n=1}^{\infty} n a_n |z|^{n-1} \lesssim M \sum_{n=1}^{\infty} \frac{|z|^n}{n \nu(1 - \frac{1}{n})}.$$

Let

$$h_x(t) = \frac{x^t}{t \nu(1 - \frac{1}{t})} \quad x \in (0, 1),$$

then h_x is decreasing in t , for sufficiently large t and each $x \in (0, 1)$. So, by Lemma 2.4 we have

$$\sum_{n=1}^{\infty} \frac{|z|^n}{n \nu(1 - \frac{1}{n})} \asymp \int_e^{\infty} \frac{e^{-t \log \frac{1}{|z|}}}{t \nu(1 - \frac{1}{t})} dt \lesssim \frac{1}{\nu(1 - \log \frac{1}{|z|})} \asymp \frac{1}{\nu(|z|)}.$$

This means that

$$\|f\|_{\mathcal{B}_\nu} \lesssim a_0 + \sup_{n \geq 1} n^2 \nu(1 - \frac{1}{n}) a_n.$$

The proof is complete. \square

Corollary 3.4. *Let $\gamma > 0$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ with a_n nonnegative and non-increasing. Then $f \in \mathcal{B}^\gamma$ if and only if*

$$\sup_{n \geq 1} n^{2-\gamma} a_n < \infty.$$

4. Generalized integral type Hilbert operator acting on weighted Bloch space

Let $\omega \in \mathcal{N}$, we write $\tilde{\omega}(t) = \int_0^t \frac{1}{\omega(s)} ds$. We begin with characterizing those measure μ for which the operator $\mathcal{I}_{\mu_{\alpha+1}}$ is well defined on \mathcal{B}_ω .

Proposition 4.1. *Let μ be a positive Borel measure on $[0, 1)$ and $\alpha > -1$. For any given $f \in \mathcal{B}_\omega$, $\mathcal{I}_{\mu_{\alpha+1}}(f)$ uniformly converges on any compact subset of \mathbb{D} if and only if*

$$\int_0^1 (\tilde{\omega}(t) + 1) d\mu(t) < \infty. \quad (4.1)$$

Proof. Let $f \in \mathcal{B}_\omega$, it is easy to verify that

$$|f(z)| \lesssim (\tilde{\omega}(|z|) + 1) \|f\|_{\mathcal{B}_\omega} \quad \text{for all } z \in \mathbb{D}. \quad (4.2)$$

If (4.1) holds, then for each $0 < r < 1$ and $z \in \mathbb{D}$ with $|z| \leq r$, we have

$$\begin{aligned} |\mathcal{I}_{\mu_{\alpha+1}}(f)(z)| &\leq \int_0^1 \frac{|f(t)|}{|1-tz|^{\alpha+1}} d\mu(t) \\ &\lesssim \frac{\|f\|_{\mathcal{B}_\omega}}{(1-r)^{\alpha+1}} \int_0^1 (\tilde{\omega}(t) + 1) d\mu(t) \\ &\lesssim \frac{\|f\|_{\mathcal{B}_\omega}}{(1-r)^{\alpha+1}}. \end{aligned}$$

This implies that $\mathcal{I}_{\mu_{\alpha+1}}(f)$ uniformly converges on any compact subset of \mathbb{D} and hence analytic in \mathbb{D} .

Suppose that the operator $\mathcal{I}_{\mu_{\alpha+1}}$ is well defined in \mathcal{B}_ω . Considering the function

$$f(z) = \int_0^z g(s) ds + 1$$

where g is the function in Lemma 2.2 with respect to ω . Then Lemma 2.2 implies that $f \in \mathcal{B}_\omega$. Since $\mathcal{I}_{\mu_{\alpha+1}}(f)(z)$ is well defined for every $z \in \mathbb{D}$, we have

$$|\mathcal{I}_{\mu_{\alpha+1}}(f)(0)| = \left| \int_0^1 f(t) d\mu(t) \right| < \infty.$$

Since μ is a positive measure and $g(s) > 0$ for all $s \in [0, 1)$, it follows from Lemma 2.2 that

$$f(t) = \int_0^t g(s) ds + 1 \asymp \tilde{\omega}(t) + 1. \quad (4.3)$$

Therefore,

$$\int_0^1 (\tilde{\omega}(t) + 1) d\mu(t) < \infty.$$

The proof is complete. \square

The sublinear generalized integral type Hilbert operator $\tilde{\mathcal{I}}_{\mu_{\alpha+1}}$ defined by

$$\tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{|f(t)|}{(1-tz)^{\alpha+1}} d\mu(t), \quad (\alpha > -1).$$

It is obvious that Proposition 4.1 is remain valid if $\mathcal{I}_{\mu_{\alpha+1}}$ is replaced by $\tilde{\mathcal{I}}_{\mu_{\alpha+1}}$. By mean of Lemma 2.1, Theorem 3.1 and the sublinear integral type Hilbert operator $\tilde{\mathcal{I}}_{\mu_{\alpha+1}}$, we have the following results.

Theorem 4.2. *Let $\omega, \nu \in \mathcal{N}$ and $\alpha > -1$. Suppose μ is a positive Borel measure on $[0, 1)$ and satisfies (4.1). Then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded;
- (b) $\tilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded;
- (c) $\sup_{n \geq 1} n^{\alpha+2} \nu(1 - \frac{1}{n}) \int_0^1 t^n (\tilde{\omega}(t) + 1) d\mu(t) < \infty$.

Proof. (a) \Rightarrow (c) : If $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded. For each $f \in \mathcal{B}_\omega$, Proposition 4.1 implies that $\mathcal{I}_{\mu_{\alpha+1}}(f)$ converges absolutely for every $z \in \mathbb{D}$ and

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n f(t) d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

Take

$$f(z) = \int_0^z g(s) ds + 1,$$

where g is the function in Lemma 2.2 with respect to ω . Then $f \in \mathcal{B}_\omega$ and

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} b_n z^n$$

where

$$b_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n \left(\int_0^t g(s) ds + 1 \right) d\mu(t).$$

It is clear that $\{b_n\}_{n=1}^{\infty}$ is a nonnegative sequence. Using Theorem 3.1, (4.3) and Stirling's formula we have

$$\begin{aligned} \|\mathcal{I}_{\mu_{\alpha+1}}(f)\|_{\mathcal{B}_\nu} &\gtrsim \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{k=1}^n k b_k \\ &\gtrsim \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \int_0^1 t^n (\tilde{\omega}(t) + 1) d\mu(t) \sum_{k=1}^n k^{\alpha+1} \\ &\asymp \sup_{n \geq 1} n^{\alpha+2} \nu(1 - \frac{1}{n}) \int_0^1 t^n (\tilde{\omega}(t) + 1) d\mu(t). \end{aligned}$$

Therefore,

$$\sup_{n \geq 1} n^{\alpha+2} \nu(1 - \frac{1}{n}) \int_0^1 t^n (\tilde{\omega}(t) + 1) d\mu(t) < \infty.$$

(c) \Rightarrow (b) : Assume (c). Then for each $n \in \mathbb{N}$, we have

$$\int_0^1 t^n (\tilde{\omega}(t) + 1) d\mu(t) \lesssim \frac{1}{n^{\alpha+2} \nu(1 - \frac{1}{n})}. \quad (4.4)$$

For a given $0 \neq f \in \mathcal{B}_\omega$,

$$\tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{|f(t)|}{(1-tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n |f(t)| d\mu(t).$$

Obviously, $\{c_n\}_{n=1}^\infty$ is a nonnegative sequence. Using (4.2), (4.4), and the definition of normal weight, we deduce that

$$\begin{aligned}
& |c_0| + \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{k=1}^n k c_k \\
& \lesssim \|f\|_{\mathcal{B}_\omega} + \|f\|_{\mathcal{B}_\omega} \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{k=1}^n (k+1)^{\alpha+1} \int_0^1 t^k (\tilde{\omega}(t) + 1) d\mu(t) \\
& \lesssim \|f\|_{\mathcal{B}_\omega} + \|f\|_{\mathcal{B}_\omega} \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{k=1}^n \frac{1}{k\nu(1 - \frac{1}{k})} \\
& \lesssim \|f\|_{\mathcal{B}_\omega} + \|f\|_{\mathcal{B}_\omega} \sup_{n \geq 1} \frac{1}{(n+1)^a} \sum_{k=1}^n (k+1)^{a-1} \\
& \lesssim \|f\|_{\mathcal{B}_\omega}.
\end{aligned}$$

Hence $\tilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded by Theorem 3.1.

(b) \Rightarrow (a) : If $\tilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded, then for each $f \in \mathcal{B}_\omega$, by Lemma 2.1 we have

$$\sup_{n \geq 1} \nu(1 - 2^{-n}) 2^n \|V_n * \tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)\|_\infty \asymp \|\tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)\|_{\mathcal{B}_\nu} \lesssim \|f\|_{\mathcal{B}_\omega} \|\tilde{\mathcal{I}}_{\mu_{\alpha+1}}\|.$$

Since the coefficients of $\tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)$ are non-negative, it is easy to check that

$$M_\infty(r, V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)) \leq M_\infty(r, V_n * \tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)) \quad \text{for all } 0 < r < 1.$$

Therefore,

$$\|V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)\|_\infty = \sup_{0 < r < 1} M_\infty(r, V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)) \leq \|V_n * \tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)\|_\infty.$$

Consequently,

$$\|\mathcal{I}_{\mu_{\alpha+1}}(f)\|_{\mathcal{B}_\nu} \asymp \sup_{n \geq 1} \nu(1 - 2^{-n}) 2^n \|V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)\|_\infty \lesssim \|f\|_{\mathcal{B}_\omega} \|\tilde{\mathcal{I}}_{\mu_{\alpha+1}}\|.$$

This implies that $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded. \square

Theorem 4.3. *Let $\omega, \nu \in \mathcal{N}$ and $\alpha > -1$. Suppose μ is a finite positive Borel measure on $[0, 1)$ and $\tilde{\omega}(1) < \infty$. Then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded;
- (b) $\tilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is bounded;
- (c) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is compact;
- (d) $\tilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is compact;
- (e) $\sup_{n \geq 1} n^{\alpha+2} \nu(1 - \frac{1}{n}) \mu_n < \infty$.

Proof. The equivalence of (a) \Leftrightarrow (b) \Leftrightarrow (e) follows from Theorem 4.2 immediately and the implications of (d) \Rightarrow (c) \Rightarrow (a) are obvious. Therefore, we only need to prove that (e) \Rightarrow (d).

Let $\{f_k\}_{k=1}^\infty$ be a bounded sequence in \mathcal{B}_ω which converges to 0 uniformly on every compact subset of \mathbb{D} . Since $\tilde{\omega}(1) < \infty$, arguing as the proof of Lemma 2.5 in [19], we have that

$$\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_k(z)| = 0.$$

For each $k \in \mathbb{N}$, we have

$$\tilde{\mathcal{I}}_{\mu_{\alpha+1}}(f_k)(z) = \int_0^1 \frac{|f_k(t)|}{(1-tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} c_{n,k} z^n,$$

where

$$c_{n,k} = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n |f_k(t)| d\mu(t).$$

It is obvious that $\{c_{n,k}\}_{n=1}^\infty$ is a nonnegative sequence for each $k \in \mathbb{N}$. To prove that $\tilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \rightarrow \mathcal{B}_\nu$ is compact, it is sufficient to prove that

$$\lim_{k \rightarrow \infty} \left(c_{0,k} + \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{j=1}^n j c_{j,k} \right) = 0$$

by using Theorem 3.1 and Lemma 2.6. If $\sup_{n \geq 1} n^{\alpha+2} \nu(1 - \frac{1}{n}) \mu_n < \infty$, then

$$\mu_n \lesssim \frac{1}{n^{\alpha+2} \nu(1 - \frac{1}{n})} \text{ for all } n \in \mathbb{N}.$$

By Stirling's formula and the above inequality, we have

$$\begin{aligned} & |c_{0,k}| + \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{j=1}^n j c_{j,k} \\ & \lesssim \int_0^1 |f_k(t)| d\mu(t) + \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{j=1}^n j^{\alpha+1} \int_0^1 t^j |f_k(t)| d\mu(t) \\ & \lesssim \sup_{t \in [0,1]} |f_k(t)| + \sup_{t \in [0,1]} |f_k(t)| \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{j=1}^n j^{\alpha+1} \mu_j \\ & \lesssim \sup_{t \in [0,1]} |f_k(t)| + \sup_{t \in [0,1]} |f_k(t)| \sup_{n \geq 1} \nu(1 - \frac{1}{n}) \sum_{j=1}^n \frac{1}{j \nu(1 - \frac{1}{j})} \\ & \lesssim \sup_{t \in [0,1]} |f_k(t)| \rightarrow 0, \quad (k \rightarrow \infty). \end{aligned}$$

Hence (d) holds. □

5. Some Applications

As a direct application of the above results, we first consider the operator $\mathcal{I}_{\mu_{\alpha+1}}$ acting from \mathcal{B}^β to \mathcal{B}^γ . If $\gamma \geq \alpha + 2$, then it is easy to see that $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}^\beta \rightarrow \mathcal{B}^\gamma$ is always a bounded operator under the condition (4.1). Therefore, we only need to consider the case $0 < \gamma < \alpha + 2$.

Corollary 5.1. *Let μ be a positive Borel measure on $[0, 1)$ and satisfies $\int_0^1 \log \frac{e}{1-t} d\mu(t) < \infty$, $\alpha > -1$. If $0 < \gamma < \alpha + 2$, then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B} \rightarrow \mathcal{B}^\gamma$ is bounded;
- (b) μ is a 1-logarithmic $\alpha + 2 - \gamma$ -Carleson measure;
- (c) $\int_0^1 t^n \log \frac{e}{1-t} d\mu(t) = O(\frac{1}{n^{\alpha+2-\gamma}})$.

Proof. Let $d\lambda(t) = \log \frac{e}{1-t} d\mu(t)$, then Lemma 2.5 in [11] shows that μ is a 1-logarithmic $\alpha + 2 - \gamma$ -Carleson measure if and only if λ is an $\alpha + 2 - \gamma$ -Carleson measure. By Theorem 2.1 in [20], λ is an $\alpha + 2 - \gamma$ -Carleson measure if and only if

$$\int_0^1 t^n d\lambda(t) = O(\frac{1}{n^{\alpha+2-\gamma}}).$$

The desired result follows from Theorem 4.2 immediately. \square

Remark 5.2. *If $\gamma = 1$ and $\alpha = 0$, the result of Theorem 5.1 have been obtained in [11](or [5]). In addition, if $\gamma = 1$ and $\alpha = 1$, the result have been given in [9].*

Corollary 5.3. *Let μ be a positive Borel measure on $[0, 1)$ and satisfies $\int_0^1 \frac{d\mu(t)}{(1-t)^{\beta-1}} < \infty$, $\alpha > -1$. If $0 < \gamma < \alpha + 2$ and $\beta > 1$, then $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}^\beta \rightarrow \mathcal{B}^\gamma$ is bounded if and only if μ is an $\alpha + 1 + \beta - \gamma$ -Carleson measure.*

Proof. It follows from Theorem 4.2 that $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}^\beta \rightarrow \mathcal{B}^\gamma$ is bounded if and only if

$$\int_0^1 t^n \frac{d\mu(t)}{(1-t)^{\beta-1}} = O(\frac{1}{n^{\alpha+2-\gamma}}).$$

This is equivalent to saying that $\frac{d\mu(t)}{(1-t)^{\beta-1}}$ is an $\alpha + 2 - \gamma$ -Carleson measure. The proof can be done by using Lemma 2.5. \square

Corollary 5.4. *Let μ be a finite positive Borel measure on $[0, 1)$ and $\alpha > -1$. If $0 < \gamma < \alpha + 2$ and $0 < \beta < 1$, then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}^\beta \rightarrow \mathcal{B}^\gamma$ is bounded;
- (b) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}^\beta \rightarrow \mathcal{B}^\gamma$ is compact;
- (c) μ is an $\alpha + 2 - \gamma$ -Carleson measure.

Proof. This is a direct consequence of Theorem 4.3. \square

Remark 5.5. *It should be mentioned that Ye and Zhou [13] have obtained some results of Corollary 5.1-5.4 by using the duality theorem. In fact, they dealt with $\gamma = \alpha$ and $\alpha \geq 1$.*

In what follows, we consider the operator $\mathcal{I}_{\mu_{\alpha+1}}$ acting between logarithmic Bloch spaces.

Corollary 5.6. *Let $\alpha > -1$, $\beta > -1$, $\gamma \in \mathbb{R}$. Suppose μ is a positive Borel measure on $[0, 1)$ and satisfies $\int_0^1 \frac{\log^\beta \frac{e}{1-t}}{1-t} d\mu(t) < \infty$. Then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^\beta} \rightarrow \mathcal{B}_{\log^\gamma}$ is bounded;
- (b) $\sup_{n \geq 1} n^{\alpha+1} \log^{-\gamma}(n+1) \int_0^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) < \infty$;
- (c) $\sup_{t \in [0,1)} \frac{\mu([t, 1)) (\log \frac{e}{1-t})^{\beta+1-\gamma}}{(1-t)^{\alpha+1}} < \infty$.

Proof. It follows from Theorem 4.2 that (a) \Leftrightarrow (b). We only need to show that (b) \Leftrightarrow (c). The implication (b) \Rightarrow (c) follows from the inequalities

$$\mu\left(\left[1 - \frac{1}{n}, 1\right)\right) \log^{\beta+1}(n+1) \lesssim \int_{1-\frac{1}{n}}^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) \lesssim \frac{\log^\gamma(n+1)}{n^{\alpha+1}}.$$

(c) \Rightarrow (b). Assume (c). Then there exists a positive constant C such that

$$\mu([t, 1)) \left(\log \frac{e}{1-t}\right)^{\beta+1-\gamma} \leq C(1-t)^{\alpha+1}, \quad 0 \leq t < 1.$$

Integrating by parts, we obtain

$$\begin{aligned} & \int_0^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) \\ &= n \int_0^1 t^{n-1} \mu([t, 1)) \log^{\beta+1} \frac{e}{1-t} dt + (\beta+1) \int_0^1 t^n \mu([t, 1)) \log^\beta \frac{e}{1-t} \frac{dt}{1-t} \\ &\lesssim n \int_0^1 t^{n-1} (1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t} dt + \int_0^1 t^n (1-t)^\alpha \log^{\gamma-1} \frac{e}{1-t} dt. \end{aligned}$$

Note that

$$\phi_1(t) = (1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t}, \quad \phi_2(t) = (1-t)^\alpha \log^{\gamma-1} \frac{e}{1-t}$$

are regular in the sense of [21]. Then, using Lemma 1.3 and (1.1) in [21], we have

$$n \int_0^1 t^{n-1} (1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t} dt \asymp \frac{\log^\gamma(n+1)}{n^{\alpha+1}}$$

and

$$\int_0^1 t^n (1-t)^\alpha \log^{\gamma-1} \frac{e}{1-t} dt \asymp \frac{\log^{\gamma-1}(n+1)}{n^{\alpha+1}}.$$

These two estimates imply that

$$\int_0^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) \lesssim \frac{\log^\gamma(n+1)}{n^{\alpha+1}}.$$

Thus, (b) holds. □

Arguing as the proof of previous theorem, one can obtain the following theorems.

Corollary 5.7. *Let $\alpha > -1$, $\beta = -1$, $\gamma \in \mathbb{R}$. Suppose μ is a positive Borel measure on $[0, 1)$ and satisfies $\int_0^1 \log \log \frac{e}{1-t} d\mu(t) < \infty$. Then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^{-1}} \rightarrow \mathcal{B}_{\log^\gamma}$ is bounded;
- (b) $\sup_{n \geq 1} n^{\alpha+1} \log^{-\gamma}(n+1) \int_0^1 t^n \log \log \frac{e}{1-t} d\mu(t) < \infty$;
- (c) $\sup_{t \in [0,1)} \frac{\mu([t, 1)) \log \log \frac{e}{1-t}}{(1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t}} < \infty$.

Corollary 5.8. *Let $\alpha > -1$, $\beta < -1$, $\gamma \in \mathbb{R}$. Suppose μ is a finite positive Borel measure on $[0, 1)$, then the following statements are equivalent.*

- (a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^\beta} \rightarrow \mathcal{B}_{\log^\gamma}$ is bounded;
- (b) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^\beta} \rightarrow \mathcal{B}_{\log^\gamma}$ is compact;
- (c) $\sup_{n \geq 1} n^{\alpha+1} \log^{-\gamma}(n+1) \mu_n < \infty$;
- (d) $\sup_{t \in [0,1)} \frac{\mu([t, 1)) \log^{-\gamma} \frac{e}{1-t}}{(1-t)^{\alpha+1}} < \infty$.

It is known that \mathcal{H} maps \mathcal{B}_{\log^β} into $\mathcal{B}_{\log^{\beta+1}}$ for all $\beta \in \mathbb{R}$ (see e.g., [22]). If μ is Lebesgue measure on $[0, 1)$, then Corollary 5.6-5.8 show that the integral type Hilbert operator $\mathcal{I} : \mathcal{B}_{\log^\beta} \rightarrow \mathcal{B}_{\log^{\beta+1}}$ is bounded if and only if $\beta > -1$.

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References

- [1] R. Zhao, On logarithmic Carleson measures, Acta Sci. Math. (Szeged) 69 (3–4) (2003) 605–618.
- [2] C. Chatzifountas, D. Girela, J. Peláez, A generalized Hilbert matrix acting on Hardy spaces, J. Math. Anal. Appl. 413 (2014) 154–168.
- [3] J. Peláez, J. Rättyä, Generalized Hilbert operators on weighted Bergman spaces, Adv. Math. 240 (2013) 227–267.
- [4] S. Li, Generalized Hilbert operator on the Dirichlet-type space, Appl. Math. Comput. 214 (2009) 304–309.
- [5] D. Girela, N. Merchán, A Hankel matrix acting on spaces of analytic functions, Integral Equ. Oper. Theory. 89 (4) (2017) 581–594.
- [6] D. Girela, N. Merchán, Hankel matrices acting on the Hardy space H^1 and on Dirichlet spaces, Rev. Mat. Comput. 32 (2019) 799–822.

- [7] M. Jevtić, B. Karapetrović, Generalized Hilbert matrices acting on spaces that are close to the Hardy space H^1 and to the space BMOA, *Complex Anal. Oper. Theory*. 13 (2019) 2357–2370.
- [8] N. Merchán, Mean Lipschitz spaces and a generalized Hilbert operator, *Collect. Math.* 70 (2019) 59–69.
- [9] S. Ye, Z. Zhou, A Derivative-Hilbert operator acting on the Bloch space, *Complex Anal. Oper. Theory*. 15 (88) (2021) <https://doi.org/10.1007/s11785-021-01135-1>.
- [10] S. Ye, Z. Zhou, A Derivative-Hilbert operator acting on Bergman spaces, *J. Math. Anal. Appl.* 506 (2021) <https://doi.org/10.1016/j.jmaa.2021.125553>.
- [11] D. Girela, N. Merchán, A generalized Hilbert operator acting on conformally invariant spaces, *Banach J. Math. Anal. Appl.* 12 (2018) 374–398.
- [12] S. Li, J. Zhou, Essential norm of generalized Hilbert matrix from Bloch type spaces to BMOA and Bloch space, *AIMS. Math.* 6 (2021) 3305–3318.
- [13] S. Ye, Z. Zhou, Generalized Hilbert operator acting on Bloch type spaces, *Acta. Math. Sinica. Ser. A.* (2022) <https://kns.cnki.net/kcms/detail/11.2038.O1.20220613.1001.018.html>.
- [14] M. Pavlović, Analytic functions with decreasing coefficients and Hardy and Bloch spaces, *Proc. Edinb. Math. Soc.* 56 (2013) 623–635.
- [15] P. Galanopoulos, D. Girela, J. Peláez, A. Siskakis, Generalized Hilbert operators, *Ann. Acad. Sci. Fenn. Math.* 39 (2014) 231–258.
- [16] Z. Hu, Composition operators between Bloch-type spaces in the polydisc, *Sci. China* 48A (sup-p) (2005) 268–282.
- [17] X. Zhang, L. Xi, H. Fan, J. Li, Atomic decomposition of μ -Bergman space in C^n , *Acta Math. Sci.* 34B (3) (2014) 779–789.
- [18] C. Cowen, B. MacCluer, *Composition operators on spaces of analytic Functions*, CRC Press, Boca Raton, 1995.
- [19] S. Stević, On an integral-type operator from zygmond-type spaces to Mixed-norm spaces on the unit ball, *Abstr. Appl. Anal.* 2010 (11) (2010) 3817–3823.
- [20] G. Bao, H. Wulan, Hankel matrices acting on Dirichlet spaces, *J. Math. Anal. Appl.* 409 (2014) 228–235.
- [21] J. Á. Peláez, J. Rättyä, Weighted bergman spaces induced by rapidly increasing weights, *Mem. Am. Math. Soc.* 227 (2014) 1–122.
- [22] B. Karapetrović, Libera and Hilbert matrix operator on logarithmically weighted Bergman, Bloch and Hardy-Bloch spaces, *Czech. Math. J.* 68 (2018) 559–576.

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