A theoretical model for buoyancy flux determination in planetary boundary layer based on endoreversible heat engine

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Key Points:

- PBL can be treated as endoreversible heat engine operating between the surface temperature and temperature at the top of the boundary layer.
- The total contribution of buoyancy to TKE in the boundary layer and heat engine efficiency increase linearly with the boundary layer height.
- The derived buoyancy flux from our theoretical model is consistent with the results from numerical simulation and dimensional analysis.

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Abstract

The determination of buoyancy flux and its contribution to turbulence kinetic energy (TKE) is a fundamental problem in planetary boundary layer (PBL). However, due to the complexity of turbulence, previous studies mainly adopted dimensional analysis and empirical formula to determine TKE budget. This study introduces the endoreversible heat engine model concept to the convective boundary layer (CBL) TKE analysis and establishes a theoretical model based on the first principles. We found that the total contribution of buoyancy to TKE and heat engine efficiency in the boundary layer increase linearly with the boundary layer height. The derived buoyancy flux from our theoretical model is consistent with the results from numerical simulation and dimensional analysis. This heat engine-based theory reveals the physical mechanism of the power of TKE generated by buoyancy. Our theoretical model can replace the empirical value and provide an ideal method for buoyancy flux determination in PBL.

Plain language summary

The determination of the efficiency and power of kinetic energy from thermal energy is a fundamental problem in atmospheric science. We introduce the heat engine model concept to the turbulence kinetic energy in planetary boundary layer (PBL) and successfully estimate the power of buoyancy work. The derived buoyancy flux from our theoretical model is consistent with the results from numerical simulation and dimensional analysis. Our theoretical model can replace the empirical value and provide an ideal method for buoyancy flux determination in PBL.

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1 Introduction

Planetary boundary layer (PBL) is the lowest part of the atmosphere closely related to human life, and its formation is driven by the turbulence. The conversion from thermal energy to kinetic energy, i.e., buoyancy contribution, is one of the two major resources generating and maintaining the turbulence in PBL. While the complexities of turbulence makes it difficult to estimate buoyancy flux and its contribution to turbulence kinetic energy (TKE). Most PBL modules are based on dimensional analysis and empirical formula to estimate buoyancy contribution to turbulence (Cheng, Li, Li, & Gentine, 2021; Li, Bou-Zeid, & De Bruin, 2012; Monin & Obukhov, 1954; Stull, 1988). Different PBL parameterization schemes have been used in numerical models (Hong, Noh, & Dudhia, 2006; Hong & Pan, 1996; Janjic, 2003; Pleim, 2007). However, studies from the first principles are rare.

To make a precise quantification of the effect of buoyancy on PBL, establishing proper thermodynamic models is a possible method since the physics of atmosphere is governed by the laws of the thermodynamics, especially in the convective boundary layer (CBL) where buoyancy is the main source of TKE (Stull, 1988). Several studies have explored atmospheric processes from the perspective of the efficiency of kinetic energy from thermal energy (Lorenz, 1967; Wulf & Davis, 1952), such as global circulation(L. Michaud, 1995; L. M. Michaud, 2000) and hurricanes(Chavas, 2017; O. M. Pauluis & Zhang, 2017; Trabing, Bell, & Brown, 2019). Previous studies normally treat the atmosphere as a Carnot heat engine, which is reversible. However, O. Pauluis (2011) pointed out that the efficiency of a heat engine is not as high as that of a Carnot engine due to the irreversible process, such as evaporation and condensation processes of water. In addition to the effect of water vapor, previous research on heat engines (Callen, 1985) also found that Carnot heat engines have zero power because all processes are quasi-static and reversible. Due to the heat transfer, the actual heat engine must involve irreversible processes, resulting in a decrease in the efficiency of the heat engine. Studies which take irreversible process of heat transfer into account in PBL meteorology are still lacking.

In order to explore the physical mechanism of CBL maintenance, this study introduces the endoreversible heat engine model to calculate the power and efficiency of the conversion from sensible heat flux to TKE. We found that PBL can be treated as a heat engine operating between the surface temperature and the temperature at the top of the boundary layer. A physical process-based theoretical model are established in this study to estimate the contribution of buoyancy to the TKE in the boundary layer and the vertical profile of the buoyancy flux.

2 The endoreversible heat engine

The contribution of buoyancy to TKE in PBL can be transformed into a heat engine power problem. The CBL can be treated as a heat engine operating between the surface and the top of the boundary layer. For a system that cannot do work on the external system (when there is no large-scale circulation and it is homogeneous in horizontal scale), it results in internal motion, which means the conversion of work into TKE. As mentioned in the previous section, this heat engine is not completely reversible due to the heat transfer process. Therefore, the endoreversible heat engine model is introduced into the analysis of the TKE budget estimation in CBL.

To determine the conversion efficiency, vertical temperature profile is needed at first. CBL is divided into the near-surface layer, the mixed layer, and the entrained layer, which accounts for 5-10%, 35-80%, 10-60% of the CBL thickness (z_i) along the vertical direction, respectively(Stull, 1988). The mixed layer thickness is assumed to be a fraction of k of PBL height, which means,

$$l_{ML} = k \cdot z_i \tag{1}$$

There is a large temperature difference in the near-surface layer and entrainment layer. Therefore, the heat transfer in the near-surface layer and the entrainment layer should be irreversible process. While, the air mass in the mixed layer can move freely due to the constant potential temperature and is always in thermal equilibrium. Thus, the heat process in the mixed layer could be treated as reversible process. As shown in Fig. 1, we assume that the temperature at the top of boundary layer is T_c (cold), the temperature at the top of mixed layer is T_t (tepid), the temperature at the bottom of mixed layer is T_w (warm), and the surface temperature is T_h (hot), which can also be written as T_s (surface). These four types of temperature follow the relationship $T_h > T_w > T_t > T_c$. The work of the endoreversible heat engine is shown in Fig. 1 (a).

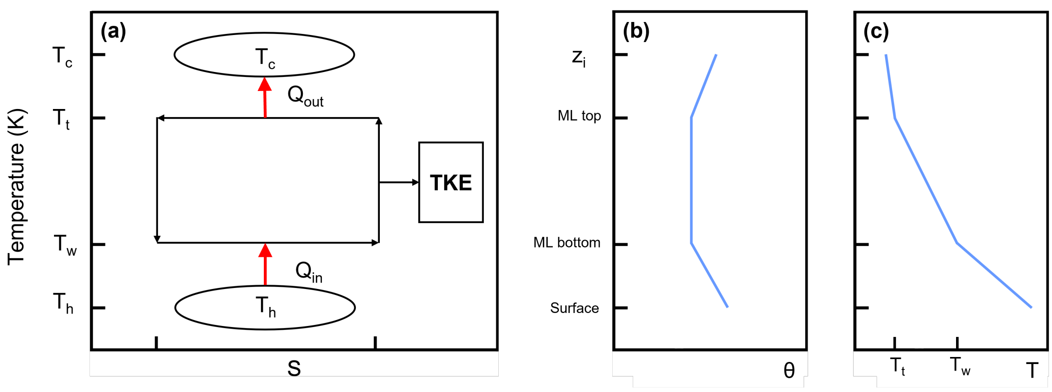

Figure 1. (a) The schematic figure of heat engine of the CBL; (b) The vertical profile of potential temperature; (c) The vertical profile of temperature.

The derivation of heat engine power in CBL follows the basic mechanism of endoreversible heat engine in pioneer studies(Callen, 1985, Curzon and Ahlborn (1975) and Hoffmann (2008)). To link the heat engine to TKE budget, the estimation of the power of heat engine is needed. We assume that the time required for the ascent and descent of air mass is negligible. Therefore, the time needed for the cycle of the heat engine is equal to the sum of the two heat transfer processes (Callen, 1985; Hoffmann, 2008) in the surface layer and the entrainment layer .

$$t = t_{in} + t_{out} \tag{2}$$

where, t_{in} is the time needed for the heat transferring from surface to the mixed layer and t_{out} is the time needed for the heat transferring out from the mixed layer. The heat transfers from surface into the CBL. The input heat per unit time (P_{in}) can be presented as,

$$P_{in} = \frac{Q_{in}}{t_{in}} \tag{3}$$

To be noted that, the heat fluxes are driven by the external forces, such as solar radiation. Therefore, P_{in} could be treated as a fixed value in analysis. Similarly, the heat exhaust per unit time (P_{out}) from CBL is,

$$P_{out} = \frac{Q_{out}}{t_{out}} \tag{4}$$

Combining Eqs. (2), (3) and (4), we can obtain

$$\begin{aligned} t &= t_{in} + t_{out} \\ &= \frac{Q_{in}}{P_{in}} + \frac{Q_{out}}{P_{out}} \end{aligned} \tag{5}$$

The output power of the heat engine, which also denote the contribution of buoyancy to TKE, is

$$P_w = \frac{W}{t} \tag{6}$$

W is the work of heat engine. The inner part of the endoreversible Carnot heat engine (in the mixed layer) is reversible, which is equivalent to Carnot heat engine working between T_w and T_t. According to the Carnot heat engine theory, we can obtain that this efficiency is

$$\varepsilon_c = \frac{W}{Q_{in}} = \frac{T_w - T_t}{T_w} \tag{7}$$

and the ratio between W and Q_{out} is

$$\frac{W}{Q_{out}} = \frac{T_w - T_t}{T_t} \tag{8}$$

Combining Eqs. (5), (7) and (8), the time needed for a cycle of endoreversible Carnot heat engine can be expressed as

$$t = \left[\frac{1}{P_{in}} \frac{T_w}{T_w - T_t} + \frac{1}{P_{out}} \frac{T_t}{T_w - T_t} \right] W \tag{9}$$

Therefore, the power of this heat engine is,

$$\begin{aligned} P_w &= \frac{W}{t} \\ &= \left[\frac{1}{P_{in}} \frac{T_w}{T_w - T_t} + \frac{1}{P_{out}} \frac{T_t}{T_w - T_t} \right]^{-1} \end{aligned} \tag{10}$$

Because the CBL is substantially thinner than the troposphere, the temperature difference in the boundary layer is small compared to the surface temperature ($T_h - T_c \ll T_h$). Therefore, the output power is small compared to the input heat or the heat exhaust, i.e., $P_{in} \approx P_{out}$. Also, for the denominator term, it can be approximated as ($T_h \approx T_w \approx T_t \approx T_c$). Thus, Eq. (10) could be rewritten as,

$$\begin{aligned} P_w &\approx P_{in} \cdot \frac{T_w - T_t}{T_w + T_t} \\ &\approx P_{in} \cdot \frac{T_w - T_t}{2T_h} \end{aligned} \tag{11}$$

Due to the intense turbulent mixing in the mixed layer, the temperature approximately decreases adiabatically. Assuming no water phase transition, the vertical temperature change conforms to the dry adiabatic lapse rate,

$$R_d = \frac{\partial T}{\partial z} = \frac{g}{C_p} = Const \tag{12}$$

where, g is the gravitational acceleration and C_p is the isobaric specific heat capacity. Therefore, the temperature difference between top and bottom of the mixed layer is

$$T_w - T_t = l_{ML} \cdot R_d = \frac{g}{C_p} \cdot l_{ML} \tag{13}$$

where, l_{ML} is the height of the mixed layer, which is assumed as kz_i. Combining Eqs. (12) and (13), Eq. (11) can be represented as

$$P_w = P_{in} \cdot \frac{g}{C_p} \cdot \frac{k}{2T_h} z_i \tag{14}$$

We can draw a conclusion that the total efficiency of the endoreversible heat engine is proportional to the height of CBL. This system has a similar concept of a wooden block being pressed against the bottom of a glass of water. The potential energy of buoyancy is proportional to the depth of the water, which is similar to the CBL thickness. The increase in TKE is approximately equal to the decrease in buoyant potential energy in CBL.

3 Vertical profile of buoyancy flux

With above theoretical derivation, we have determined the total contribution of buoyancy in CBL. The next question is to estimate the vertical profile of buoyancy flux. The quasi-steady-state approximation implies that the vertical profile of the θ is constant over time, that is,

$$\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial z}\right) = 0 \tag{15}$$

By interchanging the order of derivation, we obtain,

$$\frac{\partial}{\partial z}\left(\frac{\partial \theta}{\partial t}\right) = 0 \tag{16}$$

Also, the equation of TKE balance (Stull, 1988) is written as,

$$\frac{\partial \bar{\theta}}{\partial t} + U_j \frac{\partial \bar{\theta}}{\partial x_j} = \frac{v_\theta \partial^2 \bar{\theta}}{\partial x_j^2} - \frac{L_v E}{\bar{\rho} C_p} - \frac{\partial \left(\overline{u_j' \theta'}\right)}{\partial x_j} \tag{17}$$

where the first and second terms represent the mean storage of heat and the advection of heat by mean wind, respectively. The third term is the mean molecular conduction of heat. The fourth term is the source associated with latent heat release. The last term represents the divergence of turbulent heat flux(Stull, 1988). After ignoring the advection of heat, the mean molecular conduction of heat, and the latent heat release, the second to fourth terms turn to zero. By deriving with respect to z on both sides of Eq. (17), we can get

$$\frac{\partial}{\partial z}\frac{\partial \theta}{\partial t} + \frac{\partial}{\partial z}\frac{\partial \overline{w'\theta'}}{\partial z} = 0 \tag{18}$$

Similar to the derivation in Vilà-Guerau de Arellano (2015), by combining Eq. (15) and (18), we can get,

$$\frac{\partial^2 \overline{w'\theta'}}{\partial z^2} = 0 \tag{19}$$

It means the kinematic flux is linear in CBL, that is,

$$\overline{w'\theta'}(z) = A + Bz \tag{20}$$

Plus, we have the input heat power at surface ($z = 0$),

$$P_{in} = C_p \overline{w'\theta'_s} \tag{21}$$

where, $\overline{w'\theta'_s}$ is the kinetic flux at surface. Then, we can get,

$$A = \overline{w'\theta'_s} = \frac{P_{in}}{C_p} \tag{22}$$

With the restriction that the work power equals to the sum of buoyancy work in vertical direction,

$$P_w = \int_0^{z_i} \frac{g}{\theta(z)} \overline{w'\theta'} dz \tag{23}$$

As mentioned above, $T_h \approx T_w \approx T_t \approx T_c$, Eq. (23) can be approximated as

$$P_w \approx \frac{g}{T_h} \int_0^{z_i} (A + Bz) dz \tag{24}$$

combining Eq. (14) and (23), we can get,

$$P_w = P_{in} \cdot \frac{g}{C_p} \cdot \frac{k}{2T_h} z_i = \frac{g}{T_h} \int_0^{z_i} (A + Bz) dz \tag{25}$$

Combining Eqs. (22) and (25), we can derive,

$$B = (k-2) \frac{1}{z_i} \overline{w'\theta'_s} \tag{26}$$

Combining Eqs. (20), (22) and (26), we can obtain,

$$\frac{\overline{w'\theta'}(z)}{\overline{w'\theta'_s}} = (1 + (k-2)\frac{z}{z_i}) \tag{27}$$

Above equation determines the vertical profile of contribution of buoyancy on TKE. To be noted that $\overline{w'\theta'}(z) \neq \overline{w'\theta'_s}$ is not contradictory to the heat balance assumption. In addition to turbulent transport, there are other pathways of heat transfer, such as advection and radiation.

4 Model validation

To validate our analysis, we compared the results from endoreversible heat engine model with the traditional dimensional analysis method and simulation data (Fig. 2). Following(Stull, 1988), we can assume mixed layer height is 0.8 of z_i. Thus, (27) turns to be

$$\frac{\overline{w'\theta'}(z)}{\overline{w'\theta'_s}} = 1 - 1.2\frac{z}{z_i} \tag{28}$$

which means the ratio of buoyancy flux at the top and at the bottom (A_R) is 0.2, which is consistent with previous study(Stull, 1976). Fig. 2 shows that our analysis perfectly fit the simulation data that indicates that the endoreversible heat engine is applicable in the CBL development.

The estimation of buoyancy flux based on the Carnot heat engine assumption is,

$$P_{w_{Carnot}} = P_{in} \cdot \frac{g}{C_p} \cdot \frac{1}{T_h} z_i \tag{29}$$

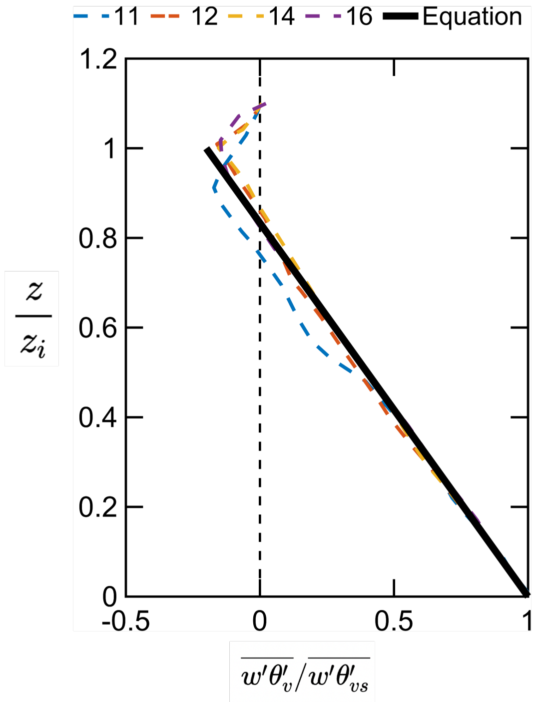

Figure 2. Heat flux data from a simulation of Wangara Day 33 (Clarke et al., 1971) and re-plot from Stull (1988)

The comparison between Eqs. (14) and (29) shows the results based on Carnot engine are more than twice the results using endoreversible engine. It implies that the Carnot engine assumption leads to a significant overestimation of the buoyant power, due to the neglect of the increase in entropy caused by the irreversible process of heat transfer.

5 Conclusions and Implications

The determination of the efficiency and power of kinetic energy from thermal energy is crucial to quantify the TKE in CBL. However, due to the complexity of turbulence, current knowledge is insufficient to derive laws based on first principles. Therefore, previous studies adopted the similarity theory and dimensional analysis method to determine the TKE budget, which needs to introduce an assumption of a constant ratio of buoyancy flux at the top and at the bottom of 0.2 for CBL TKE budget function closure. In this study, we introduce an endoreversible heat engine concept into the CBL TKE analysis and established a theoretical model based on first principles. We found that the total contribution of buoyancy to TKE in the boundary layer and heat engine efficiency increase linearly with the boundary layer height z_i. Moreover, our theoretical framework proves the linear decrease of buoyancy flux in vertical direction. The derived buoyancy flux from our theoretical model is consistent with the results from numerical simulation and dimensional analysis. This heat engine-based theory reveals the physical mechanism of the power of TKE generated by buoyancy. It may offer novel insights in dynamics under more complex situations in PBL.

Open Research

The analytic data are accessible at figshare (https://doi.org/10.6084/m9.figshare.21424917.v1).

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Author contributions

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Methodology: JDW, JPW, YBL, CGW

Investigation: JDW

Formal analysis: JDW, JPW, CL

Visualization: JPW, JDW, LC

Writing original draft: JDW, JPW

Writing review and editing: JPW, JDW, YBL, CL, LC, CGW, CSL, AJD, SXW, JMH

References

Callen, H. B. (1985). *Thermodynamics and an introduction to thermostatistics* (2nd ed ed.). New York: Wiley.

Chavas, D. R. (2017, September). A Simple Derivation of Tropical Cyclone Ventilation Theory and Its Application to Capped Surface Entropy Fluxes. *Journal of the Atmospheric Sciences*, *74*(9), 2989–2996. doi: 10.1175/JAS-D-17-0061.1

Cheng, Y., Li, Q., Li, D., & Gentine, P. (2021, March). Logarithmic profile of temperature in sheared and unstably stratified atmospheric boundary layers. *Physical Review Fluids*, *6*(3), 034606. doi: 10.1103/PhysRevFluids.6.034606

Clarke, R. H., Dyer, A. J., Brook, R. R., Reid, D. G., & Troup, A. J. (1971). The Wangara Experiment: Boundary Layer Data.

Curzon, F. L., & Ahlborn, B. (1975, January). Efficiency of a Carnot engine at maximum power output. *American Journal of Physics*, *43*(1), 22–24. doi: 10.1119/1.10023

Hoffmann, K. H. (2008, February). An introduction to endoreversible thermodynamics. *Atti della Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali*(1), 1–18. doi: 10.1478/C1S0801011

Hong, S.-Y., Noh, Y., & Dudhia, J. (2006). A New Vertical Diffusion Package with an Explicit Treatment of Entrainment Processes. *Monthly Weather Review*, *134*(9), 2318–2341. doi: 10.1175/mwr3199.1

Hong, S.-Y., & Pan, H.-L. (1996). Nonlocal Boundary Layer Vertical Diffusion in a Medium-Range Forecast Model. *Monthly Weather Review*, *124*(10), 2322–2339. doi: 10.1175/1520-0493(1996)124<2322:Nblvdi>2.0.Co;2

Janjic, Z. I. (2003, January). A nonhydrostatic model based on a new approach. *Meteorology and Atmospheric Physics*, *82*(1), 271–285. doi: 10.1007/s00703-001-0587-6

Li, D., Bou-Zeid, E., & De Bruin, H. A. R. (2012, October). Monin–Obukhov Similarity Functions for the Structure Parameters of Temperature and Humidity. *Boundary-Layer Meteorology*, *145*(1), 45–67. doi: 10.1007/s10546-011-9660-y

Lorenz, E. N. (1967). The nature and theory of the general circulation of the atmosphere..

Michaud, L. (1995, October). Heat to Work Conversion During Upward Heat Convection .1. Carnot Engine Method. *Atmospheric Research*, *39*(1-3), 157–178. doi: 10.1016/0169-8095(95)00010-O

Michaud, L. M. (2000). Thermodynamic cycle of the atmospheric upward heat convection process. *Meteorology and Atmospheric Physics*, *72*(1), 29–46. doi: 10.1007/s007030050003

Monin, A. S., & Obukhov, A. M. (1954). Basic laws of turbulent mixing in the surface layer of the atmosphere. , 30.

Pauluis, O. (2011, January). Water Vapor and Mechanical Work: A Comparison of

Carnot and Steam Cycles. *Journal of the Atmospheric Sciences*, *68*(1), 91–102. doi: 10.1175/2010JAS3530.1

Pauluis, O. M., & Zhang, F. (2017, October). Reconstruction of Thermodynamic Cycles in a High-Resolution Simulation of a Hurricane. *Journal of the Atmospheric Sciences*, *74*(10), 3367–3381. doi: 10.1175/JAS-D-16-0353.1

Pleim, J. E. (2007, September). A Combined Local and Nonlocal Closure Model for the Atmospheric Boundary Layer. Part II: Application and Evaluation in a Mesoscale Meteorological Model. *Journal of Applied Meteorology and Climatology*, *46*(9), 1396–1409. doi: 10.1175/JAM2534.1

Stull, R. B. (1976, July). The Energetics of Entrainment Across a Density Interface. *Journal of the Atmospheric Sciences*, *33*(7), 1260–1267. doi: 10.1175/1520-0469(1976)033<1260:TEOEAD>2.0.CO;2

Stull, R. B. (1988). *An introduction to boundary layer meteorology.* Dordrecht ; Boston: Kluwer Academic Publishers.

Trabing, B. C., Bell, M. M., & Brown, B. R. (2019, January). Impacts of Radiation and Upper-Tropospheric Temperatures on Tropical Cyclone Structure and Intensity. *Journal of the Atmospheric Sciences*, *76*(1), 135–153. doi: 10.1175/JAS-D-18-0165.1

Vilà-Guerau de Arellano, J. (2015). *Atmospheric boundary layer: Integrating air chemistry and land interactions.* New York, NY: Cambridge University Press.

Wulf, O. R., & Davis, L. (1952, April). ON THE EFFICIENCY OF THE ENGINE DRIVING THE ATMOSPHERIC CIRCULATION. *Journal of Meteorology*, *9*(2), 80–82. doi: 10.1175/1520-0469(1952)009<0080:OTEOTE>2.0.CO;2