Supporting Information of "lower hybrid drift waves during guide field reconnection"

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1 Derivation of $\mathbf{q}^{\parallel}$ (Eqn. 12)

From the kinetic equation in the $(t, \mathbf{r}, \mathbf{w} \equiv \mathbf{v} - \mathbf{V}))$ coordinates ($\mathbf{V}$ is the fluid velocity),

$$\frac{df}{dt} - (\mathbf{w} \cdot \nabla \mathbf{V}) \cdot \frac{\partial}{\partial \mathbf{w}} f + \nabla \cdot (\mathbf{w} f) + \frac{\partial}{\partial \mathbf{w}} \cdot (\mathbf{A} f) + \frac{q}{m} \mathbf{w} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{w}} f = C(f), \tag{1}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla, \tag{2}$$

$$\mathbf{A} = \frac{1}{m}[\mathbf{F}_* + q(\mathbf{V} \times \mathbf{B})] - \frac{d\mathbf{V}}{dt}. \tag{3}$$

For the $p^{\parallel}$ fluid equation, we need to obtain the closure

$$\mathbf{q}^{\parallel} = \int d^3 v m w_{\parallel}^2 \mathbf{w} f = q_{\parallel}^{\parallel} \hat{z} + \mathbf{q}_{\perp}^{\parallel} \tag{4}$$

where $q_{\parallel}^{\parallel} = \int d^3 v m w_{\parallel}^3 f$ has been obtained in J.-Y. Ji and Joseph (2018) and we will obtain the $\mathbf{q}_{\perp}^{\parallel}$ closure. We adopt the closure (transport) ordering $d/dt \approx 0$ and the linear response theory.

We take the moments $\int d^3 v m w_{\parallel}^2 \mathbf{w}$ of the kinetic equation:

$$\int d^3 v m w_{\parallel}^2 \mathbf{w} \frac{df}{dt} = \frac{d}{dt} \mathbf{q}^{\parallel} \ : \text{ ignored by the closure ordering}, \tag{5}$$

$$\int d^3 v m w_{\parallel}^2 \mathbf{w} (\mathbf{w} \cdot \nabla \mathbf{V}) \cdot \frac{\partial}{\partial \mathbf{w}} f \ : \text{ ignored by the linearization}, \tag{6}$$

$$\int d^3 v m w_{\parallel}^2 \mathbf{w} \nabla \cdot (\mathbf{w} f) = \nabla \cdot (\hat{z}\hat{z} : \int d^3 v m \mathbf{wwww} f). \tag{7}$$

We should decompose $\mathbf{wwww}$ into orthogonal polynomials (see J.-Y. Ji and Held (2008)) for the consistent truncation in the expansion of a distribution function.

$$\mathbf{c} = \frac{\mathbf{w}}{v_T} = \frac{\mathbf{w}}{\sqrt{2T/m}} \tag{8}$$

$$\begin{aligned} \mathbf{cccc} &= \mathsf{p}^4 + \frac{6}{7}\left(c^2 - \frac{7}{2}\right)\left\{\mathsf{p}^2\mathsf{I}\right\} + 3\left\{\mathsf{p}^2\mathsf{I}\right\} + \left(\frac{2}{5}\mathsf{p}^{02} - \mathsf{p}^{01} + \frac{3}{4}\right)\{\mathsf{II}\} \\ &= \mathsf{p}^{40} - \frac{6}{7}\left\{\mathsf{p}^{21}\mathsf{I}\right\} + 3\left\{\mathsf{p}^{20}\mathsf{I}\right\} + \left(\frac{2}{5}\mathsf{p}^{02} - \mathsf{p}^{01} + \frac{3}{4}\right)\{\mathsf{II}\} \\ &= 3\left\{\mathsf{p}^2\mathsf{I}\right\} + \frac{3}{4}\{\mathsf{II}\} + \text{higher order moments to be truncated}, \end{aligned} \tag{9}$$

where the operator $\{...\}$ is the symmetrization of the tensor (J.-Y. Ji & Held, 2008). For $\mathbf{bb}:\mathbf{cccc}$ (notation $\mathbf{b}=\hat{z}$),

$$\mathbf{bb}:\left\{\mathsf{p}^2\mathsf{I}\right\}=\frac{1}{6}\left(\mathbf{bb}:\mathsf{p}^2\mathsf{I}+2\mathbf{bb}\cdot\mathsf{p}^2+2\mathbf{b}\cdot\mathsf{p}^2\mathbf{b}+\mathsf{p}^2\right), \tag{10}$$

$$\mathbf{bb}:\{\mathsf{II}\}=\frac{1}{3}(\mathsf{I}+2\mathbf{bb}). \tag{11}$$

Therefore,

$$\begin{aligned}
\int d\mathbf{v}mw_\parallel^2\mathbf{ww}f &= mv_T^4\int d\mathbf{v}\mathbf{bb}:\left(3\left\{\mathsf{p}^2\mathsf{I}\right\}+\frac{3}{4}\{\mathsf{II}\}\right)f \qquad (12)\\
&= mv_T^4\int d\mathbf{v}\left[3\frac{1}{6}\left(\mathbf{bb}:\mathsf{p}^2\mathsf{I}+2\mathbf{bb}\cdot\mathsf{p}^2+2\mathbf{bb}\cdot\mathsf{p}^2+\mathsf{p}^2\right)+\frac{3}{4}\frac{1}{3}(\mathsf{I}+2\mathbf{bb})\right]f\\
&= \frac{v_T^2}{2}\left(\pi_\parallel\mathsf{I}+2\mathbf{bb}\cdot\boldsymbol{\pi}+2\mathbf{b}\cdot\boldsymbol{\pi}\mathbf{b}+\boldsymbol{\pi}\right)+mv_T^4\frac{1}{4}n\left(\mathsf{I}+2\mathbf{bb}\right)\\
&= \frac{T}{m}\left(\pi_\parallel\mathsf{I}+2\mathbf{bb}\cdot\boldsymbol{\pi}+2\mathbf{b}\cdot\boldsymbol{\pi}\mathbf{b}+\boldsymbol{\pi}\right)+T\frac{p}{m}\left(\mathsf{I}+2\mathbf{bb}\right)\\
&= \frac{T}{m}p^\parallel\mathsf{I}+2\frac{T}{m}\mathbf{bb}\cdot\boldsymbol{\pi}+2\frac{T}{m}\mathbf{b}\cdot\boldsymbol{\pi}\mathbf{b}+\frac{T}{m}\boldsymbol{\pi}+T\frac{p}{m}2\mathbf{bb},
\end{aligned}$$

where

$$p^\parallel = p+\pi_\parallel, \tag{13}$$

$$p^\perp = p-\frac{1}{2}\pi_\parallel, \tag{14}$$

$$\pi_\parallel=\frac{2}{3}(p^\parallel-p^\perp), \tag{15}$$

$$p=\frac{1}{3}\left(p^\parallel+2p^\perp\right), \tag{16}$$

$$\boldsymbol{\pi}=\frac{3}{2}\pi_\parallel(\mathbf{bb}-\frac{1}{3}\mathsf{I}), \tag{17}$$

$$\mathbf{b}\cdot\boldsymbol{\pi}=\pi_\parallel\mathbf{b}. \tag{18}$$

Hereafter we will drop $\mathbf{b}$ terms, which will be nullified by the $\mathbf{b}\times$ operation:

$$\nabla\cdot\boldsymbol{\pi}=\frac{3}{2}\mathbf{b}\partial_\parallel\pi_\parallel-\frac{1}{2}\nabla\pi_\parallel\to-\frac{1}{2}\nabla\pi_\parallel, \tag{19}$$

$$\begin{aligned}
\nabla\cdot\int d\mathbf{v}mw_\parallel^2\mathbf{ww}f &\approx \nabla\cdot\left(\frac{T}{m}p^\parallel\mathsf{I}+2\frac{T}{m}\mathbf{bb}\cdot\boldsymbol{\pi}+2\frac{T}{m}\mathbf{b}\cdot\boldsymbol{\pi}\mathbf{b}+\frac{T}{m}\boldsymbol{\pi}+T\frac{p}{m}2\mathbf{bb}\right) \qquad (20)\\
&= \nabla\cdot\left(\frac{T}{m}p^\parallel\mathsf{I}+4\frac{T}{m}\pi_\parallel\mathbf{bb}+\frac{T}{m}\boldsymbol{\pi}+T\frac{p}{m}2\mathbf{bb}\right)\\
&\to \frac{1}{m}p^\parallel\nabla T+\frac{1}{m}T\nabla p^\parallel-\frac{T}{2m}\nabla\pi_\parallel.
\end{aligned}$$

For the $\frac{\partial}{\partial\mathbf{w}}\cdot(\mathbf{A}f)$ term,

$$\mathbf{A}=\frac{1}{m}[\mathbf{F}_*+q(\mathbf{V}\times\mathbf{B})]-\frac{d\mathbf{V}}{dt}=\frac{1}{mn}(\nabla p+\nabla\cdot\boldsymbol{\pi}). \tag{21}$$

$$
\begin{aligned}
\int d\mathbf{v} m w_\parallel^2 \mathbf{w} \frac{\partial}{\partial \mathbf{w}} \cdot (\mathbf{A} f) &= -\int d\mathbf{v} m \mathbf{A} \cdot \frac{\partial}{\partial \mathbf{w}} (w_\parallel^2 \mathbf{w}) f \\
&= -\int d\mathbf{v} m (2\mathbf{b} \cdot \mathbf{w}\mathbf{w}\mathbf{A} \cdot \mathbf{b} + w_\parallel^2 \mathbf{A}) f \\
&= -2\mathbf{b} \cdot \mathsf{p}\mathbf{A} \cdot \mathbf{b} - p^\parallel \mathbf{A} \\
&= -2p^\parallel \mathbf{A} \cdot \mathbf{b}\mathbf{b} - p^\parallel \mathbf{A} \\
&\to -p^\parallel \frac{1}{mn} (\nabla p + \nabla \cdot \boldsymbol{\pi}) \\
&= -p^\parallel \frac{1}{mn} (\nabla p - \frac{1}{2} \nabla \pi_\parallel + \frac{3}{2} \mathbf{b} \partial_\parallel \pi_\parallel) \\
&\to -\frac{p^\parallel}{mn} \nabla p^\perp .
\end{aligned} \tag{22}
$$

All together, $\nabla \cdot (\mathbf{w} f) + \frac{\partial}{\partial \mathbf{w}} \cdot (\mathbf{A} f)$

$$
\begin{aligned}
\mathbf{all} &= \frac{1}{m} p^\parallel \nabla T + \frac{1}{m} T \nabla p^\parallel - \frac{T}{2m} \nabla \pi_\parallel - \frac{p^\parallel}{mn} \nabla p^\perp \\
&= \frac{1}{m} \left(p^\parallel \nabla T + T \nabla p^\parallel - \frac{T}{2} \nabla \pi_\parallel - \frac{p^\parallel}{n} \nabla p^\perp \right).
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\int d^3 v m w_\parallel^2 \mathbf{w} \frac{\partial}{\partial \mathbf{w}} \cdot (\mathbf{w} \times \mathbf{B} f) &= -m \int d^3 v \, (\mathbf{w} \times \mathbf{B} f) \cdot \frac{\partial}{\partial \mathbf{w}} \left(w_\parallel^2 \mathbf{w} \right) \\
&= -m \int d^3 v \, (\mathbf{w} \times \mathbf{B} f) \cdot \left(2 w_\parallel \mathbf{b} + w_\parallel^2 \mathsf{I} \right) \\
&= -m \int d^3 v w_\parallel^2 \mathbf{w} \times \mathbf{B} f \\
&= -\mathbf{q}^\parallel \times \mathbf{B}.
\end{aligned} \tag{24}
$$

$$
\frac{q}{m} \int d^3 v m w_\parallel^2 \mathbf{w} \frac{\partial}{\partial \mathbf{w}} \cdot (\mathbf{w} \times \mathbf{B} f) = -\Omega \mathbf{q}^\parallel \times \mathbf{b}. \tag{25}
$$

The final equation becomes

$$
(\text{terms dropped by closure ordering}) + \mathbf{all} + (\text{terms} \quad \propto \quad \mathbf{b}) - \Omega \mathbf{q}^\parallel \times \hat{z} = 0 \ (\text{collisionless})
$$

$$
\mathbf{q}_\perp^\parallel = \frac{1}{\Omega} \hat{z} \times \mathbf{all}. \tag{26}
$$

$$
\mathbf{q}^\parallel = \int d\mathbf{v} m w_\parallel^2 \mathbf{w} f = q_\parallel^\parallel \hat{z} + \mathbf{q}_\perp^\parallel . \tag{27}
$$

$$
\mathbf{q}_\perp^\parallel = \frac{1}{m\Omega} \mathbf{b} \times \left(p^\parallel \nabla T + T \nabla p^\parallel - \frac{T}{2} \nabla \pi_\parallel - \frac{p^\parallel}{n} \nabla p^\perp \right). \tag{28}
$$

2 Derivation of the dispersion relation

The geometry of the model is shown in 1. From the equilibrium momentum equations, the equilibrium electric field is

$$
E_0 = \frac{T_{\mathrm{i}0}}{T_{\mathrm{e}0}^\perp + T_{\mathrm{i}0}} u_{\mathrm{e}0x} B_0 . \tag{29}
$$

The inverse of the gradient scale is given by

$$
\epsilon = \frac{e u_{\mathrm{e}0x} B_0}{T_{\mathrm{e}0}^\perp + T_{\mathrm{i}0}} . \tag{30}
$$

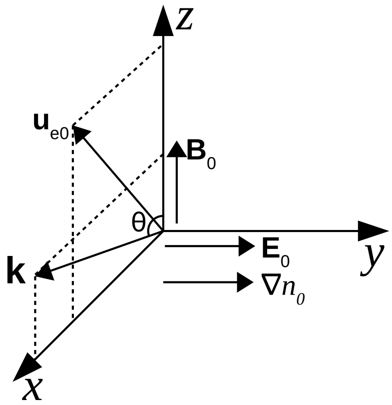

Figure 1. Geometry of the local theory for the LHDW dispersion calculation. The model is in the ion rest frame with z toward the equilibrium magnetic field ($\mathbf{B}_0$) and y along the density gradient direction. The equilibrium electric field $\mathbf{E}_0$ is also along y for the force balance. The equilibrium electron flow velocity $\mathbf{u}_{\rm e0}$ and wave vector $\mathbf{k}$ reside on the x-z plane. The angle between $\mathbf{k}$ and $\mathbf{B}_0$ is given by θ.

For the dispersion relation, the following Maxwell's equation without the displacement current term is used:

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_1) = -i\omega\mu_0 \mathbf{J}_1. \tag{31}$$

The perturbed ion current density ($\mathbf{J}_{\rm i1}$) is given by (H. Ji, Kulsrud, Fox, & Yamada, 2005)

$$\mathbf{J}_{\rm i1} = -\frac{in_0 e^2}{m_{\rm i} k v_{\rm ti}} \left[Z(\zeta)\mathbf{E}_1 + \frac{Z''(\mathbf{E}_1 \cdot \hat{\mathbf{k}})}{2}\hat{\mathbf{k}} - i\left(\frac{\epsilon}{2k}\right) Z'' E_{1y}\hat{\mathbf{k}} \right]. \tag{32}$$

The first order electron momentum equation is given by

$$im_{\rm e} n_0 \left(\omega - \mathbf{k}\cdot\mathbf{u}_{\rm e0}\right)\mathbf{u}_{\rm e1} = i\mathbf{k}\cdot\mathbf{P}_{\rm e1} + en_0(\mathbf{E}_1 + \mathbf{u}_{\rm e1}\times\mathbf{B}_0 + \mathbf{u}_{\rm e0}\times\mathbf{B}_1) + e(\mathbf{E}_0 + \mathbf{u}_{\rm e0}\times\mathbf{B}_0)n_{\rm e1}. \tag{33}$$

The perturbed electron density $n_{\rm e1}$ is given by the electron continuity equation, which is

$$(\omega - \mathbf{k}\cdot\mathbf{u}_{\rm e0})n_{\rm e1} = (\mathbf{k}\cdot\mathbf{u}_{\rm e1} - i\epsilon u_{{\rm e1}y})n_{\rm e0}. \tag{34}$$

Assuming that the perpendicular temperature perturbation is negligible, $p_{\rm e1}^{\perp}$ is just

$$p_{\rm e1}^{\perp} \approx n_{\rm e1} T_{\rm e0}^{\perp}. \tag{35}$$

For the parallel electron pressure, the following equation from the Vlasov equation is used:

$$\frac{\partial p_{\rm e}^{\parallel}}{\partial t} + \nabla\cdot(\mathbf{u}_{\rm e} p_{\rm e}^{\parallel}) + \nabla\cdot\mathbf{q}_{\rm e}^{\parallel} + 2\frac{\partial u_{\rm ez}}{\partial z} p_{\rm e}^{\parallel} = 0, \tag{36}$$

where

$$p_{\rm e}^{\parallel} = m_{\rm e}\int (v_z - u_{{\rm e}z})^2 f_{\rm e} d\mathbf{v}, \tag{37}$$

$$\mathbf{q}_{\rm e}^{\parallel} = m_{\rm e}\int (\mathbf{v} - \mathbf{u}_{\rm e})(v_z - u_{{\rm e}z})^2 f_{\rm e} d\mathbf{v}, \tag{38}$$

$$n_{\rm e}\mathbf{u}_{\rm e} = \int \mathbf{v} f_{\rm e} d\mathbf{v}. \tag{39}$$

Linearizing Eqn. 36 yields

$$-i\omega p_{\rm e1}^{\parallel} + \epsilon u_{{\rm e1}y} p_{\rm e0}^{\parallel} + i(\mathbf{k}\cdot\mathbf{u}_{\rm e0})p_{\rm e1}^{\parallel} + i(\mathbf{k}\cdot\mathbf{u}_{\rm e1})n_0 T_{\rm e0}^{\parallel} + i\mathbf{k}\cdot\mathbf{q}_{\rm e1}^{\parallel} + 2ik_{\parallel}u_{{\rm e1}z}n_0 T_{\rm e0}^{\parallel} = 0. \tag{40}$$

As shown in the previous section, the $3+1$ fluid model gives us

$$\mathbf{q}_{\rm e}^{\parallel} = \frac{\hat{z}}{m_{\rm e}\omega_{\rm ce}} \times \left(p_{\rm e}^{\parallel}\nabla T_{\rm e} + T_{\rm e}\nabla p_{\rm e}^{\parallel} - \frac{T_{\rm e}}{2}\nabla \pi_{\rm e}^{\parallel} - T_{\rm e}^{\parallel}\nabla p_{\rm e}^{\perp}\right) + q_{{\rm e}z}^{\parallel}\hat{z}, \tag{41}$$

where $\pi_{\rm e}^{\parallel} = 2(p_{\rm e}^{\parallel} - p_{\rm e}^{\perp})/3$, $T_{\rm e} = (2T_{\rm e}^{\perp} + T_{\rm e}^{\parallel})/3$. The closure for $q_{{\rm e1}z}^{\parallel}$ in the collisionless limit is given by J.-Y. Ji and Joseph (2018)

$$q_{{\rm e1}z}^{\parallel} = \frac{-i}{\sqrt{\pi}}\frac{k_{\parallel}}{|k_{\parallel}|}2n_0 v_{\rm te} T_{\rm e1}^{\parallel}, \tag{42}$$

where $T_{\rm e1}^{\parallel} = (p_{\rm e1}^{\parallel} - T_{\rm e0}^{\parallel}n_1)/n_0$ is the perturbed parallel temperature. Since $k_y = 0$, only $q_{{\rm e1}x}^{\parallel}$ is required to close Eqn. 40. By linearizing Eqn. 41 and using Eqn. 30, $q_{{\rm e1}x}^{\parallel}$ becomes

$$q_{{\rm e1}x}^{\parallel} = -\frac{2}{9}\frac{(T_{\rm e0}^{\parallel} - 4T_{\rm e0}^{\perp})T_{\rm e1}^{\parallel}}{T_{\rm e0}^{\perp} + T_{\rm i0}} n_0 u_{{\rm e0}x} = r_{\rm e}^{\parallel}T_{\rm e1}^{\parallel}n_0 u_{{\rm e0}x}, \tag{43}$$

where $r_{\rm e}^{\parallel} = 2(4T_{\rm e0}^{\perp} - T_{\rm e0}^{\parallel})/9(T_{\rm e0}^{\perp} + T_{\rm i0})$.

Then, $i\mathbf{k}\cdot\mathbf{q}_{\rm e1}^{\parallel}$ becomes

$$i\mathbf{k}\cdot\mathbf{q}_{\rm e1}^{\parallel} = i\left[k_{\perp}r_{\rm e}^{\parallel}u_{{\rm e0}x} - i(2/\sqrt{\pi})|k_{\parallel}|v_{\rm te}\right]n_0(p_{\rm e1}^{\parallel} - T_{\rm e0}^{\parallel}n_1) - ik_{\perp}r_{\rm e}^{\perp}u_{{\rm e0}x}n_0 T_{\rm e1}^{\perp}. \tag{44}$$

Then, from Eqn. 40, $p_{\rm e1}^{\parallel}$ becomes

$$p_{\rm e1}^{\parallel} = n_{\rm e1}T_{\rm e0}^{\parallel} + \frac{2k_{\parallel}n_0 T_{\rm e0}^{\parallel}u_{{\rm e1}z}}{\omega - \mathbf{k}\cdot\mathbf{u}_{\rm e0} - r_{\rm e}^{\parallel}k_{\perp}u_{{\rm e0}x} + i(2/\sqrt{\pi})|k_{\parallel}|v_{\rm te}}. \tag{45}$$

The z component of Eqn. 33 is

$$im_{\rm e}n_0(\omega - \mathbf{k}\cdot\mathbf{u}_{\rm e0})u_{{\rm e1}z} = ik_{\parallel}p_{\rm e1}^{\parallel} + en_0(E_{1z} + u_{{\rm e0}x}B_{1y}), \tag{46}$$

From the Faraday's Law ($\omega\mathbf{B}_1 = \mathbf{k}\times\mathbf{E}_1$), $B_{1y} = (k_{\parallel}E_{1x} - k_{\perp}E_{1z})/\omega$. With Eqn. 45, Eqn. 46, and $\alpha_{\rm e} = (\omega - \mathbf{k}\cdot\mathbf{u}_{\rm e0})/\omega_{\rm ce}$, $u_{{\rm e1}z}$ is expressed as

$$i\alpha_{{\rm e}z}u_{{\rm e1}z} = A_{{\rm e}z} + i\frac{\cos\theta}{2\alpha_{\rm e}}\left(\frac{kv_{\rm te}^{\parallel}}{\omega_{\rm ce}}\right)^2\left[u_{{\rm e1}x}\sin\theta - i\left(\frac{\epsilon}{k}\right)u_{{\rm e1}y}\right], \tag{47}$$

where

$$\alpha_{{\rm e}z} = \alpha_{\rm e} - \left(\frac{kv_{\rm te}^{\parallel}\cos\theta}{\omega_{\rm ce}}\right)^2\left[\frac{1}{2\alpha_{\rm e}} + \frac{1}{\alpha_{\rm e} - r_{\rm e}^{\parallel}(ku_{{\rm e0}x}/\omega_{\rm ce})\sin\theta + i(2/\sqrt{\pi})(|k_{\parallel}|v_{\rm te}/\omega_{\rm ce})}\right], \tag{48}$$

$$A_{{\rm e}z} = \frac{E_{1z}}{B_0} + \frac{ku_{{\rm e0}x}}{\omega}\frac{E_{1x}\cos\theta - E_{1z}\sin\theta}{B_0}. \tag{49}$$

The x component of Eqn. 33 is

$$im_{\mathrm{e}}n_0\left(\omega-\mathbf{k}\cdot\mathbf{u}_{\mathrm{e}0}\right)u_{\mathrm{e}1x}=ik_\perp p_{\mathrm{e}1}^\perp+en_{\mathrm{e}0}(E_{1x}+B_0u_{\mathrm{e}1y}-u_{\mathrm{e}0z}B_{1y}). \tag{50}$$

With Eqns. 35, 34, and 47, $u_{\mathrm{e}1y}$ can be expressed as

$$\gamma_{\mathrm{e}y}u_{\mathrm{e}1y}=i\alpha_{\mathrm{e}x}u_{\mathrm{e}1x}-\frac{\sin\theta\cos\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv_{\mathrm{te}}^\perp}{\omega_{\mathrm{ce}}}\right)^2A_{\mathrm{e}z}-A_{\mathrm{e}x}, \tag{51}$$

where $\gamma_{\mathrm{e}y}$, $\alpha_{\mathrm{e}x}$, and $A_{\mathrm{e}x}$ are

$$\gamma_{\mathrm{e}y}=1+\frac{\sin\theta}{2\alpha_{\mathrm{e}}}\left(\frac{\epsilon}{k}\right)\left(\frac{kv_{\mathrm{te}}^\perp}{\omega_{\mathrm{ce}}}\right)^2\left[1+\frac{\cos^2\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv_{\mathrm{te}}^\parallel}{\omega_{\mathrm{ce}}}\right)^2\right], \tag{52}$$

$$\alpha_{\mathrm{e}x}=\alpha_{\mathrm{e}}-\frac{\sin^2\theta}{2\alpha_{\mathrm{e}}}\left(\frac{kv_{\mathrm{te}}^\perp}{\omega_{\mathrm{ce}}}\right)^2\left[1+\frac{\cos^2\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv_{\mathrm{te}}^\parallel}{\omega_{\mathrm{ce}}}\right)^2\right], \tag{53}$$

$$A_{\mathrm{e}x}=\frac{E_{1x}}{B_0}-\frac{ku_{\mathrm{e}0z}}{\omega}\frac{E_{1x}\cos\theta-E_{1z}\sin\theta}{B_0}. \tag{54}$$

The y component of Eqn. 33 is

$$im_{\mathrm{e}}n_{\mathrm{e}0}\left(\omega-\mathbf{k}\cdot\mathbf{u}_{\mathrm{e}0}\right)u_{1y}=en_{\mathrm{e}0}(E_{1y}-B_0u_{\mathrm{e}1x}-u_{\mathrm{e}0x}B_{1z}+u_{\mathrm{e}0z}B_{1x})+e(E_0-u_{\mathrm{e}0x}B_0)n_{\mathrm{e}1}. \tag{55}$$

With Eqns. 34, 29, and 47, $u_{\mathrm{e}1x}$ can be expressed as

$$\gamma_{\mathrm{e}x}u_{\mathrm{e}1x}=-i\alpha_{\mathrm{e}y}u_{\mathrm{e}1y}+\frac{i\cos\theta}{\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\frac{T_{\mathrm{e}0}^\perp}{T_{\mathrm{e}0}^\perp+T_{\mathrm{i}0}}\left(\frac{ku_{\mathrm{e}0x}}{\omega_{\mathrm{ce}}}\right)A_{\mathrm{e}z}+A_{\mathrm{e}y}, \tag{56}$$

where $\gamma_{\mathrm{e}x}$, $\alpha_{\mathrm{e}y}$, and $A_{\mathrm{e}y}$ are

$$\gamma_{\mathrm{e}x}=1+\frac{\sin\theta}{\alpha_{\mathrm{e}}}\frac{T_{\mathrm{e}0}^\perp}{T_{\mathrm{e}0}^\perp+T_{\mathrm{i}0}}\left(\frac{ku_{\mathrm{e}0x}}{\omega_{\mathrm{ce}}}\right)\left[1+\frac{\cos^2\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv_{\mathrm{te}}^\parallel}{\omega_{\mathrm{ce}}}\right)^2\right], \tag{57}$$

$$\alpha_{\mathrm{e}y}=\alpha_{\mathrm{e}}-\frac{1}{\alpha_{\mathrm{e}}}\left(\frac{\epsilon}{k}\right)\frac{T_{\mathrm{e}0}^\perp}{T_{\mathrm{e}0}^\perp+T_{\mathrm{i}0}}\left(\frac{ku_{\mathrm{e}0x}}{\omega_{\mathrm{ce}}}\right)\left[1+\frac{\cos^2\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv_{\mathrm{te}}^\parallel}{\omega_{\mathrm{ce}}}\right)^2\right], \tag{58}$$

$$A_{\mathrm{e}y}=\frac{E_{1y}}{B_0}-\frac{k}{\omega}\frac{(u_{\mathrm{e}0x}\sin\theta+u_{\mathrm{e}0z}\cos\theta)E_{1y}}{B_0}, \tag{59}$$

With Eqns. 51 and 56, $u_{\mathrm{e}1y}$ is given by

$$u_{\mathrm{e}1y}=i\left(iC_{yx}^{\mathrm{e}}A_{\mathrm{e}x}+C_{yy}^{\mathrm{e}}A_{\mathrm{e}y}+iC_{yz}^{\mathrm{e}}A_{\mathrm{e}z}\right), \tag{60}$$

where

$$C_{yx}^{\mathrm{e}}=\left(\gamma_{\mathrm{e}y}-\frac{\alpha_{\mathrm{e}x}\alpha_{\mathrm{e}y}}{\gamma_{\mathrm{e}x}}\right)^{-1}, \tag{61}$$

$$C_{yy}^{\mathrm{e}}=C_{yx}^{\mathrm{e}}\frac{\alpha_{\mathrm{e}x}}{\gamma_{\mathrm{e}x}}, \tag{62}$$

$$C_{yz}^{\mathrm{e}}=C_{yx}^{\mathrm{e}}\left[\frac{\sin\theta\cos\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv_{\mathrm{te}}^\perp}{\omega_{\mathrm{ce}}}\right)^2+\frac{\alpha_{\mathrm{e}x}\cos\theta}{\gamma_{\mathrm{e}x}\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{T_{\mathrm{e}0}^\perp}{T_{\mathrm{e}0}^\perp+T_{\mathrm{i}0}}\frac{ku_{\mathrm{e}0x}}{\omega_{\mathrm{ce}}}\right)\right]. \tag{63}$$

Similarly, $u_{\mathrm{e}1x}$ is given by

$$u_{\mathrm{e}1x}=iC_{xx}^{\mathrm{e}}A_{\mathrm{e}x}+C_{xy}^{\mathrm{e}}A_{\mathrm{e}y}+iC_{xz}^{\mathrm{e}}A_{\mathrm{e}z}, \tag{64}$$

where

$$C^{\mathrm{e}}_{xy} = \left(\gamma_{\mathrm{e}x} - \frac{\alpha_{\mathrm{e}x}\alpha_{\mathrm{e}y}}{\gamma_{\mathrm{e}y}}\right)^{-1}, \tag{65}$$

$$C^{\mathrm{e}}_{xx} = C^{\mathrm{e}}_{xy}\frac{\alpha_{\mathrm{e}y}}{\gamma_{\mathrm{e}y}}, \tag{66}$$

$$C^{\mathrm{e}}_{xz} = C^{\mathrm{e}}_{xy}\left[\frac{\cos\theta}{\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{T^{\perp}_{\mathrm{e}0}}{T^{\perp}_{\mathrm{e}0}+T_{\mathrm{i}0}}\frac{ku_{\mathrm{e}0x}}{\omega_{\mathrm{ce}}}\right) + \frac{\alpha_{\mathrm{e}y}\sin\theta\cos\theta}{2\gamma_{\mathrm{e}y}\alpha_s\alpha_{\mathrm{e}z}}\left(\frac{kv^{\perp}_{\mathrm{te}}}{\omega_{\mathrm{ce}}}\right)^2\right]. \tag{67}$$

Then, u_{s1z} can be written as

$$u_{\mathrm{e}1z} = iC^{\mathrm{e}}_{zx}A_{\mathrm{e}x} + C^{\mathrm{e}}_{zy}A_{\mathrm{e}y} + iC^{\mathrm{e}}_{zz}A_{\mathrm{e}z}, \tag{68}$$

where

$$C^{\mathrm{e}}_{zz} = -\frac{1}{\alpha_{\mathrm{e}z}} + \frac{\cos\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv^{\parallel}_{\mathrm{te}}}{\omega_{\mathrm{ce}}}\right)^2\left(C^{\mathrm{e}}_{xz}\sin\theta + C^{\mathrm{e}}_{yz}\frac{\epsilon}{k}\right), \tag{69}$$

$$C^{\mathrm{e}}_{zx} = \frac{\cos\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv^{\parallel}_{\mathrm{te}}}{\omega_{\mathrm{ce}}}\right)^2\left(C^{\mathrm{e}}_{xx}\sin\theta + C^{\mathrm{e}}_{yx}\frac{\epsilon}{k}\right), \tag{70}$$

$$C^{\mathrm{e}}_{zy} = \frac{\cos\theta}{2\alpha_{\mathrm{e}}\alpha_{\mathrm{e}z}}\left(\frac{kv^{\parallel}_{\mathrm{te}}}{\omega_{\mathrm{ce}}}\right)^2\left(C^{\mathrm{e}}_{xy}\sin\theta + C^{\mathrm{e}}_{yy}\frac{\epsilon}{k}\right). \tag{71}$$

The final goal is to obtain the perturbed current density of electrons, which is given by $\mathbf{J}^{\mathrm{e}}_1 = -en_{\mathrm{e}0}\mathbf{u}_{\mathrm{e}1} - e\mathbf{u}_{\mathrm{e}0}n_{\mathrm{e}1}$. Thus, an expression for $n_{\mathrm{e}1}$ is required. From Eqns. 34, 60, 64, and 68, $n_{\mathrm{e}1}$ is given by

$$n_{\mathrm{e}1} = \frac{kn_{\mathrm{e}0}}{\omega - \mathbf{k}\cdot\mathbf{u}_{\mathrm{e}0}}\left[iC'^{\mathrm{e}}_{x}A_{\mathrm{e}x} + C'^{\mathrm{e}}_{y}A_{\mathrm{e}y} + iC'^{\mathrm{e}}_{z}A_{\mathrm{e}z}\right], \tag{72}$$

where

$$C'^{\mathrm{e}}_{x} = C^{\mathrm{e}}_{xx}\sin\theta + C^{\mathrm{e}}_{yx}\epsilon/k + C^{\mathrm{e}}_{zx}\cos\theta, \tag{73}$$
$$C'^{\mathrm{e}}_{y} = C^{\mathrm{e}}_{xy}\sin\theta + C^{\mathrm{e}}_{yy}\epsilon/k + C^{\mathrm{e}}_{zy}\cos\theta, \tag{74}$$
$$C'^{\mathrm{e}}_{z} = C^{\mathrm{e}}_{xz}\sin\theta + C^{\mathrm{e}}_{yz}\epsilon/k + C^{\mathrm{e}}_{zz}\cos\theta. \tag{75}$$

Now we are ready for computing the dispersion relation. Eqn. 31 is

$$k_{\parallel}^2E_{1x} - k_{\perp}k_{\parallel}E_{1z} - i\omega\mu_0J_{1x} = 0, \tag{76}$$
$$k^2E_{1y} - i\omega\mu_0J_{1y} = 0, \tag{77}$$
$$k_{\perp}^2E_{1z} - k_{\perp}k_{\parallel}E_{1x} - i\omega\mu_0J_{1z} = 0. \tag{78}$$

By multiplying d_{i}^2, the above equation can be written as

$$K^2\cos^2\theta E_{1x} - K^2\sin\theta\cos\theta E_{1z} - i\Omega\frac{B_0}{en_0}J_{1x} = 0, \tag{79}$$
$$K^2E_{1y} - i\Omega\frac{B_0}{en_0}J_{1y} = 0, \tag{80}$$
$$K^2\sin^2\theta E_{1z} - K^2\sin\theta\cos\theta E_{1x} - i\Omega\frac{B_0}{en_0}J_{1z} = 0, \tag{81}$$

where $K \equiv kd_{\mathrm{i}}$.

Eqns. 79–81 can be written as

$$\begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}\begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = 0. \tag{82}$$

From Eqn. 32, each component of $i\Omega B_0\mathbf{J}_{\mathrm{i}1}/en_0$ is

$$\frac{i\Omega B_0}{en_0}J_{i1x} = \zeta Z E_{1x} + \frac{\zeta Z''\sin\theta}{2}\left(E_{1x}\sin\theta - i\frac{\epsilon}{k}E_{1y} + E_{1z}\cos\theta\right), \tag{83}$$

$$\frac{i\Omega B_0}{en_0}J_{i1y} = \zeta Z E_{1y}, \tag{84}$$

$$\frac{i\Omega B_0}{en_0}J_{i1z} = \zeta Z E_{1z} + \frac{\zeta Z''\cos\theta}{2}\left(E_{1x}\sin\theta - i\frac{\epsilon}{k}E_{1y} + E_{1z}\cos\theta\right). \tag{85}$$

From Eqns. 64 and 72, $i\Omega J^{\mathrm{e}}_{1x}/en_0$ is given by

$$\frac{i\Omega B_0}{en_0}J^{\mathrm{e}}_{1x} = \Omega B_0\left[\left(C^{\mathrm{e}}_{xx} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_x\right)A_{\mathrm{e}x} - i\left(C^{\mathrm{e}}_{xy} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_y\right)A_{\mathrm{e}y} + \left(C^{\mathrm{e}}_{xz} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_z\right)A_{\mathrm{e}z}\right], \tag{86}$$

where $C^{\mathrm{e}}_k = K/(\Omega - \mathbf{K}\cdot\mathbf{u}_{\mathrm{e}0})$ and $\mathbf{u}_{\mathrm{e}0} = \mathbf{u}_{\mathrm{e}0}/V_{\mathrm{A}}$. Here $V_{\mathrm{A}} = B_0/\sqrt{\mu_0 m_{\mathrm{i}} n_0} = d_{\mathrm{i}}\omega_{\mathrm{ci}}$ is the Alfvén speed. Similarly, from Eqns. 68 and 72, $i\Omega J^{\mathrm{e}}_{1z}/en_0$ is given by

$$\frac{i\Omega B_0}{en_0}J^{\mathrm{e}}_{1z} = \Omega B_0\left[\left(C^{\mathrm{e}}_{zx} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_x\right)A_{\mathrm{e}x} - i\left(C^{\mathrm{e}}_{zy} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_y\right)A_{\mathrm{e}y} + \left(C^{\mathrm{e}}_{zz} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_z\right)A_{\mathrm{e}z}\right], \tag{87}$$

Since there is no y component in $\mathbf{u}_{\mathrm{e}0}$, $i\Omega J^{\mathrm{e}}_{1y}/en_0$ is simply

$$\frac{i\Omega B_0}{en_0}J^{\mathrm{e}}_{1y} = \Omega B_0\left(iC^{\mathrm{e}}_{yx}A_{\mathrm{e}x} + C^{\mathrm{e}}_{yy}A_{\mathrm{e}y} + iC^{\mathrm{e}}_{yz}A_{\mathrm{e}z}\right). \tag{88}$$

In terms of dimensionless parameters, $\Omega B_0 A_{\mathrm{e}x}$, $\Omega B_0 A_{\mathrm{e}y}$, and $\Omega B_0 A_{\mathrm{e}z}$ can be written as

$$\Omega B_0 A_{\mathrm{e}x} = \left(\Omega - K u_{\mathrm{e}0z}\cos\theta\right)E_{1x} + (K u_{\mathrm{e}0z}\sin\theta)E_{1z}, \tag{89}$$

$$\Omega B_0 A_{\mathrm{e}y} = \left[K u_{\mathrm{e}0z}\cos\theta - K(u_{\mathrm{e}0x}\sin\theta + u_{\mathrm{e}0z}\cos\theta)\right]E_{1y}, \tag{90}$$

$$\Omega B_0 A_{\mathrm{e}z} = (K u_{\mathrm{e}0x}\cos\theta)E_{1x} + \left(\Omega - K u_{\mathrm{e}0x}\sin\theta\right)E_{1z}. \tag{91}$$

Then, each component of the tensor $\mathbf{D}$ is

$$D_{xx} = K^2\cos^2\theta - \zeta Z - \frac{\zeta Z''}{2}\sin^2\theta \tag{92}$$
$$- \left[\left(C^{\mathrm{e}}_{xx} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_x\right)\left(\Omega - K u_{\mathrm{e}0z}\cos\theta\right) + \left(C^{\mathrm{e}}_{xz} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_z\right)K u_{\mathrm{e}0x}\cos\theta\right],$$

$$D_{xy} = i\left(\frac{\epsilon}{k}\right)\frac{\zeta Z''}{2}\sin\theta + \left(C^{\mathrm{e}}_{xy} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_y\right)\left[\Omega - K(u_{\mathrm{e}0x}\sin\theta + u_{\mathrm{e}0z}\cos\theta)\right], \tag{93}$$

$$D_{xz} = -K^2\sin\theta\cos\theta - \frac{\zeta Z''}{2}\sin\theta\cos\theta \tag{94}$$
$$- \left[\left(C^{\mathrm{e}}_{xx} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_x\right)K u_{\mathrm{e}0z}\sin\theta + \left(C^{\mathrm{e}}_{xz} + u_{\mathrm{e}0x}C^{\mathrm{e}}_k C'^{\mathrm{e}}_z\right)\left(\Omega - K u_{\mathrm{e}0x}\sin\theta\right)\right],$$

$$D_{yx} = -i\left[C^{\mathrm{e}}_{yx}\left(\Omega - K u_{\mathrm{e}0z}\cos\theta\right) + C^{\mathrm{e}}_{yz}K u_{\mathrm{e}0x}\cos\theta\right], \tag{95}$$

$$D_{yy} = K^2 - \zeta Z - C^{\mathrm{e}}_{yy}\left[\Omega - K(u_{\mathrm{e}0x}\sin\theta + u_{\mathrm{e}0z}\cos\theta)\right], \tag{96}$$

$$D_{yz} = -i\left[C^{\mathrm{e}}_{yx}K u_{\mathrm{e}0z}\sin\theta + C^{\mathrm{e}}_{yz}\left(\Omega - K u_{\mathrm{e}0x}\sin\theta\right)\right], \tag{97}$$

$$D_{zx} = -K^2\sin\theta\cos\theta - \frac{\zeta Z''}{2}\sin\theta\cos\theta \tag{98}$$
$$- \left[\left(C^{\mathrm{e}}_{zx} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_x\right)\left(\Omega - K u_{\mathrm{e}0z}\cos\theta\right) + \left(C^{\mathrm{e}}_{zz} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_z\right)K u_{\mathrm{e}0x}\cos\theta\right],$$

$$D_{zy} = i\left(\frac{\epsilon}{k}\right)\frac{\zeta Z''}{2}\cos\theta + i\left(C^{\mathrm{e}}_{zy} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_y\right)\left[\Omega - K(u_{\mathrm{e}0x}\sin\theta + u_{\mathrm{e}0z}\cos\theta)\right], \tag{99}$$

$$D_{zz} = K^2\sin^2\theta - \zeta Z - \frac{\zeta Z''}{2}\cos^2\theta \tag{100}$$
$$- \left[\left(C^{\mathrm{e}}_{zx} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_x\right)K u_{\mathrm{e}0z}\sin\theta + \left(C^{\mathrm{e}}_{zz} + u_{\mathrm{e}0z}C^{\mathrm{e}}_k C'^{\mathrm{e}}_z\right)\left(\Omega - K u_{\mathrm{e}0x}\sin\theta\right)\right].$$

Required input plasma parameters for the dispersion relation include B_0, n_0, $T^{\parallel}_{\mathrm{e}0}$, $T^{\perp}_{\mathrm{e}0}$, $T_{\mathrm{i}0}$, and $\mathbf{u}_{\mathrm{e}0}$. For $\mathbf{u}_{\mathrm{e}0}$, the coordinate transform from the LMN to local xyz coordinate system is needed. The z direction is along $\mathbf{B}_0$ and the y direction is along $\mathbf{B}_0\times$

$\mathbf{u}_{e0}$. The choice of the gradient (y) direction is based on the model geometry where there is no y component of $\mathbf{u}_{e0}$, and based on the MHD equilibrium, $\nabla p = \mathbf{B}\times\mathbf{J}$, which also indicates no y component of $\mathbf{u}_{e0}$.

References

Ji, H., Kulsrud, R., Fox, W., & Yamada, M. (2005). An obliquely propagating electromagnetic drift instability in the lower hybrid frequency range. *J. Geophys. Res.*, *110*, A08212.

Ji, J.-Y., & Held, E. D. (2008). Landau collision operators and general moment equations for an electron-ion plasma. *Physics of Plasmas*, *15*(10), 102101. Retrieved from `https://doi.org/10.1063/1.2977983` doi: 10.1063/1.2977983

Ji, J.-Y., & Joseph, I. (2018). Electron parallel closures for the 3 + 1 fluid model. *Phys. Plasmas*, *25*(3), 032117. Retrieved from `https://doi.org/10.1063/1.5014996` doi: 10.1063/1.5014996