# Sum up of the mean field game theory work

Robin NICOLE

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**Abstract**

Studying the model of agents with fixed preferences using the mean field game theory framework

## 1 Notations

The trading pice is defined as $\pi^i = \theta_i \mu_i^B + (1-\theta_i)\mu_i^S$
The probability that your trade $\gamma$ is valid (i.e. you bid higher than $\pi$ or sell lower than $\pi$) is

$$T_{S,m} = \int_{-\infty}^{\pi^m} \frac{\exp(\frac{-(x-\mu_m^S)^2}{2})}{\sqrt{2\pi}} dx \tag{1}$$

$$T_{B,m} = \int_{\pi^m}^{\infty} \frac{\exp(\frac{-(x-\mu_m^B)^2}{2})}{\sqrt{2\pi}} dx \tag{2}$$

The average preference toward action $\gamma$ is :

$$\bar{S}_{S,m} = \int_{-\infty}^{\pi^m} \frac{\exp(\frac{-(x-\mu_m^S)^2}{2})}{\sqrt{2\pi}} (\pi_m - x) dx \tag{3}$$

$$\bar{S}_{B,m} = \int_{\pi_m}^{\infty} \frac{\exp(\frac{-(x-\mu_m^B)^2}{2})}{\sqrt{2\pi}} (x - \pi_m) dx \tag{4}$$

The letter $F_m$ depends on the saturation is defined by :

$$\begin{array}{ll} \text{if } \frac{\tilde{N}_{B,m}}{\tilde{N}_{S,m}} > 1 & F_m = B \\ \text{if } \frac{\tilde{N}_{S,m}}{\tilde{N}_{B,m}} > 1 & F_m = A \end{array} \tag{5}$$

$p_b^{(i)}$ is the preference to buy of an agent from population $i$

## 2 Mean field game theory tools

In the case of a game problem with an infinite number of agents. The decision problem faced by one agent can be specified the following way. I want to maximise a pay-off function which depend on the distribution of the strategies chosen by the infinite number of other player and also on my own choice of strategy. Using this formalism, it is possible to define a Nash equilibrium. In a Nash equilibrium all the strategies that are likely to be chosen by other players should minimise their cost function. Because all the players in the game are indistinguishable this strategy should also minimise my pay-off function. In a game with a payoff function $J(S,\phi) : \mathcal{S} \times P(E) \to \mathcal{R}$ where $\phi$ is the probability distribution of the other agents strategies, and S is the strategy I choose. The distribution of a Nash equilibrium $\phi$ verify the following condition :

$$Supp(\phi) \in arg \max_S (J(S,\phi)) \tag{6}$$

Where $Supp(\phi)$ is the support of the probability distribution $\phi$

## 3 Condition on the pay-off function of the markets for non trivial Nash equilibrium

In that problem, there is two groups of agents $G_1$ and $G_2$ with fixed preferences to buy ($p_b^{(1)}$ and $p_b^{(2)}$) and sell. The only thing they can change is their preference to trade with the market 1 and 2. One can see that the average pay-off an agent will receive from a market will only depend on its preferences and the distribution of the preferences of the other agents. One can write the pay-off received by an agent :

$$J(p_1^{(i)}, \phi) = p_1^{(i)} J_1^{(i)}[\phi] + (1 - p_1^{(i)}) J_2^{(i)}[\phi] \tag{7}$$

The values of $p_\tau^{(i)}$ which will maximise the pay-off function will be 0 or 1 if $J_1^{(i)}[\phi] \neq J_2^{(i)}[\phi]$. One want the agents no to stick to only one market hence we need that the distribution $\phi$ of the strategies must check the following condition :

$$J_1^{(i)}[\phi] = J_2^{(i)}[\phi] \tag{8}$$

## 4 Explicit expression of the functions $J^i_{Mj}$

Because the interaction are not agent-agent but always take place through the market chosen by the agents, in the large number of agents limit, the quantity in the min is averaged

$$J^i_m = p_b^{(i)} T_{B,m} \min(1, \langle \frac{(1-p_b^{(1)})p_m^{(1)} + (1-p_b^{(2)})p_m^{(2)}}{p_b^{(1)} p_m^{(1)} + p_b^{(2)} p_m^{(2)}} \frac{T_{S,m}}{T_{B,m}} \rangle_{p_m}) \frac{\bar{S}^{()}_{B,m}}{T_{B,m}} + \\ (1-p_b^{(i)}) T_{S,m} \min(1, \langle \frac{p_b^{(1)} p_m^{(1)} + p_b^{(2)} p_m^{(2)}}{(1-p_b^{(1)})p_m^{(1)} + (1-p_b^{(2)})p_m^{(2)}} \frac{T_{B,m}}{T_{S,m}} \rangle_{p_m}) \frac{\bar{S}_{S,m}}{T_{S,m}} \tag{9}$$

For the conditions (8) to be verified, the distribution of strategies of the agents *phi* must be such that expression (9) do not depend of the considered market. This must be true for every group of agents.

## 5 Analytic conditions for non trivial Nash equilibriums

In (9) One can replace the expression in the minimum by a simpler one. by setting $F_m = \frac{\tilde{N}_{S,m}}{\tilde{N}_{B,m}}$

$$\min(1, \langle \frac{(1-p_b^{(1)})p_m^{(1)} + (1-p_b^{(2)})p_m^{(2)}}{p_b^{(1)} p_m^{(1)} + p_b^{(2)} p_m^{(2)}} \frac{T_{S,m}}{T_{B,m}} \rangle_{p_m}) = \min(1, F_m) \tag{10}$$

from here, we'll write $\langle p_i^{\phi} \rangle_\phi \doteq p_i$. using this notation, one also define the ratio between the number of sellers who sent a valid ask, and buyers who sent a valid offer in market $m$. Using those notations, the expression of $J^i_m$ is :

$$J_{i,m} = p_b^{(i)} (\tilde{S}_{B,m} - \tilde{S}_{S_m}) + \tilde{S}_{S,m} \tag{11}$$

where $\tilde{S}_{B,m} = \min(1, F_m)\bar{S}_{B,m}$ and $\tilde{S}_{S,m} = \min(1, \frac{1}{F_m})\bar{S}_{S,m}$ Then for both of the populations, the equation to solve is :

$$J_{i,1}(p^{(1)}, p^{(2)}) = J_{i,2}(p^{(1)}, p^{(2)}) \tag{12}$$

If the condition (12) is fulfilled for the population $(i)$ then, there will exist a segregated Nash equilibrium.

## 6 Analytical condition for fully non segregated Nash equilibrium

A nash equilibium which is not taken into account in the previous section is the case when both of the population are unsegregated. In that case, the probability of buying at market 1 $p_1^{(i)}$ will be either 0 or 1. The configuration $p_1^{(1)} = p_1^{(2)}$ is a Nash equilibrium independently of the preference to buy of each of the population and the preference toward buyer of each of the markets $\theta_m$. Indeed, if all the agents synchronise at the same market, then, if one decide to go buying at the other market, then he won't find somebody to trade with, hence it will be better for him no to change its strategy.

### 6.1 Not synchronised unsegregated Nash equilibrium

The other type of Nash equilibrium which is not taken into account in the previous study is the one where the agents will not be synchronised at the same market however they are desynchronised. In that case the probability to buy at market 1 is either 0 or 1 and the sum $p_1^{(1)} + p_1^{(2)} = 1$. First by looking at the symmetric case when $p_B^{(1)} = 1 - p_B^{(2)}$ and $\theta_1 = \theta_2$. One can get an analytic expression of the parameters for which asynchronised-unsegregated Nash equilibriums exist. The profit an agent from population i get when trading with market $m$ is $J_m^{(i)}$.

$$\frac{1}{2}\left(\Delta\theta + \frac{e^{-}}{1+}\right.$$

$$\frac{1}{2\left(1+\mathrm{Erf}\left(\frac{\Delta\theta}{\sqrt{2}}\right)\right)} e^{-\frac{1}{2}\Delta^2(1+2(-1+\theta)\theta)} \left(e^{\frac{1}{2}\Delta^2(-1+\theta)^2} \mathrm{Erfc}\left(\frac{\Delta(-1+\theta)}{\sqrt{2}}\right)\left(-3\sqrt{\frac{2}{\pi}} + e^{\frac{\Delta^2\theta^2}{2}}\Delta(1+2\theta)\left(-2+\mathrm{Erfc}\left(\frac{\Delta\theta}{\sqrt{2}}\right)\right)\right) + e\right.$$

$$e^{-\frac{1}{2}\Delta^2(-1+\theta)^2}\sqrt{\frac{2}{\pi}} + \left(\Delta + \frac{e^{-}}{1+}\right.$$