Supplementary material for: 'Unsteady Ekman–Stokes dynamics: implications for surface-wave induced drift of floating marine litter'

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Contour integration

To carry out the Laplace inversion in (9) in the main paper, we group the two terms in the round bracket, noting that the second term, which is proportional to $\exp(2kz)$, gives an exponentially-growing solution $\propto \exp(4k^2\nu t)$ if inverted by itself. This may be seen by closing the contour to the left and applying Cauchy's theorem. Using L'Hôpital's rule on the grouped terms shows that $s = 4k^2\nu - if$ is a removable singularity. Defining $S = s + if$ we perform the integration along the contour shown in Figure A1; since the function is analytic within the enclosed region, we have by Cauchy's Theorem

$$\frac{1}{2\pi i}\oint\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}}+\frac{if}{s+if-4k^2\nu}\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}}-e^{2kz}\right)\right)e^{st}ds=0. \tag{1}$$

The contribution of the arcs C_1 and C_2 disappears as $R \to \infty$, while the contribution of the small circle C_ε disappears as $\varepsilon \to 0^+$. Applying Cauchy's Theorem, the inverse Laplace transform equals minus the sum of the line integrals either side of the branch cut, L_+ and L_-. Changing variables to $b = \sqrt{|S|/\nu}$ and accounting for the behaviour of the square root when the branch cut is crossed, it is easy to see that

$$\sqrt{S/\nu}=\sqrt{-|S|/\nu}=\begin{cases}+ib & \text{above the branch cut}\\ -ib & \text{below the branch cut}\end{cases}. \tag{2}$$

The inverse transform is equal to the real integral

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}}+\frac{if}{4k^2\nu-if-s}\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}}-e^{2kz}\right)\right\}\\ &\quad=\frac{e^{-ift}}{\pi}\int_{-\infty}^{\infty}2k\sqrt{\nu}e^{-b^2t}\cos\left(bz/\sqrt{\nu}\right)db-\frac{ife^{-ift}}{\pi}\int_{-\infty}^{\infty}\frac{2k\sqrt{\nu}e^{-b^2t}\cos\left(bz/\sqrt{\nu}\right)}{b^2+4k^2\nu}db.\end{aligned} \tag{3}$$

The first of these is a Gaussian integral while the second may be evaluated explicitly by using Eq. No. 3.954 in Gradshteyn & Ryzhik (2014, p. 504), resulting in the analytic expression (10) in the main paper.

References

Gradshteyn, I. S., & Ryzhik, I. M. (2014). *Table of integrals, series, and products* (8th ed.). Elsevier.

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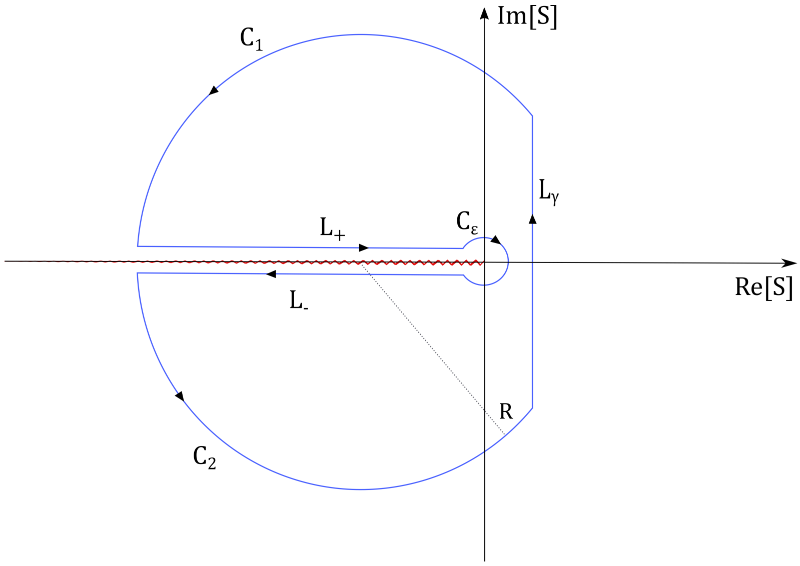

Figure 1. Integration contour for the Laplace inversion of the Ekman–Stokes kernel (**??**). The branch cut of the square root lies along the negative real axis. As $R \to \infty$, the line segment L_γ tends to the Browmwich contour used for the inverse Laplace transform.