

# PC MATRICES VIA GEOMETRIC CALCULUS: APPLICATION WITH CAYLEY FORMULA

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**ABSTRACT.** The aim of this paper is to investigate relation between PC matrices and G-orthogonal matrices. For this purpose, first of all, we define G- orthogonal and G-skew-symmetric matrices according to geometric calculus. Then, we give a relation between G-orthogonal matrices and PC matrices using the useful diagram constructed by Cayley formula in real mean. Finally we define G-Cayley formula according to G-calculus and we verify our theory with some examples.

**Keywords:** pairwise comparisons, Lie group, special orthogonal group, geometric calculus, cayley formula.

## 1. INTRODUCTION

Comparing two entities (known as pairwise comparisons) has a long history which was also used in different aspects to computer scientists, mathematicians, statisticians, engineers. The most common use of pairwise comparisons (PCs), is the multiplicative representation typically used in Satty's AHP [14]. Nevertheless, there are two types of pairwise comparisons that are commonly considered: additive and multiplicative. From the mathematical point of view, multiplicative (or ratio) PC matrices are more popular and more challenging than additive PC matrices. And also it is known that multiplicative preferences can be transformed into additive ones by quite simply applying the logarithmic function to the entries of a multiplicative PC matrix. In fact, the two representations are isomorphic so structurally equivalent [2] .

PC matrices have been studied from a different perspective for the first time in many years. Koczkodaj, Marek, Yayli [3] gave the complete mathematical description of the Lie group and Lie algebra of PC matrices. It has been shown that the set of additive PC matrices corresponds to the set of real skew-symmetric matrices of order  $n$  and is a vector subspace of  $\mathbb{R}^{n \times n}$ . But for the first time, after constructing the Lie algebra of PC matrices, they gave a theorem of an internal direct product by using the direct summability property of vector space. The property given in [3] has a problem but it is possible to solve this error in terms of G-calculus by making use of the instructions.

This paper presents some developments concerning the mentioned one of the properties, then it is reasonable to assume that relation between the PC matrix Lie group and special orthogonal group. More specifically, we shall study geometric calculus (G- calculus) compare to the special orthogonal group algebras and the relation of the Lie algebra of PC matrices Lie group.

**1.1. Problem outline.** The use of G-calculus has been mentioned independently in [8], [12], [13]. Yet, none of them provides any reasoning for it and does not mention pairwise comparisons.

## 2. PRELIMINARIES

We give two subsections to mention some basic notions in the following. The former includes some basic but important informations about PC matrices. (See for details [[1],[3],[4],[5],[6],[16]]). The latter is devoted to Lie group and Lie algebra (See for details [10],[11]).

### 2.1. Background on PC Matrices.

An  $n \times n$  pairwise comparisons matrix is defined as a square matrix  $M = [m_{ij}]$  such that  $m_{ij} > 0$  for every  $i, j \in \{1, 2, \dots, n\}$  and it is expressed in the following form,

$$M = \begin{bmatrix} 1 & m_{12} & \dots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \dots & 1 \end{bmatrix}$$

There is an explanation the PC matrix operations defined by the Hadamard product as follows:  $M = [m_{ij}]$  and  $N = [n_{ij}]$  be a two  $n \times n$  PC matrices with entries in  $\mathbb{R}^+$ , the Hadamard product of  $M$  and  $N$  is

$$[M.N]_{ij} = [M]_{ij} \cdot [N]_{ij}$$

A pairwise comparison matrix  $M$  is called reciprocal if

$$m_{ij} = \frac{1}{m_{ji}}$$

(then automatically  $m_{ii} = 1$ ). A pairwise comparison matrix  $M$  is called consistent if  $m_{ij} \cdot m_{jk} = m_{ik}$  for every  $i, j, k \in \{1, 2, \dots, n\}$ .

Let us remark that, while every consistent matrix is reciprocal, the converse is false in general. The concept of consistency can equivalently be defined through the existence of weight vectors satisfying some conditions, concerning the given PC matrices.  $M$  is consistent if and only if there exists a positive (weight) vector  $W = (w_1, w_2, \dots, w_n)$  such that  $m_{ij} = w_i/w_j$  for all  $i, j \in \{1, 2, \dots, n\}$  [14]. The justification for the use of weight vector  $W = [w_i]$  of geometric means of rows.

Instead of a PC matrix  $M = [m_{ij}] \in \mathbb{R}^+$ , the set of positive real numbers considered with multiplication, we can transform entries of  $M$  by a logarithmic function and get an additive matrix  $A = [a_{ij}] = [\log(m_{ij})]$ .

Since a PC matrix  $M$  is reciprocal, it follows that it is skew-symmetric, i.e

$$a_{ij} = -a_{ji}$$

Moreover, if  $M$  is consistent then  $A = \log(m_{ij})$  satisfies the condition of additive consistency:

$$a_{ik} + a_{kj} = a_{ij}$$

for every  $i, j, k \in T(n)$  where  $T(n) = \{(i, j, k) \in \{1, 2, \dots, n\} : i < j < k\}$ . Similarly, an additive PC matrices  $A$  is consistent if and only if there exists a real (weight) vector  $V = (v_1, v_2, \dots, v_n)$  such that  $a_{ij} = v_i - v_j$ , for all  $i, j \in \{1, 2, \dots, n\}$ .

As stated in the introduction, the set  $A_n \subset \mathbb{R}_n^n$  of additive PC matrices corresponds to set of real skew-symmetric matrices of order  $n$  and is a vector subspace of  $\mathbb{R}_n^n$ .

In view of this representation, the set  $(A_n, +)$  of all additively consistent matrices is an additive subgroup of all  $n \times n$  matrices, where whenever it is endowed with the coordinate wise matrix addition. It is a one-to-one image of the multiplicative group  $M_n = (M_n, \cdot)$  by the group isomorphism  $A = \log[M] = \log(m_{ij})$ . The inverse group isomorphism is clearly given by the formula  $M = \exp(A) = \exp[a_{ij}]$ .

Koczkodaj, Marek and Yaylı in [3], defined the sets of Lie group and Lie algebra of PC matrices. It also provides “exponential map” that allows a parametrization of the Lie group elements by simpler objects, the Lie algebra elements.

**Proposition 2.1.** *For every dimension  $n > 0$ , the following group:*

$$\mathcal{G} = \{M = [m_{ij}]_{n \times n} | M \cdot M^T = I, m_{ij} = \frac{1}{m_{ji}} > 0 \text{ for every } i, j = 1, 2, \dots, n\}$$

*is an Abelian group of  $n \times n$  PC matrices with an*

$$\cdot : \mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$$

$$(M, N) \longrightarrow M \cdot N = [m_{ij} \cdot n_{ij}]$$

*where “ $\cdot$ ” is the Hadamard product.*

**Corollary 2.2.** *The Abelian group  $\mathcal{G}$  is a PC matrix Lie group.*

**Theorem 2.3.** *The tangent space of the PC matrix Lie Group  $\mathcal{G}$  at unity  $I$  consist of all  $n \times n$  real matrices  $X$  that satisfy  $X + X^T = 0$*

**Corollary 2.4.** *The Lie Algebra of  $\mathcal{G}$ , denoted by  $T_I(\mathcal{G}) = g$ , is a Lie algebra of  $\mathcal{G}$  and  $g$  is the space of skew-symmetric  $n \times n$  matrices.*

*Remark.* The exponential map is a map from Lie algebra of a given Lie group to that group. Let  $G$  be a PC matrix Lie group and  $\mathfrak{g}$  be the Lie algebra of  $G$ . Then, the exponential map:

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ A = [a_{ij}]_{n \times n} &\longrightarrow \exp[A] = [e^{a_{ij}}] \end{aligned}$$

Koczkodaj, Marek and Yaylı in [3] proposed six properties of the exponential map that they had established to show how Lie algebra of PC matrices can be represented as exponentials of other matrices. Since these properties were already justified and defined in the original work, they are here only briefly recalled.

## 2.2. Basic Concepts on Lie Group and Lie Algebra.

The group  $SO(n, \mathbb{R})$  of rotations is the group of orientation-preserving isometries of the Euclidean space  $\mathbb{E}^n$ . The Lie algebra  $\mathfrak{so}(n, \mathbb{R})$  consisting of real skew-symmetric  $n \times n$  matrices is the corresponding set of infinitesimal rotations. The geometric transform between a Lie group and its Lie algebra is the fact that the Lie algebra can be viewed as the tangent space to the Lie group at the identity. There is a map from the tangent space to the Lie group, called the exponential map.

We begin by defining general linear group, the orthogonal group, the special orthogonal group  $SO(n)$  and the lie algebra  $\mathfrak{so}(n)$ .

**Definition 2.5.** The general linear group  $GL(n)$ ; to be the set of all invertible  $n \times n$  matrices which means a matrix is invertible if and only if its determinant is nonzero, as following,

$$GL(n) = \{A \in \mathbb{R}^{n \times n} : \det(A) \neq 0\}.$$

$GL(n)$  considered as a set, is simply the  $n^2$ -dimensional vector space  $\mathbb{R}^{n \times n}$  with the subset  $\{A : \det A = 0\}$  removed. The important point is that “nearly all”  $n \times n$  matrices are invertible; only a lower-dimensional subset of them are noninvertible. So  $GL(n)$  is a Lie group of the full dimension  $n^2$  [10].

*Remark.* We write  $GL(n, \mathbb{R})$  to stress that these are matrices whose entries are real numbers. All of the remaining groups to be introduced here will be subgroups of  $GL(n, \mathbb{R})$ .

**Definition 2.6.** The orthogonal group  $O(n)$  can be defined as following,

$$O(n) = \{A \in GL(n) : A^T A = I\}.$$

The group  $O(n)$  is the group of automorphisms of  $n$ -dimensional Euclidean space.

Taking determinants of the defining equation  $A^T A = I$  and remembering that

$$\det(A^T) = \det(A) \quad \text{and} \quad \det(AB) = \det(A) \det(B)$$

we see that a matrix  $A \in O(n)$  must satisfy  $(\det A)^2 = 1$ , or in other words

$$\det(A) = \pm 1$$

Both signs are possible. Consider, for instance,  $n \times n$  diagonal matrices with 1's and  $-1$ 's on the diagonal. All such matrices are orthogonal, hence belong to  $O(n)$ . If the number of entries  $-1$  is even, then  $\det(A) = +1$ ; but if the number of entries  $-1$  is odd, then  $\det(A) = -1$ .

The orthogonal matrices with  $\det(A) = +1$  form a subgroup of  $O(n)$  it is called the special orthogonal group  $SO(n)$  [10].

**Definition 2.7.** The special orthogonal group  $SO(n)$  can be defined as follows:

$$SO(n) = \{A \in O(n) : \det(A) = +1\}.$$

It is also called the rotation group, because the elements of  $SO(n)$  can be regarded as rotations [10].

**Definition 2.8.** The group  $O(n)$  of orthogonal matrices is called the orthogonal group, and its subgroup  $SO(n)$  is called the special orthogonal group. The vector space of real  $n \times n$  skew-symmetric matrices is denoted by  $so(n)$  [10].

Here, we use characterization of  $SO(n, \mathbb{R})$ ,  $so(n, \mathbb{R})$  and following theorem explains fundamental relationship between  $SO(n, \mathbb{R})$  and  $so(n, \mathbb{R})$ .

**Theorem 2.9.** *The exponential map*

$$\exp : so(n, \mathbb{R}) \longrightarrow SO(n, \mathbb{R})$$

*is well-defined and surjective* [11].

Since the exponential is surjective then it is possible to write preimage. The part that interests us here is that the Lie group  $SO(n, \mathbb{R})$  is known to be a Lie algebra skew-symmetric matrix space and the Lie algebra of PC matrix Lie groups, comprises of skew-symmetric matrix space [11].

### 3. A NEW APPROACH FOR PC MATRICES VIA GEOMETRIC CALCULUS

Geometric calculus is an alternative to the usual calculus of Newton and Leibniz introduced by Grossman and Katz [9]. It provides differentiation and integration tools based on multiplication instead of addition. Every property in Newtonian calculus has an analog in multiplicative calculus. We refer [8] to know basic of  $\alpha$ -generator and geometric arithmetic  $(\mathbb{R}(G), \oplus, \ominus, \odot, \oslash)$  and the sets of geometric integers and geometric real numbers  $\mathbb{Z}(G)$  and  $\mathbb{R}(G)$ , respectively,

$$\mathbb{Z}(G) = \{e^x : x \in \mathbb{Z}\} \quad \mathbb{R}(G) = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+ \setminus \{0\}$$

*Remark 3.1.*  $(\mathbb{R}(G), \oplus, \odot)$  is a field with geometric zero one and geometric identity  $e$ . We can define geometric and ordinary arithmetic operators in the following,

For all  $x, y \in \mathbb{R}(G)$

$$\begin{aligned} (3.1) \quad & x \oplus y = x \cdot y \\ & x \ominus y = x \div y \quad (y \neq 0) \\ & x \odot y = x^{\ln y} \\ & x \oslash y = x^{\frac{1}{\ln y}}, \quad (y \neq 1). \end{aligned}$$

Matrix operators can be define with an operation given by (3.1). In [15], matrix operators symbols given as  $\oplus^*$ ,  $\ominus^*$ ,  $\otimes^*$ . In order not to create confusion in our work, we will give matrix operators in the form of  $\oplus$ ,  $\ominus$ ,  $\otimes$  in the following definitions.

**Definition 3.2.** For a given  $A = [a_{ij}] \in \mathbb{R}_n^m$  and  $B = [b_{ij}] \in \mathbb{R}_n^m$  matrices,

$$A \oplus B = [a_{ij} \oplus b_{ij}]$$

are define with an operation  $\oplus$  geometric sum of matrices.

**Definition 3.3.** For a given  $A = [a_{ij}] \in \mathbb{R}_n^m$  and  $B = [b_{ij}] \in \mathbb{R}_n^m$  matrices,

$$A \ominus B = [a_{ij} \ominus b_{ij}]$$

are define with an operation  $\ominus$  geometric subtraction of matrices.

**Definition 3.4.** For a given  $A = [a_{ij}] \in \mathbb{R}_n^m$  and  $B = [b_{ij}] \in \mathbb{R}_p^n$  matrices,

$$A \otimes B = C$$

defined operation  $\otimes$  is the geometric multiplication of matrices. Here  $C = [c_{ij}]$  is define following

$$C_{ij} = \sum_G (a_{ik} \otimes b_{kj}) = \prod_{k=1}^n (a_{ik} \otimes b_{kj}).$$

**Definition 3.5.**  $\mathbf{G}_1(n)$  is a PC matrix Abelian group defining by following,

$$\mathbf{G}_1(n) = \{\mathbb{B} \in \mathbb{R}(G)_n^n \mid \mathbb{B} \oplus \mathbb{B}^T = I_{\mathbf{G}_1}\}$$

where  $\oplus$  is the Hadamard product,  $\mathbb{R}(G)$  is geometric real numbers and  $\mathbb{R}(G)_n^n$  is geometric real  $n \times n$  matrices. The identity element of the group  $(\mathbf{G}_1(n), \oplus)$  is  $I_{\mathbf{G}_1} = [1]_{n \times n}$

**Definition 3.6.**  $\mathbf{G}_2(n)$  is a PC matrix Abelian group defining by following,

$$\mathbf{G}_2(n) = \{\mathbb{A} \in \mathbb{R}(G)_n^n \mid \mathbb{A} \otimes \mathbb{A}^T = I_{\mathbf{G}_2}\}.$$

where  $\mathbb{R}(G)$  is geometric real numbers and  $\mathbb{R}(G)_n^n$  is geometric real  $n \times n$  matrices with an operation  $\otimes$ . The identity element of the group  $(\mathbf{G}_2(n), \otimes)$  is

$$I_{\mathbf{G}_2} = \begin{bmatrix} e & 1 & \dots & 1 \\ 1 & e & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & e \end{bmatrix}_{n \times n}.$$

**Theorem 3.7.** *The Lie algebra of  $SO(n, \mathbb{R}(G))$  is*

$$\mathbf{G}_1(n) = \{\mathbb{B} \in \mathbb{R}(G)_n^n \mid \mathbb{B} \oplus \mathbb{B}^T = I_{\mathbf{G}_1}\}.$$

*Proof.* Let  $e^{\mathbb{A}(t)}$  be a curve with  $t = 0$ ,  $e^{\mathbb{A}(0)} = I_{\mathbf{G}_2}$  which  $e^{\mathbb{A}(t)} = [e^{a_{ij}(t)}] \in SO(n, \mathbb{R}(G))$ . Here we get,

$$\begin{aligned} \frac{d}{dt^*}(e^{\mathbb{A}(t)} \otimes e^{\mathbb{A}^T(t)}) &= (e^{\mathbb{A}'(t)} \otimes e^{\mathbb{A}^T(t)}) \oplus (e^{\mathbb{A}(t)} \otimes e^{(\mathbb{A}')^T(t)}) \\ &= (e^{\mathbb{A}'(0)} \otimes e^{\mathbb{A}^T(0)}) \oplus (e^{\mathbb{A}(0)} \otimes e^{(\mathbb{A}')^T(0)}) \\ &= (e^{\mathbb{A}'(0)} \otimes I_{\mathbf{G}_2}) \oplus (I_{\mathbf{G}_2} \otimes e^{(\mathbb{A}')^T(0)}) \\ &= e^{\mathbb{A}'(0)} \oplus e^{(\mathbb{A}')^T(0)} \end{aligned}$$

We know that  $\frac{d}{dt^*}(I_{\mathbf{G}_2}) = I_{\mathbf{G}_1}$  for  $e^{\mathbb{A}(t)} \in SO(n, \mathbb{R}(G))$ , then it ensures the following equality,

$$e^{\mathbb{A}(t)} \otimes e^{\mathbb{A}^T(t)} = I_{\mathbf{G}_2}.$$

By differentiating the both side of equation, we get

$$\begin{aligned} \frac{d}{dt^*}(e^{\mathbb{A}(t)} \otimes e^{\mathbb{A}^T(t)}) &= \frac{d}{dt^*}(I_{\mathbf{G}_2}) \\ e^{\mathbb{A}'(0)} \oplus e^{(\mathbb{A}')^T(0)} &= I_{\mathbf{G}_1}. \end{aligned}$$

For  $\mathbb{B} \in \mathbf{G}_1$  we choose to  $\mathbb{B} = [e^{\mathbb{A}'(0)}]$  and then we obtain

$$\mathbb{B} \oplus \mathbb{B}^T = I_{\mathbf{G}_1}.$$

Which completes the proof. □

**Definition 3.8.**  $\mathbb{R}(G)$  is a field and  $\mathbf{G}_1(3)$  is a vector space and

$$\begin{aligned} [\cdot, \cdot] : \mathbf{G}_1(3) \times \mathbf{G}_1(3) &\longrightarrow \mathbf{G}_1(3) \\ (\mathbb{B}_1, \mathbb{B}_2) &\longrightarrow [\mathbb{B}_1, \mathbb{B}_2] = (\mathbb{B}_1 \otimes \mathbb{B}_2) \ominus (\mathbb{B}_2 \otimes \mathbb{B}_1) \end{aligned}$$

$[\cdot, \cdot]$  is a Lie bracket operation for geometric calculus where  $\ominus$  is  $A \ominus B = [a_{ij} \ominus b_{ij}]$ . Then we know that

$$\begin{aligned} \wedge_G : \mathbb{R}^3(G) \times \mathbb{R}^3(G) &\longrightarrow \mathbb{R}^3(G) \\ (X, Y) &\longrightarrow X \wedge_G Y = \det_G(X \wedge_G Y) \end{aligned}$$

$\wedge_G$  is a vector cross product in  $\mathbb{R}^3$  where  $\mathbb{R}^3(G) = \mathbb{R}(G) \times \mathbb{R}(G) \times \mathbb{R}(G)$ .

**Corollary 3.9.**  $(\mathbf{G}_1(3), [\cdot, \cdot])$  Lie algebra is congruent to  $(\mathbb{R}^3(G), \wedge_G)$ .

*Proof.* Let  $f$  be the Lie algebra isomorphism

$$\begin{aligned} f : \mathbf{G}_1(3) &\longrightarrow \mathbb{R}^3(G) \\ \begin{bmatrix} 1 & e^a & e^b \\ e^{-a} & 1 & e^c \\ e^{-b} & e^{-c} & 1 \end{bmatrix} &\longrightarrow f\left(\begin{bmatrix} 1 & e^a & e^b \\ e^{-a} & 1 & e^c \\ e^{-b} & e^{-c} & 1 \end{bmatrix}\right) = (e^{-c}, e^b, e^{-a}). \end{aligned}$$

First of all, let us  $\mathbb{A}$  and  $\mathbb{B}$  matrices are defined

$$\mathbb{A} = \begin{bmatrix} 1 & e^a & e^b \\ e^{-a} & 1 & e^c \\ e^{-b} & e^{-c} & 1 \end{bmatrix}$$

and

$$\mathbb{B} = \begin{bmatrix} 1 & e^x & e^y \\ e^{-x} & 1 & e^z \\ e^{-y} & e^{-z} & 1 \end{bmatrix}.$$

Then we know that  $[\mathbb{A}, \mathbb{B}] = (\mathbb{A} \otimes \mathbb{B}) \ominus (\mathbb{B} \otimes \mathbb{A})$  so that

$$\mathbb{A} \otimes \mathbb{B} = \begin{bmatrix} e^{-ax-by} & e^{-bz} & e^{az} \\ e^{-cy} & e^{-ax-cz} & e^{-ay} \\ e^{cx} & e^{-bx} & e^{-by-cz} \end{bmatrix}$$

and

$$\mathbb{B} \otimes \mathbb{A} = \begin{bmatrix} e^{-ax-by} & e^{-cy} & e^{cx} \\ e^{-bz} & e^{-ax-cz} & e^{-bx} \\ e^{az} & e^{-ay} & e^{-by-cz} \end{bmatrix}$$

from that



$$(\mathbb{A} \otimes \mathbb{B}) \ominus (\mathbb{B} \otimes \mathbb{A}) = \frac{\mathbb{A} \otimes \mathbb{B}}{\mathbb{B} \otimes \mathbb{A}} = \begin{bmatrix} 1 & e^{ay-bz} & e^{az-cx} \\ e^{-ay+bz} & 1 & e^{-ay+bx} \\ e^{cx-az} & e^{-bx+ay} & 1 \end{bmatrix} \cong (e^{ay-bz}, e^{az-cx}, e^{bx-ay})$$

it is clear from definition of  $f$ . On the other hand,  $\mathbb{A}$  and  $\mathbb{B}$  matrices are rewritten

$$\begin{aligned} \mathbb{A} &\cong (e^{-c}, e^b, e^{-a}) = \vec{a} \\ \mathbb{B} &\cong (e^{-z}, e^y, e^{-x}) = \vec{b}. \end{aligned}$$

We know that the definition of  $\wedge_G$  is

$$\vec{a} \wedge_G \vec{b} = \det_G(\vec{a} \wedge_G \vec{b}) = (e^{ay-bz}, e^{az-cx}, e^{bx-ay}).$$

Therefore, we obtain that

$$[\mathbb{A}, \mathbb{B}] \cong \vec{a} \wedge_G \vec{b}.$$

Then the proof is completed. □

**Theorem 3.10.** *For a given*

$$SO(n, \mathbb{R}) := \{A \in \mathbb{R}_n^n \mid A \cdot A^T = I_{n \times n}, \det(A) = +1\}$$

*isometry group of orthogonal matrices and the Lie algebra defined by,*

$$so(n, \mathbb{R}) := \{B \in \mathbb{R}_n^n \mid B^T + B = 0\}.$$

*Let the isometry group of  $G$ -orthogonal matrices defined by,*

$$SO(n, \mathbb{R}(G)) := \{A \in \mathbb{R}(G)_n^n \mid A \otimes A^T = I_{G_2}\}$$

*and the Lie algebra of this group will henceforth call  $G$ - skew-symmetric is following,*

$$so(n, \mathbb{R}(G)) := \{B \in \mathbb{R}(G)_n^n \mid B \oplus B^T = I_{G_1}\}.$$

*And diagram for PC matrices can be given*

$$\begin{array}{ccc} SO(n, \mathbb{R}) & \xrightleftharpoons[e]{e} & SO(n, \mathbb{R}(G)) \\ f \uparrow & & \uparrow \text{Exp} \downarrow \text{Ln} \\ so(n, \mathbb{R}) & \xrightleftharpoons[e]{e} & so(n, \mathbb{R}(G)) \end{array}$$

*Remark 3.11.* For the first time we defined and named,

$$\begin{aligned} \text{Exp} &: e \circ f \circ \text{Ln} : so(n, \mathbb{R}(G)) \longrightarrow SO(n, \mathbb{R}(G)) \\ \text{Ln} &: e \circ f^{-1} \circ \text{Ln} : SO(n, \mathbb{R}(G)) \longrightarrow so(n, \mathbb{R}(G)) \end{aligned}$$

transformations with the help of diagram. Additionally, the Cayley Formula for  $A \in SO(n, \mathbb{R})$  and  $B \in so(n, \mathbb{R})$  gives us a relation between orthogonal  $n \times n$  matrix and a real  $n \times n$  skew-symmetric matrix.

Consequently, we have the following formulas

$$\begin{aligned} A = f(B) &= (I_n + B) \cdot (I_n - B)^{-1} \\ B = f^{-1}(A) &= (I_n - A) \cdot (I_n + A)^{-1} \end{aligned}$$

But be aware of the identity element of the above formulas a real unit  $n \times n$  matrix,

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n}.$$

With the help of above diagram, we give a different approach for PC matrices by using G-calculus. In the study of Waldemar, Marek and Yaylı [3], according to proposition (6) in section five, for any  $A \in g$  it cannot always achieved  $\det(e^A) = e^{Tr(A)}$  and they presented a counterexample (Example 5.1). It is an important problem whether there is a valid structure for this property. Now it has been proved that this equality is possible with G-calculus by the proposition we give below and then we check this proposition using the example given by the study [3].

**Proposition 3.12.**  $\mathbb{B}$  is a  $G$ -skew-symmetric matrix in  $so(n, \mathbb{R}(G))$  and  $\mathbb{A}$  is a  $G$ -orthogonal matrix in  $SO(n, \mathbb{R}(G))$  then the transformation  $Exp$  gives the following equality

$$\det_{\mathbb{R}(G)}(\mathbb{A}) = e^{Tr_{\mathbb{R}(G)}(\mathbb{B})}.$$

*Proof.* For a given  $\mathbb{A} \in SO(n, \mathbb{R}(G))$  above equality is ensured.

$$\mathbb{A} \otimes \mathbb{A}^T = I_{\mathbf{G}_2}$$

by taking the determinant of both sides of the equation, we get

$$\begin{aligned} \det_{\mathbb{R}(G)}(\mathbb{A}) \otimes \det_{\mathbb{R}(G)}(\mathbb{A}^T) &= e \\ \det_{\mathbb{R}(G)}(\mathbb{A}) \otimes \det_{\mathbb{R}(G)}(\mathbb{A}) &= e \\ \det_{\mathbb{R}(G)}(\mathbb{A}) &= e \end{aligned}$$

and for  $\mathbb{B} \in so(n, \mathbb{R}(G))$  then we obtain

$$Tr_{\mathbb{R}(G)}(\mathbb{B}) = 1 \oplus 1 \oplus \dots \oplus 1 = 1$$

which completes the proof.  $\square$

**Example 3.13.** Let us consider the following PC matrix  $\mathbb{B} \in so(3, \mathbb{R}(G))$

$$\mathbb{B} = \begin{bmatrix} 1 & e & e \\ e^{-1} & 1 & 1 \\ e^{-1} & 1 & 1 \end{bmatrix}$$

and using  $\ln$  transformation over  $\mathbb{B}$ , we get

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

We know that from diagram  $B \in so(3, \mathbb{R})$  and this allows us to obtain an orthogonal matrix using Cayley formula

$$A = f(B) = (I_3 + B)(I_3 - B)^{-1}$$

which means

$$A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

Applying the exponential transformation to the orthogonal matrix, we obtain

$$\mathbb{A} = e^A = \begin{bmatrix} e^{-1/3} & e^{2/3} & e^{2/3} \\ e^{-2/3} & e^{1/3} & e^{-2/3} \\ e^{-2/3} & e^{-2/3} & e^{1/3} \end{bmatrix}$$

G-orthogonal matrix in  $\in SO(3, \mathbb{R}(G))$ . From [3] we know that  $\det(e^{\mathbb{B}}) = e^2 + e^{-2} - 2$  and  $e^{Tr(B)} = 1$  so that  $\det(e^{\mathbb{B}}) \neq e^{Tr(B)}$ .

But now let us consider

$$\det_{\mathbb{R}(G)}(\mathbb{A}) = e \quad \text{and} \quad Tr_{\mathbb{R}(G)}(\mathbb{B}) = 1.$$

It is easy to observe that  $\det_{\mathbb{R}(G)}(\mathbb{A}) = e^{Tr_{\mathbb{R}(G)}(\mathbb{B})}$ . Following example will provide for a given G-orthogonal matrix found corresponded skew-symmetric matrix by the help of the diagram.

**Example 3.14.** For a given

$$\mathbb{A} = \begin{bmatrix} 1 & e^{-t} \\ e^t & 1 \end{bmatrix} \in SO(2, \mathbb{R}(G))$$

G-orthogonal matrix, using ln transformation over  $\mathbb{A}$ , we get

$$A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}.$$

Now from diagram we know that  $A \in SO(2, \mathbb{R})$  and with Cayley formula, we will obtain the anti-symmetric matrix corresponding to orthogonal matrix

$$B = f^{-1}(A) = (I_2 - A)(I_2 + A)^{-1}$$

which means

$$B = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Now we have  $B \in so(2, \mathbb{R})$  and from diagram we know exponential map of  $B$ , we obtain

$$\mathbb{B} = \begin{bmatrix} e^{\cos t} & e^{-\sin t} \\ e^{\sin t} & e^{\cos t} \end{bmatrix} \in so(2, \mathbb{R}(G))$$

is the general rotation matrix for  $so(2, \mathbb{R}(G))$  and it easy to see that

$$\det_{\mathbb{R}(G)}(\mathbb{B}) = e^{Tr_{\mathbb{R}(G)}(\mathbb{A})}.$$

#### 4. CAYLEY FORMULA IN TERMS OF G-CALCULUS

In 1846, Arthur Cayley [7] introduced what is known as the  $B$  orthogonal matrix

$$(4.1) \quad A = \text{Cay}(B) = (I + B) \cdot (I - B)^{-1}.$$

He showed that the Cayley transform of  $A = -A^T$  is a proper-orthogonal matrix, and hence a rotation matrix. Moreover, the relation (4.1) can be given by

$$(4.2) \quad B = (I - A) \cdot (I + A)^{-1}.$$

The aim of this section is construct Cayley formula by using G-calculus. First of all, let us define inverse of any matrix  $\mathbb{A}$  according to G-calculus if the matrix  $\mathbb{A}$  invertible matrix.

From here on, we will give unit matrix  $I_{\mathbf{G}_2}$  in G-calculus as  $I$ , not to create confusion.

**Definition 4.1.** Let  $\mathbb{A} = [a_{ij}]$  be a  $m \times n$  matrices with entries  $\mathbb{R}(G)$ . Then transpose of  $\mathbb{A}$  is repressed by  $\mathbb{A}^T$  and it is defined by

$$\mathbb{A}^T = [a_{ji}] \in \mathbb{R}(G)$$

where  $a_{ij}^T = a_{ji}$ .

**Definition 4.2.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be a  $n \times n$  matrices with entries  $\mathbb{R}(G)$ .  $\mathbb{A}$  and  $\mathbb{B}$  are inverse of each other if

$$\mathbb{A} \otimes \mathbb{B} = \mathbb{B} \otimes \mathbb{A} = I.$$

A matrix has no inverse, it is said to be singular, but if it does have an inverse, it is said to be invertible or non-singular.

**Theorem 4.3.** Let  $\mathbb{B}$  be a  $G$ -skew-symmetric matrix. Then, the matrix  $(I \ominus \mathbb{B})$  is invertable matrix according to  $G$ -calculus.

*Proof.* The proof is clear since  $\det_{\mathbb{R}(G)}(I \ominus \mathbb{B}) \neq 0$ . □

**Theorem 4.4.** For every  $G$ -skew-symmetric matrix  $\mathbb{B}$ , the matrix

$$\mathbb{A} = (I \oplus \mathbb{B}) \otimes (I \ominus \mathbb{B})^{-1}$$

is a  $G$ -orthogonal matrix.

*Proof.* To show that the matrix  $\mathbb{A}$  is an orthogonal matrix, we must denote  $\mathbb{A}^T \otimes \mathbb{A} = I$ .

Since

$$\begin{aligned} \mathbb{A}^T &= (I \oplus \mathbb{B}^T)^{-1} \otimes (I \ominus \mathbb{B}^T) \\ &= (I \ominus \mathbb{B})^{-1} \otimes (I \oplus \mathbb{B}) \end{aligned}$$

and then we know that  $(I \oplus \mathbb{B}) \otimes (I \ominus \mathbb{B}) = (I \ominus \mathbb{B}) \otimes (I \oplus \mathbb{B})$ , we obtain;

$$\begin{aligned} \mathbb{A}^T \otimes \mathbb{A} &= (I \ominus \mathbb{B})^{-1} \otimes (I \oplus \mathbb{B}) \otimes (I \ominus \mathbb{B}) \otimes (I \oplus \mathbb{B})^{-1} \\ &= (I \ominus \mathbb{B})^{-1} \otimes (I \ominus \mathbb{B}) \otimes (I \oplus \mathbb{B}) \otimes (I \oplus \mathbb{B})^{-1} \\ &= I. \end{aligned}$$

Then the proof is complete.  $\square$

**Corollary 4.5.** *For every, G-orthogonal matrix  $\mathbb{A}$ , the matrix*

$$\mathbb{B} = (I \ominus \mathbb{A}) \otimes (I \oplus \mathbb{A})^{-1}$$

*is a G-skew-symmetric matrix.*

*Proof.*  $\mathbb{A}$  is an orthogonal matrix then  $\mathbb{A}^T = \mathbb{A}^{-1}$ . To show that the matrix  $\mathbb{B}$  is a G-skew-symmetric matrix, we must denote  $\mathbb{B}^T \oplus \mathbb{B} = I_{\mathbf{G}_1}$ .

$$\begin{aligned} \mathbb{B}^T &= (I \oplus \mathbb{A}^T)^{-1} \otimes (I \ominus \mathbb{A}^T), \quad (\mathbb{A}^{-1} \otimes \mathbb{A} = I) \\ &= ((\mathbb{A}^T \otimes \mathbb{A}) \oplus \mathbb{A}^T)^{-1} \otimes (I \ominus \mathbb{A}^T) \\ &= (\mathbb{A}^T \otimes (\mathbb{A} \oplus I))^{-1} \otimes (I \ominus \mathbb{A}^{-1}) \\ &= (\mathbb{A} \oplus I)^{-1} \otimes (\mathbb{A} \ominus I) \end{aligned}$$

and it is easy to observe that  $\mathbb{B}^T \oplus \mathbb{B} = I_{\mathbf{G}_1}$ , which completes the proof.  $\square$

Consequently, we can give the following definition.

**Definition 4.6.** For every G-skew symmetric matrix  $\mathbb{B}$ , the map

$$\begin{aligned} \text{Cay}_G : so(n, \mathbb{R}(G)) &\longrightarrow SO(n, \mathbb{R}(G)) \\ \mathbb{B} &\longrightarrow \text{Cay}(\mathbb{B}) = \mathbb{A} \end{aligned}$$

is named as G-Cayley transform according to G-calculus.

The formula is named as G-Cayley formula whose mean is Cayley formula in terms of G-calculus. Now we examine the examples that we have given with the help of a diagram.

**Example 4.7.** Let us consider the following G-skew symmetric PC matrix  $\mathbb{B} \in so(3, \mathbb{R}(G))$

$$\mathbb{B} = \begin{bmatrix} 1 & e & e \\ e^{-1} & 1 & 1 \\ e^{-1} & 1 & 1 \end{bmatrix}.$$

Now, using *Cay* transformation we obtain

$$\mathbb{A} = \text{Cay}_G(\mathbb{B}) = (I \ominus \mathbb{B})^{-1} \otimes (I \oplus \mathbb{B})$$

$$\mathbb{A} = \left( \begin{bmatrix} e & 1 & 1 \\ 1 & e & 1 \\ 1 & 1 & e \end{bmatrix} \ominus \begin{bmatrix} 1 & e & e \\ e^{-1} & 1 & 1 \\ e^{-1} & 1 & 1 \end{bmatrix} \right)^{-1} \otimes \left( \begin{bmatrix} e & 1 & 1 \\ 1 & e & 1 \\ 1 & 1 & e \end{bmatrix} \oplus \begin{bmatrix} 1 & e & e \\ e^{-1} & 1 & 1 \\ e^{-1} & 1 & 1 \end{bmatrix} \right)$$

$$\mathbb{A} = \begin{bmatrix} e^{-1/3} & e^{2/3} & e^{2/3} \\ e^{-2/3} & e^{1/3} & e^{-2/3} \\ e^{-2/3} & e^{-2/3} & e^{1/3} \end{bmatrix}$$

is a G-orthogonal matrix in  $SO(3, \mathbb{R}(G))$ .

**Example 4.8.** For a given

$$\mathbb{A} = \begin{bmatrix} 1 & e^{-t} \\ e^t & 1 \end{bmatrix} \in SO(2, \mathbb{R}(G))$$

G-orthogonal matrix, from Corollary 4.5, we get,

$$\begin{aligned} \mathbb{B} &= (I \ominus \mathbb{A}) \otimes (I \oplus \mathbb{A})^{-1} \\ \mathbb{B} &= \left( \begin{bmatrix} e & 1 \\ 1 & e \end{bmatrix} \ominus \begin{bmatrix} 1 & e^{-t} \\ e^t & 1 \end{bmatrix} \right) \otimes \left( \begin{bmatrix} e & 1 \\ 1 & e \end{bmatrix} \oplus \begin{bmatrix} 1 & e^{-t} \\ e^t & 1 \end{bmatrix} \right)^{-1} \\ \mathbb{B} &= \begin{bmatrix} e^{\cos t} & e^{-\sin t} \\ e^{\sin t} & e^{\cos t} \end{bmatrix} \end{aligned}$$

is the general rotation matrix for  $so(2, \mathbb{R}(G))$ .

## 5. CONCLUSIONS AND FUTURE DIRECTIONS

Various hypotheses, index values and theorems related to the stability of PC matrices have been given in the studies conducted on engineering topics, which are the most intensive application area of PC matrices. Quite recently, Koczkodaj, Marek and Yayli [3] have studied Lie algebra and Lie group of PC matrices and it provided us with a structure that can be used to establish the geometric structure of PC matrices. They examined PC matrices for the first time using differential geometry techniques. In this work, we have gained a new perspective by using in geometric arithmetic over PC matrices Lie group and Lie algebra. As the most remarkable conclusion here is that Lie bracket operator of  $so(n, \mathbb{R})$  is equal zero. We will interpret the consistency of PC matrices according to  $SO(n, \mathbb{R}(G))$ . The gains and application of *Exp* and *Ln* transformations in PC matrices will be investigated in future studies and on eliminating the shortcomings related to this topic, which includes many mathematical foundations for PC matrices that have been around for many years, will also benefit many different disciplines. With the help of G-Cayley formula we will investigate Euler–Rodrigues formula and kinematic meaning of PC matrices according to G-calculus.

## CONFLICT OF INTEREST

The authors declare that they have no known conflict of interests.

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