Bicomplex Leonardo Numbers

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Abstract

In literature until today, many authors have studied special sequences in different number systems. In this paper, using the Leonardo numbers, we introduce the bicomplex Leonardo numbers. Also, we give some algebraic properties of bicomplex Leonardo numbers such as recurrence relation, generating function, Binet's formula, D'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Keywords: Bicomplex number; Bicomplex Fibonacci number; Leonardo number; Bicomplex Leonardo number.

1. Introduction

Leonardo numbers are introduced and given some properties for example Cassini's identity, Catalan's identity and d'Ocagne's identityfor Leonardo numbers by Catarino and Borges in [1]. Leonardo sequence denoted with Le_n is defined by the following recurrence relation for $n \geq 2$

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \tag{1}$$

with the initial conditions $Le_0 = Le_1 = 1$. This sequence is also expressed as:

$$Le_{n+1} = 2\,Le_n - Le_{n-2}\,,\; n \geq 2 \tag{2}$$

Also, there are many identities between Leonardo numbers and Fibonacci and Lucas numbers, we give a few identities as follows.

$$Le_n = 2\,F_{n+1} - 1\,,\;\; n \geq 0 \tag{3}$$

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$$Le_n = L_{n+2} - F_{n+2} - 1\,, \;\; n \geq 0 \tag{4}$$

$$Le_n = 2\,(\frac{L_n + L_{n+2}}{5}) - 1\,, \;\; n \geq 0 \tag{5}$$

$$Le_n = \frac{L_{n+1} + L_{n+7}}{5} - 1\,, \;\; n \geq 0 \tag{6}$$

In [2], has defined generalized Leonardo numbers. In [3], the authors investigated the two dimensional recurrences relations of Leonardo numbers from its one-dimensional model. In [4], authors have given some properties of Leonardo numbers. In [5], the authors have defined incomplete Leonardo numbers and given some properties of these numbers.

Recently, many authors have considered special number sequences with different number systems. Bicomplex numbers are "complex numbers with complex coefficients", which explains the name bicomplex. In [6], Price introduced the set of bicomplex numbers, which can be represented as

$$\mathbb{BC} = \{\, q = q_1 + iq_2 + jq_3 + ijq_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\} \tag{7}$$

where i,j and ij satisfy the conditions (Table 1)

$$i^2 = -1, \;\; j^2 = -1, \;\; ij = ji. \tag{8}$$

The multiplication table of these numbers is below:

Table 1: Multiplication scheme of bicomplex numbers

x	1	i	j	i j
1	1	i	j	i j
i	i	-1	i j	-j
j	j	i j	-1	-i
i j	i j	-j	-i	1

In [7], the authors introduced the algebra of bicomplex numbers as a generalization of the field of complex numbers. In [8], the authors introduced algebraic properties of both bicomplex numbers and hyperbolic numbers. In [9], the authors introduced bicomplex Fibonacci and Lucas numbers. In [10], Halici has studied on bicomplex Fibonacci numbers and their generalization. In [11], Aydin has defined bicomplex Fibonacci quaternions.

2. The bicomplex Leonardo numbers

In this section, we define the bicomplex Leonardo numbers. Then, we obtain generating function, Binet's formula, d'Ocagne's identity, Cassini's identity, Catalan's identity and Honsberger identity.

Definition 2.1. *2.1. For $n \geq 1$, the n-th bicomplex Leonardo numbers $\mathcal{BL}e_n$ are defined by using the Leonardo numbers as follows*

$$\mathcal{BL}e_n = Le_n + i\, Le_{n+1} + j\, Le_{n+2} + i\, j\, Le_{n+3}. \tag{9}$$

By placing Eq.(1) in Eq.(9) we obtain that,

$$\begin{aligned}\mathcal{BL}e_n = \; & (Le_{n-1} + Le_{n-2} + 1) + i\,(Le_n + Le_{n-1} + 1) \\ & + j\,(Le_{n+1} + Le_n + 1) + i\, j\,(Le_{n+2} + Le_{n+1} + 1) \\ & = \mathcal{BL}e_{n-1} + \mathcal{BL}e_{n-2} + A\end{aligned}$$

where $A = 1 + i + j + i\, j$. Also, initial values are $\mathcal{BL}e_0 = 1 + i + 3\, j + 5\, i\, j$, $\mathcal{BL}e_1 = 1 + 3\, i + 5\, j + 9\, i\, j$. Now we give the recurrence relation corresponding to expression Eq.(2). That is

$$\mathcal{BL}e_{n+1} = 2\,\mathcal{BL}e_n - \mathcal{BL}e_{n-2}. \tag{10}$$

Using relations Eq.(2) and Eq.(9) we obtain that,

$$\begin{aligned}\mathcal{BL}e_{n+1} = \; & 2\,(Le_n - Le_{n-2}) + i\,(2\, Le_{n+1} - Le_{n-1}) \\ & + j\,(2Le_{n+2} - Le_n) + i\, j\,(2Le_{n+3} - Le_{n+1}) \\ & = 2\,\mathcal{BL}e_n - \mathcal{BL}e_{n-2}\,.\end{aligned}$$

Let $\mathcal{BL}e_n$ and $\mathcal{BL}e_m$ be two bicomplex Leonardo numbers such that

$$\mathcal{BL}e_n = Le_n + i\, Le_{n+1} + j\;\; Le_{n+2} + ij\, Le_{n+3} \tag{11}$$

and

$$\mathcal{BL}e_m = Le_m + i\, Le_{m+1} + j\;\; Le_{m+2} + ij\, Le_{m+3} \tag{12}$$

Then, the addition and subtraction of two bicomplex Leonardo numbers are defined in the obvious way,

$$\begin{aligned}\mathcal{BL}e_n \pm \mathcal{BL}e_m = \; & (Le_n + i\, Le_{n+1} + j\, Le_{n+2} + ij\, Le_{n+3}) \\ & \pm(Le_m + i\, Le_{m+1} + j\, Le_{m+2} + ij\, Le_{m+3}) \\ = \; & (Le_n \pm Le_m) + i\,(Le_{n+1} \pm Le_{m+1}) + j\,(Le_{n+2} \pm Le_{m+2}) \\ & + ij\,(Le_{n+3} \pm Le_{m+3}).\end{aligned} \tag{13}$$

Multiplication of two bicomplex Leonardo numbers is defined by

$$\begin{aligned}
\mathcal{BL}e_n \times \mathcal{BL}e_m = & \; (Le_n + i\, Le_{n+1} + j\, Le_{n+2} + ij\, Le_{n+3}) \\
& \; (Le_m + i\, Le_{m+1} + j\, Le_{m+2} + ij\, Le_{m+3}) \\
= & \; (Le_n Le_m - Le_{n+1} Le_{m+1} - Le_{n+2} Le_{m+2} + Le_{n+3} Le_{m+3}) \\
& + i\, (Le_n Le_{m+1} + Le_{n+1} Le_m - Le_{n+2} Le_{m+3} - Le_{n+3} Le_{m+2}) \\
& + j\, (Le_n Le_{m+2} + Le_{n+2} Le_m - Le_{n+1} Le_{m+3} - Le_{n+3} Le_{m+1}) \\
& + ij\, (Le_n Le_{m+3} + Le_{n+3} Le_m + Le_{n+1} Le_{m+2} + Le_{n+2} Le_{m+1}) \\
= & \; \mathcal{BL}e_m \times \mathcal{BL}e_n\, .
\end{aligned} \tag{14}$$

Three kinds of conjugation can be defined for bicomplex numbers [8]. Therefore, conjugation of the bicomplex Leonardo number is defined in three different ways as follows

$$(\mathcal{BL}e_n)_i^* = Le_n - i\, Le_{n+1} + j\, Le_{n+2} - i\, j\, Le_{n+3}, \tag{15}$$

$$(\mathcal{BL}e_n)_j^* = Le_n + i\, Le_{n+1} - j\, Le_{n+2} - i\, j\, Le_{n+3}, \tag{16}$$

$$(\mathcal{BL}e_n)_{i\,j}^* = Le_n - i\, Le_{n+1} - j\, Le_{n+2} + i\, j\, Le_{n+3}, \tag{17}$$

In the following theorem, some properties related to the conjugations of the bicomplex Leonardo numbers are given.

Theorem 2.1. *Let $(\mathcal{BL}e_n)_i^*, (\mathcal{BL}e_n)_j^*$ and $(\mathcal{BL}e_n)_{ij}^*$, be three kinds of conjugation of the bicomplex Leonardo number. In this case, we can give the following relations:*

$$\begin{aligned}
\mathcal{BL}e_n\, (\mathcal{BL}e_n)_i^* \; & = Le_n^2 + Le_{n+1}^2 - Le_{n+2}^2 - Le_{n+3}^2 \\
& \quad + 2\, j\, Le_{n+1} Le_{n+3},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathcal{BL}e_n\, (\mathcal{BL}e_n)_j^* \; & = Le_n^2 - Le_{n+1}^2 + Le_{n+2}^2 - Le_{n+3}^2 \\
& \quad + 2\, i\, [Le_n Le_{n+1} + Le_{n+2} Le_{n+3}],
\end{aligned} \tag{19}$$

$$\begin{aligned}
\mathcal{BL}e_n\, (\mathcal{BL}e_n)_{ij}^* \; & = Le_n^2 + Le_{n+1}^2 + Le_{n+2}^2 \\
& \quad + Le_{n+3}^2 + 2\, i\, j\, [Le_n Le_{n+3} - Le_{n+1} Le_{n+2}].
\end{aligned} \tag{20}$$

Proof. It can be proved easily by using Eq.(15), Eq.(16) and Eq.(17).

In the following theorem, some properties related to the bicomplex Leonardo numbers are given.

Theorem 2.2. *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number.For any integer $n \geq 0$,*

$$\mathcal{BL}e_n = 2\,\mathcal{BF}_{n+1} - A \tag{21}$$

where $\mathcal{BF}_n$ is n-th bicomplex Fibonacci number.

Proof. (21): By the Eq.(3) we obtain that,

$$\begin{aligned}
\mathcal{BL}e_n = &\; Le_n + i\,Le_{n+1} + j\,Le_{n+2} + i\,j\,Le_{n+3} \\
= &\; (2\,F_{n+1} - 1) + i\,(2\,F_{n+2} - 1) + j\,(2\,F_{n+3} - 1) + i\,j\,(2\,F_{n+4} - 1) \\
= &\; 2\,(F_{n+1} + i\,F_{n+2} + j\,F_{n+3} + i\,j\,F_{n+4}) - A \\
= &\; 2\,\mathcal{BF}_{n+1} - A\,.
\end{aligned}$$

where $A = 1 + i + j + i\,j$.

Theorem 2.3. *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number.For any integer $n \geq 0$, summation formulas as follows:*

$$\sum_{k=0}^{n} \mathcal{BL}e_k = \mathcal{BL}e_{n+2} - (n+2)\,A - (2\,i + 4\,j + 8\,i\,j\,), \tag{22}$$

$$\sum_{k=0}^{n} \mathcal{BL}e_{2\,k} = \mathcal{BL}e_{2n+1} - n\,A - (2\,i + 2\,j + 4\,i\,j\,), \tag{23}$$

$$\sum_{k=0}^{n} \mathcal{BL}e_{2\,k+1} = \mathcal{BL}e_{2n+2} - (n+2)\,A - (2\,j + 4\,i\,j\,). \tag{24}$$

For $\;n \geq 1$,

$$\sum_{k=0}^{n} (-1)^{k-1} \mathcal{BL}e_k = \begin{cases} -(\mathcal{BL}e_{n-1} + 2\; + 2\,j + 2\,i\,j\,)\,,\; n\,is\;even \\ \mathcal{BL}e_{n-1} - 1\; + i\; - j - i\,j\,,\; n\,is\;odd \end{cases} \tag{25}$$

Proof. Proof of equalities can easily be done.

(25): Using Eq.(21) and $\sum\limits_{k=1}^{n} (-1)^{k-1} F_{k+1} = (-1)^{n-1} F_n$, we obtain

$$\sum_{k=0}^{n} (-1)^{k-1} \mathcal{BL}e_k = \left\{ \begin{array}{r} -2(\mathcal{BF}e_n - 1\; + i\; - \; j - \; i\,j\,)\,,\;\text{n is even} \\ 2\mathcal{BF}e_n - 2\; - 2\,j - 2\,i\,j\,,\;\text{n is odd} \end{array} \right\}.$$

where $\mathcal{BF}_n$ is n-th bicomplex Fibonacci number. Thus, we have expressed the equation (24) with bicomplex Fibonacci numbers $\mathcal{BF}_n$ using the equation (3).

Theorem 2.4. *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number.For any integer $n \geq 0$,*

$$\mathcal{BL}e_n = \mathcal{BF}_n + \mathcal{BL}_n - A. \tag{26}$$

where $\mathcal{BL}_n$ is n-th bicomplex Lucas number.

PROOF. (26): By the Eq.(3) and $2\,F_{n+1} = f_n + L_n$ we obtain that,

$$\begin{aligned}
\mathcal{BL}e_n = \; & (2\,F_{n+1} - 1) + i\,(2\,F_{n+2} - 1) + j\,(2\,F_{n+3} - 1) \\
& + i\,j\,(2\,F_{n+4} - 1) \\
= \; & (F_n + L_n - 1) + i\,(F_{n+1} + L_{n+1} - 1) + j\,(F_{n+2} + L_{n+2} - 1) \\
& + i\,j\,(F_{n+3} + L_{n+3} - 1) \\
= \; & (F_n + i\,F_{n+1} + j\,F_{n+2} + i\,j\,F_{n+3}) + (L_n + i\,L_{n+1} + j\,L_{n+2} \\
& + i\,j\,L_{n+3}) - (1 + i + j + i\,j) \\
= \; & \mathcal{BF}_n + \mathcal{BL}_n - A\,.
\end{aligned}$$

where $A = 1 + i + j + i\,j$.

Theorem 2.5. ***Generating function***
Let $\mathcal{BL}e_n$ be bicomplex Leonardo numbers. For the generating function for these numbers is as follows:

$$g_{\mathcal{BL}e_n}(t) = \sum_{n=1}^{\infty} \mathcal{BL}e_n\,t^n = \frac{\mathcal{BL}e_0 + (\mathcal{BL}e_1 - 2\,\mathcal{BL}e_0)\,t + (\mathcal{BL}e_2 - 2\,\mathcal{BL}e_1)\,t^2}{1 - 2\,t + t^3}\,. \tag{27}$$

PROOF. Using the definition of generating function, we obtain

$$g_{\mathcal{BL}e_n}(t) = \mathcal{BL}e_0 + \mathcal{BL}e_1\,t + \ \ldots \ + \mathcal{BL}e_n\,t^n + \ \ldots\,. \tag{28}$$

Multiplying $(1 - 2\,t + t^3)$ both sides of Eq.(28) and using Eq.(10), we have

$$\begin{aligned}
(1 - 2\,t + t^3)\,g_{\mathcal{BL}e_n}(t) \; & = \mathcal{BL}e_0 + (\mathcal{BL}e_1 - 2\,\mathcal{BL}e_0)\,t \\
& + (\mathcal{BL}e_2 - 2\,\mathcal{BL}e_1)\,t^2 + (\mathcal{BL}e_3 - 2\,\mathcal{BL}e_2 + \mathcal{BL}e_0)\,t^3 + \ \ldots \\
& + (\mathcal{BL}e_{k+1} - 2\,\mathcal{BL}e_k + \mathcal{BL}e_{k-2})\,t^{k+1} + \ \ldots
\end{aligned}$$

where

$$\mathcal{BL}e_1 - 2\,\mathcal{BL}e_0 = -1 + i - j - i\,j,$$

$$\mathcal{BL}e_2 - 2\,\mathcal{BL}e_1 = 1 - i - j - 3\,i\,j$$

and

$$\mathcal{BL}e_3 - 2\,\mathcal{BL}e_2 + \mathcal{BL}e_0 = 0.$$

Thus, the proof is completed.

Theorem 2.6. ***Binet's Formula*** *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number. For any integer $n \geq 0$, the Binet's formula for these numbers is as follows:*

$$\mathcal{BL}e_n = \frac{2\,\hat{\alpha}\;\alpha^{n+1} - 2\,\hat{\beta}\;\beta^{n+1}}{\alpha - \beta} - A\,. \tag{29}$$

where

$$\hat{\alpha} = 1 + i\,\alpha + j\,\alpha^2 + i\,j\,\alpha^3, \quad \alpha = \tfrac{1+\sqrt{5}}{2}$$
$$\hat{\beta} = 1 + i\,\beta + j\,\beta^2 + i\,j\,\beta^3, \quad \beta = \tfrac{1-\sqrt{5}}{2}.$$

and

$$A = 1 + i + j + i\,j\,.$$

Proof. Binet's formula of the bicomplex Leonardo number is easily obtained by utilizing Binet's formula of bicomplex Fibonacci number [9] and using Eq.(3).

Theorem 2.7. ***D'Ocagne's identity*** *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number. For $m \geq n+1$, the following equality holds:*

$$\begin{aligned} \mathcal{BL}e_m\,\mathcal{BL}e_{n+1} - \mathcal{BL}e_{m+1}\,\mathcal{BL}e_n \; &= 12\,((-1)^{n+1}\,F_{m-n}\,(2\,j + i\,j\,)) \\ &\quad -2(1+i+j+i\,j)(-\mathcal{BF}_m + \mathcal{BF}_n). \end{aligned} \tag{30}$$

Proof. (30): Considering Eq.(21),using the commutative property of bicomplex numbers and d'Ocagne's identity of bicomplex Fibonacci numbers [9], we obtain that

$$\begin{aligned} \mathcal{BL}e_m\,\mathcal{BL}e_{n+1} - \mathcal{BL}e_{m+1}\,\mathcal{BL}e_n \; &= (2\mathcal{BF}_{m+1} - A)(2\mathcal{BF}_{n+2} - A) \\ &\quad -(2\mathcal{BF}_{m+2} - A)(2\mathcal{BF}_{n+1} - A) \\ &= 4\,(\mathcal{BF}_{m+1}\,\mathcal{BF}_{n+2} - \mathcal{BF}_{m+2}\,\mathcal{BF}_{n+1}) \\ &\quad -2\,A\,(\mathcal{BF}_{m+1} + \mathcal{BF}_{n+2} - \mathcal{BF}_{m+2} - \mathcal{BF}_{n+1}) \\ &= 4(3\,(-1)^{n+1}\,F_{m-n}\,(2\,j + i\,j\,)) \\ &\quad -2\,A\,(-\mathcal{BF}_m + \mathcal{BF}_n) \\ &= 12\,((-1)^{n+1}\,F_{m-n}\,(2\,j + i\,j\,)) \\ &\quad -2(1+i+j+i\,j)(-\mathcal{BF}_m + \mathcal{BF}_n)\,. \end{aligned}$$

where the identity $F_m F_{n+1} - F_{m+1} F_n = (-1)^n\,F_{m-n}$ is used [12], [13].

Theorem 2.8. ***Cassini's Identity*** *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number. For $n \geq 1$, the following equality holds:*

$$\begin{aligned} \mathcal{BL}e_{n-1}\,\mathcal{BL}e_{n+1} - \mathcal{BL}e_n^2 \;&= 12\,((-1)^{n+1}(2\,j + i\,j\,)) \\ &-2(1+i+j+i\,j)(\mathcal{BF}_{n-2}). \end{aligned} \tag{31}$$

Proof. (31): By (21) and using the commutative property of bicomplex numbers and Cassini's identity of bicomplex Fibonacci numbers [9, 11], we obtain that

$$\begin{aligned} \mathcal{BL}e_{n-1}\,\mathcal{BL}e_{n+1} - \mathcal{BL}e_n\,\mathcal{BL}e_n \;&= (2\mathcal{BF}_n - A)(2\mathcal{BF}_{n+2} - A) \\ &-(2\mathcal{BF}_{n+1} - A)(2\mathcal{BF}_{n+1} - A) \\ &= 4\,(\mathcal{BF}_n\,\mathcal{BF}_{n+2} - \mathcal{BF}_{n+1}\,\mathcal{BF}_{n+1}) \\ &+2\,[\,(-\mathcal{BF}_n + \mathcal{BF}_{n+1})\,A + A\,(-\mathcal{BF}_{n+2} + \mathcal{BF}_{n+1})\,] \\ &= 4(3\,(-1)^{n+1}\,(2\,j + i\,j\,)) \\ &+2\,(\mathcal{BF}_{n-1}\,2\,A(-\mathcal{BF}_n) \\ &= 12\,((-1)^{n+1}(2\,j + i\,j\,)) \\ &-2(1+i+j+i\,j)(-\mathcal{BF}_{n-1} + \mathcal{BF}_n) \\ &= 12\,((-1)^{n+1}(2\,j + i\,j\,)) \\ &-2(1+i+j+i\,j)(\mathcal{BF}_{n-2}). \end{aligned}$$

where the identity of the Fiboncci numbers $F_m F_{n+1} - F_{m+1} F_n = (-1)^n\,F_{m-n}$ is used [12], [13].

Theorem 2.9. ***Catalan's Identity*** *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number. For $n \geq 1$, the following equality holds:*

$$\begin{aligned} \mathcal{BL}e_n^2 - \mathcal{BL}e_{n-r}\,\mathcal{BL}e_{n+r} \;&= 4\,[\,3\,(-1)^{n-r+1})\,F_r^2\,(2j + i\,j)\,] \\ &-2\,A\,(2\,\mathcal{BF}_{n+1} + \mathcal{BF}_{n-r+1} + \mathcal{BF}_{n+r+1}). \end{aligned} \tag{32}$$

Proof. (32): By (21) and using the commutative property of bicomplex numbers Catalan's identity of bicomplex Fibonacci numbers [9, 11], we obtain that

$$\begin{aligned} \mathcal{BL}e_n\,\mathcal{BL}e_n - \mathcal{BL}e_{n-r}\,\mathcal{BL}e_{n+r} \;&= (2\mathcal{BF}_{n+1} - A)(2\mathcal{BF}_{n+1} - A) \\ &4\,(\mathcal{BF}_{n+1} - \mathcal{BF}_{n+1}) - \mathcal{BF}_{n-r+1}\mathcal{BF}_{n+r+1}) \\ &-2\,A\,(\mathcal{BF}_{n+1} + \mathcal{BF}_{n-r+1} + \mathcal{BF}_{n+1}\,\mathcal{BF}_{n+r+1}) \\ &= 4\,[\,3\,(-1)^{n-r+1})\,F_r^2\,(2j + i\,j)\,] \\ &-2\,A\,(2\,\mathcal{BF}_{n+1} + \mathcal{BF}_{n-r+1} + \mathcal{BF}_{n+r+1}). \end{aligned}$$

where the identities of the Fibonacci numbers $F_m F_n - F_{m+r} F_{n-r} = (-1)^{n-r} F_{m+r-n}\,F_r$ and $F_n\,F_n - F_{n-r}\,F_{n+r} = (-1)^{n-r}\,F_r^2$ are used [12], [13].

Theorem 2.10. ***Honsberger identity*** *Let $\mathcal{BL}e_n$ be the bicomplex Leonardo number. For $n, m \geq 0$, the following equality holds:*

$$\begin{aligned}\mathcal{BL}e_n\,\mathcal{BL}e_m + \mathcal{BL}e_{n+1}\,\mathcal{BL}e_{m+1} = \;& 2\,\mathcal{BF}_{n+m+3} + 2\,F_{n+m+6} - F_{n+m+3} \\ & -2\,i\,F_{n+m+8} - 2\,j\,F_{n+m+7} + 2\,i\,j\,F_{n+m+6}.\end{aligned} \tag{33}$$

Proof. (33): By (21) and using the commutative property of bicomplex numbers Honsberger identity of bicomplex Fibonacci numbers [9, 11], we obtain that

$$\begin{aligned}\mathcal{BL}e_n\,\mathcal{BL}e_m + \mathcal{BL}e_{n+1}\,\mathcal{BL}e_{m+1} &= (2\mathcal{BF}_{n+1} - A)(2\mathcal{BF}_{m+1} - A) \\ &\quad +(2\mathcal{BF}_{n+2} - A)(2\mathcal{BF}_{m+2} - A) \\ &= 4\,(\mathcal{BF}_{n+1}\,\mathcal{BF}_{m+1} - \mathcal{BF}_{n+2}\,\mathcal{BF}_{m+2}) \\ &\quad -2A[(\mathcal{BF}_{n+1} + \mathcal{BF}_{n+2}) + (\mathcal{BF}_{m+1} + \mathcal{BF}_{m+2}) - A] \\ &= 4(2\mathcal{BF}_{n+m+3} + 2\,F_{n+m+6} - F_{n+m+3} \\ &\quad -2\,i\,F_{n+m+8} - 2\,j\,F_{n+m+7} + 2\,i\,j\,F_{n+m+6}) \\ &\quad -2\,A\,[\,(\mathcal{BF}_{n+3} + \mathcal{BF}_{m+3}) - A\,] \\ &= 4(2\mathcal{BF}_{n+m+3} + 2\,F_{n+m+6} - F_{n+m+3} \\ &\quad -2\,i\,F_{n+m+8} - 2\,j\,F_{n+m+7} + 2\,i\,j\,F_{n+m+6}) \\ &\quad -2A(\mathcal{BF}_{n+3} + \mathcal{BF}_{m+3} - A)\,.\end{aligned}$$

where the identity $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ is used [12], [13].

3. Conclusion

In this paper,we introduce bicomplex Leonardo numbers. Also, using the relationship of these numbers with bicomplex Fibonacci numbers, we obtained d'Ocagne's, Cassini's, Catalan's and Honsberger identities.

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