Oresme Hybrid Quaternion Numbers

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Abstract. In literature until today, many authors have studied special sequences in different number systems. In this paper, we have introduced the Oresme hybrid quaternion numbers. We give some properties and identities such as Binet's formula, generating function, norm and characteristic equation for these quaternions. Furthermore, matrix and determinant forms for these quaternion numbers are given.

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1. Introduction

Oresme numbers are introduced and given some properties for example Cassini's identity, Catalan's identity and d'Ocagne's identity for Oresme numbers by Horadam in [6]. Oresme sequence denoted with O_n is defined by the following recurrence relation for $n \geq 0$

$$O_{n+2} = O_{n+1} - \frac{1}{4}\,O_n \tag{1.1}$$

with the initial conditions $O_0 = 0\,, O_1 = O_2 = \frac{1}{2}$. This sequence are also expressed as:

$$O_{n+2} - \frac{3}{4}\,O_n + \frac{1}{4}\,O_{n-1} = 0\,, \tag{1.2}$$

$$O_{n+2} - \frac{3}{4}\,O_{n+1} + \frac{1}{16}\,O_{n-1} = 0\,, \tag{1.3}$$

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That is, Oresme sequence O_n is

$$\frac{1}{2},\ \frac{2}{4},\ \frac{3}{8},\ \frac{4}{16}\ \frac{5}{32},\dots,\frac{n}{2^n},\dots \tag{1.4}$$

and Binet's formula and generating function for Oresme numbers respectively, as follows:

$$O_n=\frac{n}{2^n}\,, \tag{1.5}$$

$$g_{O_n}(t)=\sum_{n=1}^{\infty}O_n\,t^n=\frac{\frac{1}{2}\,t}{1-t+\frac{1}{4}\,t^2}\quad . \tag{1.6}$$

Some properties of Oresme numbers are:

$$O_{n+1}\,O_{n-1}-O_n^2=-\,(\frac{1}{4})^n\,, \tag{1.7}$$

$$O_{n+r}\,O_{n-r}-O_n^2=-\,(\frac{1}{4})^{n-r+1}\,F^2{}_{r-1}\,, \tag{1.8}$$

$$O_{n+1}^2-(\frac{1}{4})^2\,O_{n-1}^2=\frac{1}{2}\,O_{2n+1}+\frac{1}{8}\,O_{2n-1}\,, \tag{1.9}$$

$$\frac{1}{2}\,O_{m+n-1}=O_m\,O_n-\frac{1}{4}\,O_{m-1}\,O_{n-1}\,, \tag{1.10}$$

$$\frac{1}{2}\,O_{2n-1}=O_n^2-\frac{1}{4}\,O_{n-1}^2=O_{n+1}\,O_{n-1}-\frac{1}{4}\,O_n\,O_{n-2}\,, \tag{1.11}$$

$$O_{n-r}\,O_{n+r+s}-O_n\,O_{n+s}=-\,(\frac{1}{4})^{n-r+1}\,F_{r-1}\,F_{r+s-1}\,, \tag{1.12}$$

$$\sum_{j=0}^{n}O_j=4\,(O_1-O_{n+2})\,. \tag{1.13}$$

Oresme numbers were generalized by Cook in [1]. On Oresme Numbers and their connection with Fibonacci and Pell Numbers by [4]. In [3], authors have given generalization of the matrix form of the Oresme sequence and Oresme's hybrid numbers. In 2021, generalized Oresme numbers defined by [10]. In [11], the authors investigated Oresme hybrid numbers and hybrationals. In [9], authors have given dual-generalized complex component extension of Oresme numbers.

The hybrid number system can be accepted as a generalization of the complex, dual and hyperbolic number systems. In 2018, firstly, set of hybrid numbers was introduced by [8] as follows:

$$\mathbb{K}=\{\,a+b\,i\,+c\,\varepsilon\,+d\,h\,\,|a,b,c,d\in\mathbb{R}\,,i^2=-1\,,\varepsilon^2=0\,,h^2=1\,\}, \tag{1.14}$$

where units satisfy the rules

$$i\,h=-h\,i=\varepsilon+i\,.$$

The set $\mathbb{K}$ of hybrid numbers forms non-commutative ring with respect to the addition and multiplication operations.

Taking two hybrid numbers

$$z_1=a_1+b_1\,i+c_1\,\varepsilon+d_1\,h$$

,

$$z_2 = a_2 + b_2\, i + c_2\, \varepsilon + d_2\, h$$

and $s \in \mathbb{R}$ get:

- Equality $z_1 = z_2$, if and only if, $a_1 = a_2, b_1 = b_2, c_1 = c_2$, and $d_1 = d_2$;
- Sum $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\, i + (c_1 + c_2)\, \varepsilon + (d_1 + d_2)\, h$;
- Subtraction $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)\, i + (c_1 - c_2)\, \varepsilon + (d_1 - d_2)\, h$;
- Multiplication by scalar $s.z = s.a + s.b\, i + s.c\, \varepsilon + s.d\, h$.

The real number $C(z) = z.\overline{z} = \overline{z}.z = a^2 + (b-c)^2 - c^2 - d^2$ is called the character of the hybrid number z. A new expression for the character of a hybrid number z is given by

$$C(z) = (a-b)^2 - 2\, b(c-a) - d^2 \tag{1.15}$$

The real quaternions were first described by Irish mathematician William

Table 1. Multiplication scheme of hybrid numbers

x	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon + i$
ε	ε	1+h	0	$-\varepsilon$
h	h	$-\varepsilon - i$	ε	1

Rowan Hamilton in 1843. Hamilton [5] introduced the set of quaternions which can be represented as

$$H = \{\, q = q_0 + i\, q_1 + j\, q_2 + k\, q_3 \mid q_0,\, q_1,\, q_2,\, q_3 \in \mathbb{R} \,\} \tag{1.16}$$

where

$$i^2 = j^2 = k^2 = -1\,, \quad i\, j = -j\, i = k\,, \quad j\, k = -k\, j = i\,, \quad k\, i = -i\, k = j\,. \tag{1.17}$$

There are several studies on hybrid quaternions for example Horadam hyrid [2], Leonardo hybrid [7].
In this paper, we have defined the Oresme's hybrid quaternions and obtained some results.

2. Oresme's hybrid quaternion numbers

In this section, Oresme's hybrid quaternions will be obtained using the following definitions.
Definition.2.1. For $n \geq 1$, the n-th Oresme's hybrid numbers $\mathcal{HO}_n$ are defined by using the Oresme numbers as follows

$$\mathcal{HO}_n = O_n + i\, O_{n+1} + \varepsilon\, O_{n+2} + h\, O_{n+3} \tag{2.1}$$

where initial values are $\mathcal{HO}_0 = \frac{1}{2}\, i + \frac{2}{4}\, \varepsilon + \frac{3}{8}\, h,\ \mathcal{HO}_1 = \frac{1}{2} + \frac{2}{4}\, i + \frac{3}{8}\, \varepsilon + \frac{4}{16}\, h$.

Now we give the recurrence relation corresponding to expression Eq.(1.1). That is

$$\mathcal{HO}_n = \mathcal{HO}_{n-1} - \frac{1}{4}\,\mathcal{HO}_{n-2}. \tag{2.2}$$

Using relations Eq.(1.1) and Eq.(2.1) we obtain that,

$$\begin{aligned}\mathcal{HO}_n = \;& O_n + i\,O_{n+1} + \varepsilon\,O_{n+2} + h\,O_{n+3}\\ =\;& (O_{n-1} - \tfrac{1}{4}\,O_{n-2}) + i\,(O_n - \tfrac{1}{4}\,O_{n-1}) + \varepsilon\,(O_{n+1} - \tfrac{1}{4}\,O_n)\\ & + h\,(O_{n+2} - \tfrac{1}{4}\,O_{n+1})\\ =\;& \mathcal{HO}_{n-1} - \tfrac{1}{4}\,\mathcal{HO}_{n-2}\end{aligned}$$

Definition.2.2. For $n \geq 2$, the n-th Oresme's quaternion number $\mathcal{QO}_n$ are defined as follows

$$\mathcal{QO}_n = O_n + i\,O_{n+1} + j\,O_{n+2} + k\,O_{n+3} \tag{2.3}$$

Definition.2.3. The recurrence relation for Oresme's quaternion numbers $\mathcal{QO}_n$, $n \geq 2$, is defined by as follows

$$\mathcal{QO}_{n+1} = \mathcal{QO}_n - \frac{1}{4}\,\mathcal{QO}_{n-1} \tag{2.4}$$

where initial values are $\mathcal{QO}_0 = \frac{1}{2}\,i + \frac{2}{4}\,j + \frac{3}{8}\,k,\;\; \mathcal{QO}_1 = \frac{1}{2} + \frac{2}{4}\,i + \frac{3}{8}\,j + \frac{4}{16}\,k.$

Definition.2.4. Oresme's hybrid quaternion numbers $\mathcal{HQO}_n$ are defined as follows

$$\mathcal{HQO}_n = \mathcal{HO}_n + i\,\mathcal{HO}_{n+1} + j\,\mathcal{HO}_{n+2} + k\,\mathcal{HO}_{n+3} \tag{2.5}$$

where $i\,,j\,,k$ are the units of the quaternions and $\mathcal{HO}_n$ is the n-th Oresme hybrid number. Thus, Oresme's hybrid quaternions can be rewritten by as follows

$$\begin{aligned}\mathcal{HQO}_n \;&= (O_n + i\,O_{n+1} + \varepsilon\,O_{n+2} + h\,O_{n+3})\\ &+ i\,(O_{n+1} + i\,O_{n+2} + \varepsilon\,O_{n+3} + h\,O_{n+4})\\ &+ j\,(O_{n+2} + i\,O_{n+3} + \varepsilon\,O_{n+4} + h\,O_{n+5})\\ &+ k\,(O_{n+3} + i\,O_{n+4} + \varepsilon\,O_{n+5} + h\,O_{n+6})\\ &= \mathcal{QO}_n + i\,\mathcal{QO}_{n+1} + \varepsilon\,\mathcal{QO}_{n+2} + h\,\mathcal{QO}_{n+3}\end{aligned} \tag{2.6}$$

where $i\,,\varepsilon\,,h$ are the imaginary units of the hybrid numbers and $\mathcal{QO}_n = O_n + i\,O_{n+1} + j\,O_{n+2} + k\,O_{n+3}$ is Oresme quaternion number.

Definition.2.5. The recurrence relation for Oresme's hybrid quaternion numbers $\mathcal{HQO}_n$, $n \geq 1$, is defined by as follows

$$\mathcal{HQO}_{n+1} = HQO_n - \frac{1}{4}\,HQO_{n-1} \tag{2.7}$$

Definition.2.6. Oresme's hybrid quaternion numbers $\mathcal{HQO}_n$ are defined in two different ways as follows

$$\begin{aligned}\mathcal{HQO}_n &= \mathcal{HO}_n + i\,\mathcal{HO}_{n+1} + j\,\mathcal{HO}_{n+2} + k\,\mathcal{HO}_{n+3}\\ &= \mathcal{QO}_n + i\,\mathcal{QO}_{n+1} + \varepsilon\,\mathcal{QO}_{n+2} + h\,\mathcal{QO}_{n+3}\end{aligned} \tag{2.8}$$

where initial values are

$$\begin{aligned}\mathcal{HQO}_0 &= \mathcal{HO}_0 + i\,\mathcal{HO}_1 + j\,\mathcal{HO}_2 + k\,\mathcal{HO}_3\\ &= \mathcal{QO}_0 + i\,\mathcal{QO}_1 + \varepsilon\,\mathcal{QO}_2 + h\,\mathcal{QO}_3,\end{aligned}$$

TABLE 2. Oresme's hybrids and O'resme quaternion numbers

n	$\mathcal{HO}_n$	$\mathcal{QO}_n$
o	$o+\frac{1}{2}i+\frac{1}{2}\varepsilon+\frac{3}{8}h$	$0+\frac{1}{2}i+\frac{1}{2}j+\frac{3}{8}k$
1	$\frac{1}{2}+\frac{1}{2}i+\frac{3}{8}\varepsilon+\frac{1}{4}h$	$\frac{1}{2}+\frac{1}{2}i+\frac{3}{8}j+\frac{1}{4}k$
2	$\frac{1}{2}+\frac{3}{8}i+\frac{1}{4}\varepsilon+\frac{5}{32}h$	$\frac{1}{2}+\frac{3}{8}i+\frac{1}{4}j+\frac{5}{32}k$
3	$\frac{3}{8}+\frac{1}{4}i+\frac{5}{32}\varepsilon+\frac{3}{32}h$	$\frac{3}{8}+\frac{1}{4}i+\frac{5}{32}j+\frac{3}{32}k$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\mathcal{HQO}_1 &= \mathcal{HO}_1 + i\,\mathcal{HO}_2 + j\,\mathcal{HO}_3 + k\,\mathcal{HO}_4\\ &= \mathcal{QO}_1 + i\,\mathcal{QO}_2 + \varepsilon\,\mathcal{QO}_3 + h\,\mathcal{QO}_4,\end{aligned}$$

$$\begin{aligned}\mathcal{HQO}_2 &= \mathcal{HO}_2 + i\,\mathcal{HO}_3 + j\,\mathcal{HO}_4 + k\,\mathcal{HO}_5\\ &= \mathcal{QO}_2 + i\,\mathcal{QO}_3 + \varepsilon\,\mathcal{QO}_4 + h\,\mathcal{QO}_5\,.\end{aligned}$$

Definition.2.7. Let $\mathcal{HQO}_n$ and $\mathcal{HQO}_m$ be any two Oresme's hybrid quaternion numbers. The addition and subtraction of the Oresme's hybrid quaternion numbers are defined by

$$\begin{aligned}\mathcal{HQO}_n \pm \mathcal{HQO}_m = &(\mathcal{HO}_n \pm \mathcal{HO}_m) + i\,(\mathcal{HO}_{n+1} \pm +\mathcal{HO}_{m+1})\\ &+ j\,(\mathcal{HO}_{n+2} \pm \mathcal{HO}_{m+2}) + k\,(\mathcal{HO}_{n+3} \pm \mathcal{HO}_{m+3})\end{aligned} \tag{2.9}$$

or

$$\begin{aligned}\mathcal{HQO}_n \pm \mathcal{HQO}_m = &(\mathcal{QO}_n \pm \mathcal{QO}_m) + i\,(\mathcal{QO}_{n+1} \pm +\mathcal{QO}_{m+1})\\ &+ \varepsilon\,(\mathcal{QO}_{n+2} \pm \mathcal{QO}_{m+2}) + h\,(\mathcal{QO}_{n+3} \pm \mathcal{QO}_{m+3})\end{aligned} \tag{2.10}$$

Definition.2.8. Let $\mathcal{HQO}_n$ and $\mathcal{HQO}_m$ be any two Oresme's hybrid quaternion numbers. Multiplication of the Oresme's hybrid quaternion numbers are defined by

$$\begin{aligned}\mathcal{HQO}_n \times \mathcal{HQO}_m &= (\mathcal{HO}_n + i\,\mathcal{HO}_{n+1} + j\,\mathcal{HO}_{n+2} + k\,\mathcal{HO}_{n+3})\\ &\quad(\mathcal{HO}_m + i\,\mathcal{HO}_{m+1} + j\,\mathcal{HO}_{m+2} + k\,\mathcal{HO}_{m+3})\\ &= (\mathcal{HO}_n\,\mathcal{HO}_m - \mathcal{HO}_{n+1}\,\mathcal{HO}_{m+1} - \mathcal{HO}_{n+2}\,\mathcal{HO}_{m+2}\\ &\quad - \mathcal{HO}_{n+3}\,\mathcal{HO}_{m+3})\\ &\quad + i\,(\mathcal{HO}_n\,\mathcal{HO}_{m+1} + \mathcal{HO}_{n+1}\,\mathcal{HO}_m + \mathcal{HO}_{n+2}\,\mathcal{HO}_{m+3}\\ &\quad - \mathcal{HO}_{n+2}\,\mathcal{HO}_{m+2})\\ &\quad + j\,(\mathcal{HO}_n\,\mathcal{HO}_{m+2} - \mathcal{HO}_{n+1}\,\mathcal{HO}_{m+3} + \mathcal{HO}_{n+2}\,\mathcal{HO}_m\\ &\quad + \mathcal{HO}_{n+3}\,\mathcal{HO}_{m+1})\\ &\quad + k\,(\mathcal{HO}_n\,\mathcal{HO}_{m+3} + \mathcal{HO}_{n+1}\,\mathcal{HO}_{m+2} - \mathcal{HO}_{n+2}\,\mathcal{HO}_{m+1}\\ &\quad + \mathcal{HO}_{n+3}\,\mathcal{HO}_m).\end{aligned} \tag{2.11}$$

or

$$\begin{aligned}\mathcal{HQO}_n \times \mathcal{HQO}_m &= (\mathcal{QO}_n + i\,\mathcal{QO}_{n+1} + \varepsilon\,\mathcal{QO}_{n+2} + h\,\mathcal{QO}_{n+3})\\ &\quad(\mathcal{QO}_m + i\,\mathcal{QO}_{m+1} + \varepsilon\,\mathcal{QO}_{m+2} + h\,\mathcal{QO}_{m+3})\\ &= (\mathcal{QO}_n\,\mathcal{QO}_m - \mathcal{QO}_{n+1}\,\mathcal{QO}_{m+1} - \mathcal{QO}_{n+3}\,\mathcal{QO}_{m+3}\\ &\quad + \mathcal{QO}_{n+1}\,\mathcal{QO}_{m+2} + \mathcal{QO}_{n+1}\,\mathcal{QO}_{m+3} + \mathcal{QO}_{n+2}\,\mathcal{QO}_{m+1})\\ &\quad + i\,(\mathcal{QO}_n\,\mathcal{QO}_{m+1} + \mathcal{QO}_{n+1}\,\mathcal{QO}_m - \mathcal{QO}_{n+3}\,\mathcal{QO}_{m+1})\\ &\quad + \varepsilon\,(\mathcal{QO}_n\,\mathcal{QO}_{m+2} + \mathcal{QO}_{n+2}\,\mathcal{QO}_m - \mathcal{QO}_{n+2}\,\mathcal{QO}_{m+3}\\ &\quad - \mathcal{QO}_{n+3}\,\mathcal{QO}_{m+1} + \mathcal{QO}_{n+3}\,\mathcal{QO}_{m+2} + \mathcal{QO}_{n+1}\,\mathcal{QO}_{m+3})\\ &\quad + h\,(\mathcal{QO}_n\,\mathcal{QO}_{m+3} - \mathcal{QO}_{n+1}\,\mathcal{QO}_{m+2} + \mathcal{QO}_{n+2}\,\mathcal{QO}_{m+1}\\ &\quad + \mathcal{QO}_{n+3}\,\mathcal{QO}_m).\end{aligned} \tag{2.12}$$

Definition.2.9. Oresme's hybrid quaternion conjugate can be defined in three different for

$$\mathcal{HQO}_n = \mathcal{QO}_n + i\,\mathcal{QO}_{n+1} + \varepsilon\,\mathcal{QO}_{n+2} + h\,\mathcal{QO}_{n+3}$$

as follows

$$Quaternion-conjugate: \overline{\mathcal{HQO}_n} = \overline{\mathcal{QO}_n} + i\,\overline{\mathcal{QO}_{n+1}} + \varepsilon\,\overline{\mathcal{QO}_{n+2}} + h\,\overline{\mathcal{QO}_{n+3}}$$

$$Hybrid-conjugate: (\mathcal{HQO}_n)^C = \mathcal{QO}_n - i\,\mathcal{QO}_{n+1} - \varepsilon\,\mathcal{QO}_{n+2} - h\,\mathcal{QO}_{n+3}$$

$$Total-conjugate: (\mathcal{HQO}_n)^\dagger = \overline{\mathcal{QO}_n} - i\,\overline{\mathcal{QO}_{n+1}} - \varepsilon\,\overline{\mathcal{QO}_{n+2}} - h\,\overline{\mathcal{QO}_{n+3}}$$

Definition.2.10. The norm of Oresme's hybrid quaternion numbers is defined as follows

$$N(\mathcal{HQO}_n) = \mathcal{HO}_n^2 + \mathcal{HO}_{n+1}^2 + \mathcal{HO}_{n+2}^2 + \mathcal{HO}_{n+3}^2 \tag{2.13}$$

or

$$\begin{aligned}
N(\mathcal{HQO}_n) &= \mathcal{QO}_n^2 + \mathcal{QO}_{n+1}^2 - \mathcal{QO}_{n+3}^2 - 2\,\mathcal{QO}_{n+1}\,\mathcal{QO}_{n+2} \\
&= \mathcal{QO}_n^2 + (\mathcal{QO}_{n+1} - \mathcal{QO}_{n+2})^2 - \mathcal{QO}_{n+2}^2 - \mathcal{QO}_{n+3}^2 \\
&= \mathcal{QO}_n^2 + \mathcal{QO}_{n+1}^2 - 2\,\mathcal{QO}_{n+1}\mathcal{QO}_{n+2} - \mathcal{QO}_{n+2}^2 \\
&\quad + \frac{1}{2}\,\mathcal{QO}_{n+1}\mathcal{QO}_{n+1} - \frac{1}{16}\,\mathcal{QO}_{n+1}^2 \\
&= \mathcal{QO}_n^2 + \frac{15}{16}\,\mathcal{QO}_{n+1}^2 - \frac{3}{2}\mathcal{QO}_{n+1}\mathcal{QO}_{n+2} - \mathcal{QO}_{n+2}^2\,.
\end{aligned} \tag{2.14}$$

Definition.2.11. The character of Oresme's hybrid quaternion numbers is defined as follows

$$C(\mathcal{HQO}_n) = \mathcal{QO}_n^2 + (\mathcal{QO}_{n+1} - \mathcal{QO}_{n+2})^2 - \mathcal{QO}_{n+2}^2 - \mathcal{QO}_{n+3}^2 \tag{2.15}$$

Theorem 1. (Generating function)
Let $\mathcal{HQO}_n$ be Oresme's hybrid quaternion number. For the generating function for these quaternions is as follows:

$$g_{\mathcal{HQO}_n}(t) = \sum_{n=1}^{\infty} \mathcal{HQO}_n\, t^n = \frac{\mathcal{HQO}_0 + (\mathcal{HQO}_1 - \mathcal{HQO}_0)\,t}{1 - t + \frac{1}{4}\,t^2}\,. \tag{2.16}$$

Proof. Using the definition of generating function, we obtain

$$g_{\mathcal{HQO}_n}(t) = \mathcal{HQO}_0 + \mathcal{HQO}_1\,t + \ \dots\ + \mathcal{HQO}_n\,t^n + \ \dots\ . \tag{2.17}$$

Multiplying $(1 - t + \frac{1}{4}\,t^2)$ both sides of Eq.(2.17) and using Eq.(2.4), we have

$$\begin{aligned}
(1 - t + \tfrac{1}{4}\,t^2)\,g_{\mathcal{HQO}_n}(t) \quad &= \mathcal{HQO}_0 + (\mathcal{HQO}_1 - \mathcal{HQO}_0)\,t \\
&+ (\mathcal{HQO}_2 - \mathcal{HQO}_1 + \tfrac{1}{4}\,\mathcal{HQO}_0)\,t^2 \\
&+ (\mathcal{HQO}_3 - \mathcal{HQO}_2 + \tfrac{1}{4}\,\mathcal{HQO}_1)\,t^3 + \ \dots \\
&+ (\mathcal{HQO}_{k+1} - \mathcal{HQO}_k + \tfrac{1}{4}\,\mathcal{HQO}_{k-1})\,t^{k+1} + \ \dots
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{HQO}_0 \quad &= 0 + i\,(\tfrac{1}{2} + \tfrac{1}{2}\,i + \tfrac{3}{8}\,\varepsilon + \tfrac{1}{4}\,h) + j\,(\tfrac{1}{2} + \tfrac{3}{8}\,i + \tfrac{1}{4}\,\varepsilon + \tfrac{5}{32}\,h) \\
&+ k\,(\tfrac{3}{8} + \tfrac{1}{4}\,i + \tfrac{5}{32}\,\varepsilon + \tfrac{3}{32}\,h) \\
\mathcal{HQO}_1 - \mathcal{HQO}_0 \quad &= (\tfrac{1}{2} + 0\,i - \tfrac{1}{8}\,\varepsilon - \tfrac{1}{8}\,h) + i\,(0 - \tfrac{1}{8}\,i - \tfrac{1}{8}\,\varepsilon - \tfrac{3}{32}\,h) \\
&+ j\,(-\tfrac{1}{8} - \tfrac{1}{8}\,i - \tfrac{3}{32}\,\varepsilon - \tfrac{1}{16}\,h) + k(-\tfrac{1}{8} - \tfrac{3}{32}\,i - \tfrac{1}{16}\,\varepsilon - \tfrac{5}{128}\,h)
\end{aligned}$$

$$(\mathcal{HQO}_2 - \mathcal{HQO}_1 + \frac{1}{4}\,\mathcal{HQO}_0) = 0,$$

$(\mathcal{HQO}_3 - \mathcal{HQO}_2 + \frac{1}{4}\mathcal{HQO}_1) = 0, \dots, (\mathcal{HQO}_{k+1} - \mathcal{HQO}_k + \frac{1}{4}\mathcal{HQO}_{k-1}) = 0, \dots.$
Thus, the proof is completed.

Theorem 2. (Binet's Formula)
Let $\mathcal{HQO}_n$ be the Oresme hybrid quaternion. For any integer $n \geq 0$, the Binet's formula for these numbers is as follows:

$$\begin{aligned}
\mathcal{HQO}_n &= \mathcal{HO}_n + i\,\mathcal{HO}_{n+1} + j\,\mathcal{HO}_{n+2} + k\,\mathcal{HO}_{n+3} \\
&= (\tfrac{n}{2^n} + i\,\tfrac{n+1}{2^{n+1}} + \varepsilon\,\tfrac{n+2}{2^{n+2}} + h\,\tfrac{n+3}{2^{n+3}}) \\
&+i\,(\tfrac{n+1}{2^{n+1}} + i\,\tfrac{n+2}{2^{n+2}} + \varepsilon\,\tfrac{n+3}{2^{n+3}} + h\,\tfrac{n+4}{2^{n+4}}) \\
&+j\,(\tfrac{n+2}{2^{n+2}} + i\,\tfrac{n+3}{2^{n+3}} + \varepsilon\,\tfrac{n+4}{2^{n+4}} + h\,\tfrac{n+5}{2^{n+5}}) \\
&+k\,(\tfrac{n+3}{2^{n+3}} + i\,\tfrac{n+4}{2^{n+4}} + \varepsilon\,\tfrac{n+5}{2^{n+5}} + h\,\tfrac{n+6}{2^{n+6}}) \\
&= \mathcal{QO}_n + i\,\mathcal{QO}_{n+1} + \varepsilon\,\mathcal{QO}_{n+2} + h\,\mathcal{QO}_{n+3}
\end{aligned} \tag{2.18}$$

where $\mathcal{QO}_n = O_n + i\,O_{n+1} + j\,O_{n+2} + k\,O_{n+3}$ and $O_n = \frac{n}{2^n}$ [6]. *Proof.* Binet's formula of the Oresme hybrid quaternions is easily obtained by utilizing Binet's formula of Oresme hybrid numbers [11] and using

$$\mathcal{QO}_n = \frac{n}{2^n} + i\,\frac{n+1}{2^{n+1}} + j\,\frac{n+2}{2^{n+2}} + k\,\frac{n+3}{2^{n+3}}\,.$$

Also, Oresme's hybrid quaternion number can be represented in matrix form.

Theorem 3. (Matrix and Determinant Form)
For $n \in \mathbb{R}$, an array of Oresme's hybrid quaternion number is defined as

$$\varphi_{HQO_n} = \begin{pmatrix} \mathcal{QO}_n + \mathcal{QO}_{n+2} & \frac{3}{4}\,\mathcal{QO}_{n+1} \\ 2\,\mathcal{QO}_{n+2} - \frac{5}{4}\,\mathcal{QO}_{n+1} & \mathcal{QO}_n - \mathcal{QO}_{n+2} \end{pmatrix}.$$

Proof. In [8], the matrix form of a hybrid number is defined as:

$$\varphi_{a+b\,i+c\,\varepsilon+d\,h} = \begin{pmatrix} a+c & b-c+d \\ c-b+d & a-c \end{pmatrix}.$$

Making the necessary substitutions, we have:

$$\varphi_{HQO_n} = \begin{pmatrix} \mathcal{QO}_n + \mathcal{QO}_{n+2} & \mathcal{QO}_{n+1} - \mathcal{QO}_{n+2} + \mathcal{QO}_{n+3} \\ \mathcal{QO}_{n+2} - \mathcal{QO}_{n+1} + \mathcal{QO}_{n+3} & \mathcal{QO}_n - \mathcal{QO}_{n+2} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{QO}_n + \mathcal{QO}_{n+2} & \frac{3}{4}\mathcal{QO}_{n+1} \\ 2\,\mathcal{QO}_{n+2} - \frac{5}{4}\mathcal{QO}_{n+1} & \mathcal{QO}_n - \mathcal{QO}_{n+2} \end{pmatrix}$$

where $\mathcal{QO}_{n+3} = \mathcal{QO}_{n+2} - \frac{1}{4}\,\mathcal{QO}_{n+1}$ Thus, the proof is obtained.

Now, we calculate determinant of φ_{HQO_n}

$$\det(\varphi_{HQO_n}) = \begin{vmatrix} \mathcal{QO}_n + \mathcal{QO}_{n+2} & \frac{3}{4}\,\mathcal{QO}_{n+1} \\ 2\,\mathcal{QO}_{n+2} - \frac{5}{4}\,\mathcal{QO}_{n+1} & \mathcal{QO}_n - \mathcal{QO}_{n+2} \end{vmatrix}$$

$$
\begin{aligned}
&= (\mathcal{QO}_n + \mathcal{QO}_{n+2})(\mathcal{QO}_n - \mathcal{QO}_{n+2}) - \frac{3}{4}\mathcal{QO}_{n+1}\left(2\,\mathcal{QO}_{n+2} - \frac{5}{4}\mathcal{QO}_{n+1}\right) \\
&= \mathcal{QO}_n^2 - \mathcal{QO}_{n+2}^2 - \frac{3}{2}\mathcal{QO}_{n+1}\,\mathcal{QO}_{n+2} + \frac{5}{16}\mathcal{QO}_{n+1}^2 \\
&= N(\mathcal{HQO}_n)
\end{aligned}
$$

3. Conclusion

In this paper, we have introduced the Oresme hybrid quaternion numbers. We give some properties and identities such as Binet's formula, generating function, norm and characteristic equation for these quaternions. Furthermore, matrix and determinant forms for these numbers are given.

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