Least energy sign-changing solutions for a class of fractional (p, q)-Laplacian problems with critical growth in $\mathbb{R}^N$ *

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Abstract

In this paper we consider the following fractional (p, q)-Laplacian equation

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = \lambda f(u) + |u|^{q_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1), \lambda > 0, 2 < p < q < \frac{N}{s}$ and $(-\Delta)_t^s$ with $t \in \{p, q\}$ is the fractional t-Laplacian operator, potential V is a continuous function. Under suitable conditions on f, by using constrained variational methods, a quantitative Deformation Lemma and Brouwer degree theory, if λ is large enough, we prove that the above problem has a least energy sign-changing solution u_λ. Moreover, we show that the energy of u_λ is strictly larger than two times the ground state energy.

Keywords: Fractional (p, q)-Laplacian, Sign-changing solutions, Critical problem

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1 Introduction and main results

In this paper, we investigate the existence of the least energy sign-changing solution for the following fractional (p, q)-Laplacian problem:

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \lambda f(u) + |u|^{q_s^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $2 < p < q < \frac{N}{s}$, $\lambda > 0$, the potential $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$, the operator $(-\Delta)_t^s$ with $t \in \{p, q\}$ is the fractional Laplacian which, up to a normalizing constant, may be defined for

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any $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough by setting

$$(-\Delta)_t^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{t-2}(u(x) - u(y))}{|x - y|^{N+ts}} \mathrm{d}y, \ x \in \mathbb{R}^N$$

along functions $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$.

when $s = 1$, problem (1.1) boils down to a (p, q)-Laplacian problem of the type

$$-\Delta_p u - \Delta_q u + V(x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

In the last years, the main interest in this general class of problems has been since they arise from applications in biophysics, plasma physics and chemical reaction design, as it can be seen in [6] and [32]. In the last decade, many authors investigated problem (1.2), for example, Barile and Figueiredo [6] used the deformation lemma and Brouwer degree theory to prove that (1.2) possesses a least energy sign-changing solution. For more interesting results involving (p, q)-Laplacian problems set in bounded domains and in the whole of $\mathbb{R}^N$, we also mention [10, 24, 26, 32, 34, 40] and references therein.

For $s \in (0, 1)$ and $p = q = 2$, equation (1.1) appears in the study of standing wave solutions, i.e. solutions of the form $\psi(x, t) = u(x)e^{-i\omega t}$, to the following fractional Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hbar^{2s}(-\Delta)^s \psi + W(x)\psi - f(|\psi|) \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.3}$$

where $\hbar$ is the Planck constant, $W : \mathbb{R}^N \to \mathbb{R}$ is an external potential and f is a suitable nonlinear term. Equation (1.3) was derived by Laskin [30, 31] and plays a fundamental role in the study of fractional quantum mechanics. For more details, we refer the interested reader to [20] for an elementary introduction on this subject.

After that, remarkable attention has been devoted to the study of fractional Schrödinger equations, and lots of interesting results were obtained. For the existence, multiplicity and behavior of standing wave solutions to equation (1.3), we refer to [2, 11, 12, 16, 21, 23, 25, 37, 38] and the references therein.

when $p = q \neq 2$, problem (1.1) boils down to the following fractional Laplacian problem

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

Problem (1.4) piques the interest of researchers because of its nonlocal character and the operator's nonlinearity. In [15], the the authors obtained infinity many sign-changing solution of (1.4) via invariant sets of descent flow. Moreover, they also proved (1.4) possesses a least energy sign-changing solution by using deformation Lemma and Brouwer degree. It is noteworthy that Wang and Zhou [38] used the similar method to obtain the least energy sign-changing of (1.4) with $p = 2$. Besides, for equation (1.4), several existence and multiplicity results has been obtained in last decade, see for instance [3, 4, 18, 19, 35, 36] and the references therein, and [14, 27] for some interesting regularity results.

On the other hand, in the nonlocal framework, only few recent works deal with fractional (p,q)-Laplacian problems. For instance, in [17] the authors studied existence, nonexistence and multiplicity for a nonlocal (p,q)-subcritical problem. Alves et al. [1] considered the following fractional (p,q)-Laplacian problem

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) \quad \text{in}\, \mathbb{R}^N, \tag{1.5}$$

where the potential $V(x)$ satisfies the Rabinowitz conditions. Applying minimax theorems and the Ljusternik-Schnirelmann theory, they investigated the existence, multiplicity and concentration of nontrivial solutions provided that ε is sufficiently small. After that, Ambrosio and Rădulescu [5] considered (1.5) with the del pino-Felmer type potential conditions. Applying suitable variational and topological arguments, they obtained multiple positive solutions for $\varepsilon > 0$ sufficiently small as well as related concentration properties. For the other work on (1.1) or similar problems, we refer the reader to [5, 22, 28, 41] and the references therein.

Motivated by the above results, it is natural to ask, whether problem (1.1) had sign-changing solutions when the nonlinear term f has critical growth. To our knowledge, this question is open. In [25], the authors considered the following problem

$$\begin{cases} (-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u \text{ in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $2_s^* = \frac{2N}{N-2s}$ and f satisfies some suitable conditions. By using the constrained variational methods, they proved the least energy sign-changing solution of (1.6) when λ sufficiently large. However, since (1.1) contains the nonlocal and nonlinear term $(-\Delta)_p^s + (-\Delta)_q^s$, the decomposition of functional I_λ (see the definition in (1.10)) is more complicated than that in [25]. Therefore, some difficulties arise in studying the existence of a least energy sign-changing solution for problem (1.1) and this makes the study interesting.

In order to study problem (1.1), we need some assumptions on V and f as follows:

(V_1) $V(x) \in C\left(\mathbb{R}^N\right)$ and there exists $V_0 > 0$ such that $V(x) \geq V_0$ in $\mathbb{R}^N$. Moreover, $\lim_{|x|\to\infty} V(x) = +\infty$.

(f_1) $\displaystyle\lim_{|t|\to 0^+} \frac{f(t)}{|t|^{p-1}} = 0$.

(f_2) f has a "quasicritical growth" at infinity, namely,

$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{q_s^*-1}} = 0.$$

We suppose that the function f satisfies the Ambrosetti-Rabinowitz condition:

(f_3) there exists $\theta \in (q, q_s^*)$ such that

$$0 < \theta F(t) = \theta \int_0^t f(s)ds \leq f(t)t \;\; \text{for all } |t| > 0, \text{where } F(t) := \int_0^t f(\tau)d\tau,$$

and furthermore, we assume that:

(f_4) The map f and its derivative f' satisfy

$$f'(t) > (q-1)\frac{f(t)}{t} \text{ for all } t \neq 0.$$

Clearly, (f_4) implies that the map $t \mapsto \frac{f(t)}{|t|^{q-1}}$ is strictly increasing for all $|t| > 0$.

Before starting our results, we recall some useful notations. Let $1 \leq \zeta \leq \infty$, we denote by $|u|_\zeta$ the L^ζ-norm of $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^\zeta(\mathbb{R}^N)$. For $0 < s < 1$, let us define $\mathcal{D}^{s,\zeta}(\mathbb{R}^N) = \overline{\mathcal{C}_c^\infty(\mathbb{R}^N)}^{[\cdot]s,\zeta}$, where

$$[u]_{s,\zeta} := \left[\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^\zeta}{|x-y|^{N+s\zeta}} dxdy\right]^{\frac{1}{\zeta}}.$$

Let us denote by $W^{s,\zeta}(\mathbb{R}^N)$ the set of functions $u \in L^\zeta(\mathbb{R}^N)$ such that $[u]_{s,\zeta} < \infty$, endowed with the natural norm

$$\|u\|_{s,\zeta}^\zeta = [u]_{s,\zeta}^\zeta + |u|_\zeta^\zeta.$$

According to [20]. Let $s \in (0,1)$ and $N > sq$. Then there exists a sharp constant $S_q > 0$ such that for any $u \in \mathcal{D}^{s,q}(\mathbb{R}^N)$

$$|u|_{q_s^*}^q \leq S_q^{-1}[u]_{s,q}^q, \tag{1.7}$$

where $q_s^* = \frac{Nq}{N-qs}$ is the Sobolev critical exponent. Moreover, $W^{s,q}(\mathbb{R}^N)$ is continuously embedded in $L^\gamma(\mathbb{R}^N)$ for any $\gamma \in [q, q_s^*]$ and compactly in $L^\gamma(B_R(0))$, for all $R > 0$ and for any $\gamma \in [1, q_s^*)$.

In order to ensure that problem (1.1) has a variational structure, let us consider the space

$$X = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) \tag{1.8}$$

endowed with the norm

$$\|u\|_X := \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}.$$

Notice that $W^{s,r}(\mathbb{R}^N)$ is a separable reflexive Banach space for all $r \in (1, +\infty)$, then X is also a separable reflexive Banach space. We also introduce the following Banach space

$$X_V := \left\{u \in X : \int_{\mathbb{R}^N} V(x)\left(|u|^p + |u|^q\right) dx < +\infty\right\}, \tag{1.9}$$

endowed with the norm

$$\|u\| := \|u\|_{X_V} := \|u\|_{V,p} + \|u\|_{V,q},$$

where $\|u\|_{V,t}^t := [u]_{s,t}^t + \int_{\mathbb{R}^N} V(x)|u|^t dx$ for $t \in \{p, q\}$. For the weak solution to (1.1), we mean a function $u \in X_V$ such that

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx \\ &\quad + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sq}}dxdy + \int_{\mathbb{R}^N} V(x)|u(x)|^{q-2}u(x)\varphi(x)dx \\ &\quad = \int_{\mathbb{R}^N} \lambda f(u(x))\varphi(x) + |u(x)|^{q_s^*-2}u(x)\varphi(x)dx\end{aligned}$$

for all $\varphi \in X_V$.

Define the energy functional $I_\lambda : X_V \to \mathbb{R}$ by

$$\begin{aligned}I_\lambda(u) &= \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{q}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx \\ &\quad + \frac{1}{q}\int_{\mathbb{R}^N} V(x)|u(x)|^q dx - \lambda\int_{\mathbb{R}^N} F(u(x)) - \frac{1}{q_s^*}\int_{\mathbb{R}^N} |u(x)|^{q_s^*}dx. \qquad (1.10)\end{aligned}$$

By the similar arguments as in [1], we can deduce that $I_\lambda(u) \in \mathcal{C}^1(X_V, \mathbb{R})$.

For convenience, we consider the operator $\mathcal{A}_p : X_V \to X_V^*$ and $\mathcal{A}_q : X_V \to X_V^*$ given by

$$\begin{aligned}\langle \mathcal{A}_p(u), v\rangle_{X_V^*, X_V} &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx, \quad \forall u, v \in X_V\end{aligned}$$

and

$$\begin{aligned}\langle \mathcal{A}_q(u), v\rangle_{X_V^*, X_V} &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+qs}}dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)|u|^{q-2}uvdx, \quad \forall u, v \in X_V,\end{aligned}$$

where X_V^* is the dual space of X_V. In this sequel, for simplicity, we denote $\langle \cdot, \cdot \rangle_{X_V^*, X_V}$ by $\langle \cdot, \cdot \rangle$. Moreover, we denote the Nehari set $\mathcal{N}_\lambda$ by

$$\mathcal{N}_\lambda = \left\{ u \in X\backslash\{0\} : \langle I_\lambda'(u), u\rangle_{X_V^*, X_V} = 0 \right\}. \qquad (1.11)$$

Clearly, $\mathcal{N}_\lambda$ contains all the nontrivial solutions of (1.1). Denote $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \min\{u(x), 0\}$. Then, the sign-changing solutions of (1.1) stay on the following set:

$$\mathcal{M}_\lambda = \left\{ u \in X_V\backslash\{0\} : u^\pm \neq 0,\ \left\langle I_\lambda'(u), u^+\right\rangle = 0,\ \left\langle I_\lambda'(u), u^+\right\rangle = 0 \right\}. \qquad (1.12)$$

Set

$$c := \inf_{u \in \mathcal{N}_\lambda} I(u), \qquad (1.13)$$

and

$$c_\lambda := \inf_{u\in\mathcal{M}_\lambda} I(u). \tag{1.14}$$

The main results of this paper are stated in the following theorem.

Theorem 1.1. *Suppose that* $(f_1)-(f_4)$ *are satisfied. Then, there exists* $\Lambda > 0$ *such that for all* $\lambda \geq \Lambda$, *the problem* (1.1) *possess a least energy sign-changing solution* u_λ. *Moreover,* $c_\lambda > 2c$.

The proof of Theorem 1.1 is based on the arguments presented in [9]. We first check that the minimum of functional I_λ restricted on set $\mathcal{M}_\lambda$ can be achieved. Then, by using a suitable variant of the quantitative deformation Lemma, we show that it is a critical point of I. However, due to the two fractional t-Laplacian operators $(-\Delta)_t^s$ with $s\in(0,1)$ and $t\in\{p,q\}$, one cannot obtain similar equivalent definition of $(-\Delta)_t^s$ by the harmonic extension method (see [12]), and then we don't get the decomposition

$$I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) \quad \text{and} \quad \left\langle I_\lambda'(u), u^\pm\right\rangle = \left\langle I_\lambda'(u^\pm), u^\pm\right\rangle,$$

which are very useful to get sign-changing solutions of (1.1), see for instance [6–9, 13]. Furthermore, we could not adapt similar methods like in [25, 38] to conclude the set $\mathcal{M}_\lambda$ is non empty. This is because for the linear operator $(-\Delta)^s$, one can easily deduce that

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|^{N+2s}}dxdy = &\int_{\mathbb{R}^{2N}} \frac{(u^+(x)-u^+(y))^2}{|x-y|^{N+2s}}dxdy \\ &- \int_{\mathbb{R}^{2N}} \frac{(u^+(x)u^-(y)+u^-(x)u^+(y))}{|x-y|^{N+2s}}dxdy,\end{aligned}$$

which is important to prove $\mathcal{M}_\lambda$ is nonempty. But, for the nonlinear operators $(-\Delta)_p^s$ and $(-\Delta)_q^s$, the above decomposition seems invalid. Fortunately, however, we find a new way to overcome those difficulties. We use another decomposition estimation by dividing $\mathbb{R}^{2N}$ into several regions (see Lemma 2.2) as following:

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|^{N+ts}}dxdy \\ &= \int_{(\mathbb{R}^N)^+\times(\mathbb{R}^N)^+} \frac{|u^+(x)-u^+(y)|^t}{|x-y|^{N+ts}}dxdy + \int_{(\mathbb{R}^N)^+\times(\mathbb{R}^N)^-} \frac{|u^+(x)-u^-(y)|^{t-1}u^+(x)}{|x-y|^{N+ts}}dxdy \\ &\quad + \int_{(\mathbb{R}^N)^-\times(\mathbb{R}^N)^+} \frac{|u^-(x)-u^+(y)|^{t-1}u^+(y)}{|x-y|^{N+ts}}dxdy,\end{aligned}$$

where $(\mathbb{R}^N)^+ = \{x\in\mathbb{R}^N : u(x)\geq 0\}$ and $(\mathbb{R}^N)^- = \{x\in\mathbb{R}^N : u(x) < 0\}$. As we can see that it will also plays an important role in proving $\deg(\Psi_1, D, 0) = 1$ (see Section 4), and then we can get the minimizer u_λ of c_λ (that is, $I_\lambda(u_\lambda) = c_\lambda$) is exactly a sign-changing solution of Problem (1.1). Besides, another difficulty arises in verifying the compactness of the minimizing sequence in X_V since problem (1.1) includes a critical growth nonlinear term. Fortunately, thanks to the sharp constant S_q, we overcome this difficulty by choosing λ appropriately large to ensure the compactness of the minimizing sequence. Therefore, in order to obtain the least energy sign-changing solutions of (1.1), a more accurate investigation and meticulous calculations are needed in our setting.

The paper is organized as follows: In Section 2, we provide some compactness results and the decomposition properties of I_λ, which will be useful for the next sections. In Section 3, we give some technical lemmas which will be crucial in proving the main results. In Section 4, we combine the minimize arguments with a variant of Deformation Lemma and Brouwer degree theory to prove the main results.

2 Preliminaries

In this section, we outline the variational framework for the problem (1.1) and give some preliminary Lemmas. Recalling the definition of fractional Sobolev space X_V in (1.9), we have the following compactness results.

Lemma 2.1. *Suppose that* (V_1) *holds, then for all* $\gamma \in [p, q_s^*]$, *the embedding* $X_V \hookrightarrow L^\gamma\left(\mathbb{R}^N\right)$ *is continuous. For all* $\gamma \in [p, q_s^*)$, *the embedding* $X_V \hookrightarrow L^\gamma\left(\mathbb{R}^N\right)$ *is compact.*

Proof. Denote $Y = L^\gamma\left(\mathbb{R}^N\right)$ and $B_R = \left\{x \in \mathbb{R}^N : |x| < R\right\}, B_R^c = \mathbb{R}^N \backslash \overline{B_R}$. Denote $X_p := \left\{u \in W^{s,p}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(x)|u|^p dx < +\infty\right\}$.

For any $p \leq \gamma \leq q_s^*$, the space X_p is continuously embedded in Y, the space X_V is continuously embedded in X_p, so $X_V \hookrightarrow Y$ is continuous.

For any $p \leq \gamma < q_s^*$, Let $X_p(\Omega)$ and $Y(\Omega)$ be the spaces of functions $u \in X_p, u \in Y$ restricted onto $\Omega \subset \mathbb{R}^N$ respectively. Then, it follows from theorem $6.9, 6.10$ and 7.1 in [20] that $X_p\left(B_R\right) \hookrightarrow Y\left(B_R\right)$ is compact for any $R > 0$. Denote $V_R = \inf_{x \in B_R^c} V(x)$. By (V_1), we deduce that $V_R \to \infty$ as $R \to \infty$. Therefore, we have

$$\int_{B_R^c} |u|^\gamma dx \leq \frac{1}{V_R} \int_{B_R^c} V(x)|u|^\gamma dx \leq \frac{1}{V_R}\|u\|_{X_p}^\gamma,$$

which implies

$$\lim_{R \to +\infty} \sup_{u \in X \backslash \{0\}} \frac{\|u\|_{L^\gamma}\left(B_R^c\right)}{\|u\|_{X_p}} = 0.$$

By virtue of Theorem 7.9 in [29], we can see that $X_p \hookrightarrow Y$ is compact, moreover, $X_V \hookrightarrow X_p$ is compact, therefore, by interpolation inequality, the embedding $X_V \hookrightarrow Y$ is compact for any $p \leq \gamma < q_s^*$. $\square$

Remark 2.1. *It follows from Lemma 2.1 and* (f_1),(f_2) *that* I_λ *is well-defined on* X_V. *Moreover,*

$I_\lambda \in C^1\left(X_V, \mathbb{R}^N\right)$ and

$$
\begin{aligned}
\langle I_\lambda'(u), v\rangle = & \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx \\
& + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+qs}} dxdy + \int_{\mathbb{R}^N} V(x)|u|^{q-2}uvdx \\
& - \lambda \int_{\mathbb{R}^N} f(u)vdx - \int_{\mathbb{R}^N} |u|^{q_s^*-2}uvdx
\end{aligned}
\tag{2.1}
$$

for all $v \in X_V$. Consequently, the critical point of I_λ is the weak solution of problem (1.1).

Since we aim to seek the sign-changing solution of problem (1.1). As we saw in section 1, one of the difficulties is the fact that the functional I_λ does not possess the decomposition like Inspired by [15, 38], we have the following:

Lemma 2.2. *Let $u \in X_V$ with $u^\pm \neq 0$. Then,*
(i) $I_\lambda(u) > I_\lambda(u^+) + I_\lambda(u^-)$,
(ii) $\langle I_\lambda'(u), u^\pm\rangle > \langle I_\lambda'(u^\pm), u^\pm\rangle$.

Proof. Observing that

$$
\begin{aligned}
I_\lambda(u) &= \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \lambda \int_{\mathbb{R}^N} F(u)dx - \frac{1}{q_s^*}\int_{\mathbb{R}^N} |u|^{q_s^*}dx \\
&= \frac{1}{p}\left\langle \mathcal{A}_p(u), u^+\right\rangle + \frac{1}{p}\left\langle \mathcal{A}_p(u), u^-\right\rangle + \frac{1}{q}\left\langle \mathcal{A}_q(u), u^+\right\rangle + \frac{1}{q}\left\langle \mathcal{A}_q(u), u^-\right\rangle \\
&\quad - \lambda \int_{\mathbb{R}^N} F(u^+)dx - \lambda \int_{\mathbb{R}^N} F(u^-)dx - \frac{1}{q_s^*}\int_{\mathbb{R}^N} |u^+|^{q_s^*}dx - \frac{1}{q_s^*}\int_{\mathbb{R}^N} |u^-|^{q_s^*}dx
\end{aligned}
\tag{2.2}
$$

By density (see Theorem 2.4 in [20]), we can assume that u is continuous. Defining

$$
\left(\mathbb{R}^N\right)_+ = \left\{x \in \mathbb{R}^N; u^+(x) \geq 0\right\} \text{ and } \left(\mathbb{R}^N\right)_- = \left\{x \in \mathbb{R}^N; u^-(x) \leq 0\right\}.
$$

Then for $u \in X_V$ with $u^{\pm} \neq 0$, by a straightforward computation, one can see that

$$\begin{aligned}
\left\langle \mathcal{A}_p(u), u^+ \right\rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\left(u^+(x)-u^+(y)\right)}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^+|^p dx \\
&= \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_+} \frac{|u^+(x)-u^+(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{p-1}\, u^+(x)}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{p-1}\, u^+(y)}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^+|^p dx \\
&> \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_+} \frac{|u^+(x)-u^+(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^+|^p dx \\
&\quad + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x)|^p}{|x-y|^{N+ps}} dxdy + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^+(y)|^p}{|x-y|^{N+ps}} dxdy \\
&= \left\langle \mathcal{A}_p\left(u^+\right), u^+ \right\rangle
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
\left\langle \mathcal{A}_p(u), u^- \right\rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\left(u^-(x)-u^-(y)\right)}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^-|^p dx \\
&= \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_-} \frac{|u^-(x)-u^-(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{p-1}\left(-u^-(y)\right)}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{p-1}\left(-u^-(x)\right)}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^-|^p dx \\
&> \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_-} \frac{|u^-(x)-u^-(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^-|^p dx \\
&\quad + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^-(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)|^p}{|x-y|^{N+ps}} dxdy \\
&= \left\langle \mathcal{A}_p\left(u^-\right), u^- \right\rangle.
\end{aligned} \tag{2.4}$$

Similarly, we also have

$$\left\langle \mathcal{A}_q(u), u^+ \right\rangle > \left\langle \mathcal{A}_q\left(u^+\right), u^+ \right\rangle \quad \text{and} \quad \left\langle \mathcal{A}_q(u), u^- \right\rangle > \left\langle \mathcal{A}_q\left(u^-\right), u^- \right\rangle. \tag{2.5}$$

Taking into account (2.3)-(2.5), we deduce that $I_\lambda(u) > I_\lambda(u^+) + I_\lambda(u^-)$. Analogously, one can prove (ii). $\square$

The following Brézis-Lieb type Lemma will be very useful in this work, its proof is similar to Lemma 2.8 in [1] and we omit it here.

Lemma 2.3. *Let $\{u_n\} \subset X_V$ be a sequence such that $u_n \rightharpoonup u$ in X_V. Set $v_n = u_n - u$, then we have:*
(i) $[v_n]_{s,p}^p + [v_n]_{s,q}^q = \left([u_n]_{s,p}^p + [u_n]_{s,q}^q\right) - \left([u]_{s,p}^p + [u]_{s,q}^q\right) + o_n(1)$,

(ii) $\displaystyle\int_{\mathbb{R}^N} V(x)\left(|v_n|^p+|v_n|^q\right)dx=\int_{\mathbb{R}^N} V(x)\left(|u_n|^p+|u_n|^q\right)dx-\int_{\mathbb{R}^N} V(x)\left(|u|^p+|u|^q\right)dx+o_n(1)$,

(iii) $\displaystyle\int_{\mathbb{R}^N}\left(F\left(v_n\right)-F\left(u_n\right)+F(u)\right)dx=o_n(1)$,

(iv) $\displaystyle\sup_{\|w\|\leq 1}\int_{\mathbb{R}^N}\left|\left(f\left(v_n\right)-f\left(u_n\right)+f(u)\right)w\right|dx=o_n(1)$.

3 Some technical lemmas

This section aims to prove some technical lemmas related to the existence of a least energy sign-changing solution. Firstly, we collect some preliminary lemmas which will be fundamental to prove our main result.

Now, fixed $u\in X_V$ with $u^{\pm}\neq 0$, we define function $\psi_u:[0,\infty)\times[0,\infty)\to\mathbb{R}$ and mapping $T_u:[0,\infty)\times[0,\infty)\to\mathbb{R}^2$ by

$$\psi_u(\sigma,\tau):=I_\lambda\left(\sigma u^{+}+\tau u^{-}\right)$$

and

$$T_u(\sigma,\tau):=\left(\left\langle I_\lambda'\left(\sigma u^{+}+\tau u^{-}\right),\sigma u^{+}\right\rangle,\left\langle I_\lambda'\left(\sigma u^{+}+\tau u^{-}\right),\tau u^{-}\right\rangle\right).$$

Lemma 3.1. *For any $u\in X_V$ with $u^{\pm}\neq 0$, there exists a unique maximum point pair (τ_u,σ_u) of the function ψ_u such that $\tau_u u^{+}+\sigma_u u^{-}\in\mathcal{M}_\lambda$.*

Proof. Our proof will be divided into three steps.

Step 1: For any $u\in X_V$ with $u^{\pm}\neq 0$, in the following, we will prove the existence of σ_u and τ_u. Form (f_1), (f_2) and Lemma 2.2 we deduce that

$$\begin{aligned}\left\langle I_\lambda'(\sigma u^{+}+\tau u^{-}),\sigma u^{+}\right\rangle&\geq\left\langle I_\lambda'(\sigma u^{+}),\sigma u^{+}\right\rangle\\&=\sigma^p\left\|u^{+}\right\|_{V,p}^p+\sigma^q\left\|u^{+}\right\|_{V,q}^q-\lambda\int_{\mathbb{R}^N}f(\sigma u^{+})\sigma u^{+}dx-\sigma^{q_s^*}\int_{\mathbb{R}^N}|u^{+}|^{q_s^*}dx\\&\geq\sigma^p\left\|u^{+}\right\|_{V,p}^p+\sigma^q\left\|u^{+}\right\|_{V,q}^q-\lambda\varepsilon\sigma^p\int_{\mathbb{R}^N}|u^{+}|^p dx\\&\quad-\lambda C_\varepsilon\sigma^{q_s^*}\int_{\mathbb{R}^N}|u^{+}|^{q_s^*}dx-\sigma^{q_s^*}\int_{\mathbb{R}^N}|u^{+}|^{q_s^*}dx\\&\geq\left(1-\lambda C\varepsilon\right)\sigma^p\left\|u^{+}\right\|_{V,p}^p+\sigma^q\left\|u^{+}\right\|_{V,q}^q-(\lambda CC_\varepsilon+C)\sigma^{q_s^*}\left\|u^{+}\right\|^{q_s^*}.\end{aligned}\tag{3.1}$$

Similarly, we have that

$$\begin{aligned}\left\langle I_\lambda'(\sigma u^{+}+\tau u^{-}),\tau u^{-}\right\rangle&\geq\left\langle I_\lambda'(\tau u^{-}),\tau u^{-}\right\rangle\\&\geq\left(1-\lambda C\varepsilon\right)\sigma^p\left\|u^{-}\right\|_{V,p}^p+\sigma^q\left\|u^{-}\right\|_{V,q}^q-(\lambda CC_\varepsilon+C)\sigma^{q_s^*}\left\|u^{-}\right\|^{q_s^*}.\end{aligned}\tag{3.2}$$

Choose $\varepsilon>0$ such that $(1-\lambda C\varepsilon)>0$. Since $p<q<q_s^*$, there exists $r>0$ small enough such that

$$\left\langle I_\lambda'(ru^{+}+\tau u^{-}),ru^{+}\right\rangle>0\text{ for all }\tau>0\tag{3.3}$$

and

$$\left\langle I_\lambda'(\sigma u^+ + ru^-), ru^-\right\rangle > 0 \text{ for all } \sigma > 0. \tag{3.4}$$

On the other hand, by (f_3), there exists $D_1, D_2 > 0$ such that

$$F(t) \geq D_1 t^\theta - D_2 \text{ for } t > 0. \tag{3.5}$$

Then we have

$$\begin{aligned}
&\left\langle I'\left(\sigma u^+ + \tau u^-\right), \sigma u^+\right\rangle \\
&\leq \sigma^p \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_+} \frac{|u^+(x)-u^+(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma u^+(x)-\tau u^-(y)|^{p-1}\sigma u^+(x)}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau u^-(x)-\sigma u^+(y)|^{p-1}\sigma u^+(y)}{|x-y|^{N+ps}} dxdy + \sigma^q \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_+} \frac{|u^+(x)-u^+(y)|^q}{|x-y|^{N+qs}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma u^+(x)-\tau u^-(y)|^{q-1}\sigma u^+(x)}{|x-y|^{N+qs}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau u^-(x)-\sigma u^+(y)|^{q-1}\sigma u^+(y)}{|x-y|^{N+qs}} dxdy \\
&\quad + \sigma^p \int_{\mathbb{R}^N} V(x)\left|u^+\right|^p dx + \sigma^q \int_{\mathbb{R}^N} V(x)\left|u^+\right|^q dx - \lambda D_1 \sigma^\theta \int_{A^+} \left|u^+\right|^\theta dx + \lambda D_2 \left|A^+\right|.
\end{aligned}$$

where $A^+ \subset \operatorname{supp}(u^+)$ is measurable set with finite and positive measure $|A^+|$. Due to the fact $\theta > p$, for R sufficiently large, we get

$$\left\langle I_\lambda'(Ru^+ + \tau u^-), Ru^+\right\rangle < 0 \text{ for all } \tau \in [r, R]. \tag{3.6}$$

Similarly, we get

$$\left\langle I_\lambda'(\sigma u^+ + Ru^-), Ru^-\right\rangle < 0 \text{ for all } \sigma \in [r, R]. \tag{3.7}$$

Hence, by virtue of Miranda's Theorem [33], and taking (3.3), (3.4), (3.6) and (3.7) into account, we can see that there exists $(\sigma_u, \tau_u) \in [r, R]\times[r, R]$ such that $T_u(\sigma, \tau) = (0, 0)$, i.e., $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}_\lambda$.

Step 2: Now we prove the uniqueness of the pair (σ_u, τ_u).

Case 1: $u \in \mathcal{M}_\lambda$.

If $u \in \mathcal{M}_\lambda$, we have that

$$
\begin{aligned}
&\left\|u^{+}\right\|_{V,p}^{p}+\left\|u^{+}\right\|_{V,q}^{q}-\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{+}(x)|^{p}}{|x-y|^{N+ps}}dxdy-\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^{+}(y)|^{p}}{|x-y|^{N+ps}}dxdy\\
&\quad-\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{+}(x)|^{q}}{|x-y|^{N+qs}}dxdy-\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^{+}(y)|^{q}}{|x-y|^{N+qs}}dxdy\\
&\quad+\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{+}(x)-u^{-}(y)|^{p-1}\,u^{+}(x)}{|x-y|^{N+ps}}dxdy+\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^{-}(x)-u^{+}(y)|^{p-1}\,u^{+}(y)}{|x-y|^{N+ps}}dxdy\\
&\quad+\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{+}(x)-u^{-}(y)|^{q-1}\,u^{+}(x)}{|x-y|^{N+qs}}dxdy+\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^{-}(x)-u^{+}(y)|^{q-1}\,u^{+}(y)}{|x-y|^{N+qs}}dxdy\\
&=\lambda\int_{\mathbb{R}^N}f\left(u^{+}\right)u^{+}dx+\int_{\mathbb{R}^N}|u^{+}|^{q_s^*}dx
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&\left\|u^{-}\right\|_{V,p}^{p}+\left\|u^{-}\right\|_{V,q}^{q}-\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^{-}(x)|^{p}}{|x-y|^{N+ps}}dxdy-\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{-}(y)|^{p}}{|x-y|^{N+ps}}dxdy\\
&\quad-\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^{-}(x)|^{q}}{|x-y|^{N+qs}}dxdy-\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{-}(y)|^{q}}{|x-y|^{N+qs}}dxdy\\
&\quad+\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_-}\frac{|u^{-}(x)-u^{+}(y)|^{p-1}\,(-u^{-}(x))}{|x-y|^{N+ps}}dxdy+\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{+}(x)-u^{-}(y)|^{p-1}\,(-u^{-}(y))}{|x-y|^{N+ps}}dxdy\\
&\quad+\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_-}\frac{|u^{-}(x)-u^{+}(y)|^{q-1}\,(-u^{-}(x))}{|x-y|^{N+qs}}dxdy+\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^{+}(x)-u^{-}(y)|^{q-1}\,(-u^{-}(y))}{|x-y|^{N+qs}}dxdy\\
&=\lambda\int_{\mathbb{R}^N}f\left(u^{-}\right)u^{-}dx+\int_{\mathbb{R}^N}|u^{-}|^{q_s^*}dx.
\end{aligned}
\tag{3.9}
$$

We will show that $(\sigma_u,\tau_u)=(1,1)$ is the unique pair of numbers such that $\sigma_u u^{+}+\tau_u u^{-}\in\mathcal{M}_\lambda$. Let (σ_u,τ_u) be a pair of numbers such that $\sigma_u u^{+}+\tau_u u^{-}\in\mathcal{M}_\lambda$ with $0<\sigma_u\le\tau_u$, then one can see

$$
\begin{aligned}
&{\sigma_u}^p \left\|u^+\right\|_{V,p}^p + {\sigma_u}^q \left\|u^+\right\|_{V,q}^q - {\sigma_u}^p \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)|^p}{|x-y|^{N+ps}}dxdy - {\sigma_u}^p \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^+(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad - {\sigma_u}^q \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)|^q}{|x-y|^{N+qs}}dxdy - {\sigma_u}^q \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^+(y)|^q}{|x-y|^{N+qs}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{p-1} \sigma_u u^+(x)}{|x-y|^{N+ps}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{p-1} \sigma_u u^+(y)}{|x-y|^{N+ps}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{q-1} \sigma_u u^+(x)}{|x-y|^{N+qs}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{q-1} \sigma_u u^+(y)}{|x-y|^{N+qs}}dxdy \\
&= \lambda \int_{\mathbb{R}^N} f\left(\sigma_u u^+\right) \sigma_u u^+ dx + {\sigma_u}^{q_s^*} \int_{\mathbb{R}^N} |u^+|^{q_s^*} dx
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
&{\tau_u}^p \left\|u^-\right\|_{V,p}^p + {\tau_u}^q \left\|u^-\right\|_{V,q}^q - {\tau_u}^p \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^-(x)|^p}{|x-y|^{N+ps}}dxdy - {\tau_u}^p \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^-(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad - {\tau_u}^q \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^-(x)|^q}{|x-y|^{N+qs}}dxdy - {\tau_u}^q \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^-(y)|^q}{|x-y|^{N+qs}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{p-1} \left(-\tau_u u^-(x)\right)}{|x-y|^{N+ps}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{p-1} \left(-\tau_u u^-(y)\right)}{|x-y|^{N+ps}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{q-1} \left(-\tau_u u^-(x)\right)}{|x-y|^{N+qs}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{q-1} \left(-\tau_u u^-(y)\right)}{|x-y|^{N+qs}}dxdy \\
&= \lambda \int_{\mathbb{R}^N} f\left(\tau_u u^-\right) \tau_u u^- dx + {\tau_u}^{q_s^*} \int_{\mathbb{R}^N} |u^-|^{q_s^*} dx.
\end{aligned} \tag{3.11}
$$

Since $0 < \sigma_u \leq \tau_u$, it follows from (3.11) that

$$\begin{aligned}
&{\tau_u}^{p-q} \left\|u^-\right\|_{V,p}^p + \left\|u^-\right\|_{V,q}^q \\
&\quad - {\tau_u}^{p-q} \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)|^p}{|x-y|^{N+ps}} dxdy - {\tau_u}^{p-q} \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^-(y)|^p}{|x-y|^{N+ps}} dxdy \\
&\quad - \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)|^q}{|x-y|^{N+qs}} dxdy - \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^-(y)|^q}{|x-y|^{N+qs}} dxdy \\
&\quad + {\tau_u}^{p-q} \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x) - u^+(y)|^{p-1} \left(-u^-(x)\right)}{|x-y|^{N+ps}} dxdy \\
&\quad + {\tau_u}^{p-q} \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x) - u^-(y)|^{p-1} \left(-u^-(y)\right)}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x) - u^+(y)|^{q-1} \left(-u^-(x)\right)}{|x-y|^{N+qs}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x) - u^-(y)|^{q-1} \left(-u^-(y)\right)}{|x-y|^{N+qs}} dxdy \\
&\geq \lambda \int_{\mathbb{R}^N} \frac{f\left(\tau_u u^-\right) \tau_u u^-}{{\tau_u}^q} dx + {\tau_u}^{q_s^*-q} \int_{\mathbb{R}^N} |u^-|^{q_s^*} dx.
\end{aligned} \tag{3.12}$$

If $\tau_u > 1$, by (3.9) and (3.12), we get

$$\begin{aligned}
&\left({\tau_u}^{p-q} - 1\right) \left(\|u^-\|_{V,p}^p - \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)|^p}{|x-y|^{N+ps}} dxdy - \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^-(y)|^p}{|x-y|^{N+ps}} dxdy \right) \\
&\quad + \left({\tau_u}^{p-q} - 1\right) \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x) - u^+(y)|^{p-1} \left(-u^-(x)\right)}{|x-y|^{N+ps}} dxdy \\
&\quad + \left({\tau_u}^{p-q} - 1\right) \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x) - u^-(y)|^{p-1} \left(-u^-(y)\right)}{|x-y|^{N+ps}} dxdy \\
&\geq \lambda \int_{\mathbb{R}^N} \left(\frac{f\left(\tau_u u^-\right)}{|\tau_u u^-|^{q-1}} - \frac{f\left(u^-\right)}{|u^-|^{q-1}} \right) |u^-|^q dx + \left({\tau_u}^{q_s^*-q} - 1\right) \int_{\mathbb{R}^N} |u^-|^{q_s^*} dx.
\end{aligned}$$

The left side of the above inequality is negative, which is absurd because the right side is positive. Therefore, we conclude that $0 < \sigma_u \leq \tau_u \leq 1$.

Similarly, by (3.10) and $0 < \sigma_u \leq \tau_u$, we have that

$$
\begin{aligned}
&(\sigma_u^{p-q}-1)\left(\|u^+\|_{V,p}^p-\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^+(x)|^p}{|x-y|^{N+ps}}dxdy-\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^+(y)|^p}{|x-y|^{N+ps}}dxdy\right)\\
&\quad+(\sigma_u^{p-q}-1)\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^+(x)-u^-(y)|^{p-1}u^+(x)}{|x-y|^{N+ps}}dxdy\\
&\quad+(\sigma_u^{p-q}-1)\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^-(x)-u^+(y)|^{p-1}u^+(y)}{|x-y|^{N+ps}}dxdy\\
&\le\lambda\int_{\mathbb{R}^N}(\frac{f(\sigma_u u^+)}{|\sigma_u u^+|^{q-1}}-\frac{f(u^+)}{|u^+|^{q-1}})|u^+|^q dx+(\sigma_u^{q_s^*-q}-1)\int_{\mathbb{R}^N}|u^+|^{q_s^*}dx.
\end{aligned}
$$

This fact implies that $\sigma_u \geq 1$. Consequently, $\sigma_u = \tau_u = 1$.

Case 2: $u \notin \mathcal{M}_\lambda$.

Suppose that there exist $(\widetilde{\sigma}_1, \widetilde{\tau}_1),\ (\widetilde{\sigma}_2, \widetilde{\tau}_2)$ such that

$$
u_1 := \widetilde{\sigma}_1 u^+ + \widetilde{\tau}_1 u^- \in \mathcal{M}_\lambda \quad \text{and} \quad u_2 := \widetilde{\sigma}_2 u^+ + \widetilde{\tau}_2 u^- \in \mathcal{M}_\lambda\ .
$$

Hence,

$$
u_2=\left(\frac{\widetilde{\sigma}_2}{\widetilde{\sigma}_1}\right)\widetilde{\sigma}_1 u^+ + \left(\frac{\widetilde{\tau}_2}{\widetilde{\tau}_1}\right)\widetilde{\tau}_1 u^- = \left(\frac{\widetilde{\sigma}_2}{\widetilde{\sigma}_1}\right)u_1^+ + \left(\frac{\widetilde{\tau}_2}{\widetilde{\tau}_1}\right)u_1^- \in \mathcal{M}_\lambda.
$$

Since $u_1 \in \mathcal{M}_\lambda$, we deduce from case 1 that

$$
\frac{\widetilde{\sigma}_2}{\widetilde{\sigma}_1}=\frac{\widetilde{\tau}_2}{\widetilde{\tau}_1}=1,
$$

which implies $\widetilde{\sigma}_1 = \widetilde{\sigma}_2,\ \widetilde{\tau}_1 = \widetilde{\tau}_2$.

Step 3: We assert that (σ_u, τ_u) is the unique maximum point of ψ_u on $[0,+\infty)\times[0,+\infty)$. In fact, by (f_3) we can see that

$$
\begin{aligned}
I_\lambda(\sigma u^+ + \tau u^-) &= \frac{1}{p}\left\|\sigma u^+ + \tau u^-\right\|_{V,p}^p + \frac{1}{q}\left\|\sigma u^+ + \tau u^-\right\|_{V,q}^q - \lambda\int_{\mathbb{R}^N}F(\sigma u^+ + \tau u^-)dx\\
&\quad-\frac{1}{q_s^*}\int_{\mathbb{R}^N}|\sigma u^+ + \tau u^-|^{q_s^*}dx\\
&\le \frac{1}{p}\left\|\sigma u^+ + \tau u^-\right\|_{V,p}^p + \frac{1}{p}\left\|\sigma u^+ + \tau u^-\right\|_{V,q}^q - \frac{\sigma^{q_s^*}}{q_s^*}\int_{\mathbb{R}^N}|u^+|^{q_s^*}dx - \frac{\tau^{q_s^*}}{q_s^*}\int_{\mathbb{R}^N}|u^-|^{q_s^*}dx,
\end{aligned}
$$

which implies that $\lim_{|\sigma,\tau|\to\infty}\phi_u(\sigma,\tau) = -\infty$ due to $q_s^* > q$. Noticing that $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}_\lambda$, we conclude that (σ_u, τ_u) is the unique critical point of ψ_u in $(0,+\infty)\times(0,+\infty)$. Hence, it is sufficient to check that a maximum point cannot be achieved on the boundary of $[0,+\infty)\times[0,+\infty)$. By contradiction, we suppose that $(0,\tau_1)$ is a maximum point of ψ_u with $\tau_1 \geq 0$.

Then, arguing as Lemma 2.2, we have

$$
\begin{aligned}
\psi_u(\sigma,\tau_1) &= \frac{1}{p}\left\|\sigma u^+ + \tau_1 u^-\right\|_{V,p}^p + \frac{1}{q}\left\|\sigma u^+ + \tau_1 u^-\right\|_{V,q}^q - \lambda\int_{\mathbb{R}^N} F(\sigma u^+)dx \\
&\quad - \lambda\int_{\mathbb{R}^N} F(\tau_1 u^-)dx - \frac{\sigma^{q_s^*}}{q_s^*}\int_{\mathbb{R}^N}|u^+|^{q_s^*}dx - \frac{{\tau_1}^{q_s^*}}{q_s^*}\int_{\mathbb{R}^N}|u^-|^{q_s^*}dx \\
&> \frac{\sigma^p}{p}\left\|u^+\right\|_{V,p}^p + \frac{\sigma^q}{q}\left\|u^+\right\|_{V,q}^q - \lambda\int_{\mathbb{R}^N} F(\sigma u^+)dx - \frac{\sigma^{q_s^*}}{q_s^*}\int_{\mathbb{R}^N}|u^+|^{q_s^*}dx \\
&\quad + \frac{\tau_1^p}{p}\left\|u^-\right\|_{V,p}^p + \frac{\tau_1^q}{q}\left\|u^-\right\|_{V,q}^q - \lambda\int_{\mathbb{R}^N} F(\tau_1 u^-)dx - \frac{\tau_1^{q_s^*}}{q_s^*}\int_{\mathbb{R}^N}|u^-|^{q_s^*}dx \\
&= \psi_u(0,\tau_1) + \psi_u(\sigma,0).
\end{aligned} \tag{3.13}
$$

On the other hand, by the growth condition (f_1) and (f_2), one can easily check that $\psi_u(\sigma,0) > 0$ for σ sufficiently small. Combining this with (3.13), we see that

$$
\psi_u(0,\tau_1) < \psi_u(0,\tau_1) + \psi_u(\sigma,0) < \psi_u(\sigma,\tau_1)
$$

if σ is small enough, which yields a contradiction. Similarly, ψ_u can not achieve its global maximum point at $(\sigma_1, 0)$, where $\sigma_1 \geq 0$. As a result, we complete the proof of Lemma 3.1. □

Lemma 3.2. *For any $u \in X_V$ with $u^\pm \neq 0$, such that $\langle I_\lambda'(u), u^\pm\rangle \leq 0$, the unique maximum point of ψ_u in $[0,+\infty)\times[0,+\infty)$ satisfies $0 < \sigma_u, \tau_u \leq 1$.*

Proof. If $\sigma_u = 0$ or $\tau_u = 0$, according Lemma 3.1, ψ_u can not achieve maximum. Without loss of generality, we assume $\sigma_u \geq \tau_u > 0$. Since $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}_\lambda$, there holds

$$
\begin{aligned}
&{\sigma_u}^p\left\|u^+\right\|_{V,p}^p + {\sigma_u}^q\left\|u^+\right\|_{V,q}^q - {\sigma_u}^p\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^+(x)|^p}{|x-y|^{N+ps}}dxdy - {\sigma_u}^p\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^+(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad - {\sigma_u}^q\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|u^+(x)|^q}{|x-y|^{N+qs}}dxdy - {\sigma_u}^q\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|u^+(y)|^q}{|x-y|^{N+qs}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{p-1}\sigma_u u^+(x)}{|x-y|^{N+ps}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{p-1}\sigma_u u^+(y)}{|x-y|^{N+ps}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-}\frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{q-1}\sigma_u u^+(x)}{|x-y|^{N+qs}}dxdy \\
&\quad + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+}\frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{q-1}\sigma_u u^+(y)}{|x-y|^{N+qs}}dxdy \\
&= \lambda\int_{\mathbb{R}^N} f\left(\sigma_u u^+\right)\sigma_u u^+dx + {\sigma_u}^{q_s^*}\int_{\mathbb{R}^N}|u^+|^{q_s^*}dx.
\end{aligned} \tag{3.14}
$$

On the other hand, by $\langle I_\lambda'(u), u^+\rangle \le 0$, we have

$$
\begin{aligned}
&\left\|u^+\right\|_{V,p}^p + \left\|u^+\right\|_{V,q}^q - \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)|^p}{|x-y|^{N+ps}}dxdy - \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^+(y)|^p}{|x-y|^{N+ps}}dxdy \\
&- \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)|^q}{|x-y|^{N+qs}}dxdy - \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^+(y)|^q}{|x-y|^{N+qs}}dxdy \\
&+ \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{p-1}u^+(x)}{|x-y|^{N+ps}}dxdy + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{p-1}u^+(y)}{|x-y|^{N+ps}}dxdy \\
&+ \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{q-1}u^+(x)}{|x-y|^{N+qs}}dxdy + \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{q-1}u^+(y)}{|x-y|^{N+qs}}dxdy \\
&\le \lambda \int_{\mathbb{R}^N} f\left(u^+\right)u^+dx + \int_{\mathbb{R}^N} |u^+|^{q_s^*}dx.
\end{aligned}
\tag{3.15}
$$

Then it follows (3.14) and (3.15) that

$$
\begin{aligned}
&(\sigma_u^{p-q}-1)\left(\|u^+\|_{V,p}^p - \int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)|^p}{|x-y|^{N+ps}}dxdy - \int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^+(y)|^p}{|x-y|^{N+ps}}dxdy\right) \\
&+ (\sigma_u^{p-q}-1)\int_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{p-1}u^+(x)}{|x-y|^{N+ps}}dxdy \\
&+ (\sigma_u^{p-q}-1)\int_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{p-1}u^+(y)}{|x-y|^{N+ps}}dxdy \\
&\ge \lambda \int_{\mathbb{R}^N} \left(\frac{f\left(\sigma_u u^+\right)}{|\sigma_u u^+|^{q-1}} - \frac{f\left(u^+\right)}{|u^+|^{q-1}}\right)|u^+|^q dx + (\sigma_u^{q_s^*-q}-1)\int_{\mathbb{R}^N} |u^+|^{q_s^*}dx.
\end{aligned}
\tag{3.16}
$$

In view of (f_4), we conclude that $\sigma_u \le 1$. Thus, we have that $0 < \sigma_u, \tau_u \le 1$. □

Lemma 3.3. *There exists $\rho > 0$ such that $\|u^\pm\| \ge \rho$ for all $u \in \mathcal{M}_\lambda$.*

Proof. For any $u \in \mathcal{M}_\lambda$, by $(f_1), (f_2)$ and the Sobolev inequalities, we have that

$$
\begin{aligned}
\left\|u^\pm\right\|_{V,p}^p + \left\|u^\pm\right\|_{V,q}^q &\le \lambda \int_{\mathbb{R}^N} f\left(u^\pm\right)u^\pm dx + \int_{\mathbb{R}^N} |u^\pm|^{q_s^*}dx \\
&\le \lambda\varepsilon C_1 \left\|u^\pm\right\|_{V,p}^p + \lambda C_2 C_\varepsilon \|u^\pm\|^{q_s^*} + C_3\|u^\pm\|^{q_s^*}
\end{aligned}
$$

Thus we get

$$
C_0'\|u\|_{V,p}^p + \|u\|_{V,q}^q \le \widetilde{C}_2\|u\|^{q^*}, \tag{3.17}
$$

where $C_0' = (1-\lambda\varepsilon C_1)$, $\widetilde{C}_2 = (C_3 + \lambda C_2 C_\varepsilon)$ with C a Sobolev embedding constant. If $0 < \|u\| < 1$, then $\|u\|_{V,p}, \|u\|_{V,q} < 1$ and by order relations between p and q and by (3.17) we have

$$
\begin{aligned}
C''\|u\|^q &\le C''\left(\|u\|_{V,p} + \|u\|_{V,q}\right)^q \le C'\left(\|u\|_{V,p}^q + \|u\|_{V,q}^q\right) \\
&\le C_0'\|u\|_{V,p}^p + \|u\|_{V,q}^q \le \widetilde{C}_2\|u\|^{q^*},
\end{aligned}
$$

where $C' = \min\{C'_0, 1\}$ and $C'' = \frac{C'}{2^{q-1}}$. Hence, there exists a positive radius $\rho_1 > 0$ such that $\|u\| \geq \rho_1$ with $\rho_1 = \left(\frac{C''}{\widetilde{C}_\varepsilon}\right)^{\frac{1}{q^*-q}}$. Clearly we can reason analogously if $\|u\| \geq 1$ so that for some $\rho > 0$ and for every $u \in \mathcal{M}_\lambda$, we get $\rho \leq \|u\|$. □

Lemma 3.4. *Let $c_\lambda = \inf_{u\in\mathcal{M}_\lambda} I_\lambda(u)$, then we have that $\lim_{\lambda\to\infty} c_\lambda = 0$.*

Proof. Since $u \in \mathcal{M}_\lambda$, we have $\langle I'_\lambda(u), u\rangle = 0$ and then

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{\theta}\langle I'_\lambda(u), u\rangle \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u\|^p_{V,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u\|^q_{V,q}, \end{aligned} \tag{3.18}$$

thus I_λ is bounded below on $\mathcal{M}_\lambda$, which implies c_λ is well-defined.

For any $u \in X_V$ with $u^\pm \neq 0$, by Lemma 3.1, for each $\lambda > 0$, there exists $\sigma_\lambda, \tau_\lambda$ such that $\sigma_\lambda u^+ + \tau_\lambda u^- \in \mathcal{M}_\lambda$, we have

$$\begin{aligned} 0 \leq c_\lambda = \inf I_\lambda(u) &\leq I_\lambda\left(\sigma_\lambda u^+ + \tau_\lambda u^-\right) \\ &\leq \frac{1}{p}\left\|\sigma_\lambda u^+ + \tau_\lambda u^-\right\|^p_{V,p} + \frac{1}{q}\left\|\sigma_\lambda u^+ + \tau_\lambda u^-\right\|^q_{V,q} - \int\limits_{\mathbb{R}^N} F(\sigma_\lambda u^+ + \tau_\lambda u^-)dx \\ &\quad - \frac{1}{q_s^*}\int_{\mathbb{R}^N}\left|\sigma_\lambda u^+ + \tau_\lambda u^-\right|^{q_s^*}dx \\ &\leq \frac{2^{p-1}}{p}{\sigma_\lambda}^p\left\|u^+\right\|^p_{V,p} + \frac{2^{p-1}}{p}{\tau_\lambda}^p\left\|u^-\right\|^p_{V,p} + \frac{2^{q-1}}{q}{\sigma_\lambda}^q\left\|u^+\right\|^q_{V,q} + \frac{2^{q-1}}{q}{\tau_\lambda}^q\left\|u^-\right\|^q_{V,q}. \end{aligned}$$

Next, we will prove that $\sigma_\lambda \to 0$ and $\tau_\lambda \to 0$ as $\lambda \to \infty$.

Let $Q_u = \{(\sigma_\lambda, \tau_\lambda) \in [0, +\infty) \times [0, +\infty) : T_u(\sigma_\lambda, \tau_\lambda) = (0, 0), \lambda > 0\}$. Due to $\sigma_\lambda u^+ + \tau_\lambda u^- \in \mathcal{M}_\lambda$, there holds

$$\begin{aligned} &\sigma_\lambda^{q_s^*}\int_{\mathbb{R}^N}\left|u^+\right|^{q_s^*}dx + \tau_\lambda^{q_s^*}\int_{\mathbb{R}^N}\left|u^-\right|^{q_s^*}dx + \lambda\int_{\mathbb{R}^N} f(\sigma_\lambda u^+)(\sigma_\lambda u^+)dx + \lambda\int_{\mathbb{R}^N} f(\tau_\lambda u^-)(\tau_\lambda u^-)dx \\ &= \left\|\sigma_\lambda u^+ + \tau_\lambda u^-\right\|^p_{V,p} + \left\|\sigma_\lambda u^+ + \tau_\lambda u^-\right\|^q_{V,q} \\ &\leq 2^{p-1}{\sigma_\lambda}^p\left\|u^+\right\|^p_{V,p} + 2^{p-1}{\tau_\lambda}^p\left\|u^-\right\|^p_{V,p} + 2^{q-1}{\sigma_\lambda}^q\left\|u^+\right\|^q_{V,q} + 2^{q-1}{\tau_\lambda}^q\left\|u^-\right\|^q_{V,q}. \end{aligned}$$

Therefore, Q_u is bounded in $\mathbb{R}^2$. Let $\{\lambda_n\} \subset (0, \infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then there exist σ_0 and τ_0 such that $(\sigma_{\lambda_n}, \tau_{\lambda_n}) \to (\sigma_0, \tau_0)$ as $n \to \infty$.

Now, we claim $\sigma_0 = \tau_0 = 0$. By contradiction, suppose that $\sigma_0 > 0$ or $\tau_0 > 0$ by $\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^- \in \mathcal{M}_{\lambda_n}$, then for any $n \in \mathbb{N}$, there holds

$$\begin{aligned} &\left\|\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-\right\|^p_{V,p} + \left\|\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-\right\|^q_{V,q} \\ &= \lambda_n\int_{\mathbb{R}^N} f(\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-)(\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-)dx + \int_{\mathbb{R}^N}|\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-|^{q_s^*}dx. \end{aligned} \tag{3.19}$$

Thanks to $\sigma_{\lambda_n}u^+ \to \sigma_0 u^+$ and $\tau_{\lambda_n}u^- \to \tau_0 u^-$ in $X_V, (f_1), (f_2)$ and the Lebesgue dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N} f(\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-)(\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-)dx \to \int_{\mathbb{R}^N} f(\sigma_0 u^+ + \tau_0 u^-)(\sigma_0 u^+ + \tau_0 u^-)dx > 0 \quad (3.20)$$

as $n \to \infty$. It follows from $\lambda_n \to \infty$ and (3.20) that the right hand side of (3.19) tends to infty, which contradict with the boundness of $\{\sigma_{\lambda_n}u^+ + \tau_{\lambda_n}u^-\}$ in X_V. Hence, $\sigma_0 = \tau_0 = 0$. As a result, we conclude that $\lim_{\lambda\to\infty} c_\lambda = 0$. □

Lemma 3.5. *There exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the infimum c_λ is achieved.*

Proof. By the definition of $c_\lambda = \inf_{u\in\mathcal{M}_\lambda} I_\lambda(u)$, there exists a sequence $\{u_n\} \subset \mathcal{M}_\lambda$ such that

$$\lim_{\lambda\to\infty} I_\lambda(u_n) = c_\lambda.$$

Obviously, $\{u_n\}$ is bounded in X_V. Up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in X_V$ such that $u_n \rightharpoonup u$ weakly in X_V. Since the embedding $X_V \hookrightarrow L^r(\mathbb{R}^N)$ is compact for all $r \in [p, q_s^*)$, we have $u_n^\pm \to u^\pm$ in $L^r(\mathbb{R}^N)$ for all $r \in [p, q_s^*)$, $u_n^\pm(x) \to u^\pm(x)$ a.e. $x \in \mathbb{R}^N$.

Denote $\delta := \frac{s}{N} S_q^{\frac{N}{sq}}$, according to Lemma 3.4, there is $\lambda^\star > 0$ such that $c_\lambda < \delta$ for all $\lambda \geq \lambda^\star$. Fix $\lambda \geq \lambda^\star$, it follows from Lemma 3.1 that $I_\lambda(\sigma u_n^+ + \tau u_n^-) \leq I_\lambda(u_n)$ for all $\sigma, \tau \geq 0$. Then by using Brézis-Lieb type Lemma 2.3 and the Fatou's Lemma, it follows that

$$\begin{aligned}
&\liminf_{n\to\infty} I_\lambda(\sigma u_n^+ + \tau u_n^-) \\
&\quad= \liminf_{n\to\infty}\left(\frac{1}{p}\|\sigma u_n^+ + \tau u_n^-\|_{V,p}^p + \frac{1}{q}\|\sigma u_n^+ + \tau u_n^-\|_{V,q}^q - \frac{1}{q_s^*}|\sigma u_n^+ + \tau u_n^-|_{q_s^*}^{q_s^*}\right) - \lambda\int_{\mathbb{R}^N} F(\sigma u_n^+ + \tau u_n^-)dx \\
&\quad= \liminf_{n\to\infty}\left(\frac{1}{p}\|\sigma u_n^+ + \tau u_n^- - (\sigma u^+ + \tau u^-)\|_{V,p}^p + \frac{1}{q}\|\sigma u_n^+ + \tau u_n^- - (\sigma u^+ + \tau u^-)\|_{V,q}^q\right) \\
&\qquad - \frac{\sigma^{q_s^*}}{q_s^*}\lim_{n\to\infty}|u_n^+ - u^+|_{q_s^*}^{q_s^*} - \frac{\tau^{q_s^*}}{q_s^*}\lim_{n\to\infty}|u_n^- - u^-|_{q_s^*}^{q_s^*} - \frac{1}{q_s^*}|\sigma u^+ + \tau u^-|_{q_s^*}^{q_s^*} \\
&\qquad + \frac{1}{p}\|\sigma u^+ + \tau u^-\|_{V,p}^p + \frac{1}{q}\|\sigma u^+ + \tau u^-\|_{V,q}^q - \lambda\int_{\mathbb{R}^N} F(\sigma u_n^+ + \tau u_n^-)dx \\
&\quad= I_\lambda(\sigma u^+ + \tau u^-) + \lim_{n\to\infty}\left(\frac{1}{p}\|\sigma u_n^+ - \sigma u^+\|_{V,p}^p + \frac{1}{p}\|\tau u_n^- - \tau u^-\|_{V,p}^p\right) \\
&\qquad + \liminf_{n\to\infty}\left(\frac{1}{p}\|\sigma u_n^+ + \tau u_n^- - (\sigma u^+ + \tau u^-)\|_{V,p}^p - \frac{1}{p}\|\sigma u_n^+ - \sigma u^+\|_{V,p}^p - \frac{1}{p}\|\tau u_n^- - \tau u^-\|_{V,p}^p\right) \\
&\qquad + \lim_{n\to\infty}\left(\frac{1}{q}\|\sigma u_n^+ - \sigma u^+\|_{V,q}^q + \frac{1}{p}\|\tau u_n^- - \tau u^-\|_{V,q}^q\right) \\
&\qquad + \liminf_{n\to\infty}\left(\frac{1}{q}\|\sigma u_n^+ + \tau u_n^- - (\sigma u^+ + \tau u^-)\|_{V,q}^q - \frac{1}{q}\|\sigma u_n^+ - \sigma u^+\|_{V,q}^q - \frac{1}{q}\|\tau u_n^- - \tau u^-\|_{V,q}^q\right) \\
&\qquad - \frac{\sigma^{q_s^*}}{q_s^*}\lim_{n\to\infty}|u_n^+ - u^+|_{q_s^*}^{q_s^*} - \frac{\tau^{q_s^*}}{q_s^*}\lim_{n\to\infty}|u_n^- - u^-|_{q_s^*}^{q_s^*} \\
&\quad\geq I_\lambda(\sigma u^+ + \tau u^-) + \frac{1}{p}\sigma^p A_1 + \frac{1}{q}\sigma^q A_3 - \frac{\sigma^{q_s^*}}{q_s^*}B_1 + \frac{1}{p}\tau^p A_2 + \frac{1}{q}\tau^q A_4 - \frac{\tau^{q_s^*}}{q_s^*}B_2,
\end{aligned}$$

where

$$A_1=\lim_{n\to\infty}\left\|u_n^+-u^+\right\|_{V,p}^p,\quad A_2=\lim_{n\to\infty}\left\|u_n^--u^-\right\|_{V,p}^p,\quad A_3=\lim_{n\to\infty}\left\|u_n^+-u^+\right\|_{V,q}^q,$$
$$A_4=\lim_{n\to\infty}\left\|u_n^--u^-\right\|_{V,q}^q,\quad B_1=\lim_{n\to\infty}\left|u_n^+-u^+\right|_{q_s^*}^{q_s^*},\quad B_2=\lim_{n\to\infty}\left|u_n^--u^-\right|_{q_s^*}^{q_s^*}.$$

Hence, we can see that for all $\sigma\geq 0$ and $\tau\geq 0$, there holds

$$c_\lambda\geq I_\lambda\left(\sigma u^++\tau u^-\right)+\frac{1}{p}\sigma^pA_1+\frac{1}{q}\sigma^qA_3-\frac{\sigma^{q_s^*}}{q_s^*}B_1+\frac{1}{p}\tau^pA_2+\frac{1}{q}\tau^qA_4-\frac{\tau^{q_s^*}}{q_s^*}B_2. \tag{3.21}$$

Now we divide the proof into three steps.

Step 1: We prove that $u^\pm\neq 0$. Here we only prove $u^+\neq 0$ since $u^-=0$ is similar, by contradiction, we suppose $u^+=0$. Then we have the following two cases.

Case 1: $B_1=0$. If $A_1=A_3=0$, that is, $u_n^+\to u^+$ in X_V. According to Lemma 3.3, we obtain $\|u^+\|>0$, which contradicts $u^+=0$. If A_1 or $A_3>0$, By (3.21) we get $\frac{1}{p}\sigma^pA_1+\frac{\sigma^q}{q}A_3<c_\lambda$ for all $\sigma\geq 0$, which is a contradiction.

Case 2: $B_1>0$. According to definition of S_q, we have that $\delta:=\frac{s}{N}S_q^{\frac{N}{sq}}\leq\frac{s}{N}(\frac{A_3}{(B_1)^{\frac{q}{q_s^*}}})^{\frac{N}{sq}}$, by direct calculation, we have that

$$\frac{s}{N}(\frac{A_3}{(B_1)^{\frac{q}{q_s^*}}})^{\frac{N}{sq}}=\max_{\sigma\geq0}\{\frac{\sigma^q}{q}A_3-\frac{\sigma^{q_s^*}}{q_s^*}B_1\}\leq\max_{\sigma\geq0}\{\frac{\sigma^p}{p}A_1+\frac{\sigma^q}{q}A_3-\frac{\sigma^{q_s^*}}{q_s^*}B_1\}.$$

Since $c_\lambda\to 0$ as $\lambda\to\infty$, there exists $\lambda^*>0$ such that for all $\lambda>\lambda^*, c_\lambda\leq\delta$. Then, without loss of generality, we can assume $c_\lambda<\delta$. Choosing $\tau=0$, by (3.21) it follows that

$$\delta\leq\max_{\sigma\geq0}\{\frac{\sigma^q}{q}A_3-\frac{\sigma^{q_s^*}}{q_s^*}B_1\}\leq\max_{\sigma\geq0}\{\frac{\sigma^p}{p}A_1+\frac{\sigma^q}{q}A_3-\frac{\sigma^{q_s^*}}{q_s^*}B_1\}<\delta,$$

which is impossible. From the above discussion, we have that $u^+\neq 0$. Similarly, we obtain $u^-\neq 0$.

Step 2: we prove that $B_1=0,\ B_2=0$. We just prove $B_1=0$ (the proof of $B_2=0$ is analogous). By contradiction, we suppose that $B_1>0$.

Case 1: $B_2>0$, Let $\widehat{\sigma}_1$ and $\widehat{\tau}_1$ satisfy

$$\left\{\frac{\widehat{\sigma}_1^p}{p}A_1+\frac{\widehat{\sigma}_1^q}{q}A_3-\frac{\widehat{\sigma}_1^{q_s^*}}{q_s^*}B_1\right\}=\max_{\sigma\geq0}\left\{\frac{\sigma^p}{p}A_1+\frac{\sigma^q}{q}A_3-\frac{\sigma^{q_s^*}}{q_s^*}B_1\right\}$$

and

$$\left\{\frac{\widehat{\tau}_1^p}{p}A_2+\frac{\widehat{\tau}_1^q}{q}A_4-\frac{\widehat{\tau}_1^{q_s^*}}{q_s^*}B_2\right\}=\max_{\tau\geq0}\left\{\frac{\tau^p}{p}A_2+\frac{\tau^q}{q}A_4-\frac{\tau^{q_s^*}}{q_s^*}B_2\right\}.$$

According to $[0,\widehat{\sigma}_1]\times[0,\widehat{\tau}_1]$ is compact, there exist $(\sigma_u,\tau_u)\in[0,\widehat{\sigma}_1]\times[0,\widehat{\tau}_1]$ such that $\psi_u\left(\sigma_u,\tau_u\right)=\max_{(\sigma,\tau)\in[0,\widehat{\sigma}_1]\times[0,\widehat{\tau}_1]}\psi_u(\sigma,\tau)$.

In the following, we prove that $(\sigma_u,\tau_u)\in(0,\widehat{\sigma}_1)\times(0,\widehat{\tau}_1)$. Obviously, if τ is small enough, we have

$$\psi_u(\sigma,0) < I_\lambda\left(\sigma u^+\right) + I_\lambda\left(\tau u^-\right) \le I_\lambda\left(\sigma u^+ + \tau u^-\right) = \psi_u(\sigma,\tau), \quad \forall\, \sigma \in [0,\widehat{\sigma}_1].$$

Hence, there exists τ_0 such that $\psi_u(\sigma,0) \le \psi_u(\sigma,\tau_0)$, for all $\sigma \in [0,\widehat{\sigma}_1]$. That is, $(\sigma_u,\tau_u) \notin [0,\widehat{\sigma}_1] \times \{0\}$. Similarly, one can prove that $(\sigma_u,\tau_u) \notin \{0\} \times [0,\widehat{\tau}_1]$.

On the other hand, we can easily deduce that

$$\frac{\sigma^p}{p}A_1 + \frac{\sigma^q}{q}A_3 - \frac{\sigma^{q_s^*}}{q_s^*}B_1 > 0,\ \sigma \in (0,\widehat{\sigma}_1] \tag{3.22}$$

and

$$\frac{\tau^p}{p}A_2 + \frac{\tau^q}{q}A_4 - \frac{\tau^{q_s^*}}{q_s^*}B_2,\ \tau \in (0,\widehat{\tau}_1]. \tag{3.23}$$

Then, for all $\sigma \in (0,\widehat{\sigma}_1]$ and $\tau \in (0,\widehat{\tau}_1]$, we get

$$\delta \le \frac{\widehat{\sigma}_1^p}{p}A_1 + \frac{\widehat{\sigma}_1^q}{q}A_3 - \frac{\widehat{\sigma}_1^{q_s^*}}{q_s^*}B_1 + \frac{\tau^p}{p}A_2 + \frac{\tau^q}{q}A_4 - \frac{\tau^{q_s^*}}{q_s^*}B_2,$$

$$\delta \le \frac{\widehat{\tau}_1^p}{p}A_2 + \frac{\widehat{\tau}_1^q}{q}A_4 - \frac{\widehat{\tau}_1^{q_s^*}}{q_s^*}B_2 + \frac{\sigma^p}{p}A_1 + \frac{\sigma^q}{q}A_3 - \frac{\sigma^{q_s^*}}{q_s^*}B_1.$$

Together with (3.21), we obtain $\psi_u(\sigma,\widehat{\tau}_1) \le 0,\ \psi_u(\widehat{\sigma}_1,\tau) \le 0$, for all $\sigma \in [0,\widehat{\sigma}_1]$ and $\tau \in [0,\widehat{\tau}_1]$, which is absurd. Therefore, $(\sigma_u,\tau_u) \notin [0,\widehat{\sigma}_1] \times \{\widehat{\tau}_1\}$ and $(\sigma_u,\tau_u) \notin \{0,\widehat{\sigma}_1\} \times [0,\widehat{\tau}_1]$.

In conclusion, we get $(\sigma_u,\tau_u) \in (0,\widehat{\sigma}_1) \times (0,\widehat{\tau}_1)$. Hence, $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}_\lambda$. So, combining (3.21), (3.22) with (3.23), we have that

$$\begin{aligned} c_\lambda &\ge I_\lambda\left(\sigma_u u^+ + \tau_u u^-\right) + \frac{1}{p}{\sigma_u}^p A_1 + \frac{1}{q}{\sigma_u}^q A_3 - \frac{{\sigma_u}^{q_s^*}}{q_s^*}B_1 + \frac{1}{p}{\tau_u}^p A_2 + \frac{1}{q}{\tau_u}^q A_4 - \frac{{\tau_u}^{q_s^*}}{q_s^*}B_2 \\ &> I_\lambda\left(\sigma_u u^+ + \tau_u u^-\right) \ge c_\lambda. \end{aligned}$$

Therefore, we have a contradiction.

Case 2: $B_2 = 0$. In this case, we can maximize in $[0,\widehat{\sigma}_1] \times [0,\infty)$. Indeed, it is possible to show that there exists $\widehat{\tau}_0 \in [0,\infty]$ such that $I_\lambda(\sigma u^+ + \tau u^-) < 0$ for all $(\sigma,\tau) \in [0,\widehat{\sigma}_1] \times [\widehat{\tau}_0,\infty)$. Hence, there exists $(\sigma_u,\tau_u) \in [0,\widehat{\sigma}_1] \times [0,\infty)$ that satisfies $\psi_u(\sigma_u,\tau_u) = \max\limits_{\sigma \in [0,\widehat{\sigma}_1]\times[0,\infty)} \psi_u(\sigma,\tau)$.

Following, we prove that $(\sigma_u,\tau_u) \in (0,\widehat{\sigma}_1) \times (0,\infty)$.

Indeed, since $\psi_u(\sigma,0) \le \psi_u(\sigma,\tau)$ for $\sigma \in [0,\widehat{\sigma}_1]$ and τ is small enough, we have $(\sigma_u,\tau_u) \notin [0,\widehat{\sigma}_1] \times \{0\}$. Analogously, we have $(\sigma_u,\tau_u) \notin \{0\} \times [0,\infty)$. On the other hand, for all $\tau \in [0,\infty)$, it is obvious that

$$\delta \le \frac{\widehat{\sigma}_1^p}{p}A_1 + \frac{\widehat{\sigma}_1^q}{q}A_3 - \frac{\widehat{\sigma}_1^{q_s^*}}{q_s^*}B_1 + \frac{\tau^p}{p}A_2 + \frac{\tau^q}{q}A_4.$$

Hence, we have that $\psi_u(\widehat{\sigma}_1,\tau) \le 0$ for all $\tau \in [0,\infty)$, Thus, $(\sigma_u,\tau_u) \notin \{\widehat{\sigma}_1\} \times [0,\infty)$. In summary, we have $(\sigma_u,\tau_u) \in (0,\widehat{\sigma}_1) \times (0,\infty)$, namely, $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}_\lambda$. Therefore, according to (3.22), we have that

$$
\begin{aligned}
c_\lambda &\geq I_\lambda\left(\sigma_u u^+ + \tau_u u^-\right) + \frac{1}{p}{\sigma_u}^p A_1 + \frac{1}{q}{\sigma_u}^q A_3 - \frac{{\sigma_u}^{q_s^*}}{q_s^*}B_1 + \frac{1}{p}{\tau_u}^p A_2 + \frac{1}{q}{\tau_u}^q A_4 \\
&> I_\lambda\left(\sigma_u u^+ + \tau_u u^-\right) \geq c_\lambda,
\end{aligned}
$$

which is a contradiction.

Therefore, from the above discussion, we deduce that $B_1 = B_2 = 0$.

Step 3: we prove that c_λ is achieved. Since $u^\pm \neq 0$, by Lemma 3.1, there exist $\sigma_u, \tau_u > 0$ such that

$$
\widetilde{u} = \sigma_u u^+ + \tau_u u^- \in \mathcal{M}_\lambda.
$$

Furthermore, $B_1 = B_2 = 0$ and Fatou's Lemma implies $\langle I'_\lambda(u), u^\pm\rangle \leq 0$. By Lemma 3.2, we obtain $\sigma_u, \tau_u \leq 1$. Since $u_n \in \mathcal{M}_\lambda$, then according to Lemma 3.1 there holds

$$
I_\lambda(\sigma_u u_n^+ + \tau_u u_n^-) \leq I_\lambda(u_n^+ + u_n^-) = I_\lambda(u_n).
$$

Due to $\sigma_u, \tau_u \leq 1$, arguing as Lemma 2.2, one has $\|\sigma_u u^+ + \tau_u u^-\|_{V,p}^p \leq \|u\|_{V,p}^p$. Then by (f_4), Fatou's Lemma and a straightforward calculation, we deduce that

$$
\begin{aligned}
c_\lambda &\leq I_\lambda(\widetilde{u}) - \frac{1}{q}\langle I'_\lambda(\widetilde{u}), \widetilde{u}\rangle \\
&= (\frac{1}{p} - \frac{1}{q})\|\widetilde{u}\|_{V,p}^p + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{q}f(\widetilde{u})\widetilde{u} - F(\widetilde{u})\right]dx + (\frac{1}{q} - \frac{1}{q_s^*})\int_{\mathbb{R}^N}|\widetilde{u}|^{q_s^*}dx \\
&= (\frac{1}{p} - \frac{1}{q})\|\sigma_u u^+ + \tau_u u^-\|_{V,p}^p + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{q}f(\sigma_u u^+)\sigma_u u^+ - F(\sigma_u u^+)\right]dx \\
&\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{q}f(\tau_u u^-)\tau_u u^- - F(\tau_u u^-)\right]dx + (\frac{1}{q} - \frac{1}{q_s^*})\int_{\mathbb{R}^N}|\sigma_u u^+|^{q_s^*}dx \\
&\quad + (\frac{1}{q} - \frac{1}{q_s^*})\int_{\mathbb{R}^N}|\tau_u u^-|^{q_s^*}dx \\
&\leq (\frac{1}{p} - \frac{1}{q})\|u\|_{V,p}^p + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{q}f(u)u - F(u)\right]dx + (\frac{1}{q} - \frac{1}{q_s^*})\int_{\mathbb{R}^N}|u|^{q_s^*}dx \\
&\leq \liminf_{n\to\infty}\left[I_\lambda\left(u_n\right) - \frac{1}{q}\left\langle I'_\lambda\left(u_n\right), u_n\right\rangle\right] \leq c_\lambda.
\end{aligned}
$$

Therefore, $\sigma_u = \tau_u = 1$, and c_λ is achieved by $u_\lambda := u^+ + u^- \in \mathcal{M}_\lambda$. This ends the proof of Lemma 3.5. $\square$

4 Proof of Theorem 1.1

Proof of Theorem 1.1. Since $u_\lambda \in \mathcal{M}_\lambda$, we have $\langle I'_\lambda(u_\lambda), u_\lambda^+\rangle = \langle I'_\lambda(u_\lambda), u_\lambda^-\rangle = 0$. By Lemma 3.5, for $(\sigma, \tau) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus (1, 1)$, we have

$$
I_\lambda(\sigma u_\lambda^+ + \tau u_\lambda^-) < I_\lambda(u_\lambda^+ + u_\lambda^-) = c_\lambda. \tag{4.1}
$$

Now we prove u_λ is a solution of (1.1). Arguing by contradiction, we assume that $I'_\lambda(u_\lambda) \neq 0$, then there exists $\delta > 0$ and $\kappa > 0$ such that

$$|I'_\lambda(v)| \geq \kappa, \text{ for all } \|v - u_\lambda\| \leq 3\delta.$$

Define $D := [1-\delta_1, 1+\delta_1] \times [1-\delta_1, 1+\delta_1]$ and a map $g : D \to X_V$ by

$$g(\sigma, \tau) := \sigma w^+ + \tau w^-,$$

where $\delta_1 \in (0, \frac{1}{2})$ small enough such that $\|g(\sigma,\tau) - w\| \leq 3\delta$ for all $(\sigma,\tau) \in \bar{D}$. Thus, by virtue of Lemma 3.5, we can see that

$$I(g(1,1)) = c_\lambda, \ I(g(\sigma,\tau)) < c_\lambda \text{ for all } (\sigma,\tau) \in D \backslash \{(1,1)\}.$$

Therefore,

$$\beta := \max_{(\sigma,\tau)\in\partial D} I(g(\sigma,\tau)) < c_\lambda.$$

By using [39, Theorem 2.3] with

$$\mathcal{S}_\delta := \{v \in X : \|v - u_\lambda\| \leq \delta\}$$

and $c := c_\lambda$. Then, choosing $\varepsilon := \min\left\{\frac{c_\lambda - \beta}{4}, \frac{\kappa\delta}{8}\right\}$, we deduce that there exists a deformation $\eta \in C([0,1] \times X_V, X_V)$ such that:

(i) $\eta(t,v) = v$ if $v \notin I^{-1}\left([c_\lambda - 2\varepsilon, c_\lambda + 2\varepsilon]\right)$;

(ii) $I_\lambda(\eta(1,v)) \leq c_\lambda - \varepsilon$ for each $v \in X_V$ with $\|v - u\| \leq \delta$ and $I_\lambda(v) \leq c_\lambda + \varepsilon$;

(iii) $I_\lambda(\eta(1,v)) \leq I_\lambda(v)$ for all $u \in X_V$.

By (ii) and (iii) we conclude that

$$\max_{(\sigma,\tau)\in\overline{D}} I_\lambda(\eta(1, g(\sigma,\tau))) < c_\lambda. \tag{4.2}$$

Therefore, to complete the proof of this Lemma, it suffices to prove that

$$\eta(1, g(\overline{D})) \cap \mathcal{M}_\lambda \neq \emptyset. \tag{4.3}$$

Indeed, if (4.3) holds true, then by the definition of c_λ and (4.2), we get a contradiction.

In the following, we will prove (4.3). To this end, for $(\sigma,\tau) \in \overline{D}$, let $\gamma(\sigma,\tau) := \eta(1, g(\sigma,\tau))$ and

$$\begin{aligned}\Psi_0(\sigma,\tau) :=&(\langle I'_\lambda(g(\sigma,\tau)), u_\lambda^+\rangle, \langle I'_\lambda(g(\sigma,\tau)), u_\lambda^-\rangle)\\ =&(\langle I'_\lambda(\sigma u_b^+ + \tau u_\lambda^-), u_\lambda^+\rangle, \langle I'_\lambda(\sigma u_\lambda^+ + \tau u_\lambda^-), u_\lambda^-\rangle) := (\varphi_u^1(\sigma,\tau), \varphi_u^2(\sigma,\tau))\end{aligned}$$

and

$$\Psi_1(\sigma,\tau) : \ = \Big(\frac{1}{\sigma}\langle I'_\lambda(\gamma(\sigma,\tau)), (\gamma(\sigma,\tau))^+\rangle, \frac{1}{\tau}\langle I'_\lambda(\gamma(\sigma,\tau)), (\gamma(\sigma,\tau))^-\rangle\Big).$$

Firstly, let us denote

$$
\begin{aligned}
A_p &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}|u_\lambda^+(x)-u_\lambda^+(y)|^2}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(x)|u_\lambda^+|^p dx,\\
A_q &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{q-2}|u_\lambda^+(x)-u_\lambda^+(y)|^2}{|x-y|^{N+qs}}dxdy + \int_{\mathbb{R}^N} V(x)|u_\lambda^+|^q dx,\\
B_p &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}|u_\lambda^-(x)-u_\lambda^-(y)|^2}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(x)|u_\lambda^-|^p dx,\\
B_q &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{q-2}|u_\lambda^-(x)-u_\lambda^-(y)|^2}{|x-y|^{N+qs}}dxdy + \int_{\mathbb{R}^N} V(x)|u_\lambda^-|^q dx,\\
C_p &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda^-(x)-u_\lambda^-(y))(u_\lambda^+(x)-u_\lambda^+(y))}{|x-y|^{N+ps}}dxdy,\\
C_q &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{q-2}(u_\lambda^-(x)-u_\lambda^-(y))(u_\lambda^+(x)-u_\lambda^+(y))}{|x-y|^{N+qs}}dxdy,\\
D_p &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda^+(x)-u_\lambda^+(y))(u_\lambda^-(x)-u_\lambda^-(y))}{|x-y|^{N+ps}}dxdy,\\
D_q &:= \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{q-2}(u_\lambda^+(x)-u_\lambda^+(y))(u_\lambda^-(x)-u_\lambda^-(y))}{|x-y|^{N+qs}}dxdy,\\
a_1 &:= \lambda\int_{\mathbb{R}^N} f'(u_\lambda^+)|u_\lambda^+|^2dx, \qquad a_2 := \lambda\int_{\mathbb{R}^N} f(u_\lambda^+){u_\lambda}^+dx,\\
b_1 &:= \lambda\int_{\mathbb{R}^N} f'(u_\lambda^-)|u_\lambda^-|^2dx, \qquad b_2 := \lambda\int_{\mathbb{R}^N} f(u_\lambda^-)u_\lambda^-dx,\\
c_1 &:= \int_{\mathbb{R}^N} |u_\lambda^+|^{q_s^*}dx, \qquad c_2 := \int_{\mathbb{R}^N} |{u_\lambda}^-|^{q_s^*}dx.
\end{aligned}
$$

Clearly, $C_p = D_p > 0$, $C_q = D_q > 0$, $A_p, A_q, B_p, B_q > 0$ and notice that $u_\lambda \in \mathcal{M}_\lambda$, we can see that

$$
A_p + C_p + A_q + C_q = a_2 + c_1, \qquad B_p + D_p + B_q + D_q = b_2 + c_2. \tag{4.4}
$$

Moreover, (f_4) guarantees

$$
a_1 > (q-1)a_2, \qquad b_1 > (q-1)b_2. \tag{4.5}
$$

Then by direct computation, we have

$$
\begin{aligned}
\frac{\partial\varphi_u^1}{\partial\sigma}(1,1) &= (p-1)A_p + (q-1)A_q - a_1 - (q_s^*-1)c_1 < 0,\\
\frac{\partial\varphi_u^2}{\partial\tau}(1,1) &= (p-1)B_p + (q-1)B_q - b_1 - (q_s^*-1)c_2 < 0.
\end{aligned} \tag{4.6}
$$

and

$$
\frac{\partial\varphi_u^2}{\partial\tau}(1,1) = \frac{\partial\varphi_u^2}{\partial\sigma}(1,1) = (p-1)C_p + (q-1)C_q = (p-1)D_p + (q-1)D_q. \tag{4.7}
$$

Let

$$M = \begin{bmatrix} \frac{\varphi_u^1(\sigma,\tau)}{\partial\sigma}|_{1,1} & \frac{\varphi_u^2(\sigma,\tau)}{\partial\sigma}|_{1,1} \\ \frac{\varphi_u^1(\sigma,\tau)}{\partial\tau}|_{1,1} & \frac{\varphi_u^2(\sigma,\tau)}{\partial\tau}|_{1,1} \end{bmatrix}.$$

So we have

$$\begin{aligned} \det M &= [(p-1)A_p + (q-1)A_q - a_1 - (q_s^* - 1)c_1] \cdot [(p-1)B_p + (q-1)B_q - b_1 - (q_s^* - 1)c_2] \\ &\quad - [(p-1)C_p + (q-1)C_q]\,[(p-1)D_p + (q-1)D_q] \\ &> [(q-1)a_2 + (q_s^* - 1)c_1 - (p-1)A_p - (q-1)A_q] \cdot \\ &\quad [(q-1)b_2 + (q_s^* - 1)c_2 - (p-1)B_p - (q-1)B_q] \\ &\quad - [(p-1)C_p + (q-1)C_q]\,[(p-1)D_p + (q-1)D_q] \\ &= [(q-p)A_p + (q-1)C_p + (q-1)C_q(q_s^* - q)c_1] \cdot \\ &\quad [(q-p)B_p + (q-1)D_p + (q-1)D_q + (q_s^* - q)c_2] \\ &\quad - [(p-1)C_p + (q-1)C_q]\,[(p-1)D_p + (q-1)D_q] \\ &> 0. \end{aligned} \tag{4.8}$$

Since $\Psi_0(\alpha, \beta)$ is a C^1 function and (1,1) is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, D, 0) = 1$. Furthermore, combining (4.2) and (a), we obtain

$$g(\sigma, \tau) = \gamma(\sigma, \tau) \text{ on } \partial D.$$

Consequently, we deduce that $\deg(\Psi_1, D, 0) = 1$. Therefore, $\Psi_1(\sigma_0, \tau_0) = 0$ for some $(\sigma_0, \tau_0) \in D$ so that

$$\eta(1, g(\sigma_0, \tau_0)) = \gamma(\sigma_0, \tau_0) \in \mathcal{M}_\lambda,$$

which is contradicted to (4.2). From the above discussions, we deduce that u_λ is a sign-changing solution for the problem (1.1).

Next, we prove that the energy of u_b is strictly larger than two times the ground state energy.

Similar to proof of Lemma 3.1, there exists $\lambda_1^* > 0$ such that for all $\lambda \geq \lambda_1^* > 0$, there exists $v \in \mathcal{N}_\lambda$ such that $I_\lambda(v) = c^* > 0$. By standard arguments, the critical points of the functional I_λ on $\mathcal{N}_\lambda$ are critical points of I_λ in X_V, we obtain $\langle I'_\lambda(v), v\rangle = 0$, that is, v is a ground state solution of (1.1).

According to Theorem 1.1, we know that the problem (1.1) has a least energy sign-changing solution u_b when $\lambda \geq \lambda^*$. Denote $\Lambda := \max\{\lambda^*, \lambda_1^*\}$. As Proof of Lemma 3.5, there exist $\sigma_{u_\lambda^+} > 0$ and $\tau_{u_\lambda^-} > 0$ such that

$$\sigma_{u_\lambda^+} u_\lambda^+ \in \mathcal{N}_\lambda, \quad \tau_{u_\lambda^-} u_\lambda^- \in \mathcal{N}_\lambda.$$

Furthermore, Lemma 3.2 implies that $\sigma_{u_\lambda^+}, \tau_{u_\lambda^-} \in (0, 1)$.
Therefore, in view of Lemma 3.1, we have that

$$2c \leq I_\lambda(\sigma_{u_\lambda^+} u_\lambda^+) + I_\lambda(\tau_{u_\lambda^-} u_\lambda^-) < I_\lambda(\sigma_{u_\lambda^+} u_\lambda^+ + \tau_{u_\lambda^-} u_\lambda^-) < I_\lambda(u_\lambda^+ + u_\lambda^-) = c_\lambda.$$

The proof is complete. □

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