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Asymptotic estimations of eigenvalues and eigenfunctions for Nonlocal Boundary Value Problems with eigenparameter-dependent boundary conditions

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The nonlocal boundary value problem with eigenparameter dependent boundary conditions is studied in this paper. Firstly, we give the asymptotic expressions of the general solution for the equation corresponding to the initial conditions with eigenparameters, then we prove the multiplicity of eigenvalues some properties of the eigenvalues and eigenfunctions. Finally, the asymptotic formulas of eigenvalues and eigenfunctions are obtained under certain mild conditions. Our method is to incorporate the perturbation theory and asymptotic analysis in the framework of classical Sturm-Liouville problems, which provides a new sight for the investigating of the Sturm-Liouville problems with eigenparameter in boundary conditions.

Keywords: Nonlocal boundary value problem; eigenparameter dependent boundary condition; asymptotic formulas of eigenvalues and eigenfunctions

1. Introduction

Eigenvalue problems with eigenparameter dependent boundary conditions is of great significance for dealing with a lot of problems of mathematical physics and mechanics [4, 5]. It is well known that the Sturm-Liouville problem with eigenparameter dependent boundary conditions is obtained by separating variables from partial differential equation such as wave mechanics and thermodynamics [6]. The wide application of such problems in mechanics, physics and engineering has stimulated people's interest in research [6]-[9]. In [5], Fulton gave the asymptotic formulas for eigenvalues and eigenfunctions of the two-point boundary value problems involving the eigenvalue parameter in the boundary condition at one end-point.

Browne and Sleeman [2] first studied the inverse nodal problem with eigenparameter dependent boundary conditions in the following form:

$$-y''(x) + q(x)y(x) = \lambda y(x), x \in [0, \pi]$$

$$(a_0\lambda + b_0)y(0) = (c_0\lambda + d_0)y'(0),$$

$$(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1).$$

Where $q(x)$ is a real valued continuous function and $\delta_i = (-1)^i(a_i d_i - b_i c_i) < 0, i = 0, 1, c_0 c_1 \neq 0$. The asymptotic properties of eigenvalues and eigenfunctions of the Sturm-Liouville problem with eigenparameter dependent boundary conditions are introduced in detail in [1] and [3].

More detailed studies on such problems can be found in various literatures [10]-[16]. In particular, the authors in [11] considered a Sturm-Liouville operator with eigenparameter dependent boundary conditions and transmission conditions at two interior points. They got the asymptotic formulas of eigenvalues and the characteristic function, the completeness of its eigenfunctions was

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also involved. The authors in [17] considered the Sturm-Liouville problem

$$-y''(x) + q(x)y(x) = \lambda y(x), x \in [0, \pi]$$

subject to

$$y'(0) = 0, \; y'(\pi) + f(\lambda)y(\pi) = 0.$$

Here $q \in [0, \pi]$ and

$$f(\lambda) = a_1\sqrt{\lambda} + a_2\sqrt{\lambda}^2 + \cdots + a_m\sqrt{\lambda}^m, a_i \in R, a_m \neq 0, m \in Z^+.$$

They gave the asymptotic expressions of the eigenvalues λ_n for n sufficiently large, and the classical Ambarzumyan's theorem for the regular Sturm-Liouville problem was extended to the case in which the boundary conditions are eigenparameter dependent. By creating a new self-adjoint operator related to a Sturm-Liouville problem with eigenparameter dependent boundary conditions and eigenparameter dependent transmission conditions, Sen constructed fundamental solutions and obtained the asymptotic formulas for its eigenvalues and fundamental solutions in [18].

Nonlocal boundary value problems were abstracted from some practical problems in the fields of mathematical physics, biology and biotechnology [19]-[23]. In recent years, the study of Sturm-Liouville problems with different types of nonlocal boundary conditions has become a hot topic, more researchers have been devoted to such problems and obtained certain excellent results. Eigenvalue problems with nonlocal boundary conditions of Bitsadze-Samarskii or integral type were considered in [24]-[26]. The authors in [27, 28] investigated more complicated cases of the Sturm-Liouville problem with one classical boundary condition and another nonlocal boundary condition. Typically, the Sturm-Liouville problem in three cases of nonlocal two-point boundary conditions was considered in [27], Peciulyte and Stikonas proved general properties of eigenvalues and eigenfunctions. Besides, the authors also described the qualitative behavior of all eigenvalues dependent on nonlocal boundary parameters. For the case of $q(t) \equiv 0$, Stikonas in [29] obtained general properties of the characteristic function and spectrum for the Sturm-Liouville problem with one classical and another nonlocal boundary condition. In order to obtain some new results of the spectrum of the Sturm-Liouville problem with one Bitsadze-Samarskii type nonlocal boundary condition, the characteristic function method has been used in [30].

Recently, Sen and Stikonas in [31] studied the asymptotic properties of eigenvalues and eigenfunctions for the following second order nonlocal boundary value problems with potential function $q(t)$ in differential equation

$$-u''(t) + q(t)u(t) = \lambda u(t), t \in (0, 1)$$

$$u(0) = 0,$$

$$u(1) = \gamma u(\xi),$$

where $\gamma \in \mathbb{R}$ and $\xi \in (0, 1)$. The asymptotic formulas for eigenvalues and eigenfunctions of nonlocal boundary value problems were obtained in [31] where general properties of eigenvalues were proved. However, there were relatively few studies on the asymptotic distribution for eigenvalues and eigenfunctions of the Sturm-Liouville problem with one eigenparameter dependent and another nonlocal boundary condition.

In this paper, we investigate the equation

$$-u''(t) + q(t)u(t) = \lambda u(t), t \in [0, 1] \tag{1}$$

associated with three-point boundary conditions

$$\frac{u(0)}{u'(0)} = \frac{a\lambda + b}{c\lambda + d}, \tag{2}$$

$$u(1) = ru(\xi). \tag{3}$$

Here $q \in C[0, 1]$, a, b, c, d are real numbers. Suppose $r \in \mathbb{R}$, $\xi \in (0, 1)$, $\rho = ad - bc \neq 0$.

This paper is organized as follows. In Section 2, the basic solution is estimated and it is obtained that the geometric multiplicity of eigenvalues of problem (1)-(3) is simple. In Section 3, we calculate the asymptotic formulas of eigenvalues and eigenfunctions of problem (1)-(3) under certain mild conditions.

In this paper, we will use the following symbols:

$$\begin{aligned}
&\mathbb{R}_s^- = \{s = x + iy \in \mathbb{C} : x = 0, y > 0\}, \quad \mathbb{R}_s^+ = \{s = x + iy \in \mathbb{C} : x > 0, y = 0\},\\
&\mathbb{C}_s^+ = \{s = x + iy \in \mathbb{C} : x > 0, y > 0\}, \quad \mathbb{C}_s^- = \{s = x + iy \in \mathbb{C} : x > 0, y < 0\},\\
&\mathbb{R}_s^0 = \{s = 0\}.
\end{aligned}$$

where $\mathbb{R}_s = \mathbb{R}_s^- \cup \mathbb{R}_s^+ \cup \mathbb{R}_s^0$, $\mathbb{C}_s = \mathbb{R}_s \cup \mathbb{C}_s^+ \cup \mathbb{C}_s^-$.

2. Preliminaries

Lemma 1 *If* $\varphi_\lambda(t)$ *is a solution of Eq.*(1) *satisfying the initial conditions*

$$\varphi_\lambda(0) = a\lambda + b, \quad \varphi'_\lambda(0) = c\lambda + d, \tag{4}$$

then $\varphi_\lambda(t)$ *is uniquely determined by the existence and uniqueness theorem of the solution, and the function* $\varphi(t,\lambda) = \varphi_\lambda(t)$ *is an entire function with respect to* λ *by the continuous differentiability of the solution to the initial value parameter.*

We denote the Wronskians $W(y,z;t) = y'(t)z(t) - y(t)z'(t)$, according to Liouville theorem, $W(y,z;t)$ does not depend on t and it is constant on $[0,1]$.

Lemma 2 *Let* $\lambda = s^2$, $s = x + iy$. *Then*
for $a \neq 0$,

$$\frac{d^k}{dt^k}\varphi_\lambda(t) = as^2\frac{d^k}{dt^k}\cos(st) + O(|s|^{k+1}e^{|y|t}), \tag{5}$$

whereas if $a = 0$,

$$\frac{d^k}{dt^k}\varphi_\lambda(t) = sc\frac{d^k}{dt^k}\sin(st) + O(|s|^k e^{|y|t}), \ \ k = 0, 1. \tag{6}$$

Each of these estimations holds uniformly for t *as* $|\lambda| \to \infty$.

Proof: According to the constant variation formula, the solution of Eq.(1) satisfying the initial conditions (4) is

$$\varphi_\lambda(t) = (as^2 + b)\cos(st) + \frac{cs^2 + d}{s}\sin(st) + \frac{1}{s}\int_0^t \sin(s(t-\tau))q(\tau)\varphi_\lambda(\tau)\mathrm{d}\tau. \tag{7}$$

For $k = 0$, $a \neq 0$, let $\varphi_\lambda(t) = e^{|y|t}F(t,\lambda)$ and substitute it into (7), we get

$$\begin{aligned} F(t,\lambda) =& (as^2 + b)\cos(st)e^{-|y|t} + \frac{cs^2 + d}{s}\sin(st)e^{-|y|t} \\ &+ \frac{1}{s}\int_0^t \sin(s(t-\tau))q(\tau)e^{-|y|(t-\tau)}F(\tau,\lambda)\mathrm{d}\tau. \end{aligned}$$

Let $u(\lambda) = \max_{0\le t\le 1}|F(t,\lambda)|$, we note that

$$|\cos st| \le e^{|y|t}, \quad |\sin st| \le e^{|y|t}.$$

So

$$u(\lambda) \le |as^2| + |b| + |cs| + \left|\frac{d}{s}\right| + \frac{1}{|s|}\int_0^1 |q(\tau)|u(\lambda)\mathrm{d}\tau,$$

when $|s| > 2\int_0^1 |q(\tau)|\mathrm{d}\tau$, we obtain

$$u(\lambda) \le \frac{|as^2| + |b| + |cs| + \left|\frac{d}{s}\right|}{1 - \frac{1}{|s|}\int_0^1 |q(\tau)|\mathrm{d}\tau} < M|s|^2,$$

where M is a constant independent of λ, thus

$$\varphi_\lambda(t) = e^{|y|t}F(t,\lambda) = O(|s|^2 e^{|y|t}), \tag{8}$$

substituting (8) into (7)

$$\varphi_\lambda(t) = as^2\cos(st) + O(|s|e^{|y|t}),$$

then (5) can be got by differentiating the above formula with respect to t. The proof for (6) is similar.

Theorem 1 *The geometric multiplicity of eigenvalues of the problem* (1)-(3) *is simple.*

Proof: Let λ be the eigenvalue of problem (1)-(3) and $\psi_\lambda(t)$ be the corresponding eigenfunction. It follows from the boundary condition (2) and the initial condition (4) that

$$W[\psi_\lambda, \varphi_\lambda](0) = \begin{vmatrix} \psi_\lambda(0) & \varphi_\lambda(0) \\ (\psi_\lambda)'(0) & (\varphi_\lambda)'(0) \end{vmatrix} = \begin{vmatrix} a\lambda + b & a\lambda + b \\ c\lambda + d & c\lambda + d \end{vmatrix} = 0.$$

Therefore, $\varphi_\lambda(t)$ and $\psi_\lambda(t)$ are linearly dependent on $[0,1]$, so $\varphi_\lambda(t)$ is also the eigenfunctions of problem (1)-(3).

3. Main results

Suppose that $q(t) \equiv 0$. Eq.(1) becomes $u''(t) + \lambda u(t) = 0$. Accordingly, the characteristic equation is

$$b(s) = \cos s - r\cos(s\xi) = 0. \tag{9}$$

Next, we only consider the case $|r| < 1$.

Lemma 3 *The roots of equation $b(s) = 0$ are all real.*

Proof: Let $s = x + iy (y \neq 0)$. By Euler formula, we get

$$\begin{aligned} \cos x \cosh y - r\cos(\xi x)\cosh(\xi y) = 0, \\ \sin x \sinh y + r\sin(\xi x)\sinh(\xi y) = 0. \end{aligned}$$

Then we calculate

$$\left(r\frac{\cosh(\xi y)}{\cosh y}\cos(\xi x)\right)^2 + \left(r\frac{\sinh(\xi y)}{\sinh y}\sin(\xi x)\right)^2 = 1.$$

Since

$$\left|r\frac{\cosh(\xi y)}{\cos hy}\right| < 1, \quad \left|r\frac{\sinh(\xi y)}{\sinh y}\right| < 1.$$

Thus we get a contradiction, which completes the proof.

Remark 1 *Using the intermediate value theorem and Lemma 3, equation $b(s) = 0$ has countable positive single roots m_k for $|r| < 1$. We see that the roots of equation $b(s) = 0$ is unique in the interval $((k-1)\pi, \pi)$ and it can be expressed as $m_k = m_k(r) = k\pi - f_k(r)$, where $0 < f_k(r) < \pi$. Using Rolle theorem, there exists $\hat{m}_k \in (m_k, m_{k+1})$ such that $b'(\hat{m}_k) = 0$, that is, the root of $\sin x - r\xi \sin(\xi x) = 0$.*

For the case of $q(t) \neq 0$, substituting $\varphi_\lambda(t)$ into (3), we get the characteristic function of problem (1)-(3)

$$\triangle(\lambda) = \varphi_\lambda(1) - r\varphi_\lambda(\xi).$$

Then λ is the eigenvalue of problem (1)-(3) if and only if $\triangle(\lambda) = 0$.

If $a = 0$,

$$\triangle_1(s) := \triangle(s) = sc\sin s - rsc\sin(s\xi) + O(e^{|y|}).$$

Now we introduce a function

$$A(s) := \frac{1}{sc}(\varphi_\lambda(1) - r\varphi_\lambda(\xi)) = \sin s - r\sin(s\xi) + O(|s|^{-1}e^{|y|}),$$

then the set of eigenvalues of problem (1)-(3) is identical with $\{\lambda : \lambda = s^2, scA(s) = \varphi_\lambda(1) - r\varphi_\lambda(\xi) = 0\}$. When $|r| < 1$, the asymptotic formulas of eigenvalues and eigenfunctions of problem (1)-(3) can be similarly referred to [31].

If $a \neq 0$,

$$\triangle_2(s) := \triangle(s) = as^2\cos s - ras^2\cos(s\xi) + O(|s|e^{|y|}),$$

next, we introduce a function

$$B(s) := \frac{1}{as^2}(\varphi_\lambda(1) - r\varphi_\lambda(\xi)) = \cos s - r\cos(s\xi) + O(|s|^{-1}e^{|y|}),$$

we see that $B(s)$ is the analytic function of s and $\triangle_2(0) = 0$. Since we consider the asymptotic properties of eigenvalues, next we only consider the zeros of $B(s) = 0$. For convenience, we write $B(s)$ as follows:

$$B(s) = b(s) + b_0(s) = \cos s - r\cos(s\xi) + O(|s|^{-1}e^{|y|}), \tag{10}$$

where $b(s) = \cos s - r\cos(s\xi)$, $b_0(s) = O(|s|^{-1}e^{|y|})$.

Theorem 2 *The real eigenvalues of the problem (1)-(3) are bounded below.*

Proof: Suppose $\hat{B}(\lambda) = B(s)$ and $s = iy$, $y > 0$.

$$
\begin{aligned}
\hat{B}(-y^2) &= \frac{e^{is}+e^{-is}}{2} - r\frac{e^{is\xi}+e^{-is\xi}}{2} + O(|s|^{-1}e^{|y|}) \\
&= \frac{e^{-y}+e^{y}}{2} - r\frac{e^{-y\xi}+e^{y\xi}}{2} + O(y^{-1}e^{|y|}) \\
&= \frac{e^{y}(1+e^{-2y}-re^{-(\xi+1)y}-re^{(\xi-1)y})}{2} + O(y^{-1}e^{|y|}).
\end{aligned}
$$

Since $0 < \xi < 1$, we have

$$\lim_{y\to\infty} \hat{B}(-y^2) = \infty.$$

Notice that there exists a y_0 such that $\hat{B}(-y^2) \neq 0$ for $y > y_0$, i.e. if $\lambda < -y_0^2$, then $\hat{B}(\lambda) \neq 0$. Hence $\lambda > -y_0^2$

Corollary 1 *The number of negative eigenvalues of problem* (1)-(3) *is finite.*

Theorem 3 *Problem* (1)-(3) *has countable positive eigenvalues.*

Proof: Let $s = x$, $0 < x \in \mathbb{R}$. Note that, for k large enough, since $|r| < 1$, we get

$$|r\cos(s\xi) + O(x^{-1})| < 1.$$

We know that $\cos x$ takes the local maximum at $2(k-1)\pi$ and the local minimum at $(2k-1)\pi$. According to the intermediate value theorem, equation $B(s) = 0$ has at least one root in every interval $((k-1)\pi, k\pi)$ for k large enough, so equation $B(s) = 0$ has countable roots. Therefore, problem (1)-(3) has countable positive eigenvalues.

Lemma 4 *For* $0 < \xi < 1, \beta \geq 0$. *If* $\cos x - r\xi^{\beta}\cos(\xi x) = 0$, *then there exists* $k > 0$ *such that* $|\sin x| - |r||\sin(\xi x)| \geq \kappa > 0$.

Proof: Suppose $0 < v < 1$. If $\alpha := \sqrt{1-v^2}$, then $0 < \alpha < 1$. Let consider several possible cases.
(i) For the case of $\cos x = 0$, then $|\sin x| = 1$, we get

$$|\sin x| - |r||\sin(\xi x)| \geq 1 - |r| =: \kappa_1 > 0.$$

(ii) For the case of $\cos x \neq 0$, then $\cos(\xi x) \neq 0$ and $r \neq 0$, so $0 < |\cos x/\cos(\xi x)| = |r|\xi^{\beta} = 1 - \delta$, where $\delta := 1 - |r|\xi^{\beta}, 0 < \delta < 1$. Then we get $\cos^2 x = \cos^2(\xi x)(1-\delta)^2$, correspondingly,

$$\sin^2 x = \sin^2(\xi x) + \delta(2-\delta)\cos^2(\xi x) \geq \sin^2(\xi x) + \delta\cos^2(\xi x), \tag{11}$$

therefore, $|\sin x| > |\sin(\xi x)|$ and

$$0 < |\sin x| - |\sin(\xi x)| \leq |\sin x| - |r||\sin(\xi x)|. \tag{12}$$

① If $|\cos x| \geq v$, suppose $\kappa_2 := \frac{\delta v^2}{3} > 0$, since $0 < \kappa_2 < 1$ and $|\cos(\xi x)| > |\cos x|$, then

$$\delta\cos^2(\xi x) > \delta\cos^2 x \geq \delta v^2 = 3\kappa_2 \geq 2\kappa_2 + \kappa_2^2 \geq 2\kappa_2|\sin(\xi x)| + \kappa_2^2,$$

and from (11)-(12), we obtain

$$\sin^2 x \geq \sin^2(\xi x) + 2\kappa_2|\sin(\xi x)| + \kappa_2^2 = (|\sin(\xi x)| + \kappa_2)^2,$$

thus $|\sin x| - |\sin(\xi x)| \geq \kappa_2 > 0$, it follows from (12) that

$$|\sin x| - |r||\sin(\xi x)| \geq \kappa_2 > 0.$$

② If $0 < |\cos x| < v$, suppose $\kappa_3 := (1-|r|)\alpha$, by (12), we get

$$|\sin x| - |r||\sin(\xi x)| \geq (1-|r|)|\sin x| \geq \kappa_3 > 0.$$

Therefore, taking $\kappa = \min\{\kappa_1, \kappa_2, \kappa_3\} > 0$, we have $|\sin x| - |r||\sin(\xi x)| \geq \kappa > 0$.
As $|\sin m_k - r\xi\sin(\xi m_k)| \geq |\sin m_k| - |r||\sin(\xi m_k)|$, we can get the following corollary.

Corollary 2 *If* m_k *is the root of Eq.*(9), *then there exists* $\kappa > 0$ *such that* $|\sin m_k - r\xi\sin(\xi m_k)| \geq \kappa > 0$ *for* $k \in N$.

Lemma 5 *Suppose* $0 < \xi < 1, \beta \geq 0$. *If* $\sin x - r\xi^{\beta}\sin(\xi x) = 0$, *then there exists* $\hat{\kappa} > 0$ *such that* $|\cos x| - |r||\cos(\xi x)| \geq \hat{\kappa} > 0$. *(if* $\beta = \infty$, *i.e.*$\xi^{\beta} = 0$, *the conclusion still holds.)*

Corollary 3 *If* $\hat{m}_k$ *is the root of* $\sin x - r\xi^{\beta}\sin(\xi x) = 0$, *then there exists* $\hat{\kappa} > 0$ *such that* $|\cos\hat{m}_k - r\cos(\xi\hat{m}_k)| \geq \hat{\kappa} > 0$.

Corollary 4 *Suppose* $x = b_k = k\pi, k \in N$ *(in this case* $\sin b_k = 0$*), then there exists* $\hat{\kappa} > 0$ *such that* $\cos|b_k| - |r||\cos(\xi b_k)| \geq \hat{\kappa} > 0$.

Let $D_k = \{s : |x| \leq b_k = k\pi,\ |y| \leq b_k\}$, $D_k^s = D_k \cap \mathbb{C}_s$. Define a counter $\Gamma_k^s = \partial D_k \cap \mathbb{C}_s$, the corresponding counter Γ_k^{λ} is simple closed curve in the plane $\mathbb{C}_{\lambda}$, where $\lambda = s^2$ is the bijection from $\mathbb{C}_s$ to $\mathbb{C}_{\lambda}$.

Lemma 6 *If* $|r| < 1$, *then there exists* $M > 0$ *such that all eigenvalues of problem* (1)-(3) *are positive in the domain* $\{s \in \mathbb{C}_s : |s| > M\}$.

Proof: If $s = b_k + iy, y \in [-b_k, b_k]$. It follows from (12) that

$$Reb(s) = \cos b_k \cosh y - r\cos(\xi b_k)\cosh(\xi y),$$

then

$$\begin{aligned}|b(s)| \geq |Reb(s)| \geq& |\cos b_k|\cosh y - |r||\cos(\xi b_k)|\cosh(\xi y)\\ \geq&(|\cos b_k| - |r||\cos(\xi(b_k)|)\cosh y,\end{aligned}$$

From Corollary 4, we obtain $|b(s)| \geq \hat{\kappa}\cosh y \geq B_1 e^{|y|}$, where $B_1 > 0$.
If $y = \pm a_k$, $x \in [0, b_k]$. Since

$$|\cos s| = \sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} = \sqrt{\cos^2 x + \sinh^2 y} = \sqrt{\cosh^2 y - \sin^2 x},$$

we get

$$|\cos s| \geq |\sinh y|, \quad |\cos(\xi s)| \leq \cosh(\xi y),$$

so

$$|b(s)| \geq |\sinh y| - |r|\cosh(\xi y) \geq |\sinh y| - \cosh(\xi y).$$

Let consider $h(y) := (|\sinh y| - \cosh(\xi y))e^{-|y|}$. It can be found that there exists $y^*(\xi) > 0$ such that $h(y) > 1/4$ for $|y| > y^*$ by analyzing the function $h(y)$, so $|b(s)| \geq \frac{e^{|y|}}{4}$. Taking $B = \min\{B_1, 1/4\}$, we have $|b(s)| \geq Be^{|y|}$ for k large enough. And from (9), $|b_0(s)| \leq c_1|s|^{-1}e^{|y|} < Be^{|y|} \leq |b(s)|$. According to Rouche's theorem, equation $b(s) = 0$ and $B(s) = b(s) + b_0(s) = 0$ have the same number of zeros inside Γ_k. It has been discussed that $b(s) = 0$ has only one root between counters Γ_{k-1} and Γ_k, combined with Theorem 3, the roots of $B(s) = 0$ in this region are positive.

Let s_k be the root of $B(s) = 0$, according to Lemma **??**, s_k is positive for k large enough. Next, we only consider the case $s = x > 0$, so

$$B(s) = \cos s - r\cos(\xi x) + O(s^{-1}). \tag{13}$$

Since $s_k, m_k \in ((k-1)\pi, k\pi)$, and when $k \to \infty$, s_k, m_k and $k\pi$ are equivalently infinite quantities. Define $\varepsilon_k := s_k - m_k$, it is obvious that $\varepsilon_k = o(1)$. Since eigenvalues are real, from (5), we get

$$\varphi_{\lambda}(t) = as^2\cos(st) + O(|s|). \tag{14}$$

Theorem 4 *Suppose* $q \in C[0, 1]$ *and* $|r| < 1$, *then the asymptotic formulas of eigenvalues and eigenfunctions of problem* (1)-(3) *have the form*

$$s_k = m_k + O(k^{-1}), \quad u_k(t) = am_k^2\cos(m_k t) + O(k)$$

for k *large enough, respectively.*

Proof: Substituting $s_k = m_k + \varepsilon_k$ into (13), we obtain

$$\cos m_k(1 + O(\varepsilon_k^2)) - \varepsilon_k\sin m_k - r\cos(\xi m_k)(1 + O(\varepsilon_k^2)) + r\xi\varepsilon_k\sin(\xi m_k) = O(k^{-1}),$$

$$\cos m_k - r\cos(\xi m_k) - \varepsilon_k(\sin m_k - r\xi\sin(\xi m_k)) + O(\varepsilon_k^2) = O(k^{-1}).$$

Since m_k is the root of $b(s) = 0$, i.e. $\cos m_k - r\cos(\xi m_k) = 0$, we get

$$[\sin m_k - r\xi\sin(\xi m_k) + O(\varepsilon_k)]\,(-\varepsilon_k) = O(k^{-1}),$$

which implies that $\varepsilon_k = O(k^{-1})$.
Substituting $s_k = m_k + \varepsilon_k$ into (14), we get

$$\begin{aligned} u_k(t) = \varphi_{\lambda_k}(t) =& a(m_k+\varepsilon_k)^2 \cos((m_k+\varepsilon_k)t) + O(k) \\ =& (am_k^2 + 2am_k\varepsilon_k + a\varepsilon_k^2)\left[\cos(m_k t)(1+O(\varepsilon_k^2)) - \sin(m_k t)(\varepsilon_k t)\right] + O(k) \\ =& am_k^2 \cos(m_k t) + O(k). \end{aligned}$$

Next, we will normalize $u_k(t)$:

$$\begin{aligned} \alpha_k^2 = \int_0^1 u_k^2 \mathrm{d}t =& \int_0^1 \left[a^2 m_k^4 \cos^2(m_k t) + O(k^3)\right] \mathrm{d}t \\ =& \frac{a^2 m_k^4}{2}\left(1 + O(\frac{1}{k})\right), \end{aligned}$$

$$\frac{1}{\alpha_k} = \frac{\sqrt{2}}{|a| m_k^2} + O(k^{-3}).$$

Therefore, the normalized eigenfunctions have asymptotic formulas:

$$\begin{aligned} v_k(t) =& \left(\frac{\sqrt{2}}{|a|m_k^2} + O(k^{-3})\right)\left(am_k^2 \cos(m_k t) + O(k)\right) \\ =& \frac{\sqrt{2}}{|a|m_k^2} am_k^2 \cos(m_k t) + O(k^{-1}). \end{aligned}$$

If $a > 0$, then $v_k(t) = \sqrt{2}\cos(m_k t) + O(k^{-1})$; If $a < 0$, then $v_k(t) = -\sqrt{2}\cos(m_k t) + O(k^{-1})$.

In order to obtain more exact asymptotic formulas of eigenvalues and eigenfunctions, we assume that $q \in C^1[0,1]$, then the following formulas hold.

$$\int_0^t q(\tau)\cos(2s\tau)\mathrm{d}\tau = O(s^{-1}), \quad \int_0^t q(\tau)\sin(2s\tau)\mathrm{d}\tau = O(s^{-1}).$$

Suppose $Q(t) = \frac{1}{2}\int_0^t q(\tau)\mathrm{d}\tau$, $Q(t)$ is obviously bounded. Substituting (14) into (7), we get

$$\varphi_\lambda(t) = (as^2+b)\cos(st) + \frac{cs^2+d}{s}\sin(st) + \frac{1}{s}\int_0^t \sin(s(t-\tau))q(\tau)(as^2\cos(s\tau) + O(|s|))\mathrm{d}\tau,$$

where

$$\begin{aligned} & \frac{1}{s}\int_0^t \sin(s(t-\tau))q(\tau)as^2\cos(s\tau)\mathrm{d}\tau \\ =& as\sin(st)\int_0^t \cos^2(s\tau)q(\tau)\mathrm{d}\tau - \frac{1}{2}as\cos(st)\int_0^t \sin(2s\tau)q(\tau)\mathrm{d}\tau \\ =& as\sin(st)Q(t) + O(1), \end{aligned}$$

$$\frac{1}{s}\int_0^t \sin(s(t-\tau))q(\tau)O(|s|)\mathrm{d}\tau = O(1).$$

So we get

$$\varphi_\lambda(t) = (as^2+b)\cos(st) + \frac{cs^2+d}{s}\sin(st) + as\sin(st)Q(t) + O(1), \tag{15}$$

then

$$B(s) = \cos s - r\cos(s\xi) + \frac{Q(1)\sin s - rQ(\xi)\sin(s\xi)}{s} + O(s^{-1}). \tag{16}$$

Next we define

$$F_1(r,\xi,s) := \frac{Q(1)\sin s - rQ(\xi)\sin(s\xi)}{\sin s - r\xi\sin(s\xi)}.$$

Theorem 5 *If $q \in C^1[0,1]$ and $|r| < 1$, then the asymptotic formulas of eigenvalues and eigenfunctions of problem (1)-(3) have the form*

$$s_k = m_k + F_1(r,\xi,m_k)m_k^{-1} + O(k^{-1}), \tag{17}$$

$$u_k(t) = (am_k^2+b)\cos(m_k t) + \left(\frac{cm_k^2 + d + am_k^2 Q(t)}{m_k} - am_k t F_1\right)\sin(m_k t) + O(k) \tag{18}$$

for k large enough, respectively.

Proof: Substituting $s_k = m_k + \varepsilon_k$ into (16), we get

$$\cos m_k - r\cos(\xi m_k) + \frac{Q(1)\sin m_k - rQ(\xi)r\sin(\xi m_k)}{m_k}$$
$$- \left[(\sin m_k - r\xi\sin(\xi m_k) - (Q(1)\cos(m_k) - rQ(\xi)\xi\cos(\xi m_k))m_k^{-1}\right]\varepsilon_k = O(k^{-1}),$$

since $\cos m_k - r\cos(\xi m_k) = 0$, we have

$$-(\sin m_k - r\xi\sin(\xi m_k) + O(k^{-1}))\varepsilon_k = -\frac{Q(1)\sin m_k - Q(\xi)r\sin(m_k)}{m_k} + O(k^{-1}),$$

or

$$\varepsilon_k = \frac{F_1(r,\xi,m_k)}{m_k} + O(k^{-1}).$$

Substituting $s_k = m_k + \varepsilon_k$ into (15), we have

$$\begin{aligned} u_k(t) =& (am_k^2 + b)\cos(m_k t) + \left(\frac{cm_k^2 + d}{m_k}\right)\sin(m_k t) \\ &+ am_k\sin(m_k t)Q(t) - am_k^2\varepsilon_k t\sin(m_k t) + O(1), \end{aligned}$$

since $\varepsilon_k = \frac{F_1(r,\xi,m_k)}{m_k} + O(k^{-1})$, we obtain (18).

4. Conclusion

This paper studied the nonlocal boundary value problem with eigenparameter dependent boundary conditions, the general properties of the eigenvalues and eigenfunctions for such a problem were proved. Finally, we give the asymptotic formulas of eigenvalues and eigenfunctions for second-order differential operator with eigenparameter dependent boundary conditions.

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