RESEARCH

A cubic B-spline finite element method for a class of fourth order nonlinear differential equation with variable coefficient

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Abstract

In this paper, the cubic B-spline element method is proposed for a class of fourth order nonlinear parabolic problem with variable coefficient. We prove the boundness of the approximate solutions of the semi-discrete and fully discrete finite element schemes. The boundness is the basis of error analysis of nonlinear parabolic problem, especially in the case of fourth order term with variable coefficient. The error estimates are discussed by constructing the energy functional in L^2 norm and H^2 norm. Numerical results confirm our results of theoretical analysis.

Keywords: cubic B-spline finite element method; nonlinear parabolic equation; variable coefficient; boundness; error estimate

1 Introduction

Higher order nonlinear parabolic equations play an important role in natural science. As a typical nonlinear differential equation, the extended Fisher-Kolmogorov (EFK) equation can describe many phenomena which include traveling waves in reaction-diffusion systems [1, 2], propagation of domain walls in liquid crystals [3], and brain tumors dynamics [4]. The EFK equation is assumed in the following form:

$$u_t + \gamma\Delta^2 u - \Delta u + u^3 - u = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

where γ is a positive constant, and Ω is a bounded domain with boundary $\partial\Omega$. In concrete applications, researchers use the EFK equation with the initial-boundary conditions. If $\gamma = 0$, (1) turns into the standard Fisher-Kolmogorov equation [5, 6]. The fourth order derivative term was added to the Fisher-Kolmogorov equation by Dee and van Saarloos [6], Coullet et all [7], and van Saarloos [8].

The finite element method (FEM) is effective in solving partial differential equations [9–11]. Some papers, which have already been published, study the Cahn-Hilliard equations using various different forms of FEM [14–18]. In [14, 15], Qin et al. considered two different fourth order nonlinear parabolic problems with variable coefficient employing B-spline FEM respectively, and the boundness and the error estimates of the approximate solutions were proved. The nonlinear term and the fully discrete scheme in [14] were different from that of [15]. Bao et al. conducted numerical experiments to study the effect of a precursor fluid layer on the motion of two phase system in a channel [16]. The researches involved the solution of Cahn-Hillard equation with semi-implicit and mixed finite element discretization with a convex splitting scheme. Feng considered some fully discrete finite element methods for a parabolic system consisting of the Navier-Stokes equation and the Cahn-Hilliard equation in [17]. Qiao et al. proposed a mixed finite element method with Crank-Nicolson time-stepping for simulating the molecular beam epitaxy model and discussed the error analysis [18].

We also want to talk about the development and application of B-splines. In 1946, Schoenberg considered polynomial approximations and first introduced the B-spline method [19]. In [19, 20], the theoretical basis of univariate B-spline functions were studied. In 1976, C. de Boor defined the multivariate B-spline functions [21]. B-splines were widely used in scientific computing and engineering applications [22–25]. For instance, B-splines were often used as the basis functions of FEM [26–31]. Significantly, compared with Lagrange and Hermite type elements, the number of B-spline basis functions is halved for the same boundary value problem. Then the scale of matrix from B-spline FEM is smaller than that from Lagrange and Hermite elements. Moreover, B-spline shape functions are smoother. For example, cubic B-splines are in $C^2(-\infty,+\infty)$. However, the B-spline basis functions need to be modified to deal with boundary conditions [14, 15].

In this paper, we apply the cubic B-spline FEM to solve a class of fourth order nonlinear parabolic equation with variable coefficient. In section 2 of this paper, we introduce the model and some basic preliminaries. In section 3, we show the boundness and error estimates for the semi-discrete scheme. In section 4, a fully discrete scheme based on the backward Euler method is studied. In section 5, a numerical experiment is provided to confirm theoretical results.

In this work, we denote L^2, L^k, L^∞, H^k norms in I by $\|\cdot\|$, $\|\cdot\|_{L^k}$, $|\cdot|_\infty$, and $\|\cdot\|_k$, respectively.

2 Some preliminaries

We consider the following fourth order nonlinear parabolic problem:

$$\begin{cases} u_t + (\alpha(x,t)u_{xx})_{xx} - u_{xx} + u^3 - u = 0, \quad (x,t)\in I\times(0,T], \\ u(x,t) = u_x(x,t) = 0, \quad x\in\partial I, \quad t\in(0,T], \\ u(x,0) = u_0(x), \quad x\in I, \end{cases} \tag{2}$$

where $I=[0,1]$ and $u_t = \frac{\partial u}{\partial t}$. We propose the following three assumptions:

$$\alpha(x,t), \frac{\partial\alpha}{\partial t}(x,t) \in C(I\times[0,T]),$$

$$0 < s \le \alpha(x,t) \le S < +\infty, \quad \forall x\in I, \quad t\in[0,T], \tag{3}$$

$$0 \le \left|\frac{\partial\alpha}{\partial t}\right| \le M_1, \quad 0 \le \left|\frac{\partial^2\alpha}{\partial t^2}\right| \le M_2, \quad \forall x\in I, \quad t\in[0,T]. \tag{4}$$

There are four boundary conditions for the equation (2), e.g. two boundary conditions at $x = 0, 1$. Notice that the essential boundary conditions are $u(0,t) =$

$u(1,t) = u_x(0,t) = u_x(1,t) = 0$ in (2). Then we define the function space as follows:

$$H_0^2(I) = \{w; w \in H^2(I), w(0,t) = w(1,t) = w_x(0,t) = w_x(1,t) = 0\}.$$

The variational problem related to (2) is: Find $u = u(\cdot,t) \in H_0^2(I)(0 \le t \le T)$ such that

$$\begin{cases} (u_t, v) + (\alpha(x,t)D^2u, D^2v) + (Du, Dv) + (u^3 - u, v) = 0, \quad \forall v \in H_0^2(I), \\ u(x,0) = u_0(x), \quad x \in I, \end{cases} \tag{5}$$

where $Du = \frac{\partial u}{\partial x}$. We give the existence of the solution of problem (2) in the following theorem [9].

Theorem 2.1 *Suppose that $u_0 \in H_0^2(I)$, then there exists a unique global solution $u(x,t)$ for problem (2), such that*

$$u \in L^\infty([0,T]; H_0^2(I)) \cap L^2([0,T]; H^4(I)), \quad u_t \in L^2([0,T]; L^2(I)).$$

Throughout this paper, the letters C and C' denote generic constants independent of the division size not necessarily the same at different occurrences.

3 Semi-discrete approximation

The interval I is partitioned into M equal finite elements by $I_h : 0 = x_0 < x_1 < \cdots < x_M = 1$ such that $h = x_i - x_{x-1}$, $I_i = [x_{i-1}, x_i]$. Assume that I_h is shape-regular, that is, there exists a positive constant ρ such that

$$\rho h \le h_i \le h, \quad 1 \le i \le M.$$

By the affine transformation, the cubic B-spline functions with knots x_j are described as

$$\phi_j(x) = \begin{cases} \frac{1}{6}(\frac{x-x_j}{h})^3, & x \in [x_j, x_{j+1}], \\ -\frac{1}{2}(\frac{x-x_j}{h})^3 + 2(\frac{x-x_j}{h})^2 - 2(\frac{x-x_j}{h}) + \frac{2}{3}, & x \in [x_{j+1}, x_{j+2}], \\ \frac{1}{2}(\frac{x-x_j}{h})^3 - 4(\frac{x-x_j}{h})^2 + 10(\frac{x-x_j}{h}) - \frac{22}{3}, & x \in [x_{j+2}, x_{j+3}], \\ -\frac{1}{6}(\frac{x-x_j}{h} - 4)^3, & x \in [x_{j+3}, x_{j+4}], \\ 0, & otherwise. \end{cases}$$

Let the set of $\{\phi_{-3}, \phi_{-2}, \phi_{-1}, \phi_0, \cdots, \phi_{M-3}, \phi_{M-2}, \phi_{M-1}\}$ be the basis functions of FEM. In order to deal with the boundary conditions, we modify the boundary B-spline basis functions according to [14, 15]. The approximate solution can be written as follows:

$$u_h(x,t) = \sum_{j=-3}^{M-1} \delta_j(t)\phi_j(x),$$

where $\delta_j(t)$ are time dependent parameters.

The semi-discrete finite element scheme based on B-splines for problem (2) is: Find $u_h = u_h(\cdot, t) \in U_h (0 < t \leq T)$, such that

$$\begin{cases} (u_{h,t}, v_h) + (\alpha(x,t)D^2 u_h, D^2 v_h) + (Du_h, Dv_h) + (u_h^3 - u_h, v_h) = 0, \ v_h \in U_h, \\ (u_h(0) - u_0, v_h) = 0, \ v_h \in U_h. \end{cases} \tag{6}$$

To analyze the boundness and convergence of the B-spline FEM, we need to introduce the elliptic projection $R_h : u \to R_h u \in U_h$ which is defined by [9]

$$a(u - R_h u, v_h) \equiv (\alpha(x,t)D^2(u - R_h u), D^2 v_h) = 0, \quad \forall v_h \in U_h. \tag{7}$$

Lemma 3.1 *It then follows* (7) *that*

$$a(u, u) \geq C_0 \|u\|_2^2, \quad \forall u \in H_0^2(I), \tag{8}$$

where C_0 is a positive constant depending only on $\alpha(x,t)$. Hence, $a(u,v)$ is a symmetrical positive determined bilinear form, and

$$\|u - R_h u\| + h\|u - R_h u\|_1 + h^2 \|u - R_h u\|_2 \leq Ch^4 \|u\|_4. \tag{9}$$

First, we discuss the existence of the approximate solution of the semi-discrete scheme.

Theorem 3.1 *Let $u_h(0) \in H_0^2(I)$, then there exists a unique approximation solution $u_h(t) \in U_h$ for problem* (6)*, such that*

$$\|u_h(t)\|_2 \leq C\|u_h(0)\|_2, \quad 0 \leq t \leq T, \tag{10}$$

and

$$\int_0^t \|D^2 u_h\|^2 dt \leq C\|u_h(0)\|_2^2. \tag{11}$$

where C is a positive constant depending on $\alpha(x,t)$, μ and T, independent of mesh size h.

Proof According to ordinary differential equation theory, there exists a unique local solution to problem (6) in the interval $[0, t_n)$. If we have (10), then according to the extension theorem, we can also obtain the existence of unique global solution. So, we only need to prove (10).

Setting $v_h = u_h$ in (6), we get

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + s\|D^2 u_h\|^2 + \|Du_h\|^2 + \|u_h\|_{L^4}^4 \leq \|u_h\|^2. \tag{12}$$

We derive that

$$\frac{d}{dt}\left(e^{-2t}\|u_h\|^2\right) \leq 0. \tag{13}$$

Integrating (13) with respect to t, we have

$$\|u_h(t)\|^2 \le e^{2t}\|u_h(0)\|^2 \le e^{2T}\|u_h(0)\|^2 \le C\|u_h(0)\|^2, \quad 0 \le t \le T. \tag{14}$$

By (12) and (14), it is easy to see that

$$\int_0^t \|D^2 u_h\|^2 dt \le C\|u_h(0)\|^2.$$

Letting $v_h = u_{h,t}$ in (6), we get

$$\|u_{h,t}\|^2 + (\alpha(x,t)D^2 u_h, D^2 u_{h,t}) + (Du_h, Du_{h,t}) + (u_h^3 - u_h, u_{h,t}) = 0. \tag{15}$$

We introduce the following energy function

$$E(w) = \frac{1}{2}(\alpha(x,t)D^2 w, D^2 w) + \frac{1}{2}\|Dw\|^2 + \frac{1}{4}((1-w^2)^2, 1). \tag{16}$$

Then we have

$$\|u_{h,t}\|^2 + \frac{d}{dt}E(u_h) - \frac{1}{2}(\frac{\partial \alpha}{\partial t}D^2 u_h, D^2 u_h) = 0.$$

By (4), we obtain

$$\frac{d}{dt}E(u_h) \le \frac{M_1}{2}\|D^2 u_h\|^2. \tag{17}$$

Integrating (17) with respect to t, we have

$$E(u_h) - E(u_h(0)) \le \frac{M_1}{2}\int_0^t \|D^2 u_h\|^2 dt \le C\|u_h(0)\|^2. \tag{18}$$

According to (3), (16) and (18), we have

$$\begin{aligned} &s\|D^2 u_h\| + \|Du_h\|^2 + \frac{1}{2}\|u_h\|_{L^4}^4 - \|u_h\|^2 \\ \le &S\|D^2 u_h(0)\| + \|Du_h(0)\|^2 + \frac{1}{2}\|u_h(0)\|_{L^4}^4 + C\|u_h(0)\|^2 - \|u_h(0)\|^2. \end{aligned}$$

Then

$$\begin{aligned} &s\|D^2 u_h\| + \|Du_h\|^2 + \frac{1}{2}\|u_h\|_{L^4}^4 \\ \le &S\|D^2 u_h(0)\| + \|Du_h(0)\|^2 + \frac{1}{2}\|u_h(0)\|_{L^4}^4 + C\|u_h(0)\|^2. \end{aligned}$$

We have

$$\|D^2 u_h\| \le C\|D^2 u_h(0)\|. \tag{19}$$

We know that

$$\|Du_h\|^2 = -(D^2u_h, u_h) \leq \frac{1}{2}\|D^2u_h\|^2 + \frac{1}{2}\|u_h\|^2.$$

Thus we obtain the boundness of the approximate solution in H^2 norm. So, (10) and (11) hold.

Now, we give the error estimates between the exact solution and the approximate solution of the FEM in L^2 norm and H^2 semi-norm .

Theorem 3.2 *Let u be the solution of* (5), *u_h be the solution of* (6), $u(0) \in H^4(I)$, $u, u_t \in L^2(0,T;H^4(I))$, *and the initial value satisfies*

$$\|u(0) - u_h(0)\| \leq Ch^4\|u(0)\|_4. \tag{20}$$

As $0 \leq t \leq T$, we have the following error estimate

$$\|u - u_h\| \leq Ch^4\bigg(\|u(0)\|_4^2 + \int_0^t (\|u(\tau)\|_4^2 + \|u_t(\tau)\|_4^2)d\tau\bigg)^{\frac{1}{2}}. \tag{21}$$

Proof To describe the error estimate for the semi-discrete B-spline FEM, denote $\theta(t) = R_hu - u_h$, $\rho(t) = u - R_hu$, then

$$u - u_h = u - R_hu + R_hu - u_h = \theta(t) + \rho(t).$$

Therefore

$$\|u - u_h\| \leq \|\theta(t)\| + \|\rho(t)\|.$$

By (5)-(7), we know

$$\begin{aligned}&(\theta_t + \rho_t, v_h) + (\alpha(x,t)D^2\theta, D^2v_h) + (D\theta + D\rho, Dv_h)\\ &+ (u^3 - u_h^3, v_h) - (\theta + \rho, v_h) = 0.\end{aligned} \tag{22}$$

Setting $v_h = \theta$ in (22), we have

$$\begin{aligned}&(\theta_t, \theta) + (\alpha(x,t)D^2\theta, D^2\theta) + (D\theta, D\theta)\\ =&- (\rho_t, \theta) + (\rho, D^2\theta) - (u^3 - u_h^3, \theta) + (\theta + \rho, \theta).\end{aligned}$$

Using (3) and the Cauchy's inequality, we can deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + s\|D^2\theta\|^2 + \|D\theta\|^2\\ \leq&\frac{1}{2}\|\rho_t\|^2 + (\frac{1}{2s} + \frac{1}{2})\|\rho\|^2 + \frac{s}{2}\|D^2\theta\|^2 + \frac{1}{2}\|u^3 - u_h^3\|^2 + \frac{5}{2}\|\theta\|^2.\end{aligned}$$

Based on the Sobolev's space embedding theorem, notice that

$$\begin{aligned}&\|u^3 - u_h^3\| \leq |u^2 + uu_h + u_h^2|_\infty\|u - u_h\|\\ \leq&(|u|_\infty^2 + |u|_\infty|u_h|_\infty + |u_h^2|_\infty)\|u - u_h\|\\ \leq&C\|u - u_h\| \leq C(\|\theta\| + \|\rho\|).\end{aligned} \tag{23}$$

Hence

$$\frac{d}{dt}\|\theta\|^2 + s\|D^2\theta\|^2 \leq \|\rho_t\|^2 + C(\|\rho\|^2 + \|\theta\|^2). \tag{24}$$

By the Gronwall's inequality, we obtain

$$\|\theta\|^2 \leq C\big[\|\theta(0)\|^2 + \int_0^t (\|\rho_t\|^2 + \|\rho\|^2)d\tau\big]. \tag{25}$$

We know

$$\|\theta(0)\| = \|u(0) - u_h(0) + R_h u(0) - u(0)\| \leq \|u(0) - u_h(0)\| + \|\rho(0)\|. \tag{26}$$

Hence, when $0 \leq t \leq T$, it follows from (20) and (25)-(26) that (21) is obtained. This completes the proof of the theorem.

Theorem 3.3 *Let u be the solution of* (5), *u_h be the solution of* (6), $u(0) \in H^4(I)$, $u, u_t \in L^2(0,T;H^4(I))$, *and the initial value satisfies*

$$|u(0) - u_h(0)|_2 \leq Ch^2\|u(0)\|_4. \tag{27}$$

Then, we have the following error estimate

$$|u(t) - u_h(t)|_2 \leq Ch^2 \left[\|u(0)\|_4 + \left(\int_0^t (\|u(\tau)\|_4^2 + h^2\|u_t(\tau)\|_4^2)d\tau\right)^{\frac{1}{2}}\right]. \tag{28}$$

Proof Letting $v_h = \theta_t$ in (22), we have

$$\begin{aligned}
&\|\theta_t\|^2 + (\alpha(x,t)D^2\theta, D^2\theta_t) + (D\theta, D\theta_t) \\
=& -(\rho_t, \theta_t) + (D\rho, D\theta_t) - (u^3 - u_h^3, \theta_t) + (u - u_h, \theta_t),
\end{aligned}$$

where

$$(\alpha(x,t)D^2\theta, D^2\theta_t) = \frac{1}{2}\frac{d}{dt}(\alpha(x,t)D^2\theta, D^2\theta) - \frac{1}{2}(\frac{\partial\alpha}{\partial t}D^2\theta, D^2\theta).$$

It is easy to see

$$\begin{aligned}
&\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}(\alpha(x,t)D^2\theta, D^2\theta) \\
=&\frac{1}{2}(\frac{\partial\alpha}{\partial t}D^2\theta, D^2\theta) + (D^2\theta, \theta_t) - (\rho_t, \theta_t) - (D^2\rho, \theta_t) - (u^3 - u_h^3, \theta_t) + (u - u_h, \theta_t).
\end{aligned}$$

Based on (23) and the ε-inequality, we have

$$\begin{aligned}
&\|\theta_t\|^2 + \frac{s}{2}\frac{d}{dt}\|D^2\theta\|^2 \\
\leq&(4 + \frac{M}{2})\|D^2\theta\|^2 + 4(\|\rho_t\|^2 + \|D^2\rho\|^2 + \|u^3 - u_h^3\|^2 + \|u - u_h\|^2) + \frac{5}{16}\|\theta_t\|^2 \\
\leq&(4 + \frac{M}{2})\|D^2\theta\|^2 + 4(\|\rho_t\|^2 + \|D^2\rho\|^2) + C(\|\theta\|^2 + \|\rho\|^2) + \frac{1}{2}\|\theta_t\|^2.
\end{aligned}$$

Hence, we get

$$\|\theta_t\|^2 + s\frac{d}{dt}\|D^2\theta\|^2 \le C(\|D^2\theta\|^2 + \|\theta\|^2\| + \|\rho_t\|^2 + \|\rho\|^2 + \|D^2\rho\|^2). \tag{29}$$

Integrating (29) respect with to t, we have

$$\|D^2\theta\|^2 \le C\bigg(\|D^2\theta(0)\|^2 + \int_0^t (\|\theta\|^2 + \|\rho_t\|^2 + \|\rho\|_2^2)d\tau\bigg). \tag{30}$$

By the triangle inequality, it is obvious that

$$\|D^2\theta(0)\| \le \|D^2u(0) - D^2u_h(0)\| + \|D^2R_hu(0) - D^2u(0)\|. \tag{31}$$

For the reasons given above (27) and (30)-(31), we obtain (28). The error estimate in H^2 semi-norm is proven.

4 Fully discrete finite element scheme

In this section, we construct the fully discrete finite element scheme using the Crank-Nicolson type and discuss the convergence of the fully discrete scheme.

First, we define the double well potential function

$$H(u_h^n) = \frac{1}{4}(1 - |u_h^n|^2)^2, \tag{32}$$

where $H'(u_h) = |u_h|^2u_h - u_h$.

The Crank-Nicolson scheme for problem (2) is: Find $u_h^n \in U_h (n = 1, 2, \cdots, N)$ such that

$$\begin{cases} (\partial_t u_h^n, v_h) + (\alpha^{n-\frac{1}{2}} D^2 u_h^{n-\frac{1}{2}}, D^2 v_h) + (Du_h^{n-\frac{1}{2}}, Dv_h) \\ \quad + \left(\dfrac{H(u_h^n) - H(u_h^{n-1})}{u_h^n - u_h^{n-1}}, v_h\right) = 0, \quad \forall v_h \in U_h, \\ (u(0) - u_h^0, v_h) = 0, \quad \forall v_h \in U_h, \end{cases} \tag{33}$$

where N is a given positive integer, $\Delta t = T/N$ denotes the time step size, $t_n = n\Delta t$ and

$$\partial_t u_h^n = (u_h^n - u_h^{n-1})/\Delta t,$$
$$u_h^{n-\frac{1}{2}} = (u_h^n + u_h^{n-1})/2,$$
$$t^{n-\frac{1}{2}} = (t^n + t^{n-1})/2.$$

In the following theorem, the boundness of the fully discrete scheme (33) is going to be deduced. The boundness is a key step for the error analysis in the field of the nonlinear parabolic equation.

Theorem 4.1 *Let $u_h^0 \in H_0^2(I) \bigcap W^{1,4}(I)$, then there exists a unique solution u_h^n for problem* (33) *such that*

$$\|u_h^n\|_2 \le C\|u_h^0\|_2, \quad 0 \le t \le T, \tag{34}$$

where C is a positive constant depending on $\alpha(x,t)$ and T, independent of h and Δt.

Proof A direct calculation gives

$$\begin{aligned}&\frac{H(u_h^n)-H(u_h^{n-1})}{u_h^n-u_h^{n-1}}\\=&\frac{1}{4}(u_h^n+u_h^{n-1})(|u_h^n|^2+|u_h^{n-1}|^2)-\frac{1}{2}(u_h^n+u_h^{n-1}).\end{aligned} \tag{35}$$

Choosing $v_h=u_h^n+u_h^{n-1}$ in (33), we obtain

$$\begin{aligned}&\frac{1}{\Delta t}(\|u_h^n\|^2-\|u_h^{n-1}\|^2)+\frac{s}{2}\|D^2u_h^n+D^2u_h^{n-1}\|^2+\frac{1}{2}\|Du_h^n+Du_h^{n-1}\|^2\\&+\frac{1}{4}((u_h^n+u_h^{n-1})^2,|u_h^n|^2+|u_h^{n-1}|^2)\le\frac{1}{2}\|u_h^n+u_h^{n-1}\|^2.\end{aligned} \tag{36}$$

It is obvious to derive

$$\frac{1}{\Delta t}(\|u_h^n\|^2-\|u_h^{n-1}\|^2)\le\frac{1}{2}\|u_h^n+u_h^{n-1}\|^2\le\|u_h^n\|^2+\|u_h^{n-1}\|^2. \tag{37}$$

Further, we get

$$\|u_h^n\|^2\le\frac{1+\Delta t}{1-\Delta t}\|u_h^{n-1}\|^2\le\cdots\le\left(\frac{1+\Delta t}{1-\Delta t}\right)^n\|u_h^0\|^2. \tag{38}$$

It is easy to show

$$\left(\frac{1+\Delta t}{1-\Delta t}\right)^n=\left(1+\frac{2\Delta t}{1-\Delta t}\right)^{\frac{1-\Delta t}{2\Delta t}\cdot\frac{2n\Delta t}{1-\Delta t}}.$$

If Δt is small enough, we know

$$\|u_h^n\|^2\le C\|u_h^0\|^2. \tag{39}$$

Letting $v_h=\partial_t u_h^n$ in (33), we have

$$\begin{aligned}&\|\partial_t u_h^n\|^2+\frac{1}{2\Delta t}(\alpha(x,t^{n-\frac{1}{2}})(|D^2u_h^n|^2-|D^2u_h^{n-1}|^2),1)\\&-\frac{1}{2}(D^2u_h^n+D^2u_h^{n-1},\partial_t u_h^n)+\frac{1}{\Delta t}(H(u_h^n)-H(u_h^{n-1}),1)=0.\end{aligned} \tag{40}$$

With the ε-inequality, we conclude

$$\begin{aligned}&\|\partial_t u_h^n\|^2+\frac{1}{2\Delta t}(\alpha(x,t^{n-\frac{1}{2}})(|D^2u_h^n|^2-|D^2u_h^{n-1}|^2),1)+\frac{1}{\Delta t}(H(u_h^n)-H(u_h^{n-1}),1)\\\le&\frac{1}{8}\|\partial_t u_h^n\|^2+\frac{1}{2}\|D^2u_h^n+D^2u_h^{n-1}\|^2\le\frac{1}{8}\|\partial_t u_h^n\|^2+\|D^2u_h^n\|^2+\|D^2u_h^{n-1}\|^2.\end{aligned}$$

And then, we obtian

$$
\begin{aligned}
&(\frac{1}{2}\alpha(x,t^{n-\frac{1}{2}})|D^2u_h^n|^2+H(Du_h^n),1)\\
\le&(\frac{1}{2}\alpha(x,t^{n-\frac{1}{2}})|D^2u_h^{n-1}|^2+H(Du_h^{n-1}),1)+\Delta t(\|D^2u_h^n\|^2+\|D^2u_h^{n-1}\|^2).
\end{aligned}
\tag{41}
$$

We need to define the following function

$$
G(u_h^n,t^{n-\frac{1}{2}})=\left(\frac{1}{2}\alpha(x,t^{n-\frac{1}{2}})|D^2u_h^n|^2+H(Du_h^n),1\right).
\tag{42}
$$

Obviously, $G(u_h^n,t^{n-\frac{1}{2}})\ge 0$. By (41) and (42), we have

$$
\begin{aligned}
&G(u_h^n,t^{n-\frac{1}{2}})-G(u_h^{n-1},t^{n-\frac{3}{2}})\\
\le&\frac{1}{2}((\alpha(x,t^{n-\frac{1}{2}})-\alpha(x,t^{n-\frac{3}{2}}))|D^2u_h^{n-1}|^2,1)+\Delta t(\|D^2u_h^n\|^2+\|D^2u_h^{n-1}\|^2).
\end{aligned}
$$

By the Lagrange Intermediate Value Theorem and (4), we obtain

$$
\begin{aligned}
&G(u_h^n,t^{n-\frac{1}{2}})-G(u_h^{n-1},t^{n-\frac{3}{2}})\\
\le&\frac{\Delta t}{2}\left|\frac{\partial\alpha}{\partial t}(x,\xi)\right|\|D^2u_h^{n-1}\|^2+\Delta t(\|D^2u_h^n\|^2+\|D^2u_h^{n-1}\|^2)\\
\le&\frac{M_1\Delta t}{2}\|D^2u_h^{n-1}\|^2+\Delta t(\|D^2u_h^n\|^2+\|D^2u_h^{n-1}\|^2),
\end{aligned}
$$

where $t^{n-\frac{3}{2}}<\xi<t^{n-\frac{1}{2}}$. Therefore, we have

$$
G(u_h^n,t^{n-\frac{1}{2}})-G(u_h^{n-1},t^{n-\frac{3}{2}})\le\Delta t\|D^2u_h^n\|^2+(1+\frac{M_1}{2})\Delta t\|D^2u_h^{n-1}\|^2.
$$

Taking the sum over n, it is easy to obtain

$$
\begin{aligned}
&G(u_h^n,t^{n-\frac{1}{2}})-G(u_h^1,t^{\frac{1}{2}})\\
\le&\Delta t\sum_{j=2}^{n}\|D^2u_h^j\|^2+(1+\frac{M_1}{2})\Delta t\sum_{j=2}^{n}\|D^2u_h^{j-1}\|^2\le C\Delta t\sum_{j=1}^{n}\|D^2u_h^j\|^2.
\end{aligned}
\tag{43}
$$

Hence, it is obvious to get

$$
G(u_h^n,t^{n-\frac{1}{2}})\ge\frac{s}{2}\|D^2u_h^n\|^2+(H(Du_h^n),1)\ge\frac{s}{2}\|D^2u_h^n\|^2.
\tag{44}
$$

Using (43) and (44), we know

$$
G(u_h^n,t^{n-\frac{1}{2}})-G(u_h^1,t^{\frac{1}{2}})\le\frac{2C\Delta t}{s}\sum_{j=1}^{n}G(u_h^j,t^{j-\frac{1}{2}}).
$$

Based on (41)-(42) and $u_h^0 \in H_0^2(I)\bigcap W^{1,4}(I)$, we have

$$\begin{aligned} G(u_h^1, t^{\frac{1}{2}}) &= \left(\frac{1}{2}\alpha(x, t^{\frac{1}{2}})|D^2u_h^1|^2 + H(Du_h^1), 1\right) \\ &\le \left(\frac{1}{2}\alpha(x, t^{\frac{1}{2}})|D^2u_h^0|^2 + H(Du_h^0), 1\right) \le C(u_h^0), \end{aligned}$$

where $C(u_h^0)$ is a constant depending on u_h^0. Then

$$G(u_h^n, t^{n-\frac{1}{2}}) \le C(u_h^0) + \frac{2C\Delta t}{s}\sum_{j=1}^{n} G(u_h^n, t^{n-\frac{1}{2}}). \tag{45}$$

With the discrete Gronwall's inequality, we deduce

$$G(u_h^n, t^{n-\frac{1}{2}}) \le C, \quad C = C(u_h^0, s, M_1, T). \tag{46}$$

From (46), we have

$$\|D^2u_h^n\| \le C\|D^2u_h^0\|. \tag{47}$$

In addition, the following formula is known

$$\|Du_h^n\|^2 \le \frac{1}{2}(\|u_h^n\|^2 + \|D^2u_h^n\|^2).$$

Combined (39) and (47), we have (34). The proof is completed. □

Next, we analyze the convergence in L^2 norm.

Theorem 4.2 *Let u^n be the solution to problem (5), u_h^n be the solution to the fully discrete scheme (33), $u(0) \in H^4(I)$, $u_t \in L^2(0,T;H^4(I)) \cap L^2(0,T;W^{2,4}(I))$, $u_{ttt} \in L^2(0,T;L^2(I))$ and $u_h^0 \in U_h$ satisfying*

$$\|u(0) - u_h^0\| \le Ch^4\|u(0)\|_4. \tag{48}$$

Then, we have the following error estimate:

$$\|u^n - u_h^n\| \le C((\Delta t)^2 + h^4), \tag{49}$$

where C is a positive constant depending on $\alpha(x,t)$ and T, independent of mesh size h.

Proof Denote $u_t^n = u_t(x, t^n)$ and $u^n = u(x, t^n)$. Setting $t = t^{n-1}$ and $t = t^n$ in (5), respectively, we obtain

$$\begin{aligned} &\left(\frac{u_t^n + u_t^{n-1}}{2}, v_h\right) + \left(\frac{\alpha(x,t^n)D^2u^n + \alpha(x,t^{n-1})D^2u^{n-1}}{2}, D^2v_h\right) \\ &+ \left(\frac{Du^n + Du^{n-1}}{2}, Dv_h\right) + \left(\frac{(u^n)^3 + (u^{n-1})^3 - u^n - u^{n-1}}{2}, v_h\right) = 0. \end{aligned} \tag{50}$$

Denote

$$
\begin{aligned}
&\Phi(D^2u^n, D^2u^{n-1}, D^2u_h^{n-\frac{1}{2}})\\
=&\frac{\alpha(x,t^n)D^2u^n + \alpha(x,t^{n-1})D^2u^{n-1}}{2} - \alpha(x,t^{n-\frac{1}{2}})D^2u_h^{n-\frac{1}{2}},
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
&F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\\
=&\frac{(u^n)^3 + (u^{n-1})^3 - u^n - u^{n-1}}{2} - \frac{H(u_h^n) - H(u_h^{n-1})}{u_h^n - u_h^{n-1}}.
\end{aligned}
\tag{52}
$$

It follows from (50)-(52) and (33) that

$$
\begin{aligned}
&\left(\frac{u_t^n + u_t^{n-1}}{2} - \partial_t u_h^n, v_h\right) + (\Phi(D^2u^n, D^2u^{n-1}, D^2u_h^{n-\frac{1}{2}}), D^2v_h)\\
&+\left(\frac{Du^n + Du^{n-1} - Du_h^n - Du_h^{n-1}}{2}, Dv_h\right) + (F(u^n, u^{n-1}, u_h^n, u_h^{n-1}), v_h) = 0.
\end{aligned}
\tag{53}
$$

Let $\rho^n = u^n - R_h u^n$ and $\theta^n = R_h u^n - u_h^n$, then $u^n - u_h^n = \rho^n + \theta^n$. It is clear to get

$$
\begin{aligned}
&\frac{u_t^n + u_t^{n-1}}{2} - \partial_t u_h^n = \frac{u_t^n + u_t^{n-1}}{2} - \partial_t u^n + \partial_t u^n - \partial_t u_h^n\\
=&\frac{u_t^n + u_t^{n-1}}{2} - \partial_t u^n + \partial_t(u^n - R_h u^n + R_h u^n - u_h^n) = \partial_t\theta^n - r^n,
\end{aligned}
\tag{54}
$$

where

$$
r^n = \partial_t R_h u^n - \partial_t u^n + \partial_t u^n - \frac{u_t(t_j) + u_t(t_{j-1})}{2}.
$$

An easy calculation gives

$$
\begin{aligned}
&\Phi(D^2u^n, D^2u^{n-1}, D^2u_h^{n-\frac{1}{2}})\\
=&\frac{1}{2}((\alpha(x,t^n) - \alpha(x,t^{n-\frac{1}{2}}))D^2u^n + (\alpha(x,t^{n-1}) - \alpha(x,t^{n-\frac{1}{2}}))D^2u^{n-1}\\
&+\alpha(x,t^{n-\frac{1}{2}})(D^2u^n + D^2u^{n-1} - D^2u_h^n - D^2u_h^{n-1}))\\
=&\frac{1}{2}((\alpha(x,t^n) - \alpha(x,t^{n-\frac{1}{2}}))D^2u^n + (\alpha(x,t^{n-1}) - \alpha(x,t^{n-\frac{1}{2}}))D^2u^{n-1}\\
&+\alpha(x,t^{n-\frac{1}{2}})(D^2\theta^n + D^2\theta^{n-1} + D^2\rho^n + D^2\rho^{n-1})).
\end{aligned}
$$

Using the Taylor's theorem, we have

$$
\alpha(x,t^n) = \alpha(x,t^{n-\frac{1}{2}}) + \frac{\Delta t}{2}\frac{\partial\alpha}{\partial t}(x,t^{n-\frac{1}{2}}) + \frac{(\Delta t)^2}{8}\frac{\partial^2\alpha}{\partial^2 t}(x,t^{n-\frac{1}{2}} + \xi_1\frac{\Delta t}{2}),
$$

and

$$
\alpha(x,t^{n-1}) = \alpha(x,t^{n-\frac{1}{2}}) - \frac{\Delta t}{2}\frac{\partial\alpha}{\partial t}(x,t^{n-\frac{1}{2}}) + \frac{(\Delta t)^2}{8}\frac{\partial^2\alpha}{\partial^2 t}(x,t^{n-\frac{1}{2}} + \xi_2\frac{\Delta t}{2}),
$$

where $0 < \xi_1 < 1$, $-1 < \xi_2 < 0$. With (4), we get

$$\begin{aligned}
&\Phi(D^2u^n, D^2u^{n-1}, D^2u_h^{n-\frac{1}{2}}) \\
=&\frac{\Delta t}{2}\frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}})(D^2u^n - D^2u^{n-1}) + O((\Delta t)^2) \\
&+ \frac{1}{2}\alpha(x, t^{n-\frac{1}{2}})(D^2\theta^n + D^2\theta^{n-1} + D^2\rho^n + D^2\rho^{n-1}).
\end{aligned} \tag{55}$$

From (7), we have

$$\begin{aligned}
&(\partial_t\theta^n, v_h) + \frac{1}{2}(\alpha(x, t^{n-\frac{1}{2}})(D^2\theta^n + D^2\theta^{n-1}), D^2v_h) \\
&+ \frac{\Delta t}{2}\left(\frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}})(D^2u^n - D^2u^{n-1}), D^2v_h\right) + (O((\Delta t)^2), D^2v_h) \\
=&\frac{1}{2}(\theta^n + \rho^n + \theta^{n-1} + \rho^{n-1}, D^2v_h) + (r^n, v_h) - (F(u^n, u^{n-1}, u_h^n, u_h^{n-1}), v_h).
\end{aligned} \tag{56}$$

Setting $v_h = \theta^n + \theta^{n-1}$ in (56), then we get

$$\begin{aligned}
&\frac{1}{\Delta t}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s}{2}\|D^2\theta^n + D^2\theta^{n-1}\|^2 \\
\leq&\frac{s}{8}\|D^2\theta^n + D^2\theta^{n-1}\|^2 + \frac{1}{2s}\|\theta^n + \rho^n + \theta^{n-1} + \rho^{n-1}\|^2 \\
&+ \|r^n\|^2 + \frac{1}{4}\|\theta^n + \theta^{n-1}\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2 \\
&+ \frac{1}{4}\|\theta^n + \theta^{n-1}\|^2 + \frac{M_1\Delta t}{2}\|D^2u^n - D^2u^{n-1}\|\|D^2\theta^n + D^2\theta^{n-1}\| \\
\leq&\frac{s}{8}\|D^2\theta^n + D^2\theta^{n-1}\|^2 + \frac{2}{s}(\|\theta^n\|^2 + \|\rho^n\|^2 + \|\theta^{n-1}\|^2 + \|\rho^{n-1}\|^2) \\
&+ \|r^n\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2 + \frac{1}{2}\|\theta^n + \theta^{n-1}\|^2 \\
&+ \frac{s}{8}\|D^2\theta^n + D^2\theta^{n-1}\|^2 + \frac{(M_1\Delta t)^2}{2s}\|D^2u^n - D^2u^{n-1}\|^2 \\
\leq&\frac{s}{4}\|D^2\theta^n + D^2\theta^{n-1}\|^2 + (1 + \frac{2}{s})(\|\theta^n\|^2 + \|\theta^{n-1}\|^2) + \frac{2}{s}(\|\rho^n\|^2 + \|\rho^{n-1}\|^2) \\
&+ \|r^n\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2 + \frac{(M_1\Delta t)^2}{2s}\|D^2u^n - D^2u^{n-1}\|^2.
\end{aligned}$$

Based on the Newton-Leibniz formula and the Hölder's inequality, we have

$$\left|D^2u^n - D^2u^{n-1}\right| = \left|\int_{t_{n-1}}^{t_n} D^2u_t(t)dt\right| \leq \Delta t^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n} |D^2u_t(t)|^2dt\right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}
&\frac{1}{\Delta t}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s}{4}\|D^2\theta^n + D^2\theta^{n-1}\|^2 \\
\leq&(1 + \frac{2}{s})(\|\theta^n\|^2 + \|\theta^{n-1}\|^2) + \frac{2}{s}(\|\rho^n\|^2 + \|\rho^{n-1}\|^2) \\
&+ \|r^n\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2 + \frac{M_1^2(\Delta t)^3}{2s}\int_{t_{n-1}}^{t_n} \|D^2u_t(t)\|^2dt.
\end{aligned} \tag{57}$$

A direct calculation gives

$$
\begin{aligned}
&\|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\| \\
=&\bigg\|\frac{1}{2}((u^n)^3 + (u^{n-1})^3) - \frac{1}{4}(u^n + u^{n-1})(|u^n|^2 + |u^{n-1}|^2) \\
&+ \frac{1}{4}(u^n + u^{n-1})(|u^n|^2 + |u^{n-1}|^2) - \frac{1}{4}(u_h^n + u_h^{n-1})(|u_h^n|^2 + |u_h^{n-1}|^2) \\
&- \frac{1}{2}(u^n + u^{n-1}) + \frac{1}{2}(u_h^n + u_h^{n-1})\bigg\|.
\end{aligned}
$$

Since (34) and the Sobolev's embedding theorem, $H_0^2(I) \hookrightarrow H^{1,\infty}(I)$, we know

$$
\big|Du^n\big|_\infty \le C\big\|u^n\big\|_2 \le C, \quad \big|Du_h^n\big|_\infty \le C\big\|u_h^n\big\|_2 \le C. \tag{58}
$$

Using the Hölder's inequality, we obtain

$$
|u^n - u^{n-1}| = \left|\int_{t_{n-1}}^{t_n} u_t(t)dt\right| \le C(\Delta t)^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} |u_t(t)|^2 dt\right)^{\frac{1}{2}}. \tag{59}
$$

From (58) and (59), we have

$$
\begin{aligned}
&\left\|\frac{1}{2}((u^n)^3 + (u^{n-1})^3) - \frac{1}{4}(u^n + u^{n-1})(|u^n|^2 + |u^{n-1}|^2)\right\| \\
=&\frac{1}{4}\|(u^n)^3 - |u^n|^2 u^{n-1} - u^n|u^{n-1}|^2 + (u^{n-1})^3\| \\
=&\frac{1}{4}\|(u^n + u^{n-1})(u^n - u^{n-1})^2\| \\
\le&\frac{1}{4}(|u^n|_\infty + |u^{n-1}|_\infty)\|(u^n - u^{n-1})^2\| \\
\le&C\Delta t \int_{t_{n-1}}^{t_n} \|u_t(t)\|^2 dt.
\end{aligned} \tag{60}
$$

Due to (58), we get

$$
\begin{aligned}
&\|(u^n + u^{n-1})(|u^n|^2 + |u^{n-1}|^2) - (u_h^n + u_h^{n-1})(|u_h^n|^2 + |u_h^{n-1}|^2)\| \\
=&\|(u^n + u^{n-1})(|u^n|^2 + |u^{n-1}|^2) - (u_h^n + u_h^{n-1})(|u^n|^2 + |u^{n-1}|^2) \\
&+ (u_h^n + u_h^{n-1})(|u^n|^2 + |u^{n-1}|^2) - (u_h^n + u_h^{n-1})(|u_h^n|^2 + |u_h^{n-1}|^2)\| \\
\le&(|u^n|_\infty^2 + |u^{n-1}|_\infty^2)\|(u^n + u^{n-1}) - (u_h^n + u_h^{n-1})\| \\
&+ (|u_h^n|_\infty + |u_h^{n-1}|_\infty)\|(u^n + u_h^n)(u^n - u_h^n) + (u^{n-1} + u_h^{n-1})(u^{n-1} - u_h^{n-1})\| \\
\le&(|u^n|_\infty^2 + |u^{n-1}|_\infty^2)(\|\theta^n + \theta^{n-1}\| + \|\rho^n + \rho^{n-1}\|) \\
&+ (|u_h^n|_\infty + |u_h^{n-1}|_\infty)(|u^n|_\infty + |u_h^n|_\infty + |u^{n-1}|_\infty + |u_h^{n-1}|_\infty) \\
&\quad \cdot (\|\theta^n + \theta^{n-1}\| + \|\rho^n + \rho^{n-1}\|) \\
\le&C(\|\theta^n + \theta^{n-1}\| + \|\rho^n + \rho^{n-1}\|).
\end{aligned} \tag{61}
$$

By the triangle inequality, we obtain

$$
\begin{aligned}
&\|(u^n + u^{n-1}) - (u_h^n + u_h^{n-1})\| \\
=&\|\theta^n + \rho^n + \theta^{n-1} + \rho^{n-1}\| \le \|\theta^n + \theta^{n-1}\| + \|\rho^n + \rho^{n-1}\|.
\end{aligned} \tag{62}
$$

In view of (60)-(62) and (9), we have

$$
\begin{aligned}
&\|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\| \\
\leq& C\bigg(\|\theta^n + \theta^{n-1}\| + \|\rho^n + \rho^{n-1}\| + \Delta t \int_{t_{n-1}}^{t_n} \|u_t(t)\|^2 dt \bigg) \\
\leq& C\bigg(\|\theta^n + \theta^{n-1}\| + h^4 + \Delta t \int_{t_{n-1}}^{t_n} \|u_t(t)\|^2 dt \bigg).
\end{aligned}
$$

Based on the ε-inequality and the Hölder's inequality, we obtain

$$
\begin{aligned}
&\|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2 \\
\leq& C\bigg(\|\theta^n + \theta^{n-1}\|^2 + h^8 + (\Delta t)^3 \int_{t_{n-1}}^{t_n} \|u_t(t)\|^4 dt \bigg).
\end{aligned} \tag{63}
$$

Let $r^n = r_1^n + r_2^n$, where

$$
\begin{aligned}
&r_1^j = \partial_t R_h u(t_j) - \partial_t u(t_j) = \frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} (R_h - I) u_t dt, \\
&r_2^j = \partial_t u(t_j) - \frac{u_t(t_j) + u_t(t_{j-1})}{2}.
\end{aligned}
$$

It is clear to see that

$$
\|r_1^j\| \leq \frac{1}{\Delta t} C h^4 \int_{t_{j-1}}^{t_j} \|u_t\|_4 dt \leq C(\Delta t)^{-\frac{1}{2}} h^4 \bigg(\int_{t_{j-1}}^{t_j} \|u_t\|_4^2 dt \bigg)^{\frac{1}{2}}.
$$

Using the Taylor's formula, we derive

$$
\|r_2^j\| \leq C\Delta t \int_{t_{j-1}}^{t_j} \|u_{ttt}\| dt \leq C(\Delta t)^{\frac{3}{2}} \bigg(\int_{t_{j-1}}^{t_j} \|u_{ttt}\|^2 dt \bigg)^{\frac{1}{2}}.
$$

We easily get

$$
\sum_{j=1}^n \|r^j\|^2 \leq C(\Delta t)^{-1}((\Delta t)^4 + h^8) \int_0^{t_n} (\|u_t\|_4^2 + \|u_{ttt}\|^2) dt. \tag{64}
$$

Adding (57), (63) and (64), we have

$$
\begin{aligned}
&(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s\Delta t}{4} \|D^2\theta^n + D^2\theta^{n-1}\|^2 \\
\leq& C\bigg(\Delta t(\|\theta^n + \theta^{n-1}\|^2 + h^8) \\
&+ ((\Delta t)^4 + h^8) \int_{t_{n-1}}^{t_n} (\|u_t\|_4^2 + \|u_t\|^4 + \|D^2 u_t\|^2 + \|u_{ttt}\|^2) dt \bigg).
\end{aligned}
$$

We know

$$
\|\theta^n + \theta^{n-1}\|^2 \leq 2(\|\theta^n\|^2 + \|\theta^{n-1}\|^2).
$$

Then

$$
\begin{aligned}
&(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s\Delta t}{4}\|D^2\theta^n + D^2\theta^{n-1}\|^2 \\
\leq & C\bigg(\Delta t(\|\theta^n\|^2 + \|\theta^{n-1}\|^2 + h^8) \\
&+ ((\Delta t)^4 + h^8)\int_{t_{n-1}}^{t_n}(\|u_t\|_4^2 + \|u_t\|^4 + \|D^2 u_t\|^2 + \|u_{ttt}\|^2)dt\bigg).
\end{aligned}
\tag{65}
$$

Taking the sum over n, by $n\Delta t = t_n \leq T$, we have

$$
\begin{aligned}
&\|\theta^n\|^2 - \|\theta^0\|^2 + \frac{s\Delta t}{4}\sum_{i=1}^{n}\|D^2\theta^i + D^2\theta^{i-1}\|^2 \\
\leq & C\bigg(\Delta t\sum_{i=1}^{n}(\|\theta^i\|^2 + \|\theta^{i-1}\|^2) + Th^8 \\
&+ ((\Delta t)^4 + h^8)\int_0^{t_n}(\|u_t\|_4^2 + \|u_t\|^4 + \|D^2 u_t\|^2 + \|u_{ttt}\|^2)dt\bigg).
\end{aligned}
$$

Hence

$$
(1 - C\Delta t)\|\theta^n\|^2 \leq (1 + C\Delta t)\|\theta^0\|^2 + C\bigg(\Delta t\sum_{i=1}^{n-1}\|\theta^i\|^2 + (\Delta t)^4 + h^8\bigg).
$$

If Δt is small enough, we have

$$
\|\theta^n\|^2 \leq \frac{1 + C\Delta t}{1 - C\Delta t}\|\theta^0\|^2 + \frac{C}{1 - C\Delta t}\bigg(\Delta t\sum_{i=1}^{n-1}\|\theta^i\|^2 + (\Delta t)^4 + h^8\bigg).
$$

With the discrete Gronwall's inequality, it gives

$$
\|\theta^n\| \leq C((\Delta t)^2 + h^4).
$$

Using (9) and (48), we get

$$
\|\theta^0\| \leq \|u(0) - u_h(0)\| + \|u(0) - R_h u(0)\| \leq Ch^4\|u(0)\|_4.
$$

Finally, we obtain (49). The proof is completed. □

In the following theorem, we introduce the error estimate in H^2 norm.

Theorem 4.3 *Let u^n be the solution to (5), u_h^n be the solution to the fully discrete problem (33), $u(0) \in H^4(I)$, $u_t \in L^2(0,T;H^4(I)) \cap L^2(0,T;W^{2,4}(I))$, $u_{ttt} \in L^2(0,T;L^2(I))$, and $u_h^0 \in U_h$ satisfying*

$$
|u(0) - u_h^0|_2 \leq Ch^2\|u(0)\|_4.
\tag{66}
$$

Then, we have the following error estimate:

$$
|u^n - u_h^n|_2 \leq C(\Delta t + h^2).
\tag{67}
$$

Proof Taking $v_h = \partial_t\theta^n$ in (56), we obtain

$$
\begin{aligned}
&\|\partial_t\theta^n\|^2 + \frac{1}{2\Delta t}(\alpha^{n-\frac{1}{2}}(D^2\theta^n + D^2\theta^{n-1}), D^2\theta^n - D^2\theta^{n-1}) \\
&\quad + \frac{1}{2}\left(\frac{\partial\alpha}{\partial t}(x, t^{n-\frac{1}{2}})(D^2u^n - D^2u^{n-1}), D^2\theta^n - D^2\theta^{n-1}\right) \\
\leq& \frac{1}{4}(\|D^2(\theta^n + \rho^n + \theta^{n-1} + \rho^{n-1})\|^2) + \frac{1}{4}\|\partial_t\theta^n\|^2 \\
&\quad + \|r^n\|^2 + \frac{1}{4}\|\partial_t\theta^n\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2 + \frac{1}{4}\|\partial_t\theta^n\|^2 \\
\leq& \|D^2\theta^n\|^2 + \|D^2\rho^n\|^2 + \|D^2\theta^{n-1}\|^2 + \|D^2\rho^{n-1}\|^2 + \frac{3}{4}\|\partial_t\theta^n\|^2 \\
&\quad + \|r^n\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2.
\end{aligned}
$$

With the Newton-Leibniz formula and the Hölder's inequality, we get

$$
|D^2u^n - D^2u^{n-1}|^2 \leq \left|\int_{t_{n-1}}^{t_n} |D^2u_t|^2 dt\right| \leq \Delta t \int_{t_{n-1}}^{t_n} |D^2u_t|^2 dt.
$$

Based on the Cauchy's inequality, we have

$$
\begin{aligned}
&(\alpha^{n-\frac{1}{2}} D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{1}{2}} D^2\theta^{n-1}, D^2\theta^{n-1}) \\
\leq& 2\Delta t(\|D^2\theta^n\|^2 + \|D^2\rho^n\|^2 + \|D^2\theta^{n-1}\|^2 + \|D^2\rho^{n-1}\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2) \\
&\quad + 2\Delta t\|r^n\|^2 + M_1\Delta t\|D^2u^n - D^2u^{n-1}\|\|D^2\theta^n - D^2\theta^{n-1}\| \\
\leq& 2\Delta t(\|D^2\theta^n\|^2 + \|D^2\rho^n\|^2 + \|D^2\theta^{n-1}\|^2 + \|D^2\rho^{n-1}\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2) \\
&\quad + 2\Delta t\|r^n\|^2 + \frac{M_1^2\Delta t}{2}\|D^2u^n - D^2u^{n-1}\|^2 + \frac{\Delta t}{2}(\|D^2\theta^n + D^2\theta^{n-1}\|^2) \\
\leq& 2\Delta t(\|D^2\theta^n\|^2 + \|D^2\rho^n\|^2 + \|D^2\theta^{n-1}\|^2 + \|D^2\rho^{n-1}\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2) \\
&\quad + 2\Delta t\|r^n\|^2 + \frac{M_1^2(\Delta t)^2}{2}\int_{t_{n-1}}^{t_n} \|D^2u_t\|^2 dt + \Delta t(\|D^2\theta^n\|^2 + \|D^2\theta^{n-1}\|^2).
\end{aligned} \tag{68}
$$

There exists $\xi \in (t^{n-\frac{3}{2}}, t^{n-\frac{1}{2}})$ such that

$$
\begin{aligned}
&(\alpha^{n-\frac{1}{2}}(D^2\theta^n + D^2\theta^{n-1}), D^2\theta^n - D^2\theta^{n-1}) \\
=& (\alpha^{n-\frac{1}{2}} D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{3}{2}} D^2\theta^{n-1}, D^2\theta^{n-1}) \\
&\quad - ((\alpha^{n-\frac{1}{2}} - \alpha^{n-\frac{3}{2}})D^2\theta^{n-1}, D^2\theta^{n-1})) \\
=& (\alpha^{n-\frac{1}{2}} D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{3}{2}} D^2\theta^{n-1}, D^2\theta^{n-1}) - \Delta t\left(\frac{\partial\alpha}{\partial t}(x, \xi)D^2\theta^{n-1}, D^2\theta^{n-1}\right).
\end{aligned}
$$

Thus it is to get

$$
\begin{aligned}
&(\alpha^{n-\frac{1}{2}} D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{3}{2}} D^2\theta^{n-1}, D^2\theta^{n-1}) \\
\leq& 2\Delta t(\|D^2\theta^n\|^2 + \|D^2\rho^n\|^2 + \|D^2\theta^{n-1}\|^2 + \|D^2\rho^{n-1}\|^2) \\
&\quad + 2\Delta t(\|r^n\|^2 + \|F(u^n, u^{n-1}, u_h^n, u_h^{n-1})\|^2) \\
&\quad + \frac{M_1^2(\Delta t)^2}{2}\int_{t_{n-1}}^{t_n} \|D^2u_t\|^2 dt + \Delta t(\|D^2\theta^n\|^2 + (1 + M_1)\|D^2\theta^{n-1}\|^2).
\end{aligned} \tag{69}
$$

Taking the sum over n, the following formula is deduced

$$\begin{aligned} &(\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{\frac{1}{2}}D^2\theta^1, D^2\theta^1) \\ \leq& C\Delta t\sum_{j=2}^{n}(\|D^2\theta^j\|^2 + \|D^2\rho^j\|^2 + \|D^2\theta^{j-1}\|^2 + \|D^2\rho^{j-1}\|^2) \\ &+ C\Delta t\sum_{j=2}^{n}(\|r^j\|^2 + \|F(u^j, u^{j-1}, u_h^j, u_h^{j-1})\|^2) \\ &+ C(\Delta t)^2\int_0^{t_n}\|D^2u_t\|^2dt + C\Delta t\sum_{j=2}^{n}(\|D^2\theta^j\|^2 + \|D^2\theta^{j-1}\|^2). \end{aligned} \tag{70}$$

Using (3), we know

$$(\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) \geq s\|D^2\theta^n\|^2, \qquad -(\alpha^{\frac{1}{2}}D^2\theta^1, D^2\theta^1) \geq -S\|D^2\theta^1\|^2,$$

Further, one has

$$\begin{aligned} &s\|D^2\theta^n\|^2 - S\|D^2\theta^1\|^2 \\ \leq& C\Delta t\sum_{j=1}^{n}(\|D^2\theta^j\|^2 + \|D^2\rho^j\|^2 + \|r^j\|^2 + \|F(u^j, u^{j-1}, u_h^j, u_h^{j-1})\|^2) \\ &+ C(\Delta t)^2\int_0^{t^n}\|D^2u_t\|^2dt. \end{aligned} \tag{71}$$

Substituting (63) and (64) into (71), we know

$$\begin{aligned} &s\|D^2\theta^n\|^2 - S\|D^2\theta^1\|^2 \\ \leq& C\Delta t\sum_{j=1}^{n}(\|D^2\theta^j\|^2 + \|D^2\rho^j\|^2 + \|\theta^j\|^2) \\ &+ C((\Delta t)^2 + h^8)\int_0^{t_n}(\|u_t\|_4^2 + \|u_{ttt}\|^2 + \|u_t(t)\|^4 + \|D^2u_t(t)\|^2)dt. \end{aligned} \tag{72}$$

Letting $n = 1$ in (68), and using (63) and (64), one could have

$$\|D^2\theta^1\| \leq C\|D^2\theta^0\| + O(\Delta t). \tag{73}$$

By (72) and (73), we get

$$\|D^2\theta^n\|^2 \leq C(\|D^2\theta^0\|^2 + (\Delta t)^2 + h^4 + \Delta t\sum_{j=1}^{n-1}(\|D^2\theta^j\|^2 + \|\theta^j\|^2).$$

With the help of (49), we prove

$$\|D^2\theta^n\|^2 \leq C(\|D^2\theta^0\|^2 + (\Delta t)^2 + h^4 + \Delta t\sum_{j=1}^{n-1}\|D^2\theta^j\|^2).$$

If the time step is sufficiently small, the discrete Gronwall's inequality yields

$$\|D^2\theta^n\| \le C(\Delta t + h^2).$$

The proof is completed. □

Numerical approximation

In this section, to test the efficiency of the cubic B-spline finite element scheme, we consider the following problem:

$$\begin{cases} u_t + (\alpha(x,t)u_{xx})_{xx} - u_{xx} + u^3 - u = f(x,t), \quad (x,t) \in (0,1)\times(0,1], \\ u(x,t) = u_x(x,t) = 0, \quad x = 0,1, \quad t \in (0,1], \\ u(x,0) = u_0(x), \quad x \in [0,1], \end{cases} \tag{74}$$

where $u_0(x) = 0$ and $\alpha(x,t) = 1 + xt$. The exact solution of the problem (74) is chosen as $u(x,t) = t^2(1 - cos2\pi x)$. Then the concrete functional form of $f(x,t)$ is

$$\begin{aligned} f(x,t) =& (2t - t^2)(1 - cos2\pi x) + t^6(1 - cos2\pi x)^3 \\ & - 4\pi^2 t^2(1 - 4\pi^2(1 + xt))cos2\pi x - 16\pi^3 t^3 sin2\pi x. \end{aligned}$$

As shown Figure 1 and Figure 2, the numerical solution is in well accordance with the exact solution and the numerical scheme is valid and efficient.

Tables 1-2 display the corresponding errors and convergence rates of the cubic B-spline FEM.

Figures

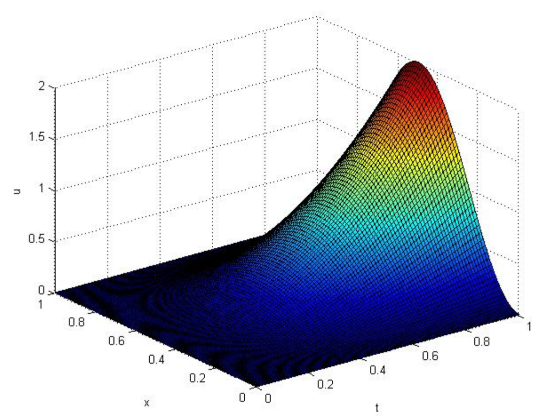

Figure 1 The exact solution to problem (74).

Tables

In Table 1, the time step is taken as $\Delta t = \frac{1}{8000}$ to get the spatial convergence order. The data demonstrate that the error decreases with the decrease of the space step.

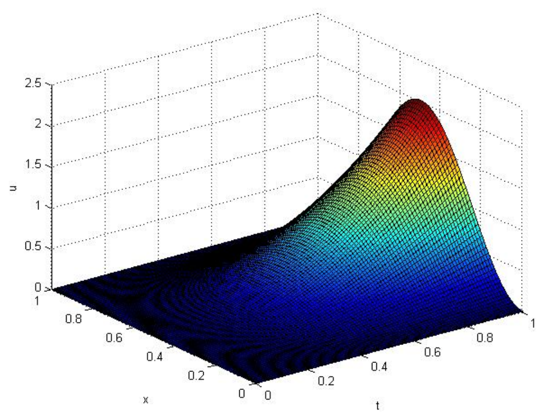

Figure 2 The approximate solution to problem (74).

The convergent rate of the numerical solution is the fourth order in L^2 norm and is the second order convergent in H^2 semi-norm.

Table 1 The errors for different space step h at $t = 1$ and convergence orders.

$(\Delta t, h)$	$\|u-u_h\|$	$rate$	$\|u-u_h\|_1$	$rate$	$\|u-u_h\|_2$	$rate$
$(1/8000, 1/10)$	$2.0139e^{-4}$		$7.0746e^{-3}$		$4.3020e^{-1}$	
$(1/8000, 1/20)$	$1.1537e^{-5}$	4.1256	$8.1497e^{-4}$	3.1178	$1.0389e^{-1}$	2.0500
$(1/8000, 1/40)$	$7.0611e^{-7}$	4.0302	$9.9733e^{-5}$	3.0306	$2.5745e^{-2}$	2.0127
$(1/8000, 1/80)$	$4.5077e^{-8}$	3.9694	$1.2400e^{-5}$	3.0077	$6.4221e^{-3}$	2.0032

In Table 2, the space step is fixed to $h = \frac{1}{1000}$, we analyze the corresponding error estimates and convergence orders in time direction. The data tell us that the convergent orders both are the second order in L^2 and H^2 norms.

Table 2 The errors for different time step Δt at $t = 1$ and convergence orders.

$(\Delta t, h)$	$\|u-u_h\|$	$rate$	$\|u-u_h\|_1$	$rate$	$\|u-u_h\|_2$	$rate$
$(1/20, 1/1000)$	$7.1681e^{-4}$		$2.5193e^{-3}$		$1.6053e^{-2}$	
$(1/40, 1/1000)$	$1.9373e^{-4}$	1.8875	$6.8532e^{-4}$	1.8781	$4.3412e^{-3}$	1.8867
$(1/80, 1/1000)$	$4.7734e^{-5}$	2.0210	$1.7138e^{-4}$	1.9996	$1.0794e^{-3}$	2.0079
$(1/160, 1/1000)$	$1.1336e^{-5}$	2.0741	$4.1135e^{-5}$	2.0588	2.6176^{-4}	2.0438

The numerical example shows that the cubic B-spline FEM is an efficient approximate calculation tool for solving the fourth order nonlinear parabolic equation.

Conclusion

In this work, we have presented the cubic B-spline FEM for solving a mathematical model consisting of a fourth order nonlinear parabolic equation and initial-boundary value conditions. The variable coefficient of the fourth order main term is function of time and space variables, which increases the difficulty of theoretical analysis and numerical experiment. To solve this problem, we introduce the elliptic projection operator and the energy function. The boundness of the semi-discrete and fully discrete schemes are proved. Thus the error estimates in L^2 norm and H^2 norm are deduced by means of the boundness, Sobolev's embedding theorem, and so on. The results of theoretical analysis are verified by numerical experiment.

Based on the following considerations, we adopt the B-spline FEM. For one thing, B-splines have better smoothness than the Lagrange and Hermite type elements. For another, B-spline finite element only has one type of basis functions, so the scale of matrix from B-spline FEM is lower.

In summary, the B-spline FEM is a powerful numerical method for solving higher order nonlinear parabolic equations. It is worthy of further study.

Availability of data and materials
Please contact author for data requests.

Competing interests
The authors declare that they have no competing interests.

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Author's contributions
DQ wrote the first draft. DQ and WH made the figure of numerical solution and errors, corrected and improved the final version. DQ and WH read and approved the final draft.

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