NORMALIZED SOLUTIONS OF TWO-COMPONENT NONLINEAR SCHRÖDINGER EQUATIONS WITH LINEAR COUPLES

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Abstract. In this paper, we focus on the following nonlinear Schrödinger equations with linear couples

$$\begin{cases} -\Delta u + V_1(x)u + \lambda_1 u = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|}\,dy|u|^{p-2}u + \beta v & \text{in } \mathbb{R}^3, \\ -\Delta v + V_2(x)v + \lambda_2 v = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|v(y)|^q}{|x-y|}\,dy|v|^{q-2}v + \beta u & \text{in } \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |u|^2\,dx = a, \ \int\limits_{\mathbb{R}^3} |v|^2\,dx = b, \end{cases}$$

where $\frac{5}{3} < p,q < \frac{7}{3}$, $\mu_1,\mu_2 > 0$, $a,b \geq 0$, $\beta \in \mathbb{R}\setminus\{0\}$, $\lambda_1,\lambda_2 \in \mathbb{R}$ are Lagrange multipliers and $V_1(x), V_2(x) : \mathbb{R}^3 \to \mathbb{R}$ are trapping potentials. We prove the existence of the solutions with prescribed $L^2(\mathbb{R})$-norm with trivial trapping potentials and nontrivial trapping potentials by applying the rearrangement inequalities.

1. Introduction and main results

The two-component nonlinear Schrödinger system with Hartree type nonlinears

$$\begin{cases} -i\frac{\partial \Phi_1}{\partial t} + V_1(x)\Phi_1 = \frac{\hbar^2}{2m}\Delta\Phi_1 + \mu_1(C(x) * |\Phi_1|^p)|\Phi_1|^{p-2}\Phi_1 + \beta\Phi_2, \\ -i\frac{\partial \Phi_2}{\partial t} + V_2(x)\Phi_2 = \frac{\hbar^2}{2m}\Delta\Phi_2 + \mu_2(C(x) * |\Phi_2|^q)|\Phi_2|^{q-2}\Phi_2 + \beta\Phi_1, \end{cases} \tag{1.1}$$

has attracted a great deal of attraction in physics recently. Here Φ_i: $\mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ for $i = 1,2$, $V_1(x)$ and $V_2(x)$ are the external potentials, $\hbar$ is the Plank constant, m is the mass of the particles, $\mu_1,\mu_2 > 0$ imply the self-focusing strength in the component of the beam and $C(x)$ is the response function and possesses information on the mutual interaction between the particles. $\beta > 0$ (or < 0) is a coupling coefficient which measures the interaction between the two components of the beam. The i-th component of the beam in Kerr-like photorefractive media is denoted by solution Φ_i for $i = 1,2$. The existence of self-trapping of incoherent beam in a nonlinear medium has been proved by experiments in [25, 26]. Due to the optical pulses propagating in a linear medium have a natural tendency to broaden respectively in time (dispersion) and space (diffraction), the observation of (1.1) is important.

It is well-known that nonlinear Schrödinger equations for the mass-subcritical case are extensive studied recently. Our study is motivated by the following nonlinear Schrödinger equations

$$\begin{cases} -\Delta u = +\lambda_1 u + \mu_1|u|^{p-2}u + \gamma_1\beta|u|^{\gamma_1-2}uv^{\gamma_2} & \text{in } \mathbb{R}^3, \\ -\Delta v = +\lambda_2 v + \mu_2|v|^{q-2}v + \gamma_2\beta|v|^{\gamma_2-2}vu^{\gamma_1} & \text{in } \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |u|^2\,dx = a, \ \int\limits_{\mathbb{R}^3} |v|^2\,dx = b. \end{cases} \tag{1.2}$$

Guo and Jeanjean [7] studied (1.2) and proved the orbital stability of the standing waves associated to the set of minimizers for the mass-subcritical case. Then Chen and Zou [3] considered the existence

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of the normalized solutions of the following elliptic system with linear coupling nonlinearities for the mass-subcritical case

$$\begin{cases} -\Delta u + V_1(x)u + \lambda_1 u = \mu_1|u|^{p-2}u + \beta v & \text{in } \mathbb{R}^3, \\ -\Delta v + V_2(x)v + \lambda_2 v = \mu_2|v|^{q-2}v + \beta u & \text{in } \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |u|^2\,dx = a, \ \int\limits_{\mathbb{R}^3} |v|^2\,dx = b. \end{cases} \tag{1.3}$$

Moreover, the nonlinear interaction can be of nonlocal nature in lots of situations. For instance, under the influence of an external potential and two-body attractive interaction between two particles for bosons or electrons, the condensate in the mean field regime is run by the nonlinear Hartree equation (see [4–6, 9])

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi + V\psi - \chi(C(x) * |\psi|^p)|\psi|^{p-2}\psi, \ x \in \mathbb{R}^3. \tag{1.4}$$

The function ψ is a radially symmetric two-body potential and V is the trapping potential, $*$ denotes the convolution in $\mathbb{R}^3$. Equation (1.4) turns to be the well-known Choquard equation [14, 17, 19] in the case $V = 0$, which arises from the model of wave propagation in a media with a large response length [1]. The nonlocal term of (1.4) describes interaction between the bosons in the condensate [22, 24].

The general singular semilinear Choquard equation problem

$$-\Delta u - \mu u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \ \mu \in \mathbb{R}, \tag{1.5}$$

has been studied in [21, 32, 33]. Moroz and Schaftingen [21] proved that any positive groundstates about some point were radially symmetric and monotone decaying for (1.5). The parameters N, α, p are essential from the theoretical point of view. The case of dimension $N = 3$ of (1.5), Xiang [32] studied the uniqueness and nondegeneracy results for ground states, provided that $p > 2$ and p is sufficiently close to 2. For $N \geq 1$ and $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha}{(N-2)_+}$ Ye [33] proved the existence of the solutions of (1.5). In the case $p = 2$, $N = 3$ and $I_\alpha = |x|^{-1}$, (1.5) is deduced to

$$-\Delta u - \mu u = (|x|^{-1} * |u|^2)u \quad \text{in } \mathbb{R}^3, \ \mu \in \mathbb{R}. \tag{1.6}$$

By using symmetrical decreasing rearrangement inequalities to (1.6), the author of [14] proposed an existence and uniqueness of the minimizing solution. Equation (1.6) is called the nonlinear Hartree or Schrödinger-Newton equation, and the problem also has been widely studied in phisics. In the physical sense, (1.6) was not only used to describe the quantum mechanics of a static polaron by Pekar [23], but also used by Choquard in a certain approximating to Hartree-Fock theory of one component plasma to describe an electron trapped in its own hole in 1976 by Lieb [14]. The readers may turn to [8, 15–17, 20] and the references therein for more mathematical and physics background.

On the one hand, the nonlinear Schrödinger system with nonlinear couples also has been intensively studied in the past twenty years. Wang and Yang in [31] proved the existence and nonexistence of $L^2(\mathbb{R}^N)$-normalized solutions of coupled Hartree equations, the system given by

$$\begin{cases} -\Delta u + V_1(x)u = \lambda_1 u + \mu_1\left(\int\limits_{\mathbb{R}^3} \frac{u(y)^2}{|x-y|^\alpha}\,dy\right)u + \beta\left(\int\limits_{\mathbb{R}^3} \frac{v(y)^2}{|x-y|^\alpha}\,dy\right)u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \lambda_2 v + \mu_2\left(\int\limits_{\mathbb{R}^3} \frac{v(y)^2}{|x-y|^\alpha}\,dy\right)v + \beta\left(\int\limits_{\mathbb{R}^3} \frac{u(y)^2}{|x-y|^\alpha}\,dy\right)v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $\beta > 0$ and $\alpha = 2$, under certain type trapping potentials. By proving some delicate energy estimates, a precise description was given on the concentration behavior of minimizer solutions of (1.7). In addition, an optimal blowing up rate for the minimizer solutions of the system was also proved.

In [30], standing wave solutions of coupled nonlinear Hartree equations with nonlocal interaction were considered for (1.7) in the case $V_1 = V_2 = 0$ and $\alpha = 1$.

On the other hand, the general two-component of nonlinear Schrödinger equations system with nonlocal Hartree type interaction has also been studied. We consider the following system of elliptic equations

$$\begin{cases} -\Delta u + V_1(x)u + \lambda_1 u = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|}\, dy|u|^{p-2}u + \beta v \quad \text{ in } \mathbb{R}^3, \\ -\Delta v + V_2(x)v + \lambda_2 v = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|v(y)|^q}{|x-y|}\, dy|v|^{q-2}v + \beta u \quad \text{ in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

where $\frac{5}{3} < p, q < \frac{7}{3}$, $\mu_1, \mu_2 > 0$, $a, b \geq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{0\}$ is a coupling constant which describes attractive or repulsive interactions, and $V_1(x), V_2(x) : \mathbb{R}^3 \to \mathbb{R}$ are trapping potentials. We investigate the solutions $(u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ under the following constraints

$$\int\limits_{\mathbb{R}^3} |u|^2\, dx = a > 0, \int\limits_{\mathbb{R}^3} |v|^2\, dx = b > 0.$$

Thus $\lambda_1, \lambda_2 \in \mathbb{R}$ are considerd as Lagrange multipliers. In fact, the function $(\Phi_1(x,t), \Phi_2(x,t)) = (e^{i\lambda t}u(x), e^{i\lambda t}v(x))$ gives a nonlinear solitary wave for (1.1) whenever (u, v) solves (1.8).

To the best of our knowledge, there is no work concerning the system (1.8) no matter whether mass-subcritical or mass-supercritical in recent works. A major difficulty in searching for the solutions of (1.8) is to get the compactness of the embedding $H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Since the presence of the linear couple terms, it is difficult to tackle the compactness of the PS sequence.

In this paper, we are devoted to study the nonlinear Schrödinger equations (1.8) for the mass-subcritical case. Firstly, we consider the existence of the solutions of (1.8) with trivial trapping potential. If $V_i(x) \equiv 0$ for $i = 1, 2$, then (1.8) is reduced to the following nonlocal system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|}\, dy|u|^{p-2}u + \beta v \quad \text{ in } \mathbb{R}^3, \\ -\Delta v + \lambda_2 v = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|v(y)|^q}{|x-y|}\, dy|v|^{q-2}v + \beta u \quad \text{ in } \mathbb{R}^3. \end{cases} \tag{1.9}$$

Here the energy functional corresponding to (1.9) is defined by

$$\begin{aligned} J_\infty(u, v) =& \frac{1}{2} \int\limits_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla v|^2)\, dx - \frac{\mu_1}{2p} \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|}\, dx\, dy \\ & - \frac{\mu_2}{2q} \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{|v(x)|^q |v(y)|^q}{|x-y|}\, dx\, dy - \beta \int\limits_{\mathbb{R}^3} uv\, dx. \end{aligned} \tag{1.10}$$

Furthermore, the case, studying the existence of normalized solutions to (1.9), can be described equivalently by considering the limiting minimization problem

$$m_\infty(a, b) := \inf_{(u,v) \in S(a,b)} J_\infty(u, v),$$

where $S(a, b)$ is defined by

$$S(a, b) := \{(u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} |u|^2\, dx = a, \int\limits_{\mathbb{R}^3} |v|^2\, dx = b\}.$$

In this sense, λ_1 and λ_2 are Lagrange multipliers to be determined. For simplicity, we suppose the following conditions are always true throughout the paper

$$\textbf{(M)}\ \frac{5}{3} < p, q < \frac{7}{3}, \mu_1, \mu_2 > 0, a, b \geq 0, \beta \in \mathbb{R} \setminus \{0\}.$$

The first main result of the present paper is concerned about the existence of minimizers of (1.10) restricted to $S(a,b)$.

Theorem 1.1. *Assume that **(M)** holds, then $m_\infty(a,b)$ is attained at $(\bar{u},\bar{v})$, that is, (1.9) exists at least one ground state solution with the constraint in $S(a,b)$ such that $dJ_\infty|_{S(a,b)}(\bar{u},\bar{v}) = 0$ and $J_\infty(\bar{u},\bar{v}) = m_\infty(a,b)$. Furthermore, $(\bar{u},\bar{v})$ satisfies*

(i) *If $\beta > 0$, then $\bar{u},\bar{v}$ are both positive and radial.*
(ii) *If $\beta < 0$, then $\bar{u},\bar{v}$ are both radial, and either $\bar{u} > 0, \bar{v} < 0$ or $\bar{u} < 0, \bar{v} > 0$.*

Remark 1.2. *$\beta > 0$ and $\beta < 0$ decide the choice of minimizing sequence and the performance of minimizers, which can be seen in lemma 2.5. Moreover, this phenomenon also occurs in situation nontrivial $V_i(x)$ for $i = 1,2$.*

Due to we are interested in studying (1.8) with the linearly coupled systems. This is a quite delicate matter and still with difficulty to obtain optimal results. Motivated by [7], the authors mainly verified the compactness of the minimizing sequences with the rearrangement results by Shibata in [28]. Also motivated by Chen and Zou in [3], they proved the existence of the normalized solutions of (1.3) with linear coupling nonlinearities. The proof was based on the refined energy estimates. We shall refer to above great approaches to prove our main results. It is much more difficult for us to exclude the dichotomy of minimizing sequences. However, we can overcome the issue by a more accurate upper bound of the energy function $m_\infty(a,b)$. The key idea to deal with the problem is using a special test function.

In the rest results, we consider (1.8) with general potential $V_i(x)$ for $i = 1,2$, supposing that $V_i(x) \in C(\mathbb{R}^3)$ satisfies
(V1) $V_i(0) = \min_{x\in\mathbb{R}^3} V_i(x) = c_i > -\infty$ for $i = 1,2$,
(V2) $\lim_{|x|\to\infty} V_i(x) = \sup_{x\in\mathbb{R}^3} V_i(x) =: V_{i,\infty} \in (c_i, +\infty]$ for $i = 1,2$.
Indeed, the system (1.8) is related to the system of Euler-Lagrange equations of the following constrained problem

$$m(a,b) := \inf_{(u,v)\in S^*(a,b)} J(u,v),$$

where energy functional is defined by

$$\begin{aligned} J(u,v) =& \frac{1}{2}\int\limits_{\mathbb{R}^3} (|\nabla u|^2 + V_1(x)u^2 + |\nabla v|^2 + V_2(x)v^2)\,dx \\ & - \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy - \frac{\mu_2}{2q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|v(x)|^q|v(y)|^q}{|x-y|}\,dx\,dy - \beta\int\limits_{\mathbb{R}^3} uv\,dx, \end{aligned}$$

and $S^*(a,b)$ is given by

$$S^*(a,b) := \{(u,v) \in \mathcal{H} : \int\limits_{\mathbb{R}^3} |u|^2\,dx = a, \int\limits_{\mathbb{R}^3} |v|^2\,dx = b\},$$

here

$$\mathcal{H} := \{(u,v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : \left|\int\limits_{\mathbb{R}^3} V_1(x)u^2\,dx\right| < \infty, \left|\int\limits_{\mathbb{R}^3} V_2(x)v^2\,dx\right| < \infty\}.$$

Since different types of trapping potentials decide different results, the purpose of this paper is to provide some results in this respect. The coerciveness is essential for verifying the compactness result. We first concern a general case $V_i = \infty$ for $i = 1,2$, which is coercive. Our first result reads as follows.

Theorem 1.3. *Assume that **(M)** holds. If $V_i(x)$ satisfies **(V1)** and **(V2)** with $V_{i,\infty}=\infty$ for $i=1,2$, then there exists $(\bar{u},\bar{v})\in S(a,b)$ such that $m(a,b)$ is attained at $(\bar{u},\bar{v})$. Therefore, (1.8) exists at least one ground state solution $(\bar{u},\bar{v})$ in $S(a,b)$. Furthermore, $(\bar{u},\bar{v})$ satisfies*

(i) *If $\beta>0$, then $\bar{u},\bar{v}$ are both positive.*
(ii) *If $\beta<0$, then either $\bar{u}>0,\bar{v}<0$ or $\bar{u}<0,\bar{v}>0$.*

Finally, we consider the case of $V_{i,\infty}<\infty$. Without loss of generality, we suppose that $c_i<V_{i,\infty}=0$. Since $(V_i(x),\lambda_i)$ can be replaced by $(\tilde{V}_i(x),\tilde{\lambda}_i):=(V_i(x)-V_{i,\infty},\lambda_i+V_{i,\infty})$. We have the following result.

Theorem 1.4. *Assume that **(M)** holds. If $V_i(x)$ satisfies **(V1)** and **(V2)** with $c_i<V_{i,\infty}=0$ for $i=1,2$, then there exists $(\bar{u},\bar{v})\in S(a,b)$ such that $m(a,b)$ is attained at $(\bar{u},\bar{v})$. Therefore, (1.8) exists at least one ground state solution $(\bar{u},\bar{v})$ in $S(a,b)$. Furthermore, $(\bar{u},\bar{v})$ satisfies*

(i) *If $\beta>0$, then $\bar{u},\bar{v}$ are both positive.*
(ii) *If $\beta<0$, then either $\bar{u}>0,\bar{v}<0$ or $\bar{u}<0,\bar{v}>0$.*

Remark 1.5. *In fact, there are many functions satisfy **(V1)** and **(V2)** with $c_i<V_{i,\infty}=0$. We give some examples*

(i) $V_i(x)=-Ce^{-|x|}$ *for C is a positive constant;*
(ii) $V_i(x)=-\frac{1}{1+|x|}$.

Notice that we use the results deduced by Lions [18] when studying the nontrivial potentials case. He proved every minimizing sequence for the energy function $m(a,b)$ has a convergent subsequence in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, which is equivalent to the subadditivity condition, i.e.,

$$m(a_1+a_2,b_1+b_2)<m(a_1,b_1)+m_\infty(a_2,b_2)$$

for any $a_2+b_2>0$. By reduction to absurdity, [10] showed the compactness of the minimizing sequence. We also can show the compactness of the minimizing sequence by reduction to absurdity. Motivated by Shibata in [27], our paper is devoted to use a new rearrangement approach. After using this new approach we find it is very easy to study the problem (1.8). We hope this new approach can be applied to more problems to simplify the proof.

Throughout this paper, we will use the following notations.

- Set $H:=H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ and $H_r=H_r^1\times H_r^1$, where H_r^1 denotes the redial symmetric with respect to 0, which is the subspace of $H^1(\mathbb{R}^3)$.
- $o(1)$ denotes a quantity which tends to 0.
- $|\cdot|_p$ means the standard norm of $L^p(\mathbb{R}^3)$.
- $S_r(a,b)$ denotes $S(a,b)\cap H_r$.
- $\|(u,v)\|_H:=\big(\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)\,dx\big)^{\frac{1}{2}}$ is the norm of H.
- $\rightharpoonup$ means weak convergence.
- u^* denotes the symmetric decreasing rearrangement of $u\in H^1(\mathbb{R}^3)$, recalling that

$$\begin{aligned}|\nabla u^*|_2&\le|\nabla u|_2,\\ \int_{\mathbb{R}^3}uv\,dx&\le\int_{\mathbb{R}^3}u^*v^*\,dx,\\ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy&\le\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u^*(x)|^p|u^*(y)|^p}{|x-y|}\,dx\,dy,\end{aligned}\tag{1.11}$$

for $p>1$ in [12].

This paper is organized as follows. Section 2 is focused on introducing some preliminary results. In Section 3 we deal with the case of trivial potentials. In Section 4 we shall address the case of nontrivial potentials.

2. Some preliminaries

In this section, we investigate some significant lemmas which will be used in proving our main results.

We first consider singular Choquard equation with $L^2(\mathbb{R}^3)$-constraint. For fixed a, $\mu>0$, $\frac{5}{3}<p<\frac{7}{3}$, we concern the existence of solution $(\lambda,u)\in\mathbb{R}\times H^1(\mathbb{R}^3)$ to the following system

$$\begin{cases}-\Delta u+\lambda u=\mu\int\limits_{\mathbb{R}^3}\frac{|u(y)|^p}{|x-y|}\,dy|u|^{p-2}u \qquad \text{in } \mathbb{R}^3,\\ \int\limits_{\mathbb{R}^3}|u|^2\,dx=a.\end{cases} \tag{2.1}$$

The energy functional of above system is given by

$$J_{\mu,p}(u)=\frac{1}{2}\int\limits_{\mathbb{R}^3}|\nabla u|^2\,dx-\frac{\mu}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy, \tag{2.2}$$

which is constrained on $S(a)$, where $S(a)=\{u\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}|u|^2\,dx=a\}$. By using variational methods, the existence of solutions of (2.1) can be described equivalently by nonnegative minimizers of the following problem

$$m_{\mu,p}(a):=\inf_{u\in S(a)}J_{\mu,p}(u). \tag{2.3}$$

Then we have the following lemma.

Lemma 2.1. *Assume that a, $\mu>0$, $\frac{5}{3}<p<\frac{7}{3}$, then up to a translation (2.1) has positive solutions with $\lambda>0$. Furthermore,*

$$m_{\mu,p}(a)=\inf_{u\in S(a)}J_{\mu,p}(u)<0.$$

Proof. The existence of positive solutions with $\lambda>0$ of (2.3) have been proved by Xiang [32]. Thus we only prove $m_{\mu,p}(a)=\inf_{u\in S(a)}J_{\mu,p}(u)<0$. For any $u\in(a)$, we have $t^{\frac{3}{2}}u(tx)\in S(a)$ for all $t>0$. After a simple calculation obtains that

$$m_{\mu,p}(a)=\inf_{t>0}J_{\mu,p}\big(t^{\frac{3}{2}}u(tx)\big)=C_1(p)\bigg(\frac{D_p(u)^2}{K(u)^{3p-5}}\bigg)^{\frac{1}{7-3p}}<0. \tag{2.4}$$

where $C_1(p)=\frac{7-3p}{3p-5}(\frac{3p-5}{2})^{\frac{2}{7-3p}}$, $K(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx$ and $D_p(u)=\frac{\mu}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy$. This finishes the proof of Lemma 2.1. $\square$

We next recall the following classical Hardy-Littlewood-Sobolev inequality.

Lemma 2.2. *[12] Assume that $f\in L^s(\mathbb{R}^N)$, $g\in L^t(\mathbb{R}^N)$, then we have*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^\lambda}\,dx\,dy\le C(s,t,\lambda)|f|_s|g|_t,$$

where $1<s,t<\infty$, $0<\lambda<N$ and $\frac{1}{s}+\frac{1}{t}+\frac{\lambda}{N}=2$.

Recalling Lemma 2.2, there exists a positive constant A such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^s|u(y)|^s}{|x-y|}\,dx\,dy \leq A|u|_{\frac{6s}{5}}^{2s}, \qquad \forall\, u \in L^{\frac{6s}{5}}(\mathbb{R}^3), \tag{2.5}$$

for $\frac{5}{3} < s < 5$. Then combining interpolation inequality, we obtain that

$$|u|_{\frac{6s}{5}} \leq |u|_2^{\theta}|u|_6^{1-\theta}, \qquad \forall\, u \in L^2(\mathbb{R}^3)\cap L^6(\mathbb{R}^3), \tag{2.6}$$

where $0 < \theta = \frac{5-s}{2s} < 1$. By Sobolev inequality, there existence a positive constant B such that

$$|u|_6 \leq B|\nabla u|_2, \qquad \forall\, u \in H^1(\mathbb{R}^3). \tag{2.7}$$

Therefore, combining (2.5), (2.6) and (2.7), we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^s|u(y)|^s}{|x-y|}\,dx\,dy \leq AB^{3s-5}|u|_2^{5-s}|\nabla u|_2^{3s-5}. \tag{2.8}$$

It is vital to verify the convergence of minimizing sequences to prove Theorems 1.1, 1.3 and 1.4. Therefore, we now recall some lemmas of this result given in [2] by Brezis and Lieb.

Lemma 2.3. *Assume that $\{(u_n,v_n)\} \subset H$ is a bounded sequence, $(u_n,v_n) \rightharpoonup (u,v)$ in H, then we obtain that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \big(|\nabla u_n|^2 - |\nabla u|^2 - |\nabla(u_n-u)|^2)\big)\,dx = 0.$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \big(|\nabla v_n|^2 - |\nabla v|^2 - |\nabla(v_n-v)|^2)\big)\,dx = 0.$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} \big(u_nv_n - uv - (u_n-u)(v_n-v)\big)\,dx = 0.$$

The following lemma is essential to verify the compactness for the nonlocal term of the functional.

Lemma 2.4. *[21] Let $N \geq 3$, $\alpha \in (0,N)$, $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$, and $\{u_n\}$ be a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. If $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to +\infty$, then*

$$\lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}(I_\alpha * |u_n|^p)|u_n|^p\,dx - \int_{\mathbb{R}^N}(I_\alpha * |u_n-u|^p)|u_n-u|^p\,dx\Big) = \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx.$$

Next, we summarize a list of basic properties of a minimizing sequence for $m(a,b)$.

Lemma 2.5. *Let $\{(u_n,v_n)\} \subset S(a,b)$ be a minimizing sequence for $m(a,b)$. Then we obtain that*

(i) *for $\beta > 0$, $\{(|u_n|,|v_n|)\}$ is also a minimizing sequence;*
(ii) *for $\beta < 0$, $\{(|u_n|,-|v_n|)\}$ or $\{(-|u_n|,|v_n|)\}$ is also a minimizing sequence.*

Proof. Combining $\int_{\mathbb{R}^3} \big|\nabla|u|\big|^2\,dx \leq \int_{\mathbb{R}^3} |\nabla u|^2\,dx$ and $\int_{\mathbb{R}^3} \big|\nabla|v|\big|^2\,dx \leq \int_{\mathbb{R}^3} |\nabla v|^2\,dx$ and noticing $\int_{\mathbb{R}^3} uv\,dx \leq \int_{\mathbb{R}^3} |u||v|\,dx$, for $\beta > 0$, we deduce that

$$\begin{aligned}
J(|u|,|v|) =&\frac{1}{2}\int\limits_{\mathbb{R}^3}(\big|\nabla|u|\big|^2 + V_1(x)u^2 + \big|\nabla|v|\big|^2 + V_2(x)v^2)\,dx \\
&- \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy - \frac{\mu_2}{2q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|v(x)|^q|v(y)|^q}{|x-y|}\,dx\,dy - \beta\int\limits_{\mathbb{R}^3}|u||v|\,dx \\
&\leq \frac{1}{2}\int\limits_{\mathbb{R}^3}(|\nabla u|^2 + V_1(x)u^2 + |\nabla v|^2 + V_2(x)v^2)\,dx \\
&- \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy - \frac{\mu_2}{2q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|v(x)|^q|v(y)|^q}{|x-y|}\,dx\,dy - \beta\int\limits_{\mathbb{R}^3}uv\,dx \\
&= J(u,v),
\end{aligned}\tag{2.9}$$

hence we have $J(|u_n|,|v_n|) \leq J(u_n,v_n)$.

Similar to the above arguments, for $\beta < 0$, we have $J(-|u_n|,|v_n|) = J(|u_n|,-|v_n|) \leq J(u_n,v_n)$. This completes the proof of Lemma 2.5. $\square$

Lemma 2.5 implies that the case $\beta < 0$ can be treated as the case $\beta > 0$. In the rest of this paper, we shall only consider the case $\beta > 0$. In next lemma we continue to summarize some properties of a minimizing sequence for $m(a,b)$.

Lemma 2.6. *Assume that $(u_0,v_0) \in H$ and $\{(u_n,v_n)\}$ is a minimizing sequence for $m(a,b)$ with $(u_n,v_n) \to (u_0,v_0)$ in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, then we obtain that $(u_n,v_n) \to (u_0,v_0)$ in H.*

Proof. Using the fact that $|u_n - u_0|_2^2 + |v_n - v_0|_2^2 \to 0$, we obtain that $|u_0|_2^2 = a$, $|v_0|_2^2 = b$ and

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^3}\left(V_1(x)u_n^2 + V_2(x)v_n^2)\right)dx = \int\limits_{\mathbb{R}^3}\left(V_1(x)u_0^2 + V_2(x)v_0^2)\right)dx.$$

According to (2.8), we derive that

$$\begin{aligned}
J_\infty(u,v) =&\frac{1}{2}\int\limits_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v|^2)\,dx - \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy \\
&- \frac{\mu_2}{2q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|v(x)|^q|v(y)|^q}{|x-y|}\,dx\,dy - \beta\int\limits_{\mathbb{R}^3}uv\,dx \\
&\geq \frac{1}{2}(|\nabla u|_2^2 + |\nabla v|_2^2) - \frac{\mu_1 a^{\frac{5-p}{2}}AB^{3p-5}}{2p}|\nabla u|_2^{3p-5} \\
&- \frac{\mu_2 b^{\frac{5-q}{2}}AB^{3q-5}}{2q}|\nabla v|_2^{3q-5} - \beta a^{\frac{1}{2}}b^{\frac{1}{2}},
\end{aligned}\tag{2.10}$$

where $(u,v) \in S(a,b)$. Since $0 < 3p-5, 3q-5 < 2$, we deduce that $J_\infty(u,v)$ is bounded from below and coercive on $S(a,b)$. By using similar arguments as in the above we can prove the coerciveness of J. By the coerciveness of J, we know that the sequence $\{(u_n,v_n)\}$ is bounded in H. Up to a subsequence, we have $(u_n,v_n) \rightharpoonup (\hat{u},\hat{v})$ in H. Moreover, combining $(u_n,v_n) \to (u_0,v_0)$ in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, then we have $(\hat{u}, \hat{v}) = (u_0, v_0)$, that is, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in H. We infer from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned} m(a,b) &\le J(u_0, v_0) \\ &\le \lim_{n\to\infty} \inf J(u_n, v_n) \\ &= m(a,b), \end{aligned}$$

which implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + |\nabla v_n|^2)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} (|\nabla u_0|^2 + |\nabla v_0|^2)\, dx. \tag{2.11}$$

Combining (2.11) with $(\nabla u_n, \nabla v_n) \rightharpoonup (\nabla u_0, \nabla v_0)$ in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, then we conclude that $(u_n, v_n) \to (u_0, v_0)$ in H. This finishes the proof. $\square$

Finally, we shall introduce a different transform to replace the rearrangement results in [27] as expressed in [11]. Here we assume that $u, v \in H^1(\mathbb{R}^3)$, then $\sqrt{u^2+v^2}$ has properties given by the following lemma.

Lemma 2.7. *For all $u, v \in H^1(\mathbb{R}^3)$, there satisfies*

(i) *If $v(x) > 0$,*

$$\int_{\mathbb{R}^3} |\nabla \sqrt{u^2+v^2}|^2\, dx \le \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla v|^2)\, dx,$$

where the equality holds if and only if $u(x) = cv(x)$, here c is a constant.

(ii) *For $\frac{5}{3} < p < \frac{7}{3}$,*

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\sqrt{u(x)^2+v(x)^2}|^p |\sqrt{u(y)^2+v(y)^2}|^p}{|x-y|}\, dx\, dy \\ &\ge \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p |u(y)|^p}{|x-y|}\, dx\, dy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^p |v(y)|^p}{|x-y|}\, dx\, dy. \end{aligned} \tag{2.12}$$

(iii) *Set $0 \le u_1, u_2, v_1, v_2 \in H^1(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3} \sqrt{u_1^2+v_1^2}\sqrt{u_2^2+v_2^2}\, dx \ge \int_{\mathbb{R}^3} (u_1 v_1 + u_2 v_2)\, dx. \tag{2.13}$$

Proof. (i) This has been proved in [12].

(ii) For $a_1, a_2, b_1, b_2 \ge 0$, $t > \frac{5}{6}$, we obtain that

$$(a_1+a_2)^t (b_1+b_2)^t \ge a_1^t b_1^t + a_2^t b_2^t,$$

and the equality holds if and only if $a_1 b_2 = a_2 b_1 = 0$. Let $u(x)^2 = a_1$, $v(x)^2 = a_2$, $u(y)^2 = b_1$, $v(y)^2 = b_2$, $t = \frac{p}{2}$, multiplying by $|x-y|^{-1}$ both sides and integrating over $\mathbb{R}^3$ both sides, then we obtain (2.12). In fact, we see that (2.12) attains the equality if and only if $u = 0$ or $v = 0$.

(iii) For $a_1, a_2, b_1, b_2 \ge 0$, we obtain that

$$(a_1^2+a_2^2)(b_1^2+b_2^2) \ge (a_1 b_1 + a_2 b_2)^2.$$

Let $u_1(x) = a_1$, $u_2(x) = a_2$, $v_1(x) = b_2$, $v_2(x) = b_2$, then we obtain (2.13). $\square$

3. Proof of Theorem 1.1

In this section, we still use the notation $m_\infty(a,b)$ for $a,b\geq 0$, which implies one component of (a,b) may be zero. Firstly, we summarize a list of basic properties of $m_\infty(a,b)$.

Lemma 3.1. *Assume that **(M)** holds, then we have the basic properties of $m_\infty(a,b)$ as follows*

(i) *For any $a,b\geq 0$, if either $a>0$ or $b>0$, then we have*

$$-\infty < m_\infty(a,b) < 0.$$

(ii) *$m_\infty(a,b)$ is continuous with respect to $a,b\geq 0$.*

(iii) *For any $a_1,a_2,b_1,b_2\geq 0$, we have*

$$m_\infty(a_1+a_2,b_1+b_2)\leq m_\infty(a_1,b_1)+m_\infty(a_2,b_2).$$

Proof. (i) For the case $ab\neq 0$, we rewrite the functional $J_\infty(u,v)$ defined in (1.10) for convenience as follows

$$\begin{aligned}J_\infty(u,v) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)\,dx - \frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy \\ &\quad - \frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v(x)|^q|v(y)|^q}{|x-y|}\,dx\,dy - \beta\int_{\mathbb{R}^3} uv\,dx \\ &= J_{\mu_1,p}(u)+J_{\mu_2,q}(v)-\beta\int_{\mathbb{R}^3} uv\,dx.\end{aligned}$$

where $J_{\mu_1,p}(u)$ and $J_{\mu_2,q}(v)$ have been defined in (2.2). If $a,b>0$, from Lemma 2.1, then we know $m_{\mu_1,p}(a)$ and $m_{\mu_2,q}(b)$ exist minimizer u_0 and v_0 respectively, such that

$$m_\infty(a,b)\leq J_\infty(u_0,v_0)=J_{\mu_1,p}(u_0)+J_{\mu_2,q}(v_0)-\beta\int_{\mathbb{R}^3} u_0v_0\,dx<0.$$

For the case $a=0$ or $b=0$, then we set $u=0$, $v=v_0$ or $u=u_0$, $v=0$, we also obtain above inequality. By (2.10), we deduce that $J_\infty(u,v)$ is bounded from below. Thus $-\infty<m_\infty(a,b)<0$ holds.

For (ii)(iii), we will prove with potentials as a more general case in Lemma (4.2). $\square$

Let $(u_n,v_n)\subset H$ be the minimizing sequence of $m_\infty(a,b)$, then we need to verify the compactness of minimizing sequence in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$. Therefore, we first consider the following argument.

Lemma 3.2. *Assume that **(M)** holds, then we have*

$$\inf_{(u,v)\in S_r(a,b)} J_\infty(u,v) = \inf_{(u,v)\in S(a,b)} J_\infty(u,v). \tag{3.1}$$

Proof. It is easy to see that $\inf_{(u,v)\in S_r(a,b)} J_\infty(u,v)\geq \inf_{(u,v)\in S(a,b)} J_\infty(u,v)$. For any $(u,v)\in H$, we study the rearrangement of (u,v) denoted by (u^*,v^*), then applying the rearrangement inequalities

(1.11), we have that

$$\begin{aligned}
J_\infty(u^*,v^*) &= \frac{1}{2}\int\limits_{\mathbb{R}^3}(|\nabla u^*|^2+|\nabla v^*|^2)\,dx - \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u^*(x)|^p|u^*(y)|^p}{|x-y|}\,dx\,dy\\
&\quad - \frac{\mu_2}{2q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|v^*(x)|^q|v^*(y)|^q}{|x-y|}\,dx\,dy - \beta\int\limits_{\mathbb{R}^3}u^*v^*\,dx\\
&\le \frac{1}{2}\int\limits_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)\,dx - \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|}\,dx\,dy\\
&\quad - \frac{\mu_2}{2q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|v(x)|^q|v(y)|^q}{|x-y|}\,dx\,dy - \beta\int\limits_{\mathbb{R}^3}uv\,dx\\
&= J_\infty(u,v).
\end{aligned}$$

Hence combining above two inequalities, we obtain (3.1). □

Since the coercive of J_∞ on $S_r(a,b)$, we conclude that the minimizing sequence (u_n,v_n) is bounded in H_r, then we deduce that $(u_n,v_n)\rightharpoonup(\bar u,\bar v)$ in H_r. In view of the compact embedding $H^1_r(\mathbb{R}^3)\hookrightarrow\hookrightarrow L^s(\mathbb{R}^3)$ for $2<s<6$(see [29]), we know that $(u_n,v_n)\to(\bar u,\bar v)$ in $L^s(\mathbb{R}^3)\times L^t(\mathbb{R}^3)$ for $2<s,t<6$. Next we give the compactness of (u_n,v_n) in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$.

Lemma 3.3. *Assume that **(M)** holds, then we have*

$$|\bar u|_2^2=a,\ \ |\bar v|_2^2=b.$$

Proof. We argue by contradiction. Suppose, on the contrary, that

$$\int\limits_{\mathbb{R}^3}|\bar u|^2\,dx := a_1\le a,\ \ \int\limits_{\mathbb{R}^3}|\bar v|^2\,dx := b_1\le b,$$

set $a_2=a-a_1$, $b_2=b-b_1$. If $a_1+b_1<a+b$, then we study the following three cases.

Case 1: $a_1<a$, $b_1<b$. In this case, we define $\tilde u_n=u_n-\bar u$, $\tilde v_n=v_n-\bar v$, by the Brezis-Lieb Lemma, then we obtain that

$$|\tilde u_n|_2^2\to a_2>0,\ \ |\tilde v_n|_2^2\to b_2>0. \tag{3.2}$$

Combining Lemma 2.3, Lemma 2.4 and $(u_n.v_n)\rightharpoonup(\tilde u,\tilde v)$ in H_r, we deduce that

$$\begin{aligned}
m_\infty(a,b) &= J_\infty(u_n,v_n)+o(1)\\
&= J_\infty(\bar u,\bar v)+J_\infty(\tilde u,\tilde v)+o(1)\\
&\ge m_\infty(a_1,b_1)+J_\infty(\tilde u,\tilde v)+o(1).
\end{aligned}\tag{3.3}$$

Since $\tilde u_n\to0$, $\tilde v_n\to0$ in $L^s(\mathbb{R}^3)$, $2<s<6$, by Lemma 2.2 and (3.2), we obtain that

$$\begin{aligned}
J_\infty(\tilde u_n,\tilde v_n) &= \frac{1}{2}\int\limits_{\mathbb{R}^3}\left(|\nabla\tilde u_n|^2+|\nabla\tilde v_n|^2\right)dx-\beta\int\limits_{\mathbb{R}^3}\tilde u_n\tilde v_n\,dx+o(1)\\
&\ge -\beta a_2^{\frac12}b_2^{\frac12}+o(1).
\end{aligned}\tag{3.4}$$

If $a_2 \geq b_2$, then $(u, \sqrt{\frac{b_2}{a_2}}u) \in S(a_2, b_2)$ when $u \in S(a_2)$, thus

$$\begin{aligned}
&J_\infty(u, \sqrt{\frac{b_2}{a_2}}u) \\
=& \frac{1}{2}\int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + \frac{b_2}{a_2}|\nabla u|^2\right) dx - \frac{\mu_1}{2p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|}\, dx\, dy \\
&- \frac{\mu_2 b_2^q}{2q a_2^q}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)|^q|u(y)|^q}{|x-y|}\, dx\, dy - \beta a_2^{\frac{1}{2}} b_2^{\frac{1}{2}} \\
\leq& 2\Big[\frac{1}{2}\int\limits_{\mathbb{R}^3} \left(|\nabla u|^2\, dx - \frac{\mu_1}{4p}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|}\, dx\, dy\Big] - \beta a_2^{\frac{1}{2}} b_2^{\frac{1}{2}} \\
=& 2J_{\frac{\mu_1}{2},p}(u) - \beta a_2^{\frac{1}{2}} b_2^{\frac{1}{2}},
\end{aligned} \tag{3.5}$$

where $J_{\frac{\mu_1}{2},p}(u)$ is defined as in (2.2). From Lemma 2.2, without loss of generality, we assume that u_0 is a positive solution of (2.3) with $|u_0|_2^2 = a_2$, then we have $J_{\frac{\mu_1}{2},p}(u_0) < 0$. Therefore,

$$\begin{aligned}
m_\infty(a_2, b_2) &\leq J_\infty(u_0, \sqrt{\frac{b_2}{a_2}}u_0) \\
&\leq 2J_{\frac{\mu_1}{2},p}(u_0) - \beta a_2^{\frac{1}{2}} b_2^{\frac{1}{2}} \\
&< -\beta a_2^{\frac{1}{2}} b_2^{\frac{1}{2}}.
\end{aligned} \tag{3.6}$$

Combining (3.3)-(3.6) and Lemma 3.1 (iii), we obtain that

$$\begin{aligned}
m_\infty(a, b) &\geq m_\infty(a_1, b_1) + m_\infty(a_2, b_2) - 2J_{\frac{\mu_1}{2},p}(u_0) \\
&> m_\infty(a_1, b_1) + m_\infty(a_2, b_2) \\
&\geq m_\infty(a, b),
\end{aligned}$$

which is a contradiction. If $a_2 < b_2$, the proof is similar to the above arguments.

Case 2: If $a_1 < a$, $b_1 = b$, then $a_2 > 0$, $b_2 = 0$. Similarly, like Case 1, we obtain that

$$\begin{aligned}
m_\infty(a, b) &= J_\infty(\bar{u}, \bar{v}) + J_\infty(\tilde{u}_n, \tilde{v}_n) + o(1) \\
&\geq m_\infty(a_1, b) + o(1) \\
&\geq m_\infty(a_1, b) + m_\infty(a_2, 0) - J_{\mu_1,p}(u_0) + o(1) \\
&> m_\infty(a_1, b) + m_\infty(a_2, 0) \\
&\geq m_\infty(a, b).
\end{aligned} \tag{3.7}$$

which is a contradiction.

Case 3: If $a_1 = a$, $b_1 < b$, the proof is similar to Case 2. Therefore, the proof of Lemma 3.3 is complete now. □

Proof of Theorem 1.1. We remark that Lemma 3.3 implies that $(u_n, v_n) \to (\bar{u}, \bar{v})$ in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, combining with the fact Lemma 2.5, then we conclude that $(u_n, v_n) \to (\bar{u}, \bar{v})$ in H. Moreover, $(\bar{u}, \bar{v})$ is a minimizer, that is, $(\bar{u}, \bar{v}) \in S(a, b)$ and $J_\infty(\bar{u}, \bar{v}) = m_\infty(a, b)$. Therefore, $(\bar{u}, \bar{v})$ satisfies

$$\begin{cases}
-\Delta \bar{u} + \lambda_1 \bar{u} = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|\bar{u}(y)|^p}{|x-y|}\, dy |\bar{u}|^{p-2}\bar{u} + \beta \bar{v} & \text{in } \mathbb{R}^3, \\
-\Delta \bar{v} + \lambda_2 \bar{v} = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|\bar{v}(y)|^q}{|x-y|}\, dy |\bar{v}|^{q-2}\bar{v} + \beta \bar{u} & \text{in } \mathbb{R}^3,
\end{cases}$$

with $|\bar{u}|_2^2 = a$, $|\bar{v}|_2^2 = b$, $\bar{u}, \bar{v} \geq 0$. Then we apply the maximum principle, $\bar{u}, \bar{v} > 0$. By Lemma 2.5, the proof of Theorem 1.8 is complete. $\square$

After finishing the proof of Theorem 1.8, we also need to prove the strict subadditivity of $m_\infty(a,b)$. This property will be used in proving Theorem 1.4.

Lemma 3.4. *Assume that **(M)** holds, then we have*

$$m_\infty(a_1 + a_2, b_1 + b_2) < m_\infty(a_1, b_1) + m_\infty(a_2, b_2),$$

where $a_1 + b_1 > 0$, $a_2 + b_2 > 0$.

Proof. Since the existence of the minimizers of $m_\infty(a,b)$, we deduce that there exists $(u_1, v_1) \in S(a_1, b_1)$ and $(u_2, v_2) \in S(a_2, b_2)$ such that $J_\infty(u_1, v_1) = m_\infty(a_1, b_1)$ and $J_\infty(u_2, v_2) = m_\infty(a_2, b_2)$. Without loss of generality, assume that $a_1 > 0$. If $a_2 > 0$, we can deduce that both u_1 and u_2 are both positive, then by Lemma 2.7, we obtain that

$$\begin{aligned}
m_\infty(a_1 + a_2, b_1 + b_2) &\leq J_\infty\left(\sqrt{u_1^2 + u_2^2}, \sqrt{v_1^2 + v_2^2}\right) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla \sqrt{u_1^2 + u_2^2}|^2 + |\nabla \sqrt{v_1^2 + v_2^2}|^2\right) dx \\
&\quad - \frac{\mu_1}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\sqrt{u_1(x)^2 + u_2(x)^2}|^p |\sqrt{u_1(y)^2 + u_2(y)^2}|^p}{|x-y|} \, dx \, dy \\
&\quad - \frac{\mu_2}{2q} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\sqrt{v_1(x)^2 + v_2(x)^2}|^q |\sqrt{v_1(y)^2 + v_2(y)^2}|^q}{|x-y|} \, dx \, dy \\
&\quad - \beta \int_{\mathbb{R}^3} \sqrt{u_1^2 + u_2^2} \sqrt{v_1^2 + v_2^2} \, dx \\
&< \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla v_1|^2 + |\nabla v_2|^2\right) dx \\
&\quad - \frac{\mu_1}{2p} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{|u_1(x)|^p |u_1(y)|^p}{|x-y|} + \frac{|u_2(x)|^p |u_2(y)|^p}{|x-y|}\right) dx \, dy \\
&\quad - \frac{\mu_2}{2q} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{|v_1(x)|^q |v_1(y)|^q}{|x-y|} + \frac{|v_2(x)|^q |v_2(y)|^q}{|x-y|}\right) dx \, dy \\
&\quad - \beta \int_{\mathbb{R}^3} (u_1 v_1 + u_2 v_2) \, dx \\
&= J_\infty(u_1, v_1) + J_\infty(u_2, v_2) \\
&= m_\infty(a_1, b_1) + m_\infty(a_2, b_2).
\end{aligned}$$

If $a_2 = 0$, then $b_2 > 0$, we first consider the case $b_1 > 0$. By repeating above process, we also have the same conclusion. Next, we consider the case $b_1 = 0$. From Lemma 2.1, there exists positive solutions

$u_0 \in S(a_1)$ and $v_0 \in S(b_2)$ such that $m_{\mu_1,p}(a_1) = J_{\mu_1,p}(u_0)$ and $m_{\mu_2,q}(b_2) = J_{\mu_2,q}(v_0)$, then we have

$$\begin{aligned} m_\infty(a,b) &= m_\infty(a_1,b_2) \\ &\le J_\infty(u_0,v_0) \\ &= J_{\mu_1,p}(u_0) + J_{\mu_2,q}(v_0) - \beta \int\limits_{\mathbb{R}^3} u_0 v_0 \, dx \\ &< m_\infty(a_1,0) + m_\infty(0,b_2). \end{aligned}$$

This is complete the proof of Lemma 3.4. □

4. Proof of Theorem 1.3 and 1.4

In this section, we mainly consider the existence of solutions of (1.8) with general potential $V_i(x)$ for $i = 1, 2$, satisfying (V_1) and (V_2). We first study the coercive case $V_{i,\infty} = \infty$, then we study the case $V_{i,\infty} = 0$ and $c_i < 0$.

4.1. The coercive case $V_{i,\infty} = \infty$.

We first consider the convergence of minimizing sequence in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

Lemma 4.1. *Assume that **(M)** holds. If $V_i(x)$ satisfies **(V1)** and **(V2)** with $V_{i,\infty} = \infty$ for $i = 1, 2$, then any minimizing sequence $\{(u_n, v_n)\} \subset S(a,b)$ such that $J(u_n, v_n) \to m(a,b)$ has a convergence subsequence in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.*

Proof. Combining with the fact that $\min_{x\in\mathbb{R}^3} V_i(x) = c_i > -\infty$ and Lemma 3.1 (i), we deduce that J is bounded below on $S^*(a,b)$. By the coerciveness of J on $S^*(a,b)$, we conclude that the sequence $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Thus we assume that there exists a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that $(u_n, v_n) \rightharpoonup (\bar{u}, \bar{v})$ in $\mathcal{H}$, which is also holds in H. Now we need to verify the compactness of (u_n, v_n). Set $(\tilde{u}_n, \tilde{v}_n) = (u_n - \bar{u}, v_n - \bar{v})$, then we deduce that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup 0$ in H. we argue by contradiction, assume that

$$\delta := \lim_{n\to\infty} \inf \int\limits_{\mathbb{R}^3} (\tilde{u}_n^2 + \tilde{v}_n^2) \, dx > 0.$$

Up to a subsequence, we suppose that $(\tilde{u}_n, \tilde{v}_n) \to 0$ in $L^2_{loc}(\mathbb{R}^3) \times L^2_{loc}(\mathbb{R}^3)$. Therefore, we have

$$\int\limits_{\mathbb{R}^3} (V_1(x)\tilde{u}_n^2 + V_2(x)\tilde{v}_n^2) \, dx \to \infty,$$

which implies that

$$\int\limits_{\mathbb{R}^3} (V_1(x)u_n^2 + V_2(x)v_n^2) \, dx \to \infty.$$

Since $\{(u_n, v_n)\}$ is bounded in H, combining above result, we have that

$$\begin{aligned} m(a,b) &= J(u_n, v_n) + o(1) \\ &= J_\infty(u_n, v_n) + \frac{1}{2} \int\limits_{\mathbb{R}^3} (V_1(x)u_n^2 + V_2(x)v_n^2) \, dx + o(1) \\ &\ge m_\infty(a,b) + \frac{1}{2} \int\limits_{\mathbb{R}^3} (V_1(x)u_n^2 + V_2(x)v_n^2) \, dx \to \infty. \end{aligned}$$

We reach a contradiction, since J is bounded below on $S(a,b)$. Thus we prove that $(u_n, v_n) \to (\bar{u}, \bar{v})$ in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. □

Proof of Theorem 1.3. By the coerciveness of J on $S^*(a,b)$, the sequence $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$. Thus by (2.8), we obtain that

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|u(x)-\bar{u}(x)|^p|u(y)-\bar{u}(y)|^p}{|x-y|}\,dx\,dy\le AB^{3p-5}|u-\bar{u}|_2^{5-p}|\nabla(u-\bar{u})|_2^{3p-5},$$

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{|v(x)-\bar{v}(x)|^q|v(y)-\bar{v}(y)|^q}{|x-y|}\,dx\,dy\le AB^{3q-5}|v-\bar{v}|_2^{5-q}|\nabla(v-\bar{v})|_2^{3q-5}.$$

In view of Lemma 4.1, we have $(u_n,v_n)\in(\bar{u},\bar{v})$ in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, together with above inequalities, by Lemma 2.6, then we obtain that $(u_n,v_n)\to(\bar{u},\bar{v})$ in H. Therefore, $(\bar{u},\bar{v})$ is a minimizer, that is, $(\bar{u},\bar{v})\in S(a,b)$, $J(\bar{u},\bar{v})=m(a,b)$ and $(\bar{u},\bar{v})$ satisfies

$$\begin{cases}-\Delta\bar{u}+V_1(x)\bar{u}+\lambda_1\bar{u}=\mu_1\int\limits_{\mathbb{R}^3}\frac{|\bar{u}(y)|^p}{|x-y|}\,dy|\bar{u}|^{p-2}\bar{u}+\beta\bar{v} & \text{in } \mathbb{R}^3,\\ -\Delta\bar{v}+V_2(x)\bar{v}+\lambda_2\bar{v}=\mu_2\int\limits_{\mathbb{R}^3}\frac{|\bar{v}(y)|^q}{|x-y|}\,dy|\bar{v}|^{q-2}\bar{v}+\beta\bar{u} & \text{in } \mathbb{R}^3,\end{cases}$$

where $\bar{u},\bar{v}\ge 0$. According to the maximum principle, we have $\bar{u},\bar{v}>0$. Combining these with Lemma 2.5, the proof of Theorem 1.3 is complete. □

4.2. **The case of $V_{i,\infty}<\infty$.**

In the rest of this paper, we shall consider the case of $V_{i,\infty}=0$ and $c_i<0$ for $i=1,2$. We first study some fundamental properties of $m(a,b)$.

Lemma 4.2. *Assume that* ***(M)*** *holds. If $V_i(x)$ satisfies* ***(V1)*** *and* ***(V2)*** *with $c_i<V_{i,\infty}=0$ for $i=1,2$, then the following results hold*

(i) $-\infty<m(a,b)\le m_\infty(a,b)\le 0$ *for $a,b\ge 0$. Furthermore, if $a+b>0$, then* $-\infty<m(a,b)<m_\infty(a,b)<0$.

(ii) *For $a,b\ge 0$, every minimizing sequence for $m(a,b)$ is bounded in H.*

(iii) *$m(a,b)$ is continuous with respect to $a,b\ge 0$.*

(iv) $m(a_1+a_2,b_1+b_2)\le m(a_1,b_1)+m_\infty(a_2,b_2)$ *for $a_1,a_2,b_1,b_2\ge 0$.*

Proof. (i) Notice that $c_i\le V_i(x)\le V_{i,\infty}=0$, where $c_i<0$, we deduce that $-\infty<\int\limits_{\mathbb{R}^3}(V_1(x)u^2+V_2(x)v^2)\,dx<0$. Without loss of generality, set $J_\infty(\bar{u},\bar{v})=m_\infty(a,b)$ with $(\bar{u},\bar{v})\in S(a,b)$. Together with Lemma 3.1 (i), we can get $-\infty<m(a,b)\le J(\bar{u},\bar{v})\le J_\infty(\bar{u},\bar{v})=m_\infty(a,b)\le 0$, i.e., $-\infty<m(a,b)\le m_\infty(a,b)\le 0$ for $a,b\ge 0$. If $a+b>0$, due to $c_i<0$, then the inequality is strict.

(ii) Since the coerciveness of $J(u,v)$ on $S(a,b)$, (ii) is easy to check.

(iii) For the case $ab\neq 0$, suppose that $(a_n,b_n)=(a,b)+o(1)$, by the definition of $m(a_n,b_n)$, there exists $(u_n,v_n)\in S(a_n,b_n)$, such that

$$J(u_n,v_n)\le m(a_n,b_n)+\epsilon.$$

Let

$$\bar{u}_n:=\frac{u_n}{|u_n|_2}a^{\frac12},\ \ \bar{v}_n:=\frac{v_n}{|v_n|_2}b^{\frac12},$$

then $(\bar{u}_n,\bar{v})\in S(a,b)$. By the continuity of $J(u,v)$, we have

$$\begin{aligned}m(a,b)&\le J(\bar{u}_n,\bar{v}_n)\\&=J(u_n,v_n)+o(1)\\&\le m(a_n,b_n)+\epsilon+o(1),\end{aligned}$$

then we conclude that $m(a,b) \le m(a_n,b_n)+o(1)$. Similar to above arguments, we have $m(a_n,b_n) \le m(a,b)+o(1)$. Therefore, $m(a_n,b_n)=m(a,b)+o(1)$. It's easy to derive the result as the case of $ab=0$.

(iv) For $\epsilon>0$, there exists $(\phi_{1,\epsilon},\psi_{1,\epsilon})$, $(\phi_{2,\epsilon},\psi_{2,\epsilon}) \in C_0^\infty(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3)$ such that

$$(\phi_{1,\epsilon},\psi_{1,\epsilon}) \in S(a_1,b_1),\ J(\phi_{1,\epsilon},\psi_{1,\epsilon}) < m(a_1,b_1)+\frac{\epsilon}{2},$$

$$(\phi_{2,\epsilon},\psi_{2,\epsilon}) \in S(a_2,b_2),\ J_\infty(\phi_{2,\epsilon},\psi_{2,\epsilon}) < m_\infty(a_2,b_2)+\frac{\epsilon}{2}.$$

where $a=a_1+a_2$ and $b=b_1+b_2$. Set $(\phi_{\epsilon,n},\psi_{\epsilon,n}) := (\phi_{1,\epsilon}(x)+\phi_{2,\epsilon}(x-ne_1), \psi_{1,\epsilon}(x)+\psi_{2,\epsilon}(x-ne_1))$, where e_1 is the unit vector $(1,0,0)$ in $\mathbb{R}^3$. Due to $(\phi_{1,\epsilon},\psi_{1,\epsilon})$ and $(\phi_{2,\epsilon},\psi_{2,\epsilon})$ have compact support, we get

$$(\phi_{\epsilon,n},\psi_{\epsilon,n}) \in S(a,b) \quad \text{as } n\to\infty,$$

and

$$m(a,b) \le J(\phi_{\epsilon,n},\psi_{\epsilon,n}) = J(\phi_{1,\epsilon}(x),\psi_{1,\epsilon}(x)) + J(\phi_{2,\epsilon}(x-ne_1),\psi_{2,\epsilon}(x-ne_1)) \quad as\ n\to\infty.$$

Since $V_{i,\infty}=0$, $J(\phi_{2,\epsilon}(x-ne_1),\psi_{2,\epsilon}(x-ne_1)) \to J_\infty(\phi_{2,\epsilon}(x),\psi_{2,\epsilon}(x))$ as $n\to\infty$, and we can deduce that

$$\begin{aligned} m(a,b) &\le \lim_{n\to\infty}\sup J(\phi_{\epsilon,n}(x),\psi_{\epsilon,n}(x)) \\ &= \lim_{n\to\infty}\sup \big(J(\phi_{1,\epsilon},\psi_{1,\epsilon}) + J(\phi_{2,\epsilon}(x-ne_1),\psi_{2,\epsilon}(x-ne_1))\big) \\ &= J(\phi_{1,\epsilon},\psi_{1,\epsilon}) + J_\infty(\phi_{2,\epsilon},\psi_{2,\epsilon}) \\ &\le m(a_1,b_1)+m_\infty(a_2,b_2)+\epsilon, \end{aligned}$$

where ϵ is arbitrary. Thus the proof of Lemma 4.2 is complete. □

In the next lemma inspired by [10], we state a behavior of minimizing sequence when the compactness doesn't hold.

Lemma 4.3. *Assume that **(M)**, **(V1)** and **(V2)** hold. Let $\{(u_n,v_n)\} \subset S(a,b)$ be a minimizing sequence for $m(a,b)$, such that $(u_n,v_n) \rightharpoonup (u_0,v_0)$ in H and set $a_1 := |u_0|_2^2$, $b_1 := |v_0|_2^2$. If $a_1+b_1<a+b$, then there exists $\{y_n\}\subset\mathbb{R}^3$ and $(\mu_0,\nu_0)\in H\backslash\{(0,0)\}$ such that*

$$(u_n(x+y_n),v_n(x+y_n)) \rightharpoonup (\mu_0,\nu_0) \text{ in } H \text{ as } |y_n|\to\infty, \tag{4.1}$$

$$\lim_{n\to\infty}\big(|u_n-u_0-\mu_0(x-y_n)|_2^2 + |v_n-v_0-\nu_0(x-y_n)|_2^2\big)=0, \tag{4.2}$$

and $a=a_1+a_2$, $b=b_1+b_2$, where $|\mu_0|_2^2=a_2$, $|\nu_0|_2^2=b_2$. Furthermore, the following results hold

$$J(u_0,v_0)=m(a_1,b_1),\ J_\infty(\mu_0,\nu_0)=m_\infty(a_2,b_2), \tag{4.3}$$

and

$$m(a,b)=m(a_1,b_1)+m_\infty(a_2,b_2). \tag{4.4}$$

Proof. We will divide the proof into several steps.

Step 1: Set $\{y_n\}\subset\mathbb{R}^3$ and $(\mu_0,\nu_0)\in H\backslash\{(0,0)\}$ such that (4.1) holds. We need to show that

$$\delta_0 := \lim_{n\to\infty}\inf\sup_{y\in\mathbb{R}^3}\int_{y+B(0,1)} (|u_n-u_0|^2+|v_n-v_0|^2)\,dx > 0, \tag{4.5}$$

where $B(0,1) := \{x\in\mathbb{R}^3 : |x|\le 1\}$. Firstly, arguing by contradiction, we assume that $\delta_0=0$, then we deduce that $(u_n,v_n)\to(u_0,v_0)$ in $L^s(\mathbb{R}^3)\times L^t(\mathbb{R}^3)$ for $2\le s,t<6$. Since $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in H and

$V_{i,\infty} = 0$ for $i = 1, 2$, we can derive that $\int_{\mathbb{R}^3}(V_1(x)(u_n - u_0)^2 + V_2(x)(v_n - v_0)^2)\,dx \to 0$. Combining with (3.6), (3.7) and Lemma 2.2, for $a - a_1 > 0$ or $b - b_1 > 0$, we obtain that

$$\begin{aligned} m(a,b) &= J(u_n, v_n) + o(1) \\ &= J(u_0, v_0) + J(u_n - u_0, v_n - v_0) + o(1) \\ &= J(u_0, v_0) + \frac{1}{2}\int\limits_{\mathbb{R}^3} \left(|\nabla(u_n - u_0)|^2 + |\nabla(v_n - v_0)|^2\right) dx - \beta \int\limits_{\mathbb{R}^3} (u_n - u_0)(v_n - v_0)\,dx + o(1) \\ &\geq J(u_0, v_0) - \beta(a - a_1)^{\frac{1}{2}}(b - b_1)^{\frac{1}{2}} \\ &> m(a_1, b_1) + m_\infty(a - a_1, b - b_1). \end{aligned}$$

We reach a contradiction with Lemma 4.2 (iv). Thus (4.5) holds.

By (4.5) and $(u_n, v_n) \to (u_0, v_0)$ in $L^2_{loc}(\mathbb{R}^3) \times L^2_{loc}(\mathbb{R}^3)$, we assume that $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{y_n + B(0,1)}(|u_n - u_0|^2 + |v_n - v_0|^2|)\,dx \to c_0 > 0$ as $|y_n| \to \infty$. Set $(u_n(x + y_n), v_n(x + y_n)) \rightharpoonup (\mu_0, \nu_0)$ in H. By considering $c_o > 0$, we deduce that $(\mu_0, \nu_0) \neq (0, 0)$. Thus $\{y_n\}$ and (μ_0, ν_0) satisfy (4.1). We finish the proof of Step 1.

Due to $|y_n| \to \infty$ and (4.1), we have

$$\begin{aligned} |u_n - u_0 - \mu_0(x - y_n)|_2^2 &= |u_n|_2^2 + |u_0|_2^2 + |\mu_0|_2^2 - 2\langle u_n, u_0\rangle_{L^2} - 2\langle u_n(x + y_n), \mu_0\rangle_{L^2} + o(1) \\ &= |u_n|_2^2 - |u_0|_2^2 - |\mu_0|_2^2 + o(1), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} |v_n - v_0 - \nu_0(x - y_n)|_2^2 &= |v_n|_2^2 + |v_0|_2^2 + |\nu_0|_2^2 - 2\langle v_n, v_0\rangle_{L^2} - 2\langle v_n(x + y_n), \nu_0\rangle_{L^2} + o(1) \\ &= |v_n|_2^2 - |v_0|_2^2 - |\nu_0|_2^2 + o(1). \end{aligned} \tag{4.7}$$

In particular, we define

$$a_2 := |\mu_0|_2^2 \leq \lim_{n\to\infty} \inf(|u_n|_2^2 - |u_0|_2^2) = a - a_1,$$

and

$$b_2 := |\nu_0|_2^2 \leq \lim_{n\to\infty} \inf(|v_n|_2^2 - |v_0|_2^2) = b - b_1.$$

By $c_0 > 0$, we obtain that $a_2 + b_2 > 0$.

Step 2: We investigate that $\{y_n\}$ and (μ_0, ν_0) satisfy (4.2)

In view of (4.6) and (4.7), we may assume that $\delta_1 := \lim_{n\to\infty} |u_n - u_0 - \mu_0(x - y_n)|_2^2$ and $\delta_2 := \lim_{n\to\infty} |v_n - v_0 - \nu_0(x - y_n)|_2^2$. Then we have $\delta_1 = a - a_1 - a_2$ and $\delta_2 = b - b_1 - b_2$. We suppose on the contrary that $\delta_1 + \delta_2 > 0$ to derive $\delta_1 = \delta_2 = 0$. Similar to (4.6) and (4.7), by direct calculations, we have

$$\begin{aligned} |\nabla u_n|_2^2 - |\nabla u_0|_2^2 - |\nabla \mu_0|_2^2 - |\nabla(u_n - u_0 - \mu_0(x - y_n))|_2^2 = o(1), \\ |\nabla v_n|_2^2 - |\nabla v_0|_2^2 - |\nabla \nu_0|_2^2 - |\nabla(v_n - v_0 - \nu_0(x - y_n))|_2^2 = o(1). \end{aligned} \tag{4.8}$$

Moreover, from Brezis Lieb Lemma, it is easy to have

$$\begin{aligned} \frac{1}{2}\int\limits_{\mathbb{R}^3} V_1(x)(|u_n|_2^2 - |u_0|_2^2 - |\mu_0(x - y_n)|_2^2 - |(u_n - u_0 - \mu_0(x - y_n))|_2^2)\,dx = o(1), \\ \frac{1}{2}\int\limits_{\mathbb{R}^3} V_2(x)(|v_n|_2^2 - |v_0|_2^2 - |\nu_0(x - y_n)|_2^2 - |(v_n - v_0 - \nu_0(x - y_n))|_2^2)\,dx = o(1). \end{aligned} \tag{4.9}$$

By Lemma 2.4, it's easy to see that

$$\begin{aligned}
&\frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|}\,dx\,dy+\frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_n(x)|^q|v_n(y)|^q}{|x-y|}\,dx\,dy+\beta\int_{\mathbb{R}^3}u_nv_n\,dx\\
=&\frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_0(x)|^p|u_0(y)|^p}{|x-y|}\,dx\,dy+\frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_0(x)|^q|v_0(y)|^q}{|x-y|}\,dx\,dy+\beta\int_{\mathbb{R}^3}u_0v_0\,dx\\
&+\frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_0(x)|^p|u_n(y)-u_0(y)|^p}{|x-y|}\,dx\,dy\\
&+\frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_n(x)-v_0(x)|^q|v_n(y)-v_0(y)|^q}{|x-y|}\,dx\,dy\\
&+\beta\int_{\mathbb{R}^3}(u_n-u_0)(v_n-v_0)\,dx+o(1)\\
=&\frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_0(x)|^p|u_0(y)|^p}{|x-y|}\,dx\,dy+\frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_0(x)|^q|v_0(y)|^q}{|x-y|}\,dx\,dy+\beta\int_{\mathbb{R}^3}u_0v_0\,dx\\
&+\frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\mu_0(x)|^p|\mu_0(y)|^p}{|x-y|}\,dx\,dy+\frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\nu_0(x)|^q|\nu_0(y)|^q}{|x-y|}\,dx\,dy+\beta\int_{\mathbb{R}^3}\mu_0\nu_0\,dx\\
&+\frac{\mu_1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-u_0(x)-\mu_0(x-y_n)|^p|u_n(y)-u_0(y)-\mu_0(y-y_n)|^p}{|x-y|}\,dx\,dy\\
&+\frac{\mu_2}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_n(x)-v_0(x)-\nu_0(x-y_n)|^q|v_n(y)-v_0(y)-\nu_0(y-y_n)|^q}{|x-y|}\,dx\,dy\\
&+\beta\int_{\mathbb{R}^3}(u_n(x)-u_0(x)-\mu_0(x-y_n))(v_n(x)-v_0(x)-\nu_0(x-y_n))\,dx+o(1)
\end{aligned}\tag{4.10}$$

Combining (4.8)-(4.10), we obtain that

$$\begin{aligned}
o(1)=&J(u_n,v_n)-J(u_0,v_0)-J(\mu_0(x-y_n),\nu_0(x-y_n))\\
&-J(u_n-u_0-\mu_0(x-y_n),v_n-v_0-\nu_0(x-y_n)).
\end{aligned}\tag{4.11}$$

Recall that $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in H and $V_i(x+y_n)\to 0$ as $|y_n|\to\infty$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}V_1(x)|u_n-u_0-\mu_0(x-y_n)|_2^2\,dx\to 0,\\
&\frac{1}{2}\int_{\mathbb{R}^3}V_2(x)|v_n-v_0-\nu_0(x-y_n)|_2^2\,dx\to 0.
\end{aligned}\tag{4.12}$$

Indeed, from Lemma 3.1 (ii), note the fact that $m_\infty(a,b)$ is a continuous with respect to $a,b\geq 0$, combining (4.12), we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\inf J(u_n-u_0-\mu_0(x-y_n),v_n-v_0-\nu_0(x-y_n))\\
=&\lim_{n\to\infty}\inf J_\infty(u_n-u_0-\mu_0(x-y_n),v_n-v_0-\nu_0(x-y_n))\\
\geq& m_\infty(\delta_1,\delta_2),
\end{aligned}\tag{4.13}$$

and

$$\lim_{n\to\infty}\inf J(\mu_0(x-y_n),\nu_0(x-y_n))\geq m_\infty(a_2,b_2).\tag{4.14}$$

By (4.11)-(4.14), we obtain that

$$m(a,b) \geq m(a_1,b_1) + m_\infty(a_2,b_2) + m_\infty(\delta_1,\delta_2).$$

Then by Lemma 4.2 (iv), we obtain that

$$\begin{aligned} m(a,b) &\geq m(a_1,b_1) + m_\infty(a_2,b_2) + m_\infty(\delta_1,\delta_2) \\ &\geq m(a_1+a_2,b_1+b_2) + m_\infty(\delta_1,\delta_2) \\ &\geq m(a_1+a_2+\delta_1,b_1+b_2+\delta_2) \\ &= m(a,b), \end{aligned}$$

which implies that $m(a,b) = m(a_1,b_1) + m_\infty(a_2,b_2) + m_\infty(\delta_1,\delta_2)$. Since $a_2+b_2>0$, $\delta_1+\delta_2>0$, by Lemma 3.4, we have $m_\infty(a_2,b_2)+m_\infty(\delta_1,\delta_2) > m_\infty(a_2+\delta_1,b_2+\delta_2)$. Then we deduce that

$$\begin{aligned} m(a,b) &= m(a_1,b_1) + m_\infty(a_2,b_2) + m_\infty(\delta_1,\delta_2) \\ &> m(a_1,b_1) + m_\infty(a_2+\delta_1,b_2+\delta_2) \\ &\geq m(a_1+a_2+\delta_1,b_1+b_2+\delta_2) \\ &= m(a,b), \end{aligned}$$

which is a contradiction. Therefore, $\delta_1+\delta_2=0$ and Step 2 is complete.

Step 3: Finally, we will prove that $\{y_n\}$ and (μ_0,ν_0) satisfy (4.3) and (4.4). From (4.11)-(4.14) and $\delta_1+\delta_2=0$, we obtain that

$$\begin{aligned} m(a,b) &= \lim_{n\to\infty} J(u_n,v_n) \\ &= \lim_{n\to\infty} \inf \big(J(u_0,v_0) + J(\mu_0(x+y_n),\nu_0(x-y_n))\big) \\ &\geq J(u_0,v_0) + J_\infty(\mu_0,\nu_0) \\ &\geq m(a_1,b_1) + m_\infty(a_2,b_2). \end{aligned} \tag{4.15}$$

Recalling from Lemma 4.2 (iv), we have $m(a,b) = m(a_1,b_1)+m_\infty(a_2,b_2)$. Therefore, by (4.15), we deduce that $J(u_0,v_0) = m(a_1,b_1)$ and $J_\infty(u_0,v_0) = m_\infty(a_2,b_2)$. The proof of Step 3 is complete, and the proof of Lemma 4.3 is finished. □

Since $\{(u_n,v_n)\} \subset S(a,b)$ is the minimizing sequence of $m(a,b)$, we deduce that $dJ_\infty|_{S(a,b)}(u_n,v_n)\to 0$ and there exists two sequences of real numbers $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ such that

$$\begin{aligned} o(1) \parallel (\phi,\psi) \parallel_H = &\int\limits_{\mathbb{R}^3} \big(\nabla u_n \nabla\phi + V_1(x)u_n\phi + \nabla v_n\nabla\psi + V_2(x)v_n\psi\big)\,dx \\ &- \mu_1 \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u_n(x)|^p|u_n(y)|^{p-2}u_n(y)\phi(y)}{|x-y|}\,dx\,dy \\ &- \mu_2 \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|v_n(x)|^q|v_n(y)|^{q-2}v_n(y)\psi(y)}{|x-y|}\,dx\,dy \\ &+ \lambda_{1,n}\int\limits_{\mathbb{R}^3} u_n\phi\,dx + \lambda_{2,n}\int\limits_{\mathbb{R}^3} v_n\psi\,dx \\ &- \beta\int\limits_{\mathbb{R}^3}(u_n\psi+v_n\phi)\,dx \to 0 \quad \text{as } n\to\infty, \end{aligned} \tag{4.16}$$

for any $(\phi,\psi)\in H$. Then we state the following lemma.

Lemma 4.4. *Under the assumption of Lemma 4.3, then both $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are bounded in H. Up to a subsequence, we still denoted by $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ converging to λ_1 and λ_2 respectively. Moreover, (u_0, v_0) and (μ_0, ν_0) satisfy*

$$\begin{cases} -\Delta u_0 + V_1(x)u_0 + \lambda_1 u_0 = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|u_0(y)|^p}{|x-y|}\, dy |u_0|^{p-2}u_0 + \beta v_0 & \text{in } \mathbb{R}^3, \\ -\Delta v_0 + V_2(x)v_0 + \lambda_2 v_0 = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|v_0(y)|^q}{|x-y|}\, dy |v_0|^{q-2}v_0 + \beta u_0 & \text{in } \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |u_0|^2\, dx = a_1, \ \int\limits_{\mathbb{R}^3} |v_0|^2\, dx = b_1, \end{cases} \tag{4.17}$$

and

$$\begin{cases} -\Delta \mu_0 + \lambda_1 \mu_0 = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|\mu_0(y)|^p}{|x-y|}\, dy |\mu_0|^{p-2}\mu_0 + \beta \nu_0 & \text{in } \mathbb{R}^3, \\ -\Delta \nu_0 + \lambda_2 \nu_0 = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|\nu_0(y)|^q}{|x-y|}\, dy |\nu_0|^{q-2}\nu_0 + \beta \mu_0 & \text{in } \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |\mu_0|^2\, dx = a - a_1, \ \int\limits_{\mathbb{R}^3} |\nu_0|^2\, dx = b - b_1. \end{cases} \tag{4.18}$$

Furthermore, if $a_1 + b_1 < a + b$, then $a_1 < a$, $b_1 < b$ and $\mu_0 > 0$, $\nu_0 > 0$.

Proof. By using $(u_n, 0)$ and $(0, v_n)$ as test functions in (4.16), the values of $\lambda_{1,n}$ and $\lambda_{2,n}$ can be achieved as follows

$$-(\lambda_{1,n})a^2 = \int\limits_{\mathbb{R}^3} (|\nabla u_n|^2 + V_1(x)u_n^2)\, dx - \mu_1 \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|}\, dx\, dy - \beta \int\limits_{\mathbb{R}^3} u_n v_n\, dx - o(1),$$

$$-(\lambda_{2,n})b^2 = \int\limits_{\mathbb{R}^3} (|\nabla v_n|^2 + V_2(x)v_n^2)\, dx - \mu_2 \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{|v_n(x)|^q |v_n(y)|^q}{|x-y|}\, dx\, dy - \beta \int\limits_{\mathbb{R}^3} u_n v_n\, dx - o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Note that (u_n, v_n) is bounded in H, then $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are bounded. Without loss of generality, up to a subsequence, assume that $\lambda_{1,n} \to \lambda_1$ and $\lambda_{2,n} \to \lambda_2$. Set $a_1 < a$ or $b_1 < b$, then by Lemma 4.3, there exists $\{y_n\} \subset \mathbb{R}^3$ and $(\mu_0, \nu_0) \neq (0,0)$ such that (4.1)-(4.4) holds. Thus by (4.16), we obtain that

$$\begin{cases} -\Delta \mu_0 + \lambda_1 \mu_0 = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|\mu_0(y)|^p}{|x-y|}\, dy |\mu_0|^{p-2}\mu_0 + \beta \nu_0 & \text{in } \mathbb{R}^3, \\ -\Delta \nu_0 + \lambda_2 \nu_0 = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|\nu_0(y)|^q}{|x-y|}\, dy |\nu_0|^{q-2}\nu_0 + \beta \mu_0 & \text{in } \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |\mu_0|^2\, dx = a - a_1, \ \int\limits_{\mathbb{R}^3} |\nu_0|^2\, dx = b - b_1, \end{cases}$$

thus we obtain (4.18). If $a_1 = a$, then $\mu_0 = 0$ and $\beta\nu_0 = 0$, that is, $\nu_0 = 0$, which is a contradiction since $b_1 < b$. Then we deduce that $a_1 < a$, $b_1 < b$. Due to $\mu_0 \geq 0$, $\nu_0 \geq 0$, we have

$$-\Delta \mu_0 + (\lambda_1)_+ \mu_0 \geq -\Delta \mu_0 + \lambda_1 \mu_0 = \mu_1 \int\limits_{\mathbb{R}^3} \frac{|\mu_0(y)|^p}{|x-y|}\, dy |\mu_0|^{p-2}\mu_0 + \beta \nu_0 \geq 0,$$

and

$$-\Delta \nu_0 + (\lambda_2)_+ \nu_0 \geq -\Delta \nu_0 + \lambda_2 \nu_0 = \mu_2 \int\limits_{\mathbb{R}^3} \frac{|\nu_0(y)|^q}{|x-y|}\, dy |\nu_0|^{q-2}\nu_0 + \beta \mu_0 \geq 0.$$

By the strong maximum principle, $|\mu_0|_2 = a - a_1 > 0$, and $|\nu_0|_2 = b - b_1 > 0$, then we obtain that $\mu_0 > 0$, $\nu_0 > 0$. Similar to above arguments, (u_0, v_0) satisfies (4.17). This completes the proof. $\square$

Our aim is to show the compactness of the minimizing sequence, we will prove the compactness of minimizing sequence as follows.

Lemma 4.5. *Assume that **(M)** holds. If $V_i(x)$ satisfies **(V1)** and **(V2)** with $V_{i,\infty} = 0$ for $i = 1, 2$, then any minimizing sequence $\{(u_n, v_n)\} \subset S(a, b)$ has a strong convergent subsequence in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.*

Proof. Assume that $a_1 + b_1 < a + b$, combining with Lemma 4.4, then we deduce that $a_1 < a$, $b_1 < b$, $\mu_0 > 0$ and $\nu_0 > 0$. Since $c_i \leq V_i(x) \leq 0$, here $c_i < 0$, we obtain that

$$\int_{\mathbb{R}^3} \left(V_1(x)\mu_0^2 + V_2(x)\nu_0^2\right) dx < 0. \tag{4.19}$$

Without loss of generality, set $J(u_0, v_0) = m(a_1, b_1)$, $J(\mu_0, \nu_0) = m_\infty(a_2, b_2)$, where $a = a_1 + a_2$, $b = b_1 + b_2$. By Lemmas 2.7, 4.3 and (4.19), we have

$$\begin{aligned}
m(a, b) &\leq J\left(\sqrt{u_0^2 + \mu_0^2}, \sqrt{v_0^2 + \nu_0^2}\right) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla \sqrt{u_0^2 + \mu_0^2}|^2 + V_1(x)(u_0^2 + \mu_0^2) + |\nabla \sqrt{v_0^2 + \nu_0^2}|^2 + V_2(x)(v_0^2 + \nu_0^2)\right) dx \\
&\quad - \frac{\mu_1}{2p} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\sqrt{u_0(x)^2 + \mu_0(x)^2}|^p |\sqrt{u_0(y)^2 + \mu_0(y)^2}|^p}{|x - y|} \, dx\, dy \\
&\quad - \frac{\mu_2}{2q} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\sqrt{v_0(x)^2 + \nu_0(x)^2}|^q |\sqrt{v_0(y)^2 + \nu_0(y)^2}|^q}{|x - y|} \, dx\, dy \\
&\quad - \beta \int_{\mathbb{R}^3} \sqrt{u_0^2 + \mu_0^2}\sqrt{v_0^2 + \nu_0^2} \, dx \\
&\leq J(u_0, v_0) + J_\infty(\mu_0, \nu_0) + \frac{1}{2} \int_{\mathbb{R}^3} \left(V_1(x)\mu_0^2 + V_2(x)\nu_0^2\right) dx \\
&< m(a_1, b_1) + m_\infty(a_2, b_2) \\
&= m(a, b),
\end{aligned}$$

which is a contradiction. This finishes the proof of Lemma 4.5. □

Proof of Theorem 1.4. From Lemma 4.5, we obtain $(u_n, v_n) \to (u_0, v_0)$ in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. By Lemma 2.6, we have $(u_n, v_n) \to (u_0, v_0)$ in H. Therefore, (u_0, v_0) is a minimizer, that is, $(u_0, v_0) \in S(a, b)$, $J(u_0, v_0) = m(a, b)$ and (u_0, v_0) satisfies

$$\begin{cases}
-\Delta u_0 + V_1(x)u_0 + \lambda_1 u_0 = \mu_1 \int_{\mathbb{R}^3} \frac{|u_0(y)|^p}{|x-y|} \, dy |u_0|^{p-2}u_0 + \beta v_0 & \text{in } \mathbb{R}^3, \\
-\Delta v_0 + V_2(x)v_0 + \lambda_2 v_0 = \mu_2 \int_{\mathbb{R}^3} \frac{|v_0(y)|^q}{|x-y|} \, dy |v_0|^{q-2}v_0 + \beta u_0 & \text{in } \mathbb{R}^3,
\end{cases}$$

where $u_0, v_0 \geq 0$. According to the maximum principle, we have $u_0, v_0 > 0$. Combining these with Lemma 2.5, the proof of Theorem 1.4 is complete. □

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References

[1] S. Abe, A. Ogura, Solitary waves and their critical behavior in a nonlinear nonlocal medium with power-law response, *Phys. Rev. E* **57** (1998), 6066–6070. 2

[2] H. Brézis, and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Am. Math. Soc* **88** (1983), 486-490. 7

[3] Z. Chen and W. Zou,Normalized solutions for nonlinear Schrödinger systems with linear couples, *J. Math. Anal. Appl.* **499** (2021), 125013. 1, 4

[4] J. Fröhlich, T. Tsai and H. Yau, On a classical limit of quantum theory and the non-linear Hartree equation, *Geom. Funct. Anal.* (Special Volume, Part I): (2000). GAFA 2000 (Tel Aviv, 1999), 57–78. 2

[5] J. Fröhlich, T. Tsai and H. Yau, On the point-particle (Newtonian) limit of the non-linear Hartree equation, *Commun. Math. Phys.* **225** (2002),223–274. 2

[6] J. Ginibre and G. Velo The classical field limit of scattering theory for nonrelativistic many-boson systems, *I. Commun. Math. Phys.* **66** (1979), 37–76. 2

[7] T. Gou and L. Jeanjean, Existence and orbital stability of standing waves for nonlinear Schrödinger systems, *Nonlinear Anal.* **144** (2016), 10–22. 1, 4

[8] E. Gross, Physics of Many-Particle systems, vol. 1. *Gordon Breach, New York* (1996). 2

[9] K. Hepp The classical limit for quantum mechanical correlation functions, *Commun. Math. Phys.* **35** (1974), 265–277. 2

[10] N. Ikoma and Y. Miyamoto, Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities, *Calc. Var. Partial Differ. Equ.* **59** (2020), 48. 5, 16

[11] N. Ikoma and Norihisa, Compactness of Minimizing Sequences in Nonlinear Schrödinger Systems Under Multiconstraint Conditions, *Adv. Nonlinear Stud.* **14** (2014), 115-136. 9

[12] E. Lieb and M. Loss, Analysis, second edition, Grad. Stud. Math., American Mathematical Society, Providence, RI, **14** 2001. 6, 9

[13] E. Lieb, Sharp constants in the hardy-littlewood-sobolev and related inequalities, *Ann. Math.* **118** (1983), 349-374.

[14] E. Lieb, Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation, *Stud. Appl. Math.* **57** (1976/1977), 93-105. 2

[15] E. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Stud. Appl. Math.* **57** (1976/77), 93–105. 2

[16] E. Lieb and B.Simon, : The Hartree–Fock theory for Coulomb systems, *Commun. Math. Phys.* **53** (1977), 185–194. 2

[17] P. Lions, The Choquard equation and related questions, *Nonlinear Anal.***4** (1980), 1063–1072. 2

[18] P. Lions, The concentration-compactness principle in the Calculus of Variations. The locally compact case. II, *Ann. Inst. Henri Poincaré*, Anal. Non Linéaire **1** (1984), 223-283. 5

[19] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation. Arch. Ration, *Mech. Anal.* **195** (2010), 455–467. 2

[20] I. Moroz, Penrose, R., Tod, P.: Spherically-symmetric solutions of the Schrödinger–Newton equations, *Class. Quantum Gravity* **15** (1998), 2733–2742. 2

[21] V. Moroz and J. Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.* **265** (2013), 153-184. 2, 7

[22] P. Pedri and L. Santos, Two-dimensional bright solitons in dipolar Bose–Einstein condensates, *Phys. Rev. Lett.* **95** (2005), 200404. 2

[23] S. Pekar, Untersuchung über die Elektronentheorie der Kristalle, *Akademie Verlag, Berlin* (1954). 2

[24] L. Santos, G. Shlyapnikov, P. Zoller and M. Lewenstein, Bose–Einstein condensation in trapped dipolar gases, *Phys. Rev. Lett.* **85** (2000), 1791–1794. 2

[25] M. Mitchell, Z.G. Ghen, M.F. Shih and M. Segev, Self-Trapping of Partially Spatially Incoherent Light, *Phys. Rev. Lett.* **77** (3) (1996), 490–493. 1

[26] M. Mitchell and M. Segev, Self-trapping of incoherent white light, *Nature* **387** (6636) (1997), 880-883. 1

[27] M. Shibata, A new rearrangement inequality and its application for L^2-constraint minimizing problems, *Math. Z.* **287** (2017), 341–359. 5, 9

[28] M. Shibata, A new rearrangement inequality and its application for L2-constraint minimizing problems, *Math. Z.* **287** (2017), 341–359. 4

[29] W. Strauss, Existence of solitary waves in higher dimensions, *Commun. Math. Phys.* **55** (1997), 149–162. 11

[30] J. Wang and J. Shi, Standing waves for a coupled nonlinear Hartree equations with nonlocal interaction, *Calc. Var. Partial Differential Equations* **56** (2017), 168. 3

[31] J. Wang and W. Yang, Normalized solutions and asymptotical behavior of minimizer for the coupled Hartree equations, *J. Differ. Equ.* **265** (2018), 501-544. 2

[32] C. Xiang, Uniqueness and Nondegeneracy of Ground States for Choquard Equations in three dimensions, *Calc. Var. Partial Differential Equations* **55** (2016), Art. 143,25pp. 2, 6

[33] H. Ye, Mass minimizers and concentration for nonlinear Choquard equations in $\mathbb{R}^N$, *Mathematics* **48** (2016), 393-417. 2

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