A Journey into the Inverse Problem of Diffusion-Wave Equation Equipped with Nonlocal Damping and Samarskii-Ionkin Boundary Conditions

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Abstract

This paper explores the extraction of the temporal source term in the fractional diffusion-wave equation. Addressing a complex inverse problem, it incorporates a non-local damping term featuring a two parameter Mittag-Leffler type function and a set of Samarskii-Ionkin boundary conditions. To validate the solution's existence, we establish estimates for infinite series involving the convolution of a three parameter Mittag-Leffler function. Our research contributes valuable insights at the intersection of mathematical analysis and fractional calculus providing a robust foundation for understanding and solving complex problems in this domain.

Keywords: Caputo fractional derivative, Non-self-adjoint operator, Bi-orthogonal systems, Three parameter Mittag-Leffler function

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1. Introduction and Problem Formulation

Integro-differential fractional operators have attracted the attention of scientists due to their vast applications in the diverse field of engineering and sciences. Mathematicians who played their role in the theoretical development of fractional calculus and pioneers who initiated its first applications are nicely summarized in [1]. In a review article [2], authors have gathered some novel applications of non-integer (fractional) order operators. The nonlocal nature of the fractional operator is the main reason for the success of these operators, as they are ideal to model memory dependent phenomenon that arise frequently in physical processes.

When we talk about any physical system, some factors are compelling the system to produce certain effects, the inverse problem is the study of these effects, such problems arise in many cuttingedge applications where required information can not be obtained from direct measurements. There are numerous applications of inverse problems e.g., magnetic resonance imaging [3, 4], medical imaging [5], computerized tomography [6], signal processing [7] and many other applications [8, 9].

Let us provide a brief literature review on the inverse problems associated with the fractional differential equations. In [10], authors discussed two inverse problems one inverse evolutionary problem and other inverse source problem for time fractional differential equation, while the inverse coefficient problem and inverse source problem are considered in [11] and [12], respectively. Usually, the reconstruction of the spectral source term is based on the overdetermination condition at final time [13, 14], while the temporal source term is reconstructed from additional boundary measurements [15] or from integral type condition [16],[17]. Investigation of the inverse spectral problems for the fractional diffusion equation was done by Tuan in [18, 19].

The partial differential equations involving nonlocal damping terms arise in different fields of sciences such as fiberglass, boron, and graphite composites [20, 21]. Non-Existence of the fractionally damped fractional differential problem is considered in [22]. Our current research investigation is motivated by the study of [23] in which they presented the analytical solution to the direct problem associated with fractional diffusion-wave equation involving nonlocal damping.

We are going to investigate the inverse problem for the following time fractional diffusion-wave equation involving nonlocal damping:

$$ {}^{c}D^{\beta}_{0_{+},t}u(x,t) + \mu e_{\gamma,\xi}(t;\omega) * u_{xx}(x,t) = u_{xx}(x,t) + a(t)f(x,t), \tag{1.1} $$

associated with the nonlocal family of boundary datum

$$ u(0,t) = 0,\ u_x(0,t) = u_x(1,t) + \alpha u(1,t), \quad \alpha > 0, \quad t \in (0\,,T], \tag{1.2} $$

the initial conditions

$$ u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \quad x \in (0,1), \tag{1.3} $$

and the overdetermination condition

$$ \int_0^1 u(x,t)dx = h(t), \quad t \in (0,T], \tag{1.4} $$

where $(x,t) \in \Pi := (0,1)\times(0,T]$, ${}^{c}D^{\beta}_{0_{+},t}$ represents left-sided β ordered Caputo fractional derivative; $1 < \beta < 2$, $\mu > 0$, $0 < \gamma < \xi < 1$, and "$*$" represents the Laplace convolution.

The main characteristics of this research article are as follows:

- For $\mu=0$ and $\beta=1$, the Equation (1.1) reduces to the classical diffusion equation

$$\frac{\partial u}{\partial t}(x,t)=u_{xx}(x,t)+a(t)f(x,t),$$

  with the initial condition

$$u(x,0)=\varphi(x),\quad x\in(0,1),$$

  overdetermination condition given by (1.4).

- While for $\mu=0$ and $\beta=2$, it represents the integer order wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t)-u_{xx}(x,t)=a(t)f(x,t),$$

  alongside the initial and overdetermination conditions given by (1.3) and (1.4) respectively.

- Presence of nonzero parameter α in the boundary data.

- Nonlocality of one of the boundary conditions makes the respective spectral operator non-self adjoint.

- The presence of $e_{\gamma,\xi}(t;\omega)*u_{xx}(x,t)$ represents the nonlocal damping term where $e_{\gamma,\xi}(t;\omega)$ represents the Mittag-Leffler type function.

Our main focus is on the recovery of the component of the source term, depending only on time, in the model (1.1)-(1.3), from extra data (1.4). Hence, our main aim is to prove the existence and uniqueness of the regular solution of the inverse problem (1.1)-(1.4). The regular solution represents a pair $\{u(x,t),\, a(t)\}$, such that $u\in C^{2,0}_{x,t}(\Pi)$, ${}^{c}D^{\beta}_{0+,t}u\in C(\Pi)$, $e_{\gamma,\xi}(t;\omega)*u_{xx}(x,t)\in\ C(\Pi)$ and $a(t)\in C(0,T]$.

The rest of the manuscript is structured as: we state the basic definitions of fractional calculus, Prabhakar-type Mittag-Leffler function and related basic results in the Section 2. Results related to the Sturm-Liouville problem are presented in the Section 3. In the Section 4, the construction of the solution is presented followed by existence and uniqueness results. Number of lemmata involving Prabhakar-type Mittag-Leffler and their convolution are also proved at the start of Section 4. Numeric examples are presented in the Section 5 to support our analysis. The Section 6 is devoted to concluding remarks.

2. Preliminaries

Definition 1. *[24] For $-\infty < a < z < b < \infty$, $m-1 < \sigma \leq m$ and $g \in AC^m(a,b)$ where $m = \lceil \sigma \rceil$, then the left sided σ ordered Caputo fractional derivative of is*

$$ {}^{c}D^{\sigma}_{a+,z} g(z) := \frac{1}{\Gamma(m-\sigma)} \int_a^z \frac{g^{(m)}(\tau)}{(z-\tau)^{1-m+\sigma}} d\tau. $$

Definition 2. *[25] The Prabhakar generalized Mittag-Leffler function is defined as*

$$ E^{\rho}_{\sigma,\eta}(z) := \sum_{j=0}^{\infty} \frac{(\rho)_j}{j!\,\Gamma(\sigma j+\eta)} z^j, \qquad \sigma > 0,\ \eta > 0,\ \rho > 0,\ z, \in \mathbb{R}, $$

where $(\rho)_j$ represents the Pochhammer symbol and defined as $(\rho)_j := \dfrac{\Gamma(\rho+j)}{\Gamma(\rho)}$.

For $\rho = 1$,

$$ E^{1}_{\sigma,\eta}(z) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(\sigma j+\eta)} z^j := E_{\sigma,\eta}(z). $$

For $\rho = 1 = \eta$,

$$ E^{1}_{\sigma,1}(z) = \sum_{j=0}^{\infty} \frac{1}{\Gamma(\sigma j+1)} z^j := E_{\sigma}(z). $$

For the sake of convenience, we define the following notion

$$ \mathcal{E}^{\rho}_{\sigma,\eta}(z;\nu) := z^{\eta-1} E^{\rho}_{\sigma,\eta}(-\nu z^{\sigma}), $$

for $\rho = 1$, we denote

$$ e_{\sigma,\eta} := \mathcal{E}^{1}_{\sigma,\eta}(z;\nu) := z^{\eta-1} E_{\sigma,\eta}(-\nu z^{\sigma}). $$

The solution, of the system of fractional differential equations, is obtained in the Section 4.6 by using Laplace transform, hence for reference some formulae for Laplace transform are mentioned here.

- $\mathcal{L}\big({}^{c}D^{\sigma}_{0+,z} g(z)\big) = s^{\sigma}\mathcal{L}(g(z)) - \sum_{k=0}^{n-1} s^{n-\sigma-1} g^{(k)}(0).$

- For $\sigma > 0, \eta, \rho, \nu, z \in \mathbb{R}, s > 0, |s|^{\sigma} > |z|$, we have

$$ \mathcal{L}(\mathcal{E}^{\rho}_{\sigma,\eta}(z;\nu)) = \frac{s^{\sigma\rho-\eta}}{(s^{\sigma}-z)^{\rho}}. \tag{2.1} $$

Lemma 2.1. *[26] If $g \in C[a,b]$ and $K(z,y)$ is bounded and continuous for $a \le y \le z \le b$, then the equation*

$$g(z) = g(z) + \int_0^z K(z,y)g(y)dy,$$

has one and only one bounded and continuous solution $g(z)$ in the interval $a \le z \le b$.

3. Riesz Basis

To make the article self-contained, we are to mention spectral analysis corresponding to the system (1.1)-(1.2). Detailed analysis can be found in [27].

The spectral and its conjugate problems corresponding to (1.1)-(1.2) are

$$Y''(x) = -\lambda Y(x), \quad Y(0) = 0, \quad Y'(0) = Y'(1) + \alpha Y(1), \tag{3.1}$$

$$Z''(x) = -\lambda Z, \quad Z(0) = Z(1), \quad Z'(1) + \alpha Z(1) = 0. \tag{3.2}$$

Eigenvalues of (3.1) and (3.2) are

$$\lambda_0 = (2y_0)^2, \qquad \lambda_{1i} = (2i\pi)^2, \qquad \lambda_{2i} = (2y_i)^2, \qquad i \in \mathbb{N},$$

such that y_i satisfies the non-linear algebraic equation $\tan(w) = \alpha/2w$.

Moreover, y_i satisfy the following inequalities

$$\pi i < y_i < \pi i + \pi/2, \quad i \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Note that

$$\frac{1}{\lambda_{li}} < \frac{1}{i^2}, \quad l = 1, 2. \tag{3.4}$$

The eigenfunctions of (3.1) and (3.2) are

$$\Big\{ u_0 = \sin(2y_0 x), \quad u_{1i} = \sin(2\pi i x), \quad u_{2i} = \sin(2y_i x) \Big\}, \tag{3.5}$$

$$\Big\{ v_0 = C_0 \cos(y_0(1-2x)),\ v_{1i} = C_{1i}\cos(2\pi i x + \psi_i), v_{2i} = C_{2i}\cos(y_i(1-2x)) \Big\}, \tag{3.6}$$

respectively, where

$$\begin{aligned} \psi_i &= \arctan\left(\frac{\alpha}{2i\pi}\right), \qquad C_0 = \frac{2}{\sin(y_0)\Big(1 + \operatorname{sinc}(2y_0)\Big)}, \\ C_{1i} &= -\frac{2}{\sin(\psi_i)}, \qquad C_{2i} = \frac{2}{\sin(y_i)\Big(1 + \operatorname{sinc}(2y_i)\Big)}, \end{aligned}$$

and $\operatorname{sinc}(a) := \sin(a)/a$.

The set of eigenfunction (3.5) and (3.6) are complete but not orthogonal (see [27]). The following bi-orthogonal set is constructed from eigenfunctions of (3.1) and (3.2).

$$Q_\alpha = \{q_0,\ q_{1i},\ q_{2i} : i \in \mathbb{N}\}, \qquad R_\alpha = \{r_0,\ r_{1i},\ r_{2i} : i \in \mathbb{N}\}, \tag{3.7}$$

where

$$\begin{aligned} q_0(x) &:= u_0/2y_0,\ q_{1i}(x) := \frac{u_{2i} - u_{1i}}{2\delta_i},\ q_{2i}(x) := u_{1i}, \\ r_0(x) &:= 2y_0v_0,\ r_{1i}(x) := 2\delta_i v_{2i},\ r_{2i}(x) := v_{2i} + v_{1i}, \end{aligned}$$

and $\delta_i = y_i - \pi i$.

The sets Q_α and R_α given by 3.7 are proved to be Riesz bases of $L^2(0,1)$ in [27].

4. Main Results

This section is devoted to presenting the research contributions of this article. To start with we are going to prove few key lemmata that will aid in proving the regularity of the solution (1.1)-(1.4).

Lemma 4.1. *The Fourier's coefficients g_{li}, obtained by using the Riesz bases* (3.7), *of $g(x) \in L^2(0,1)$ satisfy the following relations*

- *If $g \in C^1(0,1)$, we have*
$$|g_{li}| \le \frac{D_1}{i}(1 + \|g'\|),$$
- *If $g \in C^2(0,1)$ and $g(0) = 0,\ g'(0) = g'(1) + \alpha\, g(1)$. Then, we have*
$$|g_{li}| \le \frac{D_1}{i^2}\|g''\|,$$
- *If $g \in C^3(0,1)$ and $g(0) = 0,\ g'(0) = g'(1) + \alpha\, g(1)$. Then, we have*
$$|g_{li}| \le \frac{D_1}{i^3}(1 + \|g'''\|),$$

where $g_{li} = \langle g(x), r_i(x)\rangle,\ l = 1, 2,\ i \in \mathbb{N}$ and D_1 is a positive constant.

Proof. We are going to give the proof of the inequalities satisfied by g_{1i}, similar relation can be established for g_{2i}.

Since

$$g_{1i} = 2\delta_i C_{2i} \int_0^1 g(x)\cos(y_i(1-2x))dx.$$

Integration by parts implies us to

$$g_{1i} = \frac{2\delta_i C_{2i}}{i}\big\{ - g(x)\sin(y_i(1-2x))\big|_{x=0}^{x=1} + \int_0^1 g'(x)\sin(y_i(1-2x))dx\big\},$$

the give condition $g(0) = 0$ together with integration by parts, further modifies the results as

$$g_{1i} \quad = \quad \frac{2\delta_i C_{2i}}{2y_i}\big\{g(1)\sin(y_i) + \frac{\cos(y_i(1-2x))}{2y_i} g'(x)\big|_{x=0}^{x=1} - \frac{1}{2y_i}\int_0^1 g''(x)\cos(y_i(1-2x))dx\big\}.$$

Due to the fact $\tan(y_i) = \dfrac{\alpha}{2y_i}$ and using the second boundary condition $g'(0) = g'(1) + \alpha\, g(1)$, we obtain

$$g_{1i} = -\frac{2\delta_i C_{2i}}{(2y_i)^2}\int_0^1 g''(x)\cos(y_i(1-2x))dx,$$

again integration by parts gives

$$g_{1i} = -\frac{2\delta_i C_{2i}}{(2y_i)^3}\big\{(g''(0) + g''(1))\sin(y_i) + \int_0^1 g'''(x)\sin(y_i(1-2x))dx\big\}.$$

Using Cauchy-Bunkovsky-Schwarz, we obtain

$$|g_{1i}| \leq \frac{D_1}{i^3}(1 + \|g'''\|),$$

where

$$\begin{aligned} D_1 = &\max\{D, 2\delta_i C_{2i},\ 2\delta_i C_{2i}(|g(1)| + |g(0)|),\ 2\delta_i C_{2i} + C_{1i}\ , (2\delta_i C_{2i} + C_{1i})(|g(1)| + g(0)| \\ &\quad 2\delta_i C_{2i}(|g''(0)| + |g''(1)|), (2\delta_i C_{2i} + C_{1i})(|g''(0)| + |g''(1)|)\}, \\ D = &\max\{\|r_0\|,\ \|r_{1i}\|,\ \|r_{2i}\|\}. \end{aligned}$$

Lemma 4.2. *The Fourier's coefficients g_{li}, obtained by using the Riesz bases* (3.7), *of $g(x,t) \in L^2(\Pi)$ satisfy the following relations*

- *If $g \in C^{1,0}_{x,t}(\Pi)$, we have*

$$|g_{li}(t)| \leq \frac{D_2}{i}(1 + \|g_x\|), \quad \forall \quad t \in (0, T].$$

- *If $g \in C^{2,0}_{x,t}(\Pi)$ and $g(0,t) = 0,\ g_x(0,t) = g_x(1,t) + \alpha\, g(1,t)$. Then,*

$$|g_{li}(t)| \leq \frac{D_2}{i^2}\|g_{xx}\|, \quad \forall \quad t \in (0, T].$$

- *If $g \in C^{3,0}_{x,t}(\Pi)$ and $g(0,t) = 0,\ g_x(0,t) = g_x(1,t) + \alpha\, g(1,t)$. Then,*

$$|g_{li}(t)| \leq \frac{D_2}{i^3}(1 + \|g_{xxx}\|), \quad \forall \quad t \in (0, T].$$

where $g_{li} = \langle g(x,t), r_i(x)\rangle,\ l = 1,\ 2,\ i \in \mathbb{N}$ and $D_2\ >\ 0$.

Proof. Proof can be done on the same lines as we did for the Lemma 4.1.

Lemma 4.3. *For $\sigma > -1,\ \rho, \kappa, \eta, \nu\ > 0$ and $t \in \mathbb{R}$, we have*

$$t^{\sigma} * \mathcal{E}^{\rho}_{\kappa,\eta}(t;\nu) = \Gamma(\sigma+1)\mathcal{E}^{\rho}_{\kappa,\eta+\sigma+1}(t;\nu).$$

Proof. Above result can be obtained by using definition of convolution integral along side definition of Euler integral of first kind.

Remark 1. By taking $\sigma = 0$, we have

$$1 * \mathcal{E}^{\rho}_{\kappa,\eta}(t;\nu) = \mathcal{E}^{\rho}_{\kappa,\eta+1}(t;\nu).$$

Lemma 4.4. *If $\theta, \Lambda, \nu > 0,\ 0 < \kappa < \sigma < 1,\ 1 < \eta < 2,\ \phi \geq 0$ and $p = 0, 1$ then the following triple series involving Mittag-Leffler type functions of three parameters satisfy the following inequalities*

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{n}\,{}^{n}C_{m}\,{}^{k}C_{n}\Lambda^{k}\nu^{n-m}\theta^{k-n}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi}(t;\nu)\\ &\leq \frac{C_1 t^{\phi+\kappa p-1}}{1 + \nu t^{\kappa} + \Lambda(\theta t^{\sigma+\eta} + \nu t^{\eta+\kappa} + t^{\eta})}.\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{n}\,{}^{n}C_{m}\,{}^{k}C_{n}\Lambda^{k}\nu^{n-m}\theta^{k-n}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi}(t;\nu)\\ =&\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{n}\,{}^{n}C_{m}\,{}^{k}C_{n}\Lambda^{k}\nu^{n-m}\theta^{k-n}\sum_{j=0}^{\infty}\frac{(k+1)_j(-\nu t^{\kappa})^j t^{\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi-1}}{j!\Gamma(\kappa j + \kappa(n-m+p) + \sigma(k-n) + \eta k + \phi)}.\end{aligned}$$

Changing the order of summation yields

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{nn}C_m{}^kC_n\Lambda^k\nu^{n-m}\theta^{k-n}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi+1}(t;\nu)$$
$$=\sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{n=0}^{k}\sum_{m=0}^{n}\left[\left(\frac{(-1)^{n+s-k}\Lambda^k\nu^{n-m+s-k}\theta^{k-n}\Gamma(s+1)(k+1)_{s-k}}{\Gamma(s-k+1)\Gamma(m+1)\Gamma(n-m+1)\Gamma(k-n+1)}\right)\right.$$
$$\left.\left(\frac{t^{\kappa(s-k)+\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi-1}}{\Gamma(\kappa(s-k)+\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi)}\right)\right]. \tag{4.1}$$

Since $m\le n\le k\le s \implies n-m+p\ge 0,\ k-n\ge 0,\ s-k\ge 0$, and $\Gamma(\cdot)$ is an increasing function.

So, we can write

$$\frac{1}{\Gamma(\kappa(s-k)+\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi)}<\frac{1}{\Gamma(\kappa s+\phi)},$$

so, Equation (4.1), takes the form

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{nn}C_m{}^kC_n\Lambda^k\nu^{n-m}\theta^{k-n}\left(\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi}(t;\nu)\right)$$
$$\le\left(\sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{n=0}^{k}\frac{(-1)^{n+s-k}\Lambda^k\nu^{n+s-k}\theta^{k-n}t^{\kappa(s-k)+\kappa(n+p)+\sigma(k-n)+\eta k+\phi-1}\Gamma(s+1)}{\Gamma(k+1)\Gamma(s-k+1)\Gamma(n+1)\Gamma(k-n+1)\Gamma(\kappa s+\phi)}\right)$$
$$\left(\sum_{m=0}^{n}\frac{\Gamma(n+1)\nu^{-m}t^{-\kappa m}}{\Gamma(m+1)\Gamma(n-m+1)}\right).$$

By using the fact that

$$\sum_{j=0}^{l}\frac{\Gamma(l+1)}{\Gamma(j+1)\Gamma(l-j+1)}r^j=(1+r)^l,\ \ l\in\mathbb{N},\ r\in\mathbb{R}.$$

Hence, we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{nn}C_m{}^kC_n\Lambda^k\nu^{n-m}\theta^{k-n}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi+1}(t;\nu)$$
$$\le t^{\phi+\kappa p-1}E_{\kappa,\phi}\left(\Lambda\theta t^{\sigma+\eta}-\nu t^{\kappa}-\Lambda\nu t^{\eta+\kappa}-\Lambda t^{\eta}\right)\le t^{\phi+\kappa p-1}E_{\kappa,\phi}\left(\Lambda\theta t^{\sigma+\eta}+\nu t^{\kappa}+\Lambda\nu t^{\eta+\kappa-1}+\Lambda t^{\eta}\right)$$

By using Theorem 1.6 of [28], we have

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{nn}C_m{}^kC_n\Lambda^k\nu^{n-m}\theta^{k-n}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi+1}(t;\nu)$$
$$\le\frac{C_1t^{\phi+\kappa p-1}}{1+\Lambda\theta t^{\sigma+\eta}+\nu t^{\kappa}+\Lambda\nu t^{\eta+\kappa}+\Lambda t^{\eta}},$$

as required.

Lemma 4.5. *If $\theta, \Lambda, \nu > 0$, $0 < \kappa < \sigma < 1$, $1 < \eta < 2$, $\phi \geq 0$ and $p = 0, 1$ then the convolution of following triple series involving three parameter Mittag-Leffler type functions with two parameter Mittag-Leffler type function satisfy the following inequalities*

$$e_{\kappa,\sigma}(t;\nu) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{n\,n}C_m{}^kC_n\Lambda^k\nu^{n-m}\theta^{k-n}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p)+\sigma(k-n)+\eta k+\phi}(t;\nu)$$
$$\leq \frac{\Gamma(\sigma)C_1^2 t^{\phi+\kappa p+\sigma-1}}{1+\nu t^{\kappa}+\Lambda(\theta t^{\sigma+\eta}+\nu t^{\eta+\kappa}+t^{\eta})},$$

Proof. *Required inequalities can be obtained by using Lemma 4.3 followed by Lemma 4.4.*

The direct consequence of the Lemma 4.4, is following remark:

Remark 2. If $\theta, \Lambda_1, \Lambda_2, \nu > 0$, $0 < \kappa < \sigma < 1$, $1 < \eta < 2$, ϕ_1, $\phi_2 \geq 0$ and $p_j = 0, 1$, $j = 1, 2$ then the triple series involving the convolution of Prabhakar-type Mittag-Leffler function satisfy the following inequalities

(a)

$$\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\Lambda_1,\nu,\theta}\Bigg[\left(\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\Lambda_2,\nu,\theta}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p_1)+\sigma(k-n)+\eta k+\phi_1}(t;\nu)\right)$$
$$*\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p_2)+\sigma(k-n)+\eta k+\phi_2}(t;\nu)\Bigg] \leq \frac{C_1^2\Gamma(\phi_2+\kappa p_2)t^{\phi_1+\phi_2+\kappa p_1+\kappa p_2}}{1+\nu t^{\kappa}+\Lambda_2(\theta t^{\sigma+\eta}+\nu t^{\eta+\kappa}+t^{\eta})},$$

(b)

$$e_{\kappa,\sigma}(t;\nu) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\Lambda,\nu,\theta}\Bigg[\left(\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\Lambda,\nu,\theta}\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p_1)+\sigma(k-n)+\eta k+\phi_1}(t;\nu)\right)$$
$$*\mathcal{E}^{k+1}_{\kappa,\kappa(n-m+p_2)+\sigma(k-n)+\eta k+\phi_2}(t;\nu)\Bigg] \leq \frac{C_1^3\Gamma(\sigma)\Gamma(\phi_2+\kappa p_2)t^{\phi_1+\phi_2+\kappa p_1+\kappa p_2+\sigma-1}}{1+\nu t^{\kappa}+\Lambda_2(\theta t^{\sigma+\eta}+\nu t^{\eta+\kappa}+t^{\eta})},$$

where $\Delta^{k,n,m}_{\Lambda,\nu,\theta} := (-1)^{n\,n}C_m{}^kC_n\Lambda^k\nu^{n-m}\theta^{k-n}$.

4.6. Regularity of the solution

In this subsection, we will state the optimum conditions that will ensure the unique existence of the regular solution.

Theorem 1. *If the given data $\{\varphi,\ f,\ g\}$ satisfy the following regularity constraints*

- $\varphi \in C^3(0,1)$ *such that* $\varphi(0)=0,\ \varphi'(0)=\varphi'(1)+\alpha\,\varphi(1)$,

- $f \in C^{3,0}_{x,t}(\Pi)$ *such that* $f(0,t)=0,\ f_x(0,t)=f_x(1,t)+\alpha\, f(1,t)$.

  Moreover,

$$\langle f(x,t), r_0(x)\rangle \neq 0.$$

- $g(t) \in AC(0,T]$, *such that* $g(0)=\displaystyle\int_0^1 \varphi(x)dx$

the unique solution of the inverse source problem (1.1)-(1.4) *is regular in nature*

Proof. The proof of the theorem consists of three parts. First part contains the construction of the solution then we are going to present existence results and finally we are going to conclude the proof with uniqueness results.

Construction of the Solution

The unknown function $u(x,t)$ and the given part of source term $f(x,t)$ can be written as:

$$u(x,t) = T_0(t)q_0(x) + \sum_{i=1}^{+\infty}\left[T_{1i}(t)q_{1i}(x) + T_{2i}(t)q_{2i}(x)\right], \tag{4.2}$$

$$f(x,t) = f_0(t)q_0(x) + \sum_{i=1}^{+\infty}\left[f_{1i}(t)q_{1i}(x) + f_{2i}(t)q_{2i}(x)\right], \tag{4.3}$$

where $T_0(t)$, $T_{1i}(t), T_{2i}(t), f_0(t), f_{1i}(t)$ and $f_{2i}(t)$ are to be determined by solving the following differential equations of fractional order:

$$^cD^{\beta}_{0+,t}T_0(t) = -\lambda_0 T_0(t) + \lambda_0\mu e_{\gamma,\xi}(t;\omega) * T_0(t) + a(t)f_0(t), \tag{4.4}$$

$$^cD^{\beta}_{0+,t}T_{1i}(t) = -\lambda_{2i} T_{1i}(t) + \lambda_{2i}\mu e_{\gamma,\xi}(t;\omega) * T_{1i}(t) + a(t)f_{1i}(t), \tag{4.5}$$

$$^cD^{\beta}_{0+,t}T_{2i}(t) = -\lambda_{1i} T_{2i}(t) + \frac{\lambda_{2i}-\lambda_{1i}}{2\delta_i}T_{1i}(t) + \lambda_{1i}\mu e_{\gamma,\xi}(t;\omega) * T_{2i}(t) + a(t)f_{2i}(t). \tag{4.6}$$

By using Laplace transform, we obtained the following expression for (4.4)

$$s^{\beta}\mathcal{L}\{T_0(t)\} - s^{\beta-1}T_0(0) - s^{\beta-2}T_0'(0) = -\lambda_0\mathcal{L}\{T_0(t)\} + \frac{\mathcal{L}\{T_0(t)\}\lambda_0\mu s^{\gamma-\xi}}{s^{\gamma}+\omega} + \mathcal{L}\{a(t)f_0(t)\}$$

$$\mathcal{L}\{T_0(t)\} = \frac{s^{\beta-1}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\xi-\gamma}}{s^{\gamma}+\omega}}\varphi_0 + \frac{s^{\beta-2}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\xi-\gamma}}{s^{\gamma}+\omega}}\psi_0 + \frac{\mathcal{L}\{a(t)f_0(t)\}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\gamma-\xi}}{s^{\gamma}+\omega}}.$$

By taking the inverse Laplace transform, we get

$$
\begin{aligned}
T_0(t) = \quad & \mathcal{L}^{-1}\left(\frac{s^{\beta-1}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\xi-\gamma}}{s^{\gamma}+\omega}}\varphi_0\right)+\mathcal{L}^{-1}\left(\frac{s^{\beta-2}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\xi-\gamma}}{s^{\gamma}+\omega}}\psi_0\right)\\
& +\mathcal{L}^{-1}\left(\frac{\mathcal{L}\{a(t)f_0(t)\}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\gamma-\xi}}{s^{\gamma}+\omega}}\right).
\end{aligned}
$$

Consider

$$
\begin{aligned}
\frac{s^{\beta-1}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\gamma-\xi}}{s^{\gamma}+\omega}} =& \frac{s^{\beta+\gamma-1}+\omega s^{\beta-1}}{s^{\beta}(s^{\gamma}+\omega)\left\{1+\dfrac{s^{\gamma}\lambda_0+\omega\lambda_0-\lambda_0\mu s^{\gamma-\xi}}{s^{\beta}(s^{\gamma}+\omega)}\right\}}\\
=& (s^{\beta+\gamma-1}+\omega s^{\beta-1})\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\frac{\Delta^{k,n,m}_{\lambda_0,\omega,\mu}s^{\gamma m+(\gamma-\xi)(k-n)}}{s^{\beta(k+1)}(s^{\gamma}+\omega)^{k+1}}.
\end{aligned}
$$

Using the Equation (2.1), we have

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^{\beta-1}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\gamma-\xi}}{(s^{\gamma}+\omega)}}\right\} =& \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\lambda_0,\omega,\mu}\bigg(\mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta k+1}(t;\omega)\\
& +\omega\mathcal{E}^{k+1}_{\gamma,\gamma(n-m+1)+\xi(k-n)+\beta k+1}(t;\omega)\bigg).
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^{\beta}}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\gamma-\xi}}{s^{\gamma}+\omega}}\right\} =& \Delta^{k,n,m}_{\lambda_0,\omega,\mu}\bigg(\mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta k}(t;\omega)\\
& +\omega\mathcal{E}^{k+1}_{\gamma,\gamma(n-m+1)+\xi(k-n)+\beta k}(t;\omega)\bigg).\\
\mathcal{L}^{-1}\left\{\frac{1}{s^{\beta}+\lambda_0-\dfrac{\lambda_0\mu s^{\gamma-\xi}}{s^{\gamma}+\omega}}\right\} =& \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\lambda_0,\omega,\mu}\mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta(k+1)}(t;\omega).
\end{aligned}
$$

Hence, the solution of (4.4) takes the form

$$
T_0(t)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\lambda_0,\omega,\mu}\big[\big(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta}+\omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta}\big)\varphi_0-\big(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta}+\omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\big)\psi_0+a(t)f_0(t)*\mathcal{B}^{t;\omega}_{\gamma,\xi,\beta}\big], \tag{4.7}
$$

Similarly, solution of (4.5) and (4.6) are given by

$$T_{1i}(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta^{k,n,m}_{\lambda_{2i},\omega,\mu}\big[(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta})\varphi_{1i} - \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\psi_{1i} + a(t)f_{1i}(t) * \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta}\big], \tag{4.8}$$

$$\begin{aligned} T_{2i}(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta^{k,n,m}_{\lambda_{1i},\omega,\mu}\big[&(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta})\varphi_{2i} - \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\psi_{2i} + a(t)f_{2i}(t) * \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta} \\ &+ \frac{\lambda_{2i}-\lambda_{1i}}{2\delta_i} T_{1i}(t) * \left(\mathcal{B}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{G}^{t;\omega}_{\gamma,\xi,\beta}\right)\big], \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta k+2}(t;\omega), & \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta(k+1)}(t;\omega), \\ \mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta k+1}(t;\omega), & \mathcal{D}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m)+\xi(k-n)+\beta(k+1)+1}(t;\omega), \\ \mathcal{F}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m+1)+\xi(k-n)+\beta k+2}(t;\omega), & \mathcal{G}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m+1)+\xi(k-n)+\beta(k+1)}(t;\omega), \\ \mathcal{H}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m+1)+\xi(k-n)+\beta k+1}(t;\omega), & \mathcal{I}^{t;\omega}_{\gamma,\xi,\beta} &:= \mathcal{E}^{k+1}_{\gamma,\gamma(n-m+1)+\xi(k-n)+\beta(k+1)+1}(t;\omega). \end{aligned}$$

Hence, $u(x,t)$ can be written as

$$\begin{aligned} u(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Bigg[&\Delta^{k,n,m}_{\lambda_0,\omega,\mu}\Big\{(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta})\varphi_0 - (\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta})\psi_0 \\ &+ a(t)f_0(t) * \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta}\Big\}\frac{\sin(2y_0x)}{2y_0} + \Delta^{k,n,m}_{\lambda_{2i},\omega,\mu}\Big\{(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta})\varphi_{1i} \\ &- \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\psi_{1i} + a(t)f_{1i}(t) * \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta}\Big\}\frac{\sin(2y_ix) - \sin(2\pi ix)}{2\delta_i} \\ &+ \Delta^{k,n,m}_{\lambda_{1i},\omega,\mu}\Big\{(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta})\varphi_{2i} - \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\psi_{2i} + a(t)f_{2i}(t) * \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta} \\ &+ \frac{\lambda_{2i}-\lambda_{1i}}{2\delta_i}\Bigg(\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta^{k,n,m}_{\lambda_{2i},\omega,\mu}\big[(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta})\varphi_{1i} - \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\psi_{1i} \\ &+ a(t)f_{1i}(t) * \mathcal{B}^{t;\omega}_{\gamma,\xi,\beta}\big]\Bigg) * \left(\mathcal{B}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{G}^{t;\omega}_{\gamma,\xi,\beta}\right)\Big\}\sin(2\pi ix)\Bigg]. \end{aligned} \tag{4.10}$$

To determine $a(t)$, we will make use of the Equation 1.4, we can have

$$\int_0^1 {}^cD^{\beta}_{0_+,t}u(x,t)dx = {}^cD^{\beta}_{0_+,t}g(t). \tag{4.11}$$

By using (1.1), we have

$$a(t) = \left(\int_0^1 f(x,t)dx\right)^{-1}\left({}^cD^{\beta}_{0_+,t}g(t) + \alpha u(1,t) + \mu\int_0^1 e_{\gamma,\xi}(t;\omega) * u_{xx}(x,t)dx\right), \tag{4.12}$$

where

$$u(1,t) = \alpha\frac{\sin 2y_0}{2y_0}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\mu}^{k,n,m}\Big\{\big(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}\big)\varphi_0-\big(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}\big)\psi_0\Big\}$$
$$+\sum_{i=1}^{\infty}\alpha\frac{\sin 2y_i}{2\delta_i}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\mu}^{k,n,m}\Big\{\big(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}\big)\varphi_{1i}-\big(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}\big)\psi_{1i}\Big\}.$$

The Equation (4.12) can be written as

$$a(t) = \mathcal{F}(t) + \int_0^t K(t,\tau)a(\tau)d\tau. \tag{4.13}$$

where

$$\begin{aligned}
\mathcal{F}(t) = \Big(\int_0^1 f(x,t)dx\Big)^{-1}&\Big[{}^cD_{0_+,t}^{\beta}g(t)+\alpha\frac{\sin 2y_0}{2y_0}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\mu}^{k,n,m}\\
&\times\Big\{\big(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}\big)\varphi_0-\big(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}\big)\psi_0\Big\}\\
&+\sum_{i=1}^{\infty}\alpha\frac{\sin 2y_i}{2\delta_i}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\mu}^{k,n,m}\Big\{\big(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}\big)\varphi_{1i}\\
&-\big(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}\big)\psi_{1i}\Big\}\\
&+\mu\Big\{e_{\gamma,\xi}(t;\omega)*\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\mu}^{k,n,m}\big(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}\big)\\
&\times\Big(-2\varphi_0y_0sin^2y_0+\sum_{i=1}^{\infty}\big(-2\varphi_{1i}y_i\frac{sin^2\delta_i}{\delta_i})\Big)\\
&+e_{\gamma,\xi}(t;\omega)*\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\mu}^{k,n,m}\big(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}\big)\\
&\times\Big(-2\psi_0y_0sin^2y_0+\sum_{i=1}^{\infty}\big(-2y_i\psi_{1i}\frac{sin^2\delta_i}{\delta_i})\Big)\Big\}\Big],
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
K(t,\tau) = \Big(\int_0^1 f(x,t)dx\Big)^{-1}&\Big[f_0(\tau)y_0sin^2y_0\int_0^{t-\tau}e_{\gamma,\xi}(t-\tau-s;\omega)\mathcal{B}_{\gamma,\xi,\beta}^{s;\omega}ds\\
&+\sum_{i=0}^{\infty}f_{1i}(\tau)y_i\frac{\sin^2\delta_i}{\delta_i}\int_0^{t-\tau}e_{\gamma,\xi}(t-\tau-s;\omega)\mathcal{B}_{\gamma,\xi,\beta}^{s;\omega}ds\Big].
\end{aligned}\tag{4.15}$$

Existence of the Solution

Following are the requirements for the existence of the solution:

- $a(t) \in C(0,T)$
- $u(x,t) \in C(\Pi)$
- ${}^{c}D_{0+,t}u(x,t) \in C(\Pi)$
- $u_{xx}(x,t) \in C(\Pi)$
- $e_{\gamma,\xi}(t;\omega) * u_{xx}(x,t) \in C(\Pi)$

To show that $a(t) \in C(0,T)$, we will show the continuity of $\mathcal{F}(t)$ given by (4.14)and $K(t,\tau)$ i.e., (4.15). We will use following estimates that are obtained by making use of Lemma 4.4 and given by

$$\left.\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\theta}^{k,n,m}(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}) \le C_1(1+\omega t^{\gamma}),\\
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\theta}^{k,n,m}(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}) \le C_1 t(1+\omega t^{\gamma}),\\
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\theta}^{k,n,m}(1*\mathcal{B}_{\gamma,\xi,\beta}^{t;\omega}) \le C_1 t^{\beta}.
\end{aligned}\right\} \tag{4.16}$$

$$\left.\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\theta}^{k,n,m}(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}) \le \frac{C_1(1+\omega t^{\gamma})}{i^2 t^{\beta}},\\
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\theta}^{k,n,m}(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}) \le \frac{C_1(1+\omega t^{\gamma})}{i^2 t^{\beta-1}},\\
\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{2i},\omega,\theta}^{k,n,m}(1*\mathcal{B}_{\gamma,\xi,\beta}^{t;\omega}) \le \frac{C_1}{i^2}.
\end{aligned}\right\} \tag{4.17}$$

Due to Lemma 4.5, we have the following inequalities

$$\left.\begin{aligned}
e_{\gamma,\xi}(t;\omega) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\theta}^{k,n,m}(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}) \le \Gamma(\xi)C_1^2 t^{\xi}(1+\omega t^{\gamma})\\
e_{\gamma,\xi}(t;\omega) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\theta}^{k,n,m}(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega}+\omega\mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}) \le \Gamma(\xi)C_1^2 t^{\xi+1}(1+\omega t^{\gamma}),\\
e_{\gamma,\xi}(t-\tau;\omega) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_0,\omega,\theta}^{k,n,m}\mathcal{B}_{\gamma,\xi,\beta}^{t-\tau;\omega} \le \Gamma(\xi)C_1^2 (t-\tau)^{\xi+\beta-1}.
\end{aligned}\right\} \tag{4.18}$$

$$\left.\begin{aligned}
e_{\gamma,\xi}(t;\omega) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta_{\lambda_{2i},\omega,\theta}^{k,n,m}(\mathcal{C}_{\gamma,\xi,\beta}^{t;\omega} + \omega \mathcal{H}_{\gamma,\xi,\beta}^{t;\omega}) &\leq \frac{\Gamma(\xi) C_1^2 t^{\xi}(1+\omega t^{\gamma})}{i^2 t^{\beta}},\\
e_{\gamma,\xi}(t;\omega) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta_{\lambda_{2i},\omega,\theta}^{k,n,m}(\mathcal{A}_{\gamma,\xi,\beta}^{t;\omega} + \omega \mathcal{F}_{\gamma,\xi,\beta}^{t;\omega}) &\leq \frac{\Gamma(\xi) C_1^2 t^{\xi}(1+\omega t^{\gamma})}{i^2 t^{\beta-1}},\\
e_{\gamma,\xi}(t-\tau;\omega) * \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta_{\lambda_{2i},\omega,\theta}^{k,n,m}\mathcal{B}_{\gamma,\xi,\beta}^{t-\tau;\omega} &\leq \frac{\Gamma(\xi) C_1^2 (t-\tau)^{\xi-1}}{i^2}.
\end{aligned}\right\} \tag{4.19}$$

Consequently, by using (4.16), (4.17),(4.18) and (4.19) we have the following relation for (4.14) and (4.15)

$$\begin{aligned}
|\mathcal{F}(t)| \leq M_1 \Bigg[& {}^{c}D_{0_+,t}^{\beta} g(t) + \alpha C_1 D_1 (1+\omega t^{\gamma})\big(\|\varphi\| + \frac{\|\psi\|}{t}\big)\\
&+ \sum_{i=1}^{\infty} \frac{\alpha C_1 D_1 (1+\omega t^{\gamma})}{i^2 t^{\beta}}\big(\|\varphi\| + \frac{\|\psi\|}{t}\big)\\
&+ \mu \Big\{ \pi C_1^2 D_1 \Gamma(\xi) t^{\xi} (1+\omega t^{\gamma})\big(\|\varphi\| + \frac{\|\psi\|}{t}\big)\\
&+ \sum_{i=1}^{\infty} \frac{3\pi C_1^2 D_1 \Gamma(\xi) t^{\xi}(1+\omega t^{\gamma})}{i^2 t^{\beta}} \Big((1+\|\varphi'\|) + \frac{1}{t}(1+\|\psi'\|) \Big) \Big\} \Bigg].
\end{aligned}$$

$$|K(t,\tau)| \leq M_1 \Bigg[\frac{\pi C_1^2 D_2 (t-\tau)^{\xi+\beta-1}}{2}\|f\| + \sum_{i=1}^{\infty} \frac{3\pi \Gamma(\xi) C_1^2 D_2 (t-\tau)^{\xi-1}}{i^2}\big(1 + \|f_x\|\big) \Bigg],$$

where $\left(\int_0^1 f(x,t)dx \right)^{-1} < M_1$.

Hence by Weierstrass M-test $\mathcal{F}(t)$ and $K(t,\tau)$ represent uniformly convergent function. Consequently, by using Lemma 2.1, the Equation 4.13 unique continuous solution i.e., $a(t) \in C(0,T]$

In order to establish the continuity of $u(x,t)$, we need to present the convergent estimates for $T_{ji}\,; j = 0,1,2,\ i \in \mathbb{N}$.

To show that $u(x,t)$ represents a continuous function, first we check the boundedness of $T_0(t)$, $T_{1i}(t)$ and $T_{2i}(t)$.

Due to (4.7), we can write

$$|T_0(t)| \leq C_1(1+\omega t^{\gamma})|\varphi_0| + C_1 t(1+\omega t^{\gamma})|\psi_0| + C_1 t^{\beta}|f_0|.$$

Cauchy-Bankovsky-Schwarz inequality allows us to write

$$|T_0(t)| \leq C_1 D(1+\omega t^{\gamma})\bigg(\|\varphi\| + t\|\psi\|\bigg) + C_1 D t^{\beta}\|f\|. \tag{4.20}$$

Next, by using (3.4) and (4.16) in (4.8), we have

$$|T_{1i}(t)| \leq \frac{C_1(1+\omega t^{\gamma})}{i^2 t^{\beta}}\bigg(|\varphi_{1i}| + t\|\psi_{1i}|\bigg) + \frac{MC_1}{i^2}|f_{1i}(t)|. \tag{4.21}$$

Due to Cauchy-Bankovsky-Schwarz inequality and Lemma (4.1), above relation can be written as

$$|T_{1i}(t)| \leq \frac{C_1 D(1+\omega t^{\gamma})}{i^2 t^{\beta}}\bigg(\|\varphi\| + t\|\psi\|\bigg) + \frac{MC_1 D}{i^2}\|f\|. \tag{4.22}$$

Hence by Weierstrass M-test $\sum\limits_{i=1}^{\infty}|T_{1i}(t)|$ is uniformly convergent.

Before giving an upper bound for $|T_{2i}(t)|$, we will establish inequality for convolution of three parameter Mittag-Leffler function.

Consider

$$\begin{aligned}
&\bigg|\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{1i},\omega,\mu}^{k,n,m}T_{1i}(t) * \Big(\mathcal{B}_{\gamma,\xi,\beta}^{t;\omega} + \omega\mathcal{G}_{\gamma,\xi,\beta}^{t;\omega}\Big)\bigg| \\
&\leq \bigg|\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{1i},\omega,\mu}^{k,n,m}T_{1i}(t) * \mathcal{B}_{\gamma,\xi,\beta}^{t;\omega}\bigg| + \omega\bigg|\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{1i},\omega,\mu}^{k,n,m}T_{1i}(t) * \mathcal{G}_{\gamma,\xi,\beta}^{t;\omega}\bigg|.
\end{aligned}$$

By using Remark 2, we have

$$\begin{aligned}
&\bigg|\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{1i},\omega,\mu}^{k,n,m}T_{1i}(t) * \Big(\mathcal{B}_{\gamma,\xi,\beta}^{t;\omega} + \omega\mathcal{G}_{\gamma,\xi,\beta}^{t;\omega}\Big)\bigg| \\
&\leq \ \frac{C_1^2(\Gamma(\beta) + \omega\Gamma(\beta+1)t^{\gamma})}{i^2}(1+\omega t^{\gamma})\bigg(|\varphi_{2i}| + \frac{1}{t}|\psi_{2i}|\bigg) + \frac{MC_1^2\Gamma(\beta)}{i^2}(1+\omega t^{\gamma})|f_{1i}(t)|t^{\beta}.
\end{aligned}$$

Due to Lemmata 4.1 and 4.2, we can write

$$\begin{aligned}
&\bigg|\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}\Delta_{\lambda_{1i},\omega,\mu}^{k,n,m}T_{1i}(t) * \Big(\mathcal{B}_{\gamma,\xi,\beta}^{t;\omega} + \omega\mathcal{G}_{\gamma,\xi,\beta}^{t;\omega}\Big)\bigg| \\
&\leq \ \frac{C_1^2 D_1(\Gamma(\beta) + \omega\Gamma(\beta+1)t^{\gamma})}{i^3}(1+\omega t^{\gamma})\bigg((1+\|\varphi'\|) + \frac{1}{t}(1+\|\psi'\|)\bigg) \\
&+ \frac{MC_1^2 D_2\Gamma(\beta)}{i^3}(1+\omega t^{\gamma})(1+\|f_x\|)t^{\beta}.
\end{aligned} \tag{4.23}$$

For estimate of $T_{2i}(t)$, we will make use of Remark 2 and equation (4.23) apart from relation (4.16), to get the following inequality

$$
\begin{aligned}
|T_{2i}(t)| \leq{}& \frac{C_1 D_1 (1+\omega t^{\gamma})}{i^2 t^{\beta}} \Big(\|\varphi\| + t\|\psi\| \Big) + \frac{M C_1 D_2}{i^2} \|f\| \\
&+ \frac{5\pi C_1^2 D_1 (\Gamma(\beta) + \omega \Gamma(\beta+1) t^{\gamma})(1+\omega t^{\gamma})}{i^2} \Big((1 + \|\varphi'\|) + t(1+\|\psi'\|) \Big) \\
&+ \frac{5\pi M C_1^2 D_2 \Gamma(\beta)(1+\omega t^{\gamma})}{i^2} (1 + \|f_x\|) t^{\beta} \qquad (4.24)
\end{aligned}
$$

By using (4.22) and (4.24), together with Weierstrass M-test ensure the continuity of $u(x,t)$. To present the continuity of $u_{xx}(x,t)$. observe that

$$
\left| \frac{\partial^2 q_0}{\partial x^2} \right| \leq \pi^2, \quad \left| \frac{\partial^2 q_{1i}}{\partial x^2} \right| \leq 18\pi^2 i^2, \quad \left| \frac{\partial^2 q_{2i}}{\partial x^2} \right| \leq 4\pi^2 i^2. \qquad (4.25)
$$

Hence, for continuity of $u_{xx}(x,t)$ we need to show convergence of

$$
\sum_{i=1}^{\infty} i^2 T_{li}(t), \quad l = 1, 2,
$$

which can be deduced by following relations

$$
|i^2 T_{1i}(t)| \leq \frac{C_1 D_1 (1+\omega t^{\gamma})}{i^2 t^{\beta}} \Big(\|\varphi''\| + t\|\psi''\| \Big) + \frac{M C_1 D_2}{i^2} \|f_{xx}\|. \qquad (4.26)
$$

$$
\begin{aligned}
|i^2 T_{2i}(t)| \leq{}& \frac{C_1 D_1 (1+\omega t^{\gamma})}{i^2 t^{\beta}} \Big(\|\varphi''\| + t\|\psi''\| \Big) + \frac{M C_1 D_2}{i^2} \|f_{xx}\| \\
&+ \frac{5\pi C_1^2 D_1 (\Gamma(\beta) + \omega \Gamma(\beta+1) t^{\gamma})(1+\omega t^{\gamma})}{i^2} \Big((1 + \|\varphi'''\|) + t(1+\|\psi'''\|) \Big) \\
&+ \frac{5\pi M C_1^2 D_2 \Gamma(\beta)(1+\omega t^{\gamma})}{i^2} (1 + \|f_{xxx}\|) t^{\eta}. \qquad (4.27)
\end{aligned}
$$

In order to establish the continuity of $e_{\gamma,\xi}(t;\omega) * u_{xx}(x,t)$, we need to study the convergence of

$$
\sum_{i=1}^{\infty} i^2 e_{\gamma,\xi}(t;\omega) * T_{li}(t), \ l = 1, 2.
$$

By using Lemma 4.5 and Cauchy-Bankovsky-Schwarz inequality, we have

$$
|i^2 e_{\gamma,\xi}(t;\omega) * T_{1i}(t)| \leq \frac{\Gamma(\xi) C_1^2 D_1 t^{\xi} (1+\omega t^{\gamma})}{i^2 t^{\beta}} (1+\omega t^{\gamma}) \Big(\|\varphi''\| + t\|\psi''\| \Big) + \frac{\Gamma(\xi) M C_1^2 D_2 t^{\xi}}{i^2} \|f_{xx}\| \qquad (4.28)
$$

Lemma 4.5 and Remark 2 together with Cauchy-Bankovsky-Schwarz inequality, support us to write

$$|i^2 e_{\gamma,\xi}(t;\omega) * T_{2i}(t)| \leq \frac{\Gamma(\xi)C_1^2 D_1 t^{\xi}(1+\omega t^{\gamma})}{i^2 t^{\beta}}\bigg(\|\varphi''\| + t\|\psi''\|\bigg) + \frac{\Gamma(\xi)MC_1^2 D_2 t^{\xi}}{i^2}\|f_{xx}\|$$
$$+ 5\pi i(1+\omega t^{\gamma})(\Gamma(\beta)+\omega\Gamma(\beta+1)t^{\gamma})\Gamma(\xi)C_1^3 D_1 t^{\xi}\bigg((1+\|\varphi'''\|) + t(1+\|\psi'''\|\bigg)$$
$$+ (1+\omega t^{\gamma})MC_1^3 D_2\Gamma(\xi)\Gamma(\beta)t^{\xi+\beta}(1+\|f_{xxx}\|). \tag{4.29}$$

Finally, we are going to present the estimates that ensure the continuity of ${}^cD^{\beta}_{0_+,t}u(x,t)$. By using [29], in order to ensure the continuity of ${}^cD^{\beta}_{0_+,t}u(x,t)$, we need to establish the uniform convergence of

$$\sum_{i=1}^{\infty} T_{li}, \quad \text{and} \sum_{i=1}^{\infty} {}^cD^{\beta}_{0_+,t}T_{li}, \quad l=1,2.$$

Uniform convergence of $\sum_{i=1}^{\infty} T_{li}, \quad l=1,2$ has already been proved (see inequalities (4.26) and (4.27)). By using Equation (4.5) and (4.6), we can see that in order to show the uniform convergence of $\sum_{i=1}^{\infty} {}^cD^{\beta}_{0_+,t}T_{li}$, we need to present the convergent estimates of

$$i^2 T_{li}(t),\ i^2 e_{\gamma,\xi} * T_{li}(t);\ \ l=1,2,$$

which we have already established in (4.26) and (4.29).

Uniqueness of the Solution

Lastly, we will prove the uniqueness of the obtained solution i.e., (4.10) and (4.12).

The expression, given by (4.12), involves definite integral of known function $f(x,t)$ and the β ordered Caputo fractional derivative of given function $h(t)$. One-one nature of these operators (integral and fractional derivative) proves the uniqueness of $a(t)$.

On contrary suppose $u(x,t)$ and $v(x,t)$ be two different solutions, and define

$$\bar{u}(x,t) = u(x,t) - v(x,t). \tag{4.30}$$

Using (4.30) in (1.1)-(1.4), we have

$${}^cD^{\beta}_{0_+,t}\bar{u}(x,t) + \bar{u}_{xx}(x,t) + \mu e_{\gamma,\xi}(t;\omega) * \bar{u}_x x(x,t) = 0, \qquad (x,t) \in \Pi.$$

the boundary datum

$$\bar{u}(0,t) = 0 = \bar{u}_{xxx}(0,y,t),\ \bar{u}_x(0,t) = \bar{u}_x(1,t) + \alpha u(1,t).$$

and initial condition

$$\bar{u}(x,0) = 0, \quad \bar{u}_t(x,0) = 0. \tag{4.31}$$

Consider the functions

$$\begin{aligned}
\bar{T}_0(t) &= \int_0^1 \int_0^1 \bar{u}(x,t) r_0(x,y)dxdy, \\
\bar{T}_{1i}(t) &= \int_0^1 \int_0^1 \bar{u}(x,t) r_{1i}(x,y)dxdy, \\
\bar{T}_{2i}(t) &= \int_0^1 \int_0^1 \bar{u}(x,t) r_{2i}(x,y)dxdy.
\end{aligned}$$

Taking the fractional derivative, we get

$$\begin{aligned}
{}^cD^{\beta}_{0_+,t}\bar{T}_0(t) &= -\lambda_k \bar{T}_0(t) + \lambda_0 \mu e_{\gamma,\xi}(t;\omega) * \bar{T}_0(t), \\
{}^cD^{\beta}_{0_+,t}\bar{T}_{1i}(t) &= -\lambda_{2i} \bar{T}_{1i}(t) + \lambda_{2i} \mu e_{\gamma,\xi}(t;\omega) * \bar{T}_{1i}(t), \\
{}^cD^{\beta}_{0_+,t}\bar{T}_{2i}(t) &= -\lambda_{1i} \bar{T}_{2i}(t) + \frac{\lambda_{2i}-\lambda_{1i}}{2\delta_i}\bar{T}_{1i}(t) + \lambda_{1i} \mu e_{\gamma,\xi}(t;\omega) * \bar{T}_{2i}(t).
\end{aligned}$$

The solution of the above system is

$$\begin{aligned}
\bar{T}_0(t) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta^{k,n,m}_{\lambda_0,\omega,\mu}\big[\big(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta}\big)\bar{T}_0(0) - \big(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\big)\bar{T}_0'(0)\big], \\
\bar{T}_{1i}(t) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta^{k,n,m}_{\lambda_{2i},\omega,\mu}\big[\big(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta}\big)\bar{T}_{1i}(0) - \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\bar{T}_{1i}'(0)\big], \\
\bar{T}_{2i}(t) &= \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n} \Delta^{k,n,m}_{\lambda_{1i},\omega,\mu}\big[\big(\mathcal{C}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{H}^{t;\omega}_{\gamma,\xi,\beta}\big)\bar{T}_{2i}(0) - \left(\mathcal{A}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{F}^{t;\omega}_{\gamma,\xi,\beta}\right)\bar{T}_{2i}'(0) \\
&\quad + \frac{\lambda_{2i}-\lambda_{1i}}{2\delta_i} T_{1i}(t) * \left(\mathcal{B}^{t;\omega}_{\gamma,\xi,\beta} + \omega\mathcal{G}^{t;\omega}_{\gamma,\xi,\beta}\right)\big],
\end{aligned}$$

The condition (4.31), leads to

$$\bar{T}_0(t) = 0, \quad \bar{T}_{1i}(t) = 0, \quad \bar{T}_{2i}(t) = 0, \quad t \in (0,T].$$

The completeness of the set $\{r_0, r_{1i}, r_{2i}\}, \quad i \in \mathbb{N}$ guarantees that obtained solution is unique.

5. Example

This section is devoted to some examples for particular values of $\mu,\ f(x,t),\ \phi(x)$ and $\psi(x)$.

Example 1: By taking $\mu = 1 = \omega,\ \gamma = 0.25,\ \xi = 0.5,\ \beta = 1.5,\ \phi(x) = \sin(2\pi x),\ \psi(x) = 0,\ \alpha = 1$ and $f(x,t) = 0.7654\sin(1.30654x)t^{0.25}$.

Solution of the inverse source problem (1.1)-(1.4) has the following form:

$$a(t) = t^{1.5}$$

$$\begin{aligned} u(x,t) = &\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{n}\,{}^{n}C_{m}\,{}^{k}C_{n}\Big[(2\pi)^{2k}\big(\mathcal{E}^{k+1}_{0.25,0.25(n-m)+0.5(k-n)+1.5k+1}(t;1) \\ &+ (1.7075)^{k}1.0174\mathcal{E}^{k+1}_{0.25,0.25(n-m)+0.5(k-n)+1.5(k+2)+1}(t;1)\sin(0.3065x) \\ &+ \mathcal{E}^{k+1}_{0.25,0.25(n-m+1)+0.5(k-n)+1.5k+1}(t;1)\big)\sin(2\pi x)\Big]. \end{aligned}$$

Example 2:

By taking $\mu = 1 = \omega,\ \gamma = 0.25,\ \xi = 0.5,\ \beta = 1.5,\ \phi(x) = 0.7654\sin(1.30654x),\ \psi(x) = \sin(2\pi x),\ \alpha = 1$ and $f(x,t) = 0.7654\sin(1.30654x) + \sin(\pi x)$.

Solution of the inverse source problem (1.1)-(1.4) has the following form:

$$a(t) = 2$$

$$\phi_0 = 1,\ \psi_{21} = 1,\ f_0 = 1,\ f_{21} = 1$$

$$\begin{aligned} u(x,t) = &\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^{n}\,{}^{n}C_{m}\,{}^{k}C_{n}\Big[\big\{\mathcal{E}^{k+1}_{0.25,0.25(n-m)+0.5(k-n)+1.5k+1}(t;1) \\ &+ \mathcal{E}^{k+1}_{0.25,0.25(n-m+1)+0.5(k-n)+1.5k+1}(t;1) \\ &+ 2\mathcal{E}^{k+1}_{0.25,0.25(n-m)+0.5(k-n)+1.5(k+1)+1}(t;1)\big\}0.7654\sin(1.30654x) \\ &+ \big(\mathcal{E}^{k+1}_{0.25,0.25(n-m)+0.5(k-n)+1.5k}(t;1) \\ &+ \mathcal{E}^{k+1}_{0.25,0.25(n-m+1)+0.5(k-n)+1.5k}(t;1) \\ &+ 2\mathcal{E}^{k+1}_{0.25,0.25(n-m)+0.5(k-n)+1.5(k+1)+1}\big)\sin(2\pi x)\Big]. \end{aligned}$$

6. Conculsion

In summary, our investigation has navigated the complexities of fractional diffusion-wave equations, focusing on the extraction of the temporal component of the source term. Addressing a challenging inverse problem, we incorporate a nonlocal damping term with a two-parameter Mittag-Leffler function and include Samarskii-Ionkin type boundary conditions. The establishment of a three-parameter Mittag-Leffler function not only affirms the existence of solutions but also contributes valuable insights to the domains of mathematical analysis and fractional calculus.

Our research significantly advances the understanding of these equations, providing a solid foundation for nuanced problem-solving methodologies. The attained insights not only confirm solution existence but also pave the way for future exploration and application, marking a substantive contribution to the dynamic intersection of mathematics and physics. As we navigate the complexities of fractional diffusion-wave equation, this work opens new avenues for continued advancements and deeper understanding in this ever-evolving field.

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