

# Error estimates of a two-grid penalty finite element method for the Smagorinsky model\*

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## Abstract

In the paper, we consider the penalty finite element methods (FEMs) for the stationary Smagorinsky model. Firstly, a one-grid penalty FEM is proposed and analyzed. Since this method is nonlinear, a novel linearized iteration scheme is derived for solving it. We also derived the stability and convergence of numerical solutions for this iteration scheme. Furthermore, a two-grid penalty FEM is developed for Smagorinsky model. Under  $\varepsilon \ll h$ , this method consist of solving a nonlinear Smagorinsky model by the one-grid penalty FEM with the proposed linearized iteration scheme on a coarse mesh with mesh width  $H$ , and then solving a linearized Smagorinsky model based on the Newton iteration on a fine mesh with mesh width  $h = \mathcal{O}(H^2)$ , respectively. Stability and error estimates of numerical solutions for two-grid penalty FEM are presented. Finally, some numerical tests are provided to confirm the theoretical analysis and the effectiveness of the developed methods.

**Keywords:** Smagorinsky model, Penalty finite element method, Two-grid method, Stability, Convergence.

**Mathematics Subject Classification:** 65N15; 65M60; 65M12.

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## 1 Introduction

In this paper, we consider the penalty FEMs for the stationary Smagorinsky model

$$-Re^{-1}\Delta \mathbf{u} - \nabla \cdot ((C_S\delta)^2 |\nabla \mathbf{u}| \nabla \mathbf{u}) + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1a)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1b)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega. \quad (1c)$$

where  $\Omega \subset \mathbb{R}^d$  ( $d=2$  or  $3$ ) is a bounded and regular domain with a Lipschitz continuous boundary  $\partial\Omega$ ,  $\mathbf{u}$  represents the velocity,  $p$  the pressure,  $\mathbf{f}$  the spatially filtered forcing term,  $C_S$  the

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Smagorinsky constant,  $\delta$  the radius of the spatial filter radius employed in large eddy simulation (LES),  $|\sigma| := \sqrt{\sum_{i,j=1}^d |\sigma_{ij}|^2}$  the Frobenius norm of the tensor  $\sigma$  and  $Re$  the Reynolds number which is defined as  $Re = UL/\nu$ , where  $U$ ,  $L$  and  $\nu$  represent the characteristic velocity, length and the viscosity of fluid, respectively.

Nowadays, numerical simulation of turbulence is one of the most important but challenging research topics in computational fluid dynamics since it is widely used in engineering and environmental fields. The Smagorinsky model [1–5] is one of the most popular large eddy simulation (LES) models [6–10]. This model has been widely used in many application fields, such as gas dynamics [11] and geophysical flow [1]. The analysis of the model (1) can be found in [2,3], and we can see [8–10] for the challenging simulations. Comparing the classical Navier-Stokes equations, it is added an artificial viscosity term  $-\nabla \cdot ((C_S \delta)^2 |\nabla \mathbf{u}| \nabla \mathbf{u})$ , which induces the dissipated energy in the large scale structures at the same rate as the discarded small scale structures in model (1).

In the last decades, more and more studies have been attracted for the numerical methods of the Smagorinsky model (1). Among the studies, finite element method (FEM) is one of the most popular methods. For example, in [5], the authors applied a two-level FEM to the Smagorinsky model, in which a nonlinear problem was solved on the coarse mesh firstly and then solve a Newton linearization problem on the fine mesh. In [12], the authors combined the lowest equal-order stabilized FEM with the two-level Newton iteration to solve the steady Smagorinsky model. In [13], the stabilized FEM based on Gaussian quadrature rule is used to penalize the instability induced by the domination of convection term in the Smagorinsky model for simulating large Reynolds numbers. In [14], three iterative stabilized FEMs for the Smagorinsky model were proposed and analyzed. In [15], a low order nonconforming mixed FEM for the Smagorinsky model was studied. In [16], a two-step stabilized FEM for solving the Smagorinsky model was established.

Two-grid method is an efficient numerical scheme for the nonlinear partial differential equations, this method was pioneered by Marion and Xu [17–19]. The basic idea is to solve a nonlinear problem on a very coarse mesh, and then solve one linearized system on a fine mesh. It is a good strategy to reduce computing costs. So, two-grid method has been massively studied in recent years. For example, we can refer to [20–26] for the research of the incompressible flow. Another main difficulty is that velocity and pressure are coupled, while the penalty method is an effective method to overcome this difficulty. There are more and more researches devoted to study the penalty method in different problems. For example, we can refer to [27] for the pure Neumann problem, [28, 29] for the Stokes equations, and [30–37] for the Navier-Stokes equations. From above literature, we know that combining two-grid method and penalty method is quite efficient for the nonlinear and multi-physical quantity coupling problem. In this paper, we consider the two-grid penalty FEMs for solving the Smagorinsky model (1). Setting the penalty parameter  $0 < \varepsilon \ll 1$  as a real number. Firstly, a one-grid penalty FEM and the corresponding linearized iteration scheme are proposed and analyzed. Furthermore, we develop a two-grid penalty FEM for solving the Smagorinsky model, which consists of solving a nonlinear Smagorinsky model by the one-grid penalty FEM with the proposed linearized iteration scheme on a coarse mesh with mesh width  $H$ , and then solving a linearized Smagorinsky model based on the Newton iteration on a fine mesh with mesh width  $h = \mathcal{O}(H^2)$ , respectively. Stability and error estimates of numerical solutions for two-grid penalty FEM are derived. Some numerical tests are provided to

confirm the theoretical analysis and the effectiveness of the proposed methods.

The rest of the article is organized as follows. In the next section, some basic statements are provide. In Section 3, a two-grid penalty FEM for the Smagorinsky model is proposed and analyzed. Meanwhile, a one-grid penalty FEM and the corresponding linearized iteration scheme are also given out and analyzed. The numerical experiments are presented to validate the theoretical predictions and the efficiency of the proposed method in Section 4. Finally, we conclude the article.

## 2 Mathematical preliminaries

We first generalize some notations, definitions and preliminary lemmas which will be used in the analysis. Let  $W^{k,p}(\Omega)$  and  $W_0^{k,p}(\Omega)$  ( $k \in \mathbb{N}$ ,  $1 \leq p \leq +\infty$ ) denote the standard Sobolev spaces [38]. The norm and seminorm on  $W^{k,p}(\Omega)$  are denoted by  $\|\cdot\|_{k,p}$  and  $|\cdot|_{k,p}$ , respectively. The space  $H^k(\Omega)$  is the standard Hilbertian Sobolev space of order  $k$  with norm  $\|\cdot\|_k$ . All other norms will be clearly labeled. The inner product and norm in  $L^2(\Omega)$  are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$ , respectively. In addition, the vector spaces and vector functions will be indicated by boldface type letters, e.g., the spaces  $\mathbf{H}^k(\Omega)$ ,  $\mathbf{W}^{k,p}(\Omega)$  and  $\mathbf{L}^p(\Omega)$  represent the vector Sobolev spaces  $H^k(\Omega)^d$ ,  $W^{k,p}(\Omega)^d$  and  $L^p(\Omega)^d$ , respectively.

We introduce the following Sobolev spaces:

$$\begin{aligned}\mathbf{X} &:= \mathbf{W}_0^{1,3}(\Omega) = \{\mathbf{v} \in \mathbf{W}^{1,3}(\Omega) : \mathbf{v} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{V} &:= \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = 0 \text{ on } \partial\Omega\}, \\ Q &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\}.\end{aligned}$$

Due to  $\mathbf{X} \subset \mathbf{V}$ , the weak formulation of the Smagorinsky model (1) is given by: Find  $(\mathbf{u}, p) \in (\mathbf{X}, Q)$  satisfying for all  $(\mathbf{v}, q) \in (\mathbf{X}, Q)$

$$\begin{aligned}a(\mathbf{u}, \mathbf{v}) + (C_S \delta)^2 (|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - d(p, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \\ d(q, \mathbf{u}) &= 0,\end{aligned}\tag{2}$$

where

$$\begin{aligned}a(\mathbf{u}, \mathbf{v}) &= Re^{-1} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}, \\ d(p, \mathbf{v}) &= \int_{\Omega} p \operatorname{div} \mathbf{v} d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{X}, p \in Q, \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d\mathbf{x} + \frac{1}{2} \int_{\Omega} (\nabla \cdot \mathbf{u}) \mathbf{v} \cdot \mathbf{w} d\mathbf{x} \\ &= \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d\mathbf{x} - \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} d\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\end{aligned}$$

with  $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i,j=1}^d \frac{\partial \mathbf{u}_i}{\partial x_j} \cdot \frac{\partial \mathbf{v}_i}{\partial x_j}$ .

Define the following divergence-free function spaces:

$$\mathbf{V}_0 := \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0\}, \quad \mathbf{X}_0 := \{\mathbf{v} \in \mathbf{W}_0^{1,3}(\Omega) : \nabla \cdot \mathbf{v} = 0\}.$$

An equivalent weak formulation of the Smagorinsky model (1) reads as follows: Find  $\mathbf{u} \in \mathbf{X}_0$  satisfying for all  $\mathbf{v} \in \mathbf{X}_0$

$$a(\mathbf{u}, \mathbf{v}) + (C_S \delta)^2 (|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}). \quad (3)$$

Following [5], the following three finite quantities are defined by

$$\|\mathbf{f}\|_* := \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{|(\mathbf{f}, \mathbf{v})|}{\|\nabla \mathbf{v}\|}, \quad \|\mathbf{f}\|_{*3} := \sup_{\mathbf{v} \in \mathbf{X}} \frac{|(\mathbf{f}, \mathbf{v})|}{|\mathbf{v}|_{1,3}}, \quad \gamma_3 := \sup_{\mathbf{v} \in \mathbf{X}} \frac{\|\nabla \mathbf{v}\|}{|\mathbf{v}|_{1,3}}.$$

It is easy to verify that the the trilinear  $b(\cdot, \cdot, \cdot)$  has the following properties [42]:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}, \quad (4)$$

and

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq N \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\| \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}, \quad (5)$$

where

$$N = \sup_{\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}} \frac{|b(\mathbf{u}, \mathbf{v}, \mathbf{w})|}{\|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|}.$$

We will use the following strong monotonicity and Lipschitz continuity of the  $r$ -Laplacian [5, 39]:

**Lemma 2.1** *For all  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbf{W}^{1,r}(\Omega)$ , there exists a generic constant  $C_1$  depending on  $d, r$  and  $\Omega$ , but not on  $\mathbf{u}_1, \mathbf{u}_2$  or  $\mathbf{v}$ , such that the following inequalities hold:*

$$\begin{aligned} (|\nabla \mathbf{u}_1|^{r-2} \nabla \mathbf{u}_1, \nabla(\mathbf{u}_1 - \mathbf{u}_2)) - (|\nabla \mathbf{u}_2|^{r-2} \nabla \mathbf{u}_2, \nabla(\mathbf{u}_1 - \mathbf{u}_2)) &\geq C_1 \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{0,r}^r, \\ (|\nabla \mathbf{u}_1|^{r-2} \nabla \mathbf{u}_1, \nabla \mathbf{v}) - (|\nabla \mathbf{u}_2|^{r-2} \nabla \mathbf{u}_2, \nabla \mathbf{v}) &\leq C_1 M \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{0,r} \|\nabla \mathbf{v}\|_{0,r}, \end{aligned}$$

where  $M := \max\{\|\nabla \mathbf{u}_1\|_{0,r}^{r-2}, \|\nabla \mathbf{u}_2\|_{0,r}^{r-2}\}$ .

Then we recall the well-posedness of the solution for the problem (3) in the following lemma [2, 4, 5]:

**Lemma 2.2** *There exists a weak solution  $\mathbf{u} \in \mathbf{X}_0$  to problem (3) satisfying*

$$|\mathbf{u}|_{1,3} \leq (C_S \delta)^{-1} \|\mathbf{f}\|_{*3}^{1/2},$$

$$\|\nabla \mathbf{u}\| \leq \Psi(\|\mathbf{f}\|_*),$$

where  $\Psi$  is defined as the inverse function of  $\Phi : (0, +\infty) \rightarrow \mathbb{R}$ :

$$\Phi(x) := Re^{-1}x + (C_S \delta)^2 \gamma_3^{-3} x^2. \quad (6)$$

Furthermore, if the following inequality holds,

$$N \Psi(\|\mathbf{f}\|_*) \leq Re^{-1}, \quad (7)$$

then the problem (3) has a unique solution.



Next, we recall the strong monotonicity, for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$ , we define

$$(F(\mathbf{u}), \mathbf{v}) = (C_S \delta)^2 (|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{v}).$$

Then  $F$  is strongly monotone and satisfies the following property [5, 39]:

$$(F'(\mathbf{w})\mathbf{u}, \mathbf{u}) \geq 0, \quad (8)$$

where

$$(F'(\mathbf{w})\mathbf{u}, \mathbf{v}) = (C_S \delta)^2 (|\nabla \mathbf{w}| \nabla \mathbf{u}, \nabla \mathbf{v}) + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{w} : \nabla \mathbf{u}]}{|\nabla \mathbf{w}|} \nabla \mathbf{w}, \nabla \mathbf{v} \right).$$

Let  $\tau_\mu = \{\Omega_\mu\}$  is a quasi-uniform family of triangular partition of  $\Omega$  with mesh size  $\mu$ . The real parameter  $\mu > 0$  takes  $h$  or  $H$  ( $h \ll H$ ) tending to 0. We take the fine grid partition  $\tau_h$  as a mesh refinement generated from the coarse grid  $\tau_H$ . Define the following conforming finite element subspaces of  $\mathbf{X}$  and  $Q$ , respectively, by

$$\begin{aligned} \mathbf{W}_\mu &= \{\mathbf{v}_\mu \in \mathbf{C}(\bar{\Omega}) : \mathbf{v}_\mu|_K \in \mathbf{P}_2(K), \forall K \in \tau_h\}, \\ \mathbf{V}_\mu &= \mathbf{W}_\mu \cap \mathbf{V}, \quad \mathbf{X}_\mu = \mathbf{W}_\mu \cap \mathbf{X}, \\ Q_\mu &= \{q_\mu \in \mathbf{C}(\bar{\Omega}) : q_\mu|_K \in P_1(K), \forall K \in \tau_h\} \cap Q, \end{aligned}$$

where  $P_r(K)$  ( $r = 1, 2$ ) is the space of the  $r$ -th order polynomial on  $K$ . With the choices of the finite element spaces  $(\mathbf{V}_\mu, Q_\mu)$ , we know that the spaces  $(\mathbf{V}_\mu, Q_\mu)$  is a pair of conforming finite element space which satisfy the discrete inf-sup condition [43, 44], i.e., there exists a constant  $\beta > 0$  independent of  $\mu$  such that

$$\inf_{q_\mu \in Q_\mu} \sup_{\mathbf{v}_\mu \in \mathbf{V}_\mu} \frac{(q_\mu, \nabla \cdot \mathbf{v}_\mu)}{\|q_\mu\| \|\nabla \mathbf{v}_\mu\|} \geq \beta. \quad (9)$$

The discrete divergence-free function spaces is defined as:

$$\mathbf{V}_{0\mu} := \{\mathbf{v}_\mu \in \mathbf{X}_\mu : (q_h, \nabla \cdot \mathbf{v}_\mu) = 0 \quad \forall q_h \in M_h\}.$$

We define the projection operations  $R_\mu : \mathbf{V} \longrightarrow \mathbf{V}_\mu$  and  $Q_\mu : Q \longrightarrow Q_\mu$  by

$$\begin{aligned} a(\mathbf{u} - R_\mu \mathbf{u}, \mathbf{v}_\mu) - d(p - Q_\mu p, \mathbf{v}_\mu) &= 0 \quad \forall \mathbf{v}_\mu \in \mathbf{V}_\mu, \\ d(q_\mu, \mathbf{u} - R_\mu \mathbf{u}) &= 0 \quad \forall q_\mu \in Q_\mu. \end{aligned} \quad (10)$$

Then the following approximation properties hold [40–42]:

$$\begin{aligned} \|\mathbf{v} - R_\mu \mathbf{v}\| + \mu \|\nabla(\mathbf{v} - R_\mu \mathbf{v})\| + \|q - Q_\mu q\| &\leq C\mu^3 (\|\mathbf{v}\|_3 + \|p\|_2), \\ \|\mathbf{v} - R_\mu \mathbf{v}\|_{1,3} &\leq C\mu^{2-\frac{d}{6}} \|\mathbf{v}\|_3, \end{aligned} \quad (11)$$

for any  $\mathbf{v} \in \mathbf{H}^3(\Omega) \cup \mathbf{V}$  and  $q \in H^2(\Omega) \cup Q$ .

Furthermore, the Young's inequality and the Poincaré's inequality as follows will be used frequently

$$\begin{aligned} ab &\leq \frac{\epsilon}{p} a^p + \frac{\epsilon^{-q/p}}{q} b^q, \quad a, b, p, q, \epsilon \in \mathbb{R}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q \in (1, \infty), \quad \epsilon > 0, \\ \|\mathbf{v}\| &\leq C_p \|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{V}, \quad C_p = C_p(\Omega). \end{aligned}$$

To this end, we recall the following inverse inequality from [40]:

$$|\mathbf{v}_\mu|_{1,3} \leq C_{\text{inv}} \mu^{-\frac{d}{6}} \|\nabla \mathbf{v}_\mu\| \quad \forall \mathbf{v}_\mu \in \mathbf{X}_\mu. \quad (12)$$

### 3 Two-grid penalty FEM for the Smagorinsky model

The proposed method consist of solving a nonlinear Smagorinsky model by the one-grid penalty FEM on a coarse mesh, and then solving a linearized Smagorinsky model based on the Newton iteration on a fine mesh. Before presenting the two-grid penalty FEM, we first give out the one-grid penalty FEM.

#### 3.1 One-grid penalty FEM

The one-grid penalty FEM for the problem (2) reads as the following algorithm.

**Algorithm 3.1** (*One-grid penalty FEM*) Find  $(\mathbf{u}_{\varepsilon\mu}, p_{\varepsilon\mu}) \in (\mathbf{X}_\mu, Q_\mu)$  such that for all  $(\mathbf{v}_\mu, q_\mu) \in (\mathbf{X}_\mu, Q_\mu)$

$$\begin{aligned} & a(\mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla\mathbf{v}_\mu) + b(\mathbf{u}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) - d(p_{\varepsilon\mu}, \mathbf{v}_\mu) \\ & + d(q_\mu, \mathbf{u}_{\varepsilon\mu}) + \varepsilon(p_{\varepsilon\mu}, q_\mu) = (\mathbf{f}, \mathbf{v}_\mu), \end{aligned} \quad (13)$$

where  $0 < \varepsilon \ll 1$  is a penalty parameter.

Now, we give and derive the stability and error estimations for Algorithm 3.1.

**Theorem 3.2** Under the condition of (7), the discrete problem (13) admits a unique solution  $(\mathbf{u}_{\varepsilon\mu}, p_{\varepsilon\mu}) \in (\mathbf{X}_\mu, Q_\mu)$ , which satisfies

$$|\mathbf{u}_{\varepsilon\mu}|_{1,3} \leq (C_S\delta)^{-1} \|\mathbf{f}\|_{*3}^{\frac{1}{2}}, \quad \|\nabla\mathbf{u}_{\varepsilon\mu}\| \leq \Psi(\|\mathbf{f}\|_*), \quad \|p_{\varepsilon\mu}\| \leq \varepsilon^{-\frac{1}{2}} (C_S\delta)^{-\frac{1}{2}} \|\mathbf{f}\|_{*3}^{\frac{3}{4}}. \quad (14)$$

**Proof.** Choosing  $(\mathbf{v}_\mu, q_\mu) = (\mathbf{u}_{\varepsilon\mu}, p_{\varepsilon\mu})$  in (13), using (4) gives

$$Re^{-1} \|\nabla\mathbf{u}_{\varepsilon\mu}\|^2 + (C_S\delta)^2 |\mathbf{u}_{\varepsilon\mu}|_{1,3}^3 + \varepsilon \|p_{\varepsilon\mu}\|^2 \leq \|\mathbf{f}\|_{*3} |\mathbf{u}_{\varepsilon\mu}|_{1,3} \quad (\text{or } \|\mathbf{f}\|_* \|\nabla\mathbf{u}_{\varepsilon\mu}\|). \quad (15)$$

Thus,

$$|\mathbf{u}_{\varepsilon\mu}|_{1,3} \leq (C_S\delta)^{-1} \|\mathbf{f}\|_{*3}^{\frac{1}{2}},$$

and

$$\varepsilon \|p_{\varepsilon\mu}\|^2 \leq \|\mathbf{f}\|_{*3} |\mathbf{u}_{\varepsilon\mu}|_{1,3} \leq (C_S\delta)^{-1} \|\mathbf{f}\|_{*3}^{\frac{3}{2}},$$

which yields  $\|p_{\varepsilon\mu}\| \leq \varepsilon^{-\frac{1}{2}} (C_S\delta)^{-\frac{1}{2}} \|\mathbf{f}\|_{*3}^{\frac{3}{4}}$ .

From  $Re^{-1} \|\nabla\mathbf{u}_{\varepsilon\mu}\|^2 + (C_S\delta)^2 |\mathbf{u}_{\varepsilon\mu}|_{1,3}^3 \leq \|\mathbf{f}\|_* \|\nabla\mathbf{u}_{\varepsilon\mu}\|$ , we can derive that

$$\|\nabla\mathbf{u}_{\varepsilon\mu}\| \leq \Psi(\|\mathbf{f}\|_*),$$

where  $\Psi$  is defined in (6). To sum up, the proof is completed.  $\square$

**Theorem 3.3** Under the discrete inf-sup condition (9) and the uniqueness condition (7), if the solution of (2) satisfies  $\mathbf{u} \in \mathbf{X} \cap \mathbf{H}^3(\Omega)$ ,  $p \in Q \cup H^2(\Omega)$ , and  $\|\nabla\mathbf{u}\|_\infty$  does not depend on  $\delta$ , then the solution of problem (13) satisfies the following estimates:

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon\mu})\| + \mu^{\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon\mu}|_{1,3} \leq C(\mu^2 + \varepsilon + \delta\mu^{2-\frac{d}{3}}), \quad (16)$$

$$\|p - p_{\varepsilon\mu}\| \leq C(\mu^2 + \varepsilon + \delta\mu^{2-\frac{d}{3}} + \delta^4\varepsilon\mu^{-\frac{2d}{3}} + \delta^3\mu^{2-\frac{2d}{3}}). \quad (17)$$

**Proof.** Subtracting (13) from (2), we obtain the following error equation

$$\begin{aligned} & a(\mathbf{u} - \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u}, \nabla\mathbf{v}_\mu) - (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla\mathbf{v}_\mu) \\ & + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_\mu) - b(\mathbf{u}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) - d(p - p_{\varepsilon\mu}, \mathbf{v}_\mu) \\ & + d(q_\mu, \mathbf{u} - \mathbf{u}_{\varepsilon\mu}) - \varepsilon(p_{\varepsilon\mu}, q_\mu) = 0, \end{aligned} \quad (18)$$

for all  $(\mathbf{v}_\mu, p_\mu) \in (\mathbf{X}_\mu, M_\mu)$ . Set

$$\mathbf{e}_{\varepsilon\mu} = \mathbf{u}_{\varepsilon\mu} - R_\mu\mathbf{u}, \quad \mathbf{e}_u = \mathbf{u} - R_\mu\mathbf{u}, \quad \xi_{\varepsilon\mu} = p_{\varepsilon\mu} - Q_\mu p, \quad \eta_p = p - Q_\mu p.$$

Choose  $(\mathbf{v}_\mu, p_\mu) = (\mathbf{e}_{\varepsilon\mu}, \xi_{\varepsilon\mu})$  in (18), using (4) we have

$$\begin{aligned} & a(\mathbf{e}_{\varepsilon\mu}, \mathbf{e}_{\varepsilon\mu}) + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla\mathbf{e}_{\varepsilon\mu}) - (C_S\delta)^2(|\nabla R_\mu\mathbf{u}|\nabla R_\mu\mathbf{u}, \nabla\mathbf{e}_{\varepsilon\mu}) \\ & + b(\mathbf{e}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{e}_{\varepsilon\mu}) + \varepsilon\|\xi_{\varepsilon\mu}\|^2 \\ & = a(\mathbf{e}_u, \mathbf{e}_{\varepsilon\mu}) + b(\mathbf{u}, \mathbf{e}_u, \mathbf{e}_{\varepsilon\mu}) + (C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u}, \nabla\mathbf{e}_{\varepsilon\mu}) - (C_S\delta)^2(|\nabla R_\mu\mathbf{u}|\nabla R_\mu\mathbf{u}, \nabla\mathbf{e}_{\varepsilon\mu}) \\ & + b(\mathbf{e}_u, \mathbf{u}_{\varepsilon\mu}, \mathbf{e}_{\varepsilon\mu}) - d(p - \chi_\mu, \mathbf{e}_{\varepsilon\mu}) - \varepsilon(Q_\mu p, \xi_{\varepsilon\mu}), \end{aligned} \quad (19)$$

where  $\chi_\mu \in M_\mu$ . By (5) and Theorem 3.2, the terms on the left-hand side of (19) can be bounded as:

$$\begin{aligned} & a(\mathbf{e}_{\varepsilon\mu}, \mathbf{e}_{\varepsilon\mu}) = Re^{-1}\|\nabla\mathbf{e}_{\varepsilon\mu}\|^2, \\ & b(\mathbf{e}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{e}_{\varepsilon\mu}) \geq -N\|\nabla\mathbf{u}_{\varepsilon\mu}\|\|\nabla\mathbf{e}_{\varepsilon\mu}\|^2 \geq -N\Psi(\|\mathbf{f}\|_*)\|\nabla\mathbf{e}_{\varepsilon\mu}\|^2. \end{aligned} \quad (20)$$

By Lemma 2.1, we find

$$(C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla\mathbf{e}_{\varepsilon\mu}) - (C_S\delta)^2(|\nabla R_\mu\mathbf{u}|\nabla R_\mu\mathbf{u}, \nabla\mathbf{e}_{\varepsilon\mu}) \geq C_1(C_S\delta)^2|\mathbf{e}_{\varepsilon\mu}|_{1,3}^3 > 0. \quad (21)$$

Next, we bound the terms on the right-hand side of (19) as follows. Using (5) and Theorem 3.2, we have

$$\begin{aligned} & a(\mathbf{e}_u, \mathbf{e}_{\varepsilon\mu}) \leq Re^{-1}\|\nabla\mathbf{e}_u\|\|\nabla\mathbf{e}_{\varepsilon\mu}\|, \\ & b(\mathbf{u}, \mathbf{e}_u, \mathbf{e}_{\varepsilon\mu}) \leq N\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}_u\|\|\nabla\mathbf{e}_{\varepsilon\mu}\|, \\ & b(\mathbf{e}_u, \mathbf{u}_{\varepsilon\mu}, \mathbf{e}_{\varepsilon\mu}) \leq N\|\nabla\mathbf{u}_{\varepsilon\mu}\|\|\nabla\mathbf{e}_u\|\|\nabla\mathbf{e}_{\varepsilon\mu}\| \leq N\Psi(\|\mathbf{f}\|_*)\|\nabla\mathbf{e}_u\|\|\nabla\mathbf{e}_{\varepsilon\mu}\|. \end{aligned} \quad (22)$$

Following (4.21) in [5], we have

$$\begin{aligned} & (C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u}, \nabla\mathbf{e}_{\varepsilon\mu}) - (C_S\delta)^2(|\nabla R_\mu\mathbf{u}|\nabla R_\mu\mathbf{u}, \nabla\mathbf{e}_{\varepsilon\mu}) \\ & \leq C_1(C_S\delta)^2(\|\nabla\mathbf{u}\|_\infty + \|\nabla R_\mu\mathbf{u}\|_\infty)\|\nabla\mathbf{e}_u\|\|\nabla\mathbf{e}_{\varepsilon\mu}\| \\ & \leq C(C_S\delta)^2\|\nabla\mathbf{e}_u\|\|\nabla\mathbf{e}_{\varepsilon\mu}\|. \end{aligned} \quad (23)$$

For the next term, we know

$$d(p - \chi_\mu, \mathbf{e}_{\varepsilon\mu}) \leq \sqrt{d}\|p - \chi_\mu\|\|\nabla\mathbf{e}_{\varepsilon\mu}\|, \quad (24)$$

$$\varepsilon(Q_\mu p, \xi_{\varepsilon\mu}) \leq \varepsilon\|Q_\mu p\|\|\xi_{\varepsilon\mu}\| \leq \varepsilon\|Q_\mu p\|(\|p - Q_\mu p\| + \|p - p_{\varepsilon\mu}\|). \quad (25)$$

Choosing  $q_\mu = 0$  in (18), we have

$$\begin{aligned} & (p - p_{\varepsilon\mu}, \nabla \cdot \mathbf{v}_\mu) \\ & \leq Re^{-1}\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon\mu})\|\|\nabla\mathbf{v}_\mu\| + N(\|\nabla\mathbf{u}\| + \|\nabla\mathbf{u}_{\varepsilon\mu}\|)\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon\mu})\|\|\nabla\mathbf{v}_\mu\| \\ & + (C_S\delta)^2(|\nabla\mathbf{u}|\nabla\mathbf{u}, \nabla\mathbf{v}_\mu) - (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla\mathbf{v}_\mu). \end{aligned}$$

By Lemma 2.1 and the inverse inequality (12), we can derive that

$$\begin{aligned}
& (C_S\delta)^2(|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{v}_\mu) - (C_S\delta)^2(|\nabla \mathbf{u}_{\varepsilon\mu}| \nabla \mathbf{u}_{\varepsilon\mu}, \nabla \mathbf{v}_\mu) \\
& \leq C_1(C_S\delta)^2 \max\{|\mathbf{u}|_{1,3}, |\mathbf{u}_{\varepsilon\mu}|_{1,3}\} |\mathbf{u} - \mathbf{u}_{\varepsilon\mu}|_{1,3} |\mathbf{v}_\mu|_{1,3} \\
& \leq C_1 C_{\text{inv}} \mu^{-\frac{d}{6}} (C_S\delta)^2 \max\{|\mathbf{u}|_{1,3}, |\mathbf{u}_{\varepsilon\mu}|_{1,3}\} |\mathbf{u} - R_\mu \mathbf{u}|_{1,3} \|\nabla \mathbf{v}_\mu\| \\
& \quad + C_1 C_{\text{inv}}^2 \mu^{-\frac{d}{3}} (C_S\delta)^2 \max\{|\mathbf{u}|_{1,3}, |\mathbf{u}_{\varepsilon\mu}|_{1,3}\} \|\nabla(\mathbf{u}_{\varepsilon\mu} - R_\mu \mathbf{u})\| \|\nabla \mathbf{v}_\mu\|.
\end{aligned}$$

By the discrete inf-sup condition (9), one has

$$\begin{aligned}
\beta \|p - p_{\varepsilon\mu}\| & \leq R e^{-1} \|\nabla \mathbf{e}_u\| + R e^{-1} \|\nabla \mathbf{e}_{\varepsilon\mu}\| + C \|\nabla \mathbf{e}_u\| + C \|\nabla \mathbf{e}_{\varepsilon\mu}\| \\
& \quad + C \mu^{-\frac{d}{6}} (C_S\delta)^2 |\mathbf{e}_u|_{1,3} + C (C_S\delta)^2 \mu^{-\frac{d}{3}} \|\nabla \mathbf{e}_{\varepsilon\mu}\|.
\end{aligned} \tag{26}$$

Combining the estimates (20)-(26) into (19), we have

$$\begin{aligned}
& \sigma \|\nabla \mathbf{e}_{\varepsilon\mu}\|^2 + \varepsilon \|\xi_{\varepsilon\mu}\|^2 \\
& \leq (R e^{-1} \|\nabla \mathbf{e}_u\| + N \|\nabla \mathbf{u}\| \|\nabla \mathbf{e}_u\| + N \Psi(\|\mathbf{f}\|_*) \|\nabla \mathbf{e}_u\| + C (C_S\delta)^2 \|\nabla \mathbf{e}_u\| + \sqrt{d} \|p - \chi_\mu\| \\
& \quad + C \beta^{-1} \varepsilon \|Q_\mu p\| + C \beta^{-1} \varepsilon (C_S\delta)^2 \mu^{-\frac{d}{3}}) \|\nabla \mathbf{e}_{\varepsilon\mu}\| + \varepsilon \|Q_\mu p\| \|\eta_p\| + C \beta^{-1} \varepsilon \|Q_\mu p\| \|\nabla \mathbf{e}_u\| \\
& \quad + C \beta^{-1} \varepsilon \|Q_\mu p\| \mu^{-\frac{d}{6}} (C_S\delta)^2 |\mathbf{e}_u|_{1,3} \\
& \leq \frac{\sigma}{2} \|\nabla \mathbf{e}_{\varepsilon\mu}\|^2 + C (\|\nabla \mathbf{e}_u\|^2 + (C_S\delta)^4 \|\nabla \mathbf{e}_u\|^2 + \|p - \chi_\mu\|^2 + \varepsilon^2 + \varepsilon^2 (C_S\delta)^4 \mu^{-\frac{2d}{3}}) \\
& \quad + C \varepsilon \|\eta_p\| + C \varepsilon \|\nabla \mathbf{e}_u\| + C \varepsilon \mu^{-\frac{d}{6}} (C_S\delta)^2 |\mathbf{e}_u|_{1,3},
\end{aligned} \tag{27}$$

where  $\sigma = R e^{-1} - N \Psi(\|\mathbf{f}\|_*) > 0$ .

From (27), we find

$$\begin{aligned}
\|\nabla \mathbf{e}_{\varepsilon\mu}\|^2 & \leq C (\|\nabla \mathbf{e}_u\|^2 + (C_S\delta)^4 \|\nabla \mathbf{e}_u\|^2 + \|p - \chi_\mu\|^2 + \varepsilon^2 + \varepsilon^2 (C_S\delta)^4 \mu^{-\frac{2d}{3}}) \\
& \quad + C \varepsilon \|\eta_p\| + C \varepsilon \|\nabla \mathbf{e}_u\| + C \varepsilon \mu^{-\frac{d}{6}} (C_S\delta)^2 |\mathbf{e}_u|_{1,3} \\
& \leq C (\|\nabla \mathbf{e}_u\| + \|\eta_p\| + \varepsilon + \varepsilon \delta + \delta \mu^{-\frac{d}{6}} |\mathbf{e}_u|_{1,3})^2 \\
& \leq C (\|\nabla \mathbf{e}_u\| + \|\eta_p\| + \varepsilon + \delta \mu^{-\frac{d}{6}} |\mathbf{e}_u|_{1,3})^2.
\end{aligned} \tag{28}$$

By the approximation properties (11), we have

$$\begin{aligned}
\|\nabla \mathbf{e}_{\varepsilon\mu}\| & \leq C (\|\nabla \mathbf{e}_u\| + \|\eta_p\| + \varepsilon + \delta \mu^{-\frac{d}{6}} |\mathbf{e}_u|_{1,3}) \\
& \leq C (\mu^2 + \varepsilon + \delta \mu^{2-\frac{d}{3}}).
\end{aligned} \tag{29}$$

Applying the triangle inequality we obtain

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon\mu})\| \leq C (\mu^2 + \varepsilon + \delta \mu^{2-\frac{d}{3}}). \tag{30}$$

Combining (26) with (29), the pressure error can be bounded by

$$\|p - p_{\varepsilon\mu}\| \leq C (\mu^2 + \varepsilon + \delta \mu^{2-\frac{d}{3}} + \delta^4 \varepsilon \mu^{-\frac{2d}{3}} + \delta^3 \mu^{2-\frac{2d}{3}}). \tag{31}$$

Finally, by the triangle inequality and the inverse inequality (12) we gain

$$\begin{aligned}
|\mathbf{u} - \mathbf{u}_{\varepsilon\mu}|_{1,3} & \leq |\mathbf{u} - R_\mu \mathbf{u}|_{1,3} + |R_\mu \mathbf{u} - \mathbf{u}_{\varepsilon\mu}|_{1,3} \\
& \leq |\mathbf{u} - R_\mu \mathbf{u}|_{1,3} + C_{\text{inv}} \mu^{-\frac{d}{6}} \|\nabla(R_\mu \mathbf{u} - \mathbf{u}_{\varepsilon\mu})\| \\
& \leq C \mu^{-\frac{d}{6}} (\mu^2 + \varepsilon + \delta \mu^{2-\frac{d}{3}})
\end{aligned} \tag{32}$$

and complete the proof.  $\square$

**Remark 1** If we choose  $\varepsilon \ll \mu^2$  and  $\delta = \mathcal{O}(\mu^{\frac{d}{3}})$  in Theorem 3.3, then we can derive

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon\mu})\| + \mu^{\frac{d}{6}}|\mathbf{u} - \mathbf{u}_{\varepsilon\mu}|_{1,3} + \|p - p_{\varepsilon\mu}\| \leq C\mu^2. \quad (33)$$

### 3.2 The linearized iteration scheme of one-grid penalty FEM

Since the one-grid penalty FEM (13) is a nonlinear scheme. In practice computation, iterative methods are needed to solve it. In this section, we give a linearized iteration scheme for solving the solution  $(\mathbf{u}_{\varepsilon\mu}, p_{\varepsilon\mu})$  of the one-grid penalty FEM (13). The iteration initial value  $(\mathbf{u}_{\varepsilon\mu}^0, p_{\varepsilon\mu}^0)$  is selected by solving the following Stokes problem:

$$a(\mathbf{u}_{\varepsilon\mu}^0, \mathbf{v}_\mu) - d(p_{\varepsilon\mu}^0, \mathbf{v}_\mu) + d(q_\mu, \mathbf{u}_{\varepsilon\mu}^0) + \varepsilon(p_{\varepsilon\mu}^0, q_\mu) = (\mathbf{f}, \mathbf{v}_\mu) \quad \forall (\mathbf{v}_\mu, q_\mu) \in (\mathbf{X}_\mu, Q_\mu). \quad (34)$$

Then we solve  $(\mathbf{u}_{\varepsilon\mu}^n, p_{\varepsilon\mu}^n), n = 1, 2, \dots, M$ , by the following linearized problem:

$$\begin{aligned} a(\mathbf{u}_{\varepsilon\mu}^n, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|\nabla\mathbf{u}_{\varepsilon\mu}^n, \nabla\mathbf{v}_\mu) + (C_S\delta)^2\left(\frac{[\nabla\mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla\mathbf{u}_{\varepsilon\mu}^n]}{|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|}\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla\mathbf{v}_\mu\right) \\ + b(\mathbf{u}_{\varepsilon\mu}^{n-1}, \mathbf{u}_{\varepsilon\mu}^n, \mathbf{v}_\mu) - d(p_{\varepsilon\mu}^n, \mathbf{v}_\mu) + d(q_\mu, \mathbf{u}_{\varepsilon\mu}^n) + \varepsilon(p_{\varepsilon\mu}^n, q_\mu) \\ = (\mathbf{f}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla\mathbf{v}_\mu) \quad \forall (\mathbf{v}_\mu, q_\mu) \in (\mathbf{X}_\mu, Q_\mu). \end{aligned} \quad (35)$$

The stability of solution of above iteration scheme (34)-(35) is given out in the following theorem.

**Theorem 3.4** If  $C_S, \delta, C_{inv}, N$  and  $\mu$  satisfy  $(C_S\delta)^2 C_{inv}^3 \mu^{-\frac{d}{2}} \leq N$  and the following condition holds that

$$7NRe^2\|\mathbf{f}\|_* < 1, \quad (36)$$

then we have

$$\|\nabla\mathbf{u}_{\varepsilon\mu}^n\| \leq 2Re\|\mathbf{f}\|_*, \quad \|p_{\varepsilon\mu}^n\| \leq 2(Re/\varepsilon)^{\frac{1}{2}}\|\mathbf{f}\|_*. \quad (37)$$

**Proof.** We prove it by mathematical induction method. Firstly, we choose  $(\mathbf{v}_\mu, q_\mu) = (\mathbf{u}_{\varepsilon\mu}^0, p_{\varepsilon\mu}^0)$  in (34) and deduce that

$$Re^{-1}\|\nabla\mathbf{u}_{\varepsilon\mu}^0\|^2 + \varepsilon\|p_{\varepsilon\mu}^0\|^2 \leq \|\mathbf{f}\|_*\|\nabla\mathbf{u}_{\varepsilon\mu}^0\|.$$

Thus,  $\|\nabla\mathbf{u}_{\varepsilon\mu}^0\| \leq Re\|\mathbf{f}\|_*$  and  $\varepsilon\|p_{\varepsilon\mu}^0\|^2 \leq Re\|\mathbf{f}\|_*^2$ , which implies that the conclusions hold for  $n = 0$ .

Next, assuming the conclusions hold for  $n \leq k-1$ , that

$$\|\nabla\mathbf{u}_{\varepsilon\mu}^n\| \leq 2Re\|\mathbf{f}\|_*, \quad \|p_{\varepsilon\mu}^n\| \leq 2(Re/\varepsilon)^{\frac{1}{2}}\|\mathbf{f}\|_* \quad \text{for } n \leq k-1. \quad (38)$$

Finally, we prove that the conclusion hold for  $n = k$ . Set  $n = k$  in (35) and choose  $(\mathbf{v}_\mu, q_\mu) = (\mathbf{u}_{\varepsilon\mu}^k, p_{\varepsilon\mu}^k)$ , we have

$$\begin{aligned} Re^{-1}\|\nabla\mathbf{u}_{\varepsilon\mu}^k\|^2 + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{k-1}|\nabla\mathbf{u}_{\varepsilon\mu}^k, \nabla\mathbf{u}_{\varepsilon\mu}^k) + (C_S\delta)^2\left(\frac{[\nabla\mathbf{u}_{\varepsilon\mu}^{k-1} : \nabla\mathbf{u}_{\varepsilon\mu}^k]}{|\nabla\mathbf{u}_{\varepsilon\mu}^{k-1}|}\nabla\mathbf{u}_{\varepsilon\mu}^{k-1}, \nabla\mathbf{u}_{\varepsilon\mu}^k\right) \\ + \varepsilon\|p_{\varepsilon\mu}^k\|^2 = (\mathbf{f}, \mathbf{u}_{\varepsilon\mu}^k) + (C_S\delta)^2(|\mathbf{u}_{\varepsilon\mu}^{k-1}|\mathbf{u}_{\varepsilon\mu}^{k-1}, \mathbf{u}_{\varepsilon\mu}^k). \end{aligned} \quad (39)$$

By (8) and the inverse inequality (12), we have

$$\begin{aligned}
& Re^{-1} \|\nabla \mathbf{u}_{\varepsilon\mu}^k\|^2 + \varepsilon \|p_{\varepsilon\mu}^k\|^2 \\
& \leq \|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| + (C_S \delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \|\nabla \mathbf{u}_{\varepsilon\mu}^{k-1}\|^2 \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| \\
& \leq \|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| + 4Re^2 (C_S \delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \|\mathbf{f}\|_*^2 \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| \\
& \leq \|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| + 4NRe^2 \|\mathbf{f}\|_*^2 \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| \\
& \leq 2\|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon\mu}^k\|.
\end{aligned} \tag{40}$$

From (40), we know that

$$\|\nabla \mathbf{u}_{\varepsilon\mu}^k\| \leq 2Re \|\mathbf{f}\|_*,$$

and

$$\varepsilon \|p_{\varepsilon\mu}^k\|^2 \leq 2\|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon\mu}^k\| \leq 4Re \|\mathbf{f}\|_*^2.$$

To sum up, the proof is completed.  $\square$

**Remark 2** According to Lemma 6.1 in [4], we know  $\Psi(\|\mathbf{f}\|_*) < Re\|\mathbf{f}\|_*$ , thus, from Theorem 3.2, we can get another upper bound of  $\|\nabla \mathbf{u}_{\varepsilon\mu}\|$  as follows:

$$\|\nabla \mathbf{u}_{\varepsilon\mu}\| \leq Re\|\mathbf{f}\|_*. \tag{41}$$

The iteration error estimates of scheme (34)-(35) is presented in the following theorem.

**Theorem 3.5** Under the assumptions of Theorem 3.4, then the solution of (13) and the solution of (35) satisfy

$$\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| + \mu^{\frac{d}{6}} |\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n|_{1,3} \leq \frac{2}{7} Re \|\mathbf{f}\|_* (7NRe^2 \|\mathbf{f}\|_*)^n. \tag{42}$$

$$\|p_{\varepsilon\mu} - p_{\varepsilon\mu}^n\| \leq \frac{(18 + 2C_1)\beta^{-1} Re^{-1}}{7} (7NRe^2 \|\mathbf{f}\|_*)^n. \tag{43}$$

**Proof.** Subtracting (34) from (13), we obtain

$$\begin{aligned}
& a(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0, \mathbf{v}_{\mu}) - d(p_{\varepsilon\mu} - p_{\varepsilon\mu}^0, \mathbf{v}_{\mu}) + d(q_{\mu}, \mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0) + \varepsilon(p_{\varepsilon\mu} - p_{\varepsilon\mu}^0, q_{\mu}) \\
& = -b(\mathbf{u}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_{\mu}) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}| |\nabla \mathbf{u}_{\varepsilon\mu}, \nabla \mathbf{v}_{\mu}),
\end{aligned} \tag{44}$$

Setting  $(\mathbf{v}_{\mu}, q_{\mu}) = (\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0, p_{\varepsilon\mu} - p_{\varepsilon\mu}^0)$  and using (5), the Hölder inequality and the inverse inequality (12), we get

$$\begin{aligned}
& Re^{-1} \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\|^2 + \varepsilon \|p_{\varepsilon\mu} - p_{\varepsilon\mu}^0\|^2 \\
& \leq N \|\nabla \mathbf{u}_{\varepsilon\mu}\|^2 \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| + (C_S \delta)^2 |\mathbf{u}_{\varepsilon\mu}|_{1,3}^2 |\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0|_{1,3} \\
& \leq N \|\nabla \mathbf{u}_{\varepsilon\mu}\|^2 \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| + (C_S \delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \|\nabla \mathbf{u}_{\varepsilon\mu}\|^2 \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| \\
& \leq 2N \|\nabla \mathbf{u}_{\varepsilon\mu}\|^2 \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\|.
\end{aligned} \tag{45}$$

This lead to

$$\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| \leq \frac{2}{7} Re (7NRe^2 \|\mathbf{f}\|_*) \|\mathbf{f}\|_* \leq \frac{2}{7} Re \|\mathbf{f}\|_*. \tag{46}$$

Choosing  $q_\mu = 0$  in (44), by the inf-sup condition (9), the inverse inequality (12) and (41), we have

$$\begin{aligned}
\|p_{\varepsilon\mu} - p_{\varepsilon\mu}^0\| &\leq \beta^{-1}(Re^{-1}\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| + N\|\nabla\mathbf{u}_{\varepsilon\mu}\|^2 + (C_S\delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \|\nabla\mathbf{u}_{\varepsilon\mu}\|^2) \\
&\leq \beta^{-1}(Re^{-1}\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| + 2N\|\nabla\mathbf{u}_{\varepsilon\mu}\|^2) \\
&\leq \frac{4}{7}\beta^{-1}\|\mathbf{f}\|_*.
\end{aligned} \tag{47}$$

Next, we consider iteration error of (35). Subtracting (35) from (13), we obtain

$$\begin{aligned}
&a(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n, \mathbf{v}_\mu) - d(p_{\varepsilon\mu} - p_{\varepsilon\mu}^n, \mathbf{v}_\mu) + d(q_\mu, \mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n) + \varepsilon(p_{\varepsilon\mu} - p_{\varepsilon\mu}^n, q_\mu) \\
&= b(\mathbf{u}_{\varepsilon\mu}^{n-1}, \mathbf{u}_{\varepsilon\mu}^n, \mathbf{v}_\mu) - b(\mathbf{u}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|\nabla\mathbf{u}_{\varepsilon\mu}^n, \nabla\mathbf{v}_\mu) \\
&\quad + (C_S\delta)^2\left(\frac{[\nabla\mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla\mathbf{u}_{\varepsilon\mu}^n]}{|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|}\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla\mathbf{v}_\mu\right) - (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla\mathbf{v}_\mu) \\
&\quad - (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla\mathbf{v}_\mu).
\end{aligned} \tag{48}$$

Setting  $(\mathbf{v}_\mu, q_\mu) = (\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n, p_{\varepsilon\mu} - p_{\varepsilon\mu}^n)$  in (48), we obtain

$$\begin{aligned}
&Re^{-1}\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\|^2 + \varepsilon\|p_{\varepsilon\mu} - p_{\varepsilon\mu}^n\|^2 \\
&= -b(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1}, \mathbf{u}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n) + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|\nabla(\mathbf{u}_{\varepsilon\mu}^n - \mathbf{u}_{\varepsilon\mu}), \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)) \\
&\quad + (C_S\delta)^2\left(\frac{[\nabla\mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla(\mathbf{u}_{\varepsilon\mu}^n - \mathbf{u}_{\varepsilon\mu})]}{|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|}\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\right) \\
&\quad + (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1}), \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)) \\
&\quad + (C_S\delta)^2\left(\frac{[\nabla\mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla\mathbf{u}_{\varepsilon\mu}]}{|\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}|}\nabla\mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\right) \\
&\quad - (C_S\delta)^2(|\nabla\mathbf{u}_{\varepsilon\mu}|\nabla\mathbf{u}_{\varepsilon\mu}, \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6.
\end{aligned} \tag{49}$$

Using (5) and (41), we know

$$\begin{aligned}
\mathcal{J}_1 &\leq N\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\|\|\nabla\mathbf{u}_{\varepsilon\mu}\|\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \\
&\leq NRe\|\mathbf{f}\|_*\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\|\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\|.
\end{aligned}$$

By (8), we find

$$\mathcal{J}_2 + \mathcal{J}_3 \leq 0.$$

From the Frobenius norm inequality  $|A||B| - [A : B] \leq |A - B|^2$ , the inequality  $|a| - |b| \leq |a - b|$ , the inverse inequality (12), Theorem 3.4 and (41), we have

$$\begin{aligned}
\mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6 &= (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}| \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1}), \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)) \\
&\quad - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}| \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1}), \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)) \\
&\quad - (C_S \delta)^2 \left( \frac{|\nabla \mathbf{u}_{\varepsilon\mu}| |\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|}{|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|} \nabla \mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n) \right) \\
&\quad + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla \mathbf{u}_{\varepsilon\mu}]}{|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|} \nabla \mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n) \right) \\
&\leq (C_S \delta)^2 (|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})|^2, |\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)|) \\
&\quad + (C_S \delta)^2 \left( \frac{|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})|^2}{|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|} |\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|, |\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)| \right) \\
&\leq 2(C_S \delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\|^2 \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \\
&\leq 2N (\|\nabla \mathbf{u}_{\varepsilon\mu}\| + \|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}\|) \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \\
&\leq 6N Re \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\|.
\end{aligned} \tag{50}$$

Combining these estimates for  $\mathcal{J}_1$  to  $\mathcal{J}_6$  into (49), we derive

$$\begin{aligned}
\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| &\leq (7N Re^2 \|\mathbf{f}\|_*) \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \\
&\leq \dots \leq (7N Re^2 \|\mathbf{f}\|_*)^n \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^0)\| \leq \frac{2}{7} Re \|\mathbf{f}\|_* (7N Re^2 \|\mathbf{f}\|_*)^n.
\end{aligned}$$

Choosing  $q_\mu = 0$  in (48), we know

$$\begin{aligned}
&(p_{\varepsilon\mu} - p_{\varepsilon\mu}^n, \nabla \cdot \mathbf{v}_\mu) \\
&\leq Re^{-1} \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\| + [b(\mathbf{u}_{\varepsilon\mu}, \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) - b(\mathbf{u}_{\varepsilon\mu}^{n-1}, \mathbf{u}_{\varepsilon\mu}^n, \mathbf{v}_\mu)] \\
&\quad + [(C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}| \nabla \mathbf{u}_{\varepsilon\mu}, \nabla \mathbf{v}_\mu) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}| \nabla \mathbf{u}_{\varepsilon\mu}^n, \nabla \mathbf{v}_\mu)] \\
&\quad + \left[ (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}| \nabla \mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla \mathbf{v}_\mu) - (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla \mathbf{u}_{\varepsilon\mu}]}{|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|} \nabla \mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla \mathbf{v}_\mu \right) \right] \\
&= Re^{-1} \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\| + \mathcal{J}_7 + \mathcal{J}_8 + \mathcal{J}_9.
\end{aligned} \tag{51}$$

Using (5) and (41), we have

$$\begin{aligned}
\mathcal{J}_7 &= b(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1}, \mathbf{u}_{\varepsilon\mu}, \mathbf{v}_\mu) + b(\mathbf{u}_{\varepsilon\mu}^{n-1}, \mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n, \mathbf{v}_\mu) \\
&\leq N \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \|\nabla \mathbf{u}_{\varepsilon\mu}\| \|\nabla \mathbf{v}_\mu\| + N \|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}\| \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\| \\
&\leq N Re \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \|\nabla \mathbf{v}_\mu\| + 2N Re \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\|.
\end{aligned} \tag{52}$$

By the Hölder inequality and the inverse inequality (12) and Lemma 2.1, one has

$$\begin{aligned}
\mathcal{J}_8 &= (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}| \nabla \mathbf{u}_{\varepsilon\mu}, \nabla \mathbf{v}_\mu) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}^n| \nabla \mathbf{u}_{\varepsilon\mu}^n, \nabla \mathbf{v}_\mu) \\
&\quad + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}^n| \nabla \mathbf{u}_{\varepsilon\mu}^n, \nabla \mathbf{v}_\mu) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}| \nabla \mathbf{u}_{\varepsilon\mu}^n, \nabla \mathbf{v}_\mu) \\
&\leq C_1 (C_S \delta)^2 \max\{|\mathbf{u}_{\varepsilon\mu}|_{1,3}, |\mathbf{u}_{\varepsilon\mu}^n|_{1,3}\} |\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n|_{1,3} |\mathbf{v}_\mu|_{1,3} \\
&\quad + (C_S \delta)^2 |\mathbf{u}_{\varepsilon\mu}^n|_{1,3} |\mathbf{u}_{\varepsilon\mu}^n - \mathbf{u}_{\varepsilon\mu}^{n-1}|_{1,3} |\mathbf{v}_\mu|_{1,3} \\
&\leq C_1 (C_S \delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \max\{\|\nabla \mathbf{u}_{\varepsilon\mu}\|, \|\nabla \mathbf{u}_{\varepsilon\mu}^n\|\} \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\| \\
&\quad + (C_S \delta)^2 C_{\text{inv}}^3 \mu^{-\frac{d}{2}} \|\nabla \mathbf{u}_{\varepsilon\mu}^n\| \|\nabla(\mathbf{u}_{\varepsilon\mu}^n - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \|\nabla \mathbf{v}_\mu\| \\
&\leq 2C_1 N Re \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\| + 2N Re \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \|\nabla \mathbf{v}_\mu\| \\
&\quad + 2N Re \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \|\nabla \mathbf{v}_\mu\|,
\end{aligned} \tag{53}$$



and

$$\begin{aligned}
\mathcal{J}_9 &= (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon\mu}^{n-1} : \nabla(\mathbf{u}_{\varepsilon\mu}^{n-1} - \mathbf{u}_{\varepsilon\mu}^n)]}{|\nabla \mathbf{u}_{\varepsilon\mu}^{n-1}|} \nabla \mathbf{u}_{\varepsilon\mu}^{n-1}, \nabla \mathbf{v}_\mu \right) \\
&\leq (C_S \delta)^2 |\mathbf{u}_{\varepsilon\mu}^{n-1}|_{1,3} |\mathbf{u}_{\varepsilon\mu}^{n-1} - \mathbf{u}_{\varepsilon\mu}^n|_{1,3} |\mathbf{v}_\mu|_{1,3} \\
&\leq 2NRe \|\mathbf{f}\|_* (\|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| + \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\|) \|\nabla \mathbf{v}_\mu\|,
\end{aligned} \tag{54}$$

Combining estimates  $\mathcal{J}_7, \mathcal{J}_8$  and  $\mathcal{J}_9$  into (51), and using the discrete inf-sup condition (9), one has

$$\begin{aligned}
\|p_{\varepsilon\mu} - p_{\varepsilon\mu}^n\| &\leq \beta^{-1} (Re^{-1} + 6NRe \|\mathbf{f}\|_* + 2C_1 NRe \|\mathbf{f}\|_*) \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \\
&\quad + 5\beta^{-1} ReN \|\mathbf{f}\|_* \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \\
&\leq \beta^{-1} (Re^{-1} + \frac{6}{7} Re^{-1} + \frac{2C_1}{7} Re^{-1}) \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n)\| \\
&\quad + \frac{5}{7} \beta^{-1} Re^{-1} (7NRe^2 \|\mathbf{f}\|_*) \|\nabla(\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^{n-1})\| \\
&\leq \frac{(18 + 2C_1)\beta^{-1} Re^{-1}}{7} (7NRe^2 \|\mathbf{f}\|_*)^n.
\end{aligned} \tag{55}$$

Finally, we derive the estimate  $|\mathbf{u}_{\varepsilon\mu} - \mathbf{u}_{\varepsilon\mu}^n|_{1,3}$  by the inverse inequality (12) and complete the proof.  $\square$

As a direct consequence of Theorems 3.3 and 3.5, we immediately obtain the following theorem.

**Theorem 3.6** *Under the assumptions of Theorems 3.3 and 3.5, then we have the following estimates:*

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon\mu}^n)\| + \mu^{\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon\mu}^n|_{1,3} \leq \left( \mu^2 + \varepsilon + \delta \mu^{2-\frac{d}{3}} + \frac{2}{7} Re \|\mathbf{f}\|_* (7NRe^2 \|\mathbf{f}\|_*)^n \right), \tag{56}$$

$$\|p - p_{\varepsilon\mu}^n\| \leq \left( \mu^2 + \varepsilon + \delta \mu^{2-\frac{d}{3}} + \delta^4 \varepsilon \mu^{-\frac{2d}{3}} + \delta^3 \mu^{2-\frac{2d}{3}} + \frac{(18 + 2C_1)\beta^{-1} Re^{-1}}{7} (7NRe^2 \|\mathbf{f}\|_*)^n \right). \tag{57}$$

### 3.3 Two-grid penalty FEM

The two-grid penalty FEM for the problem (2) reads as the following algorithm.

**Algorithm 3.7** (*Two-grid penalty FEM*)

**Step 1:** Solve a nonlinear Smagorinsky model on the coarse mesh  $\tau_H$ : Find  $(\mathbf{u}_{\varepsilon H}, p_{\varepsilon H}) \in (\mathbf{X}_H, Q_H)$  such that for all  $(\mathbf{v}_H, q_H) \in (\mathbf{X}_H, Q_H)$

$$\begin{aligned}
a(\mathbf{u}_{\varepsilon H}, \mathbf{v}_H) &+ (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}| |\nabla \mathbf{u}_{\varepsilon H}, \nabla \mathbf{v}_H) + b(\mathbf{u}_{\varepsilon H}, \mathbf{u}_{\varepsilon H}, \mathbf{v}_H) - d(p_{\varepsilon H}, \mathbf{v}_H) \\
&+ d(q_H, \mathbf{u}_{\varepsilon H}) + \varepsilon(p_{\varepsilon H}, q_H) = (\mathbf{f}, \mathbf{v}_H).
\end{aligned} \tag{58}$$

**Step 2:** Solve the following linearized Smagorinsky model on the fine mesh  $\tau_h$ : Find  $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h}) \in (\mathbf{X}_h, Q_h)$  such that for all  $(\mathbf{v}_h, q_h) \in (\mathbf{X}_h, Q_h)$

$$\begin{aligned}
a(\mathbf{u}_{\varepsilon h}, \mathbf{v}_h) &+ (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| |\nabla \mathbf{u}_{\varepsilon h}, \nabla \mathbf{v}_h) + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla \mathbf{u}_{\varepsilon h}]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h \right) \\
&+ b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon H}^M, \mathbf{v}_h) - d(p_{\varepsilon h}, \mathbf{v}_h) + d(q_h, \mathbf{u}_{\varepsilon h}) + \varepsilon(p_{\varepsilon h}, q_h) \\
&= (\mathbf{f}, \mathbf{v}_h) + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| |\nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h) + b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon H}^M, \mathbf{v}_h) + \varepsilon(p_{\varepsilon H}^M, q_h),
\end{aligned} \tag{59}$$

where  $\mathbf{u}_{\varepsilon H}^M$  is the solution of (58) solved by the iteration scheme (35).

The stability and error estimates of solution of Algorithm 3.7 in step 1 are obtained in Theorems 3.2 and 3.3. Here, we only need to consider the theoretical results of the solution of Algorithm 3.7 in step 2. We first give out the stability of solution in the following theorem.

**Theorem 3.8** *Under the assumptions of Theorem 3.6, the solution solved by (59) satisfies*

$$\sigma_1 \|\nabla \mathbf{u}_{\varepsilon h}\|^2 + \varepsilon \|p_{\varepsilon h}\|^2 \leq \frac{1}{\sigma_1} (2\|\mathbf{f}\|_* + (h/H)^{-\frac{d}{6}} \|\mathbf{f}\|_*)^2 + 4Re\|\mathbf{f}\|_*^2, \quad (60)$$

where  $\sigma_1 = Re^{-1} - 2NRe\|\mathbf{f}\|_* > 0$ .

**Proof.** Setting  $(\mathbf{v}_h, q_h) = (\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$  in (59), we have

$$\begin{aligned} & a(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon h}) + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{u}_{\varepsilon h}, \nabla \mathbf{u}_{\varepsilon h}) + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla \mathbf{u}_{\varepsilon h}]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{u}_{\varepsilon h} \right) \\ & + b(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}) + \varepsilon (p_{\varepsilon h}, p_{\varepsilon h}) \\ & = (\mathbf{f}, \mathbf{u}_{\varepsilon h}) + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{u}_{\varepsilon h}) + b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}) + \varepsilon (p_{\varepsilon H}^M, p_{\varepsilon h}). \end{aligned} \quad (61)$$

By (8), (5), Theorem 3.4, the Hölder inequality and the inverse inequality (12), one has

$$\begin{aligned} & \sigma_1 \|\nabla \mathbf{u}_{\varepsilon h}\|^2 + \varepsilon \|p_{\varepsilon h}\|^2 \\ & \leq \|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon h}\| + (C_S \delta)^2 |\mathbf{u}_{\varepsilon H}^M|_{1,3}^2 |\mathbf{u}_{\varepsilon h}|_{1,3} + N \|\nabla \mathbf{u}_{\varepsilon H}^M\|^2 \|\nabla \mathbf{u}_{\varepsilon h}\| + \varepsilon \|p_{\varepsilon H}^M\| \|p_{\varepsilon h}\| \\ & \leq \|\mathbf{f}\|_* \|\nabla \mathbf{u}_{\varepsilon h}\| + N (h/H)^{-\frac{d}{6}} \|\nabla \mathbf{u}_{\varepsilon H}^M\|^2 \|\nabla \mathbf{u}_{\varepsilon h}\| + N \|\nabla \mathbf{u}_{\varepsilon H}^M\|^2 \|\nabla \mathbf{u}_{\varepsilon h}\| + \varepsilon \|p_{\varepsilon H}^M\| \|p_{\varepsilon h}\| \\ & \leq (2\|\mathbf{f}\|_* + (h/H)^{-\frac{d}{6}} \|\mathbf{f}\|_*) \|\nabla \mathbf{u}_{\varepsilon h}\| + \varepsilon \|p_{\varepsilon H}^M\| \|p_{\varepsilon h}\| \\ & \leq \frac{\sigma_1}{2} \|\nabla \mathbf{u}_{\varepsilon h}\|^2 + \frac{1}{2\sigma_1} (2\|\mathbf{f}\|_* + (h/H)^{-\frac{d}{6}} \|\mathbf{f}\|_*)^2 + \frac{\varepsilon}{2} \|p_{\varepsilon h}\|^2 + \frac{\varepsilon}{2} \|p_{\varepsilon H}^M\|^2, \end{aligned} \quad (62)$$

where  $\sigma_1 = Re^{-1} - 2NRe\|\mathbf{f}\|_* > 0$ . From the bound of pressure in Theorem 3.4, we can derive estimate (60) and complete the proof.  $\square$

Next, the error estimate of solution of Algorithm 3.7 in step 2 is presented in the following theorem.

**Theorem 3.9** *Under the assumptions of Theorem 3.6, and the following condition holds*

$$(C_S \delta)^2 C_{inv}^3 H^{-\frac{d}{6}} h^{-\frac{d}{3}} \leq C_0, \quad (63)$$

then we have

$$\begin{aligned} & \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon h})\| + h^{\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{1,3} + \|p - p_{\varepsilon h}\| \\ & \leq C(h^2 + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \varepsilon \|p - p_{\varepsilon H}^M\| + (C_S \delta)^2 h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 \\ & + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| + (C_S \delta)^2 H^{-\frac{d}{6}} h^{2-\frac{d}{3}} + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}). \end{aligned} \quad (64)$$

**Proof.** Subtracting (59) from (2), we obtain

$$\begin{aligned} & a(\mathbf{u} - \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + (C_S \delta)^2 (|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{v}_h) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{u}_{\varepsilon h}, \nabla \mathbf{v}_h) \\ & - (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla \mathbf{u}_{\varepsilon h}]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h \right) + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h) \\ & + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) - b(\mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon H}^M, \mathbf{v}_h) + b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u}_{\varepsilon H}^M, \mathbf{v}_h) \\ & - d(p - p_{\varepsilon h}, \mathbf{v}_h) + d(q_h, \mathbf{u} - \mathbf{u}_{\varepsilon h}) - \varepsilon (p_{\varepsilon h}, q_h) + \varepsilon (p_{\varepsilon H}, q_h) = 0. \end{aligned} \quad (65)$$

Set

$$\mathbf{e}_h = \mathbf{u}_{\varepsilon h} - R_h \mathbf{u}, \quad \mathbf{e}_u = \mathbf{u} - R_h \mathbf{u}, \quad \xi_h = p_{\varepsilon h} - Q_h p, \quad \eta_p = p - Q_h p.$$

Choose  $(\mathbf{v}_h, q_h) = (\mathbf{e}_h, \xi_h)$  in (65) gives

$$\begin{aligned} & Re^{-1} \|\nabla \mathbf{e}_h\|^2 + \varepsilon \|\xi_h\|^2 + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{e}_h, \nabla \mathbf{e}_h) + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla \mathbf{e}_h]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{e}_h \right) \\ &= a(\eta_u, \mathbf{e}_h) + b(\eta_u, \mathbf{u}_{\varepsilon H}^M, \mathbf{e}_h) - b(\mathbf{e}_h, \mathbf{u}_{\varepsilon H}^M, \mathbf{e}_h) + b(\mathbf{u}_{\varepsilon H}^M, \eta_u, \mathbf{e}_h) - b(\mathbf{u}_{\varepsilon H}^M, \mathbf{e}_h, \mathbf{e}_h) \\ &\quad + b(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M, \mathbf{u} - \mathbf{u}_{\varepsilon H}^M, \mathbf{e}_h) - d(p - \chi_h, \mathbf{e}_h) + (C_S \delta)^2 (|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{e}_h) \\ &\quad - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla R_h \mathbf{u}, \nabla \mathbf{e}_h) + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla (\mathbf{u}_{\varepsilon H}^M - R_h \mathbf{u})]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{e}_h \right) \\ &\quad - \varepsilon (Q_h p - p_{\varepsilon H}^M, \xi_h) = \sum_{i=1}^{11} \mathcal{I}_i. \end{aligned} \tag{66}$$

From (8), we know

$$(C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{e}_h, \nabla \mathbf{e}_h) + (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla \mathbf{e}_h]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{e}_h \right) \geq 0. \tag{67}$$

Next, we estimate  $\mathcal{I}_i, i = 1, \dots, 11$ , one by one below. It is easy to verify the following estimates:

$$\begin{aligned} \mathcal{I}_1 &\leq Re^{-1} \|\nabla \eta_u\| \|\nabla \mathbf{e}_h\|, \\ \mathcal{I}_2 + \mathcal{I}_4 &\leq 2N \|\nabla \eta_u\| \|\nabla \mathbf{u}_{\varepsilon H}^M\| \|\nabla \mathbf{e}_h\| \leq 4NRe \|\mathbf{f}\|_* \|\nabla \eta_u\| \|\nabla \mathbf{e}_h\|, \\ \mathcal{I}_3 &\leq N \|\nabla \mathbf{u}_{\varepsilon H}^M\| \|\nabla \mathbf{e}_h\|^2 \leq 2NRe \|\mathbf{f}\|_* \|\nabla \mathbf{e}_h\|^2, \\ \mathcal{I}_5 &= 0, \\ \mathcal{I}_6 &\leq N \|\nabla (\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 \|\nabla \mathbf{e}_h\|, \\ \mathcal{I}_7 &\leq \sqrt{d} \|p - \chi_h\| \|\nabla \mathbf{e}_h\|. \end{aligned} \tag{68}$$

Following (6.11)-(6.16) in [5], we have

$$\begin{aligned} \mathcal{I}_8 + \mathcal{I}_9 + \mathcal{I}_{10} &\leq 2(C_S \delta)^2 |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 |\mathbf{e}_h|_{1,3} + 2(C_S \delta)^2 |\mathbf{u}_{\varepsilon H}^M|_{1,3} |\eta_u|_{1,3} |\mathbf{e}_h|_{1,3} \\ &\leq 2C_{\text{inv}} (C_S \delta)^2 h^{-\frac{d}{6}} (|\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 + 2C_{\text{inv}} H^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\eta_u|_{1,3}) \|\nabla \mathbf{e}_h\|. \end{aligned} \tag{69}$$

For the term  $\mathcal{I}_{11}$ , we have

$$\mathcal{I}_{11} \leq \varepsilon (\|p - Q_h p\| + \|p - p_{\varepsilon H}^M\|) (\|p - p_{\varepsilon h}\| + \|p - Q_h p\|). \tag{70}$$

Choosing  $q_h = 0$  in 65, we have

$$\begin{aligned} & (p - p_{\varepsilon h}, \nabla \cdot \mathbf{v}_h) \\ &= a(\mathbf{u} - \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + (C_S \delta)^2 (|\nabla \mathbf{u}| \nabla \mathbf{u}, \nabla \mathbf{v}_h) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{u}_{\varepsilon h}, \nabla \mathbf{v}_h) \\ &\quad + (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h) - (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla \mathbf{u}_{\varepsilon h}]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h \right) \\ &\quad + b(\mathbf{u} - \mathbf{u}_{\varepsilon h}, \mathbf{u}_{\varepsilon H}^M, \mathbf{v}_h) + b(\mathbf{u}_{\varepsilon H}^M, \mathbf{u} - \mathbf{u}_{\varepsilon h}, \mathbf{v}_h) + b(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M, \mathbf{u} - \mathbf{u}_{\varepsilon H}^M, \mathbf{v}_h) \\ &= \sum_{i=1}^8 \mathcal{A}_i, \end{aligned}$$

where  $A_i, i = 1, \dots, 8$ , can be bounded below. Firstly,

$$\mathcal{A}_1 \leq Re^{-1} \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon h})\| \|\nabla \mathbf{v}_h\|.$$

By the Hölder inequality and the inverse inequality (12), we get

$$\begin{aligned} \mathcal{A}_2 + \mathcal{A}_3 &\leq (C_S \delta)^2 (|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)| |\nabla \mathbf{u}, \nabla \mathbf{v}_h) - (C_S \delta)^2 (|\nabla \mathbf{u}_{\varepsilon H}^M| |\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon h}), \nabla \mathbf{v}_h) \\ &\leq (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| \|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \mathbf{v}_h\| + (C_S \delta)^2 |\mathbf{u}_{\varepsilon H}^M|_{1,3} |\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{1,3} \|\mathbf{v}_h\|_{1,3} \\ &\leq (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| \|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \mathbf{v}_h\| + 2(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\eta_u|_{1,3} \|\nabla \mathbf{v}_h\| \\ &\quad + 2(C_S \delta)^2 C_{\text{inv}}^3 H^{-\frac{d}{6}} h^{-\frac{d}{3}} Re \|\mathbf{f}\|_* \|\nabla \mathbf{e}_h\| \|\nabla \mathbf{v}_h\|, \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_4 + \mathcal{A}_5 &= (C_S \delta)^2 \left( \frac{[\nabla \mathbf{u}_{\varepsilon H}^M : \nabla(\mathbf{u}_{\varepsilon H}^M - \mathbf{u}_{\varepsilon h})]}{|\nabla \mathbf{u}_{\varepsilon H}^M|} \nabla \mathbf{u}_{\varepsilon H}^M, \nabla \mathbf{v}_h \right) \\ &\leq (C_S \delta)^2 |\mathbf{u}_{\varepsilon H}^M|_{1,3} |\mathbf{u}_{\varepsilon H}^M - \mathbf{u}_{\varepsilon h}|_{1,3} \|\mathbf{v}_h\|_{1,3} \\ &\leq (C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} \|\nabla \mathbf{u}_{\varepsilon H}^M\| (|\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3} + |\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{1,3}) \|\nabla \mathbf{v}_h\| \\ &\leq 2(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3} \|\nabla \mathbf{v}_h\| + 2(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\eta_u|_{1,3} \|\nabla \mathbf{v}_h\| \\ &\quad + 2(C_S \delta)^2 C_{\text{inv}}^3 H^{-\frac{d}{6}} h^{-\frac{d}{3}} Re \|\mathbf{f}\|_* \|\nabla \mathbf{e}_h\| \|\nabla \mathbf{v}_h\|. \end{aligned}$$

From (5), we know

$$\begin{aligned} \mathcal{A}_6 + \mathcal{A}_7 &\leq 2N \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon h})\| \|\nabla \mathbf{u}_H^M\| \|\nabla \mathbf{v}_h\| \\ &\leq 4N Re \|\mathbf{f}\|_* \|\nabla \eta_u\| \|\nabla \mathbf{v}_h\| + 4N Re \|\mathbf{f}\|_* \|\nabla \mathbf{e}_u\| \|\nabla \mathbf{v}_h\|, \end{aligned}$$

and

$$\mathcal{A}_8 \leq N \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 \|\nabla \mathbf{v}_h\|.$$

Now, using the discrete inf-sup condition (9), we have

$$\begin{aligned} &\|p - p_{\varepsilon h}\| \\ &\leq \beta^{-1} (Re^{-1} \|\nabla \eta_u\| + Re^{-1} \|\nabla \mathbf{e}_u\| + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| \|\nabla \mathbf{u}\|_{L^\infty} \\ &\quad + 4(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\eta_u|_{1,3} + 4(C_S \delta)^2 C_{\text{inv}}^3 H^{-\frac{d}{6}} h^{-\frac{d}{3}} Re \|\mathbf{f}\|_* \|\nabla \mathbf{e}_h\| \\ &\quad + 2(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3} + Re^{-1} \|\nabla \eta_u\| \\ &\quad + Re^{-1} \|\nabla \mathbf{e}_u\| + N \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2, \end{aligned} \tag{71}$$

Combining the estimates (67)-(71) into (66), we have

$$\begin{aligned}
& \sigma_1 \|\nabla \mathbf{e}_h\|^2 + \varepsilon \|\xi_h\|^2 \\
& \leq [2Re^{-1} \|\nabla \eta_u\| + N \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \sqrt{d} \|p - \chi_h\| \\
& \quad + 2C_{\text{inv}}(C_S \delta)^2 h^{-\frac{d}{6}} (|\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 + 2C_{\text{inv}} H^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\eta_u|_{1,3}) \\
& \quad + \beta^{-1} \varepsilon (\|p - Q_h p\| + \|p - p_{\varepsilon H}^M\|) (2Re^{-1} + 4(C_S \delta)^2 C_{\text{inv}}^3 H^{-\frac{d}{6}} h^{-\frac{d}{3}} Re \|\mathbf{f}\|_*)] \|\nabla \mathbf{e}_h\| \\
& \quad + \beta^{-1} \varepsilon (\|p - Q_h p\| + \|p - p_{\varepsilon H}^M\|) (Re^{-1} \|\nabla \eta_u\| + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|) \|\nabla \mathbf{u}\|_{L^\infty} \\
& \quad + 4(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\eta_u|_{1,3} + 2(C_S \delta)^2 C_{\text{inv}}^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} Re \|\mathbf{f}\|_* |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3} \\
& \quad + N \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \|p - Q_h p\|) \\
& \leq \frac{\sigma_1}{2} \|\nabla \mathbf{e}_h\|^2 + C(\|\nabla \eta_u\|^2 + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^4 + \|p - \chi_h\|^2 \\
& \quad + (C_S \delta)^4 h^{-\frac{d}{3}} (|\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^4 + H^{-\frac{d}{3}} |\eta_u|_{1,3}^2) + \varepsilon^2 (\|p - Q_h p\|^2 + \|p - p_{\varepsilon H}^M\|^2) \\
& \quad + C\varepsilon (\|p - Q_h p\| + \|p - p_{\varepsilon H}^M\|) (\|\nabla \eta_u\| + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\eta_u|_{1,3} \\
& \quad + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3} + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \|p - Q_h p\|) \\
& \leq \frac{\sigma_1}{2} \|\nabla \mathbf{e}_h\|^2 + C(\|\nabla \eta_u\| + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \|p - \chi_h\| + \varepsilon (\|p - Q_h p\| + \|p - p_{\varepsilon H}^M\|) \\
& \quad + (C_S \delta)^2 h^{-\frac{d}{6}} (|\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 + H^{-\frac{d}{6}} |\eta_u|_{1,3}) + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| \\
& \quad + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\eta_u|_{1,3} + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3})^2,
\end{aligned} \tag{72}$$

here (63) is used. Thus, By the approximation properties (11), we have

$$\begin{aligned}
\|\nabla \mathbf{e}_h\| & \leq C(\|\nabla \eta_u\| + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \|p - \chi_h\| + \varepsilon (\|p - Q_h p\| + \|p - p_{\varepsilon H}^M\|) \\
& \quad + (C_S \delta)^2 h^{-\frac{d}{6}} (|\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 + H^{-\frac{d}{6}} |\eta_u|_{1,3}) + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| \\
& \quad + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\eta_u|_{1,3} + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}) \\
& \leq C(h^2 + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \varepsilon \|p - p_{\varepsilon H}^M\| + (C_S \delta)^2 h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 \\
& \quad + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| + (C_S \delta)^2 H^{-\frac{d}{6}} h^{2-\frac{d}{3}} + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}).
\end{aligned} \tag{73}$$

Combining (73) with (71), we get the pressure error bound.

$$\begin{aligned}
\|p - p_{\varepsilon h}\| & \leq C(h^2 + \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\|^2 + \varepsilon \|p - p_{\varepsilon H}^M\| + (C_S \delta)^2 h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}^2 \\
& \quad + (C_S \delta)^2 \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| + (C_S \delta)^2 H^{-\frac{d}{6}} h^{2-\frac{d}{3}} + (C_S \delta)^2 H^{-\frac{d}{6}} h^{-\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3}).
\end{aligned} \tag{74}$$

Finally, by the triangle inequality and the inverse inequality (12) we obtain

$$\begin{aligned}
|\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{1,3} & \leq |\mathbf{u} - R_h \mathbf{u}|_{1,3} + |\mathbf{e}_h|_{1,3} \\
& \leq |\mathbf{u} - R_h \mathbf{u}|_{1,3} + C_{\text{inv}} h^{-\frac{d}{6}} \|\nabla \mathbf{e}_h\| \\
& \leq C h^{-\frac{d}{6}} \|\nabla \mathbf{e}_h\|,
\end{aligned} \tag{75}$$

and complete the proof.  $\square$

**Remark 3** If we choose  $\varepsilon \ll H^2$  and  $\delta = \mathcal{O}(H^{\frac{d}{3}})$  in Theorem 3.6, for sufficient large iterations  $M$ , thanks to the stable condition (36), then we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon H}^M)\| + H^{\frac{d}{6}} |\mathbf{u} - \mathbf{u}_{\varepsilon H}^M|_{1,3} + \|p - p_{\varepsilon H}^M\| \leq CH^2. \tag{76}$$

Thus, we can derive that

$$\begin{aligned} & \|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon h})\| + h^{\frac{d}{6}}|\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{1,3} + \|p - p_{\varepsilon h}\| \\ & \leq C(h^2 + H^4 + \varepsilon H^2 + \delta^2 h^{-\frac{d}{6}} H^{4-\frac{d}{3}} + \delta^2 H^2 + \delta^2 H^{-\frac{d}{6}} h^{2-\frac{d}{3}} + \delta^2 H^{2-\frac{d}{3}} h^{-\frac{d}{6}}), \end{aligned} \quad (77)$$

if we choose  $\varepsilon \ll h$  and  $H = \mathcal{O}(h^{\frac{1}{2}})$ ,  $\delta = \mathcal{O}(h^{\frac{3+d}{6}})$ , then we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon h})\| + h^{\frac{d}{6}}|\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{1,3} + \|p - p_{\varepsilon h}\| \leq Ch^2. \quad (78)$$

## 4 Numerical experiments

In this section, we will present some numerical experiments to confirm our theoretical analysis and to verify the stability and effectiveness of the presented methods. We first present some numerical experiments to verify the optimal rate of convergence derived in this paper and verify the high efficiency of the proposed two-grid penalty FEM. Next, we will test a popular benchmark problem lid driven cavity flow in both two and three dimensional. In all the experiments, we choose the Smagorinsky constant  $Cs = 0.17$ , which is the most commonly used choice in practice for simulating turbulence, and the iteration tolerance as  $10^{-8}$ . All computations are carried out by the public finite element software package Freefem++ [45].

### 4.1 Rates of convergence study

In this test, we take  $\Omega = [0, 1]^2$  and the analytical solution for the velocity  $\mathbf{u} = (u_1, u_2)$  and the pressure  $p$  are given as follows:

$$\begin{aligned} u_1 &= 10x^2(x-1)^2y(y-1)(2y-1), \\ u_2 &= -10x(x-1)(2x-1)y^2(y-1)^2, \\ p &= x^2 - y^2, \end{aligned}$$

where the forcing function  $\mathbf{f} = (f_1, f_2)$  and the boundary values of  $(\mathbf{u}, p)$  are determined by (1). We consider the case of the Reynolds number  $Re = 1.0$ . In order to verify the optimal rates of convergence, we select  $\delta = 0.1h^{2/3}$  and  $\varepsilon = 0.0001h^2$  for one-grid penalty method, and  $\delta = h^{5/6}$ ,  $\varepsilon = 0.0001h^2$  and  $h = H^2$  for two-grid penalty FEM, respectively. The numerical results of the one-grid penalty FEM and two-grid penalty FEM are displayed in Tables 1 and 2, respectively. We can see from Tables 1 and 2 that these results are in good agreement with the theoretical convergence rates predictions for the proposed methods. What's more, corresponding the mesh size  $h = \frac{1}{4}, \frac{1}{16}, \frac{1}{36}, \frac{1}{64}, \frac{1}{100}, \frac{1}{144}$ , the two-grid penalty FEM can save 11.32%, 41.84%, 50.01%, 50.68%, 51.80%, 52.23% CPU time comparing with the one-grid penalty FEM.

Further more, we shall discuss the dependency upon the spatial filter radius  $\delta$ . The convergence rates of the solutions for the velocity and pressure computed by two-grid penalty FEM with  $\sigma = h^2, h^{5/6}, h^{1/2}, 1, 2, 5, 10$  and 30 are displayed in Fig. 1, which shows that no obvious difference was observed between the accuracy of the solutions when  $\delta < 1$ , while when  $\delta > 1$ , the accuracy and the convergence rates of the solutions are getting worse and worse. From this numerical experiment we summarize that, the proposed two-grid penalty FEM seems to be less sensitive to the choices of  $\delta$  when  $\delta < 1$ . However, when the value of  $\delta$  is large, the proposed method can not do well. One guess reason is that the small data condition (7) may be dissatisfied.

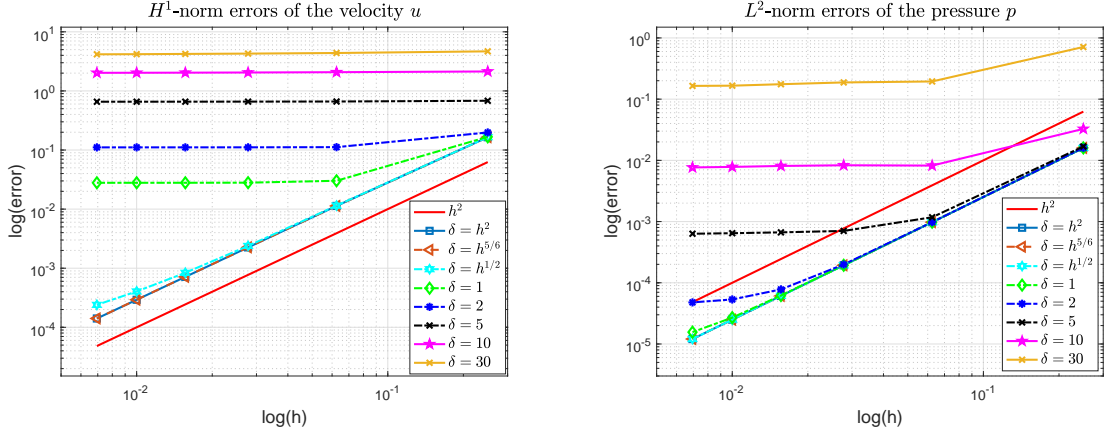


Fig. 1: Convergence rates of the velocity and the pressure by two-grid penalty FEM with different  $\sigma$ .

Table 1: Numerical results by using one-grid penalty FEM.

$1/h$	$\frac{\ \mathbf{u}-\mathbf{u}_h\ _1}{\ \mathbf{u}\ _1}$	$\mathbf{u}_{H^1}$ -Rate	$\frac{\ p-p_h\ }{\ p\ }$	$p_{L^2}$ -Rate	it	CPU(s)
4	0.164752		0.0159364		2	0.053
16	0.0113162	1.93191	0.000976974	2.01393	2	0.717
36	0.00225199	1.99083	0.000192908	2.00048	2	3.639
64	0.000713519	1.99762	6.10356e-005	2.00005	2	11.484
100	0.000292371	1.99913	2.50001e-005	2.00001	2	28.867
144	0.000141017	1.99961	1.20563e-005	2.00000	2	62.585

## 4.2 The 2D lid-driven cavity flow

The 2D lid-driven cavity flow is a popular benchmark problem for testing the numerical schemes of incompressible flow, which has been analyzed in [46]. In this problem, computations are carried out in the domain  $\Omega = [0, 1]^2$ . Flow is driven by the tangential velocity field on the top boundary and imposed no-slip boundary conditions on other boundaries. The presented numerical results are compared to the benchmark datum of Ghia et al [46].

We use the present two-grid penalty FEM to compute solution for the lid-driven cavity flow at  $Re = 1000, 3200, 5000$  and  $7500$ , where  $h = 1/100$  and  $H = 1/50$  are used. We have tested various values of  $\delta = 4h, h, 0.1h$  and  $0.1h^2$  in this experiment. This test shows that at  $Re = 5000, 7500$  with both  $\delta = 0.1h$  and  $0.1h^2$ , the two-grid penalty FEM failed to compute a solution and diverges; see Table 3 for details. With  $\delta = h$ , the computed streamlines at  $Re = 1000, 3200, 5000$  and  $7500$  are plotted in Fig. 2, showing that our results are comparable to those of Ghia et al. [46]. Figs. 3 and 4 draw the computed  $u_1$ -velocity along the vertical centerline and  $u_2$ -velocity along the horizontal centerlines by the two-grid penalty FEM compared with the benchmark data of Ghia et al. [46]. From Figs. 2-4 one can observe that good consistency with the data of Ghia et al. [46] verifies effectiveness of the proposed method.

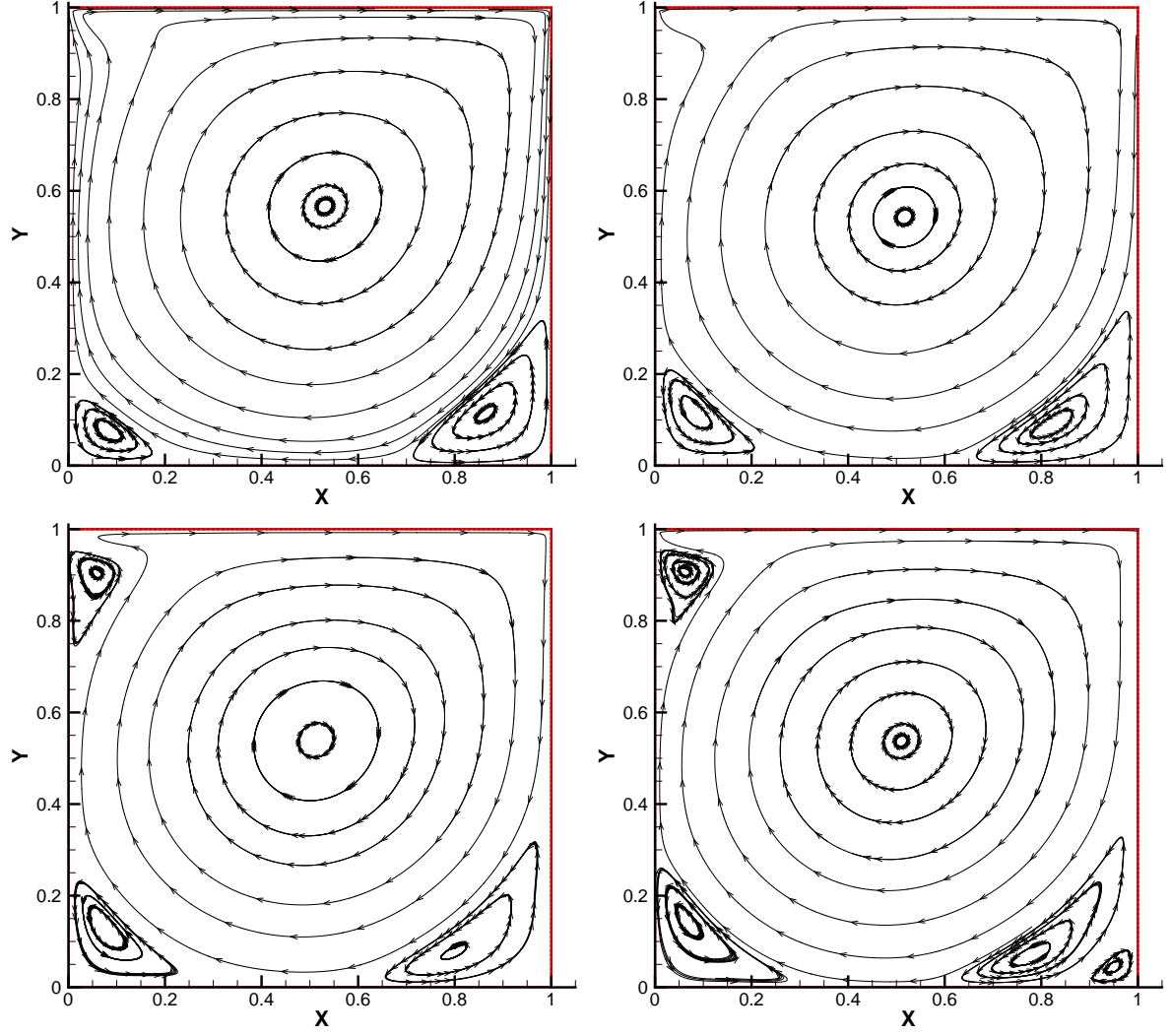


Fig. 2: The streamline of velocity of the 2D lid-driven cavity flow at  $Re = 1000, 3200, 5000$  and  $7500$ .



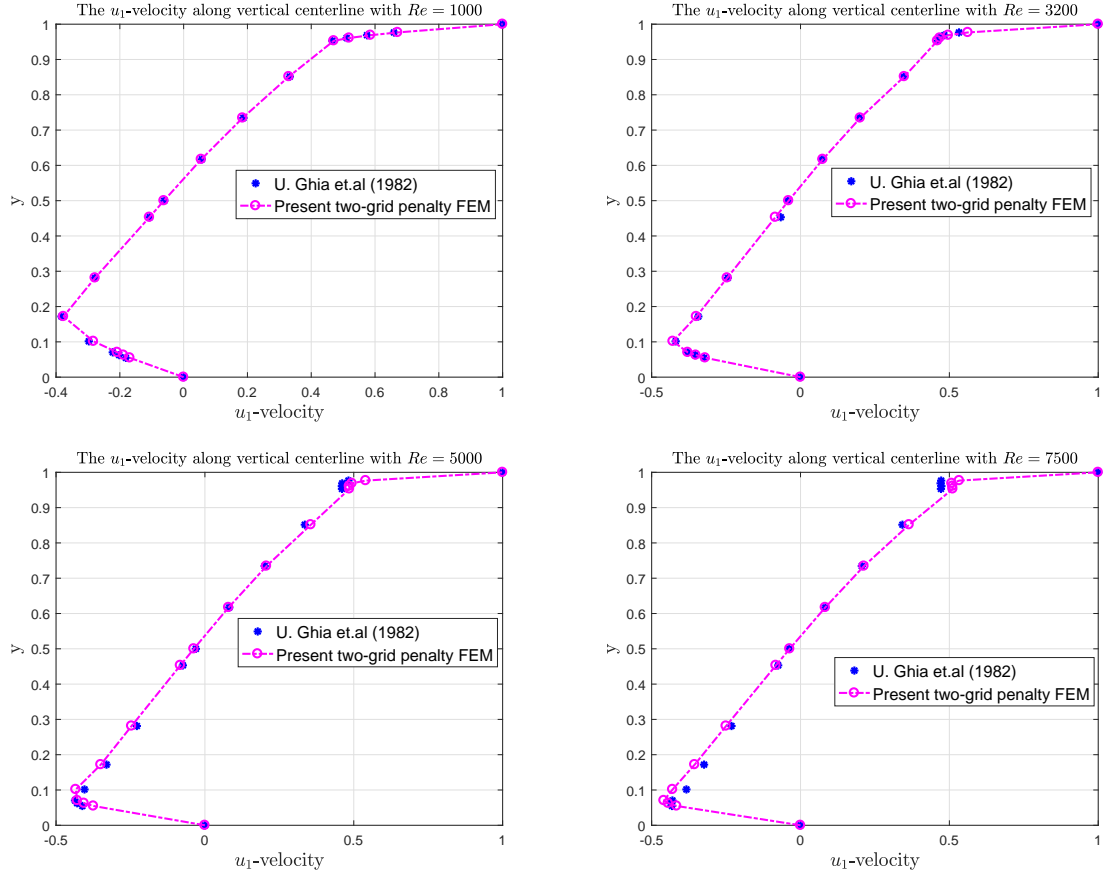


Fig. 3: A comparison of the  $u_1$ -velocity along vertical centerline for  $Re = 1000, 3200, 5000$  and  $7500$ .

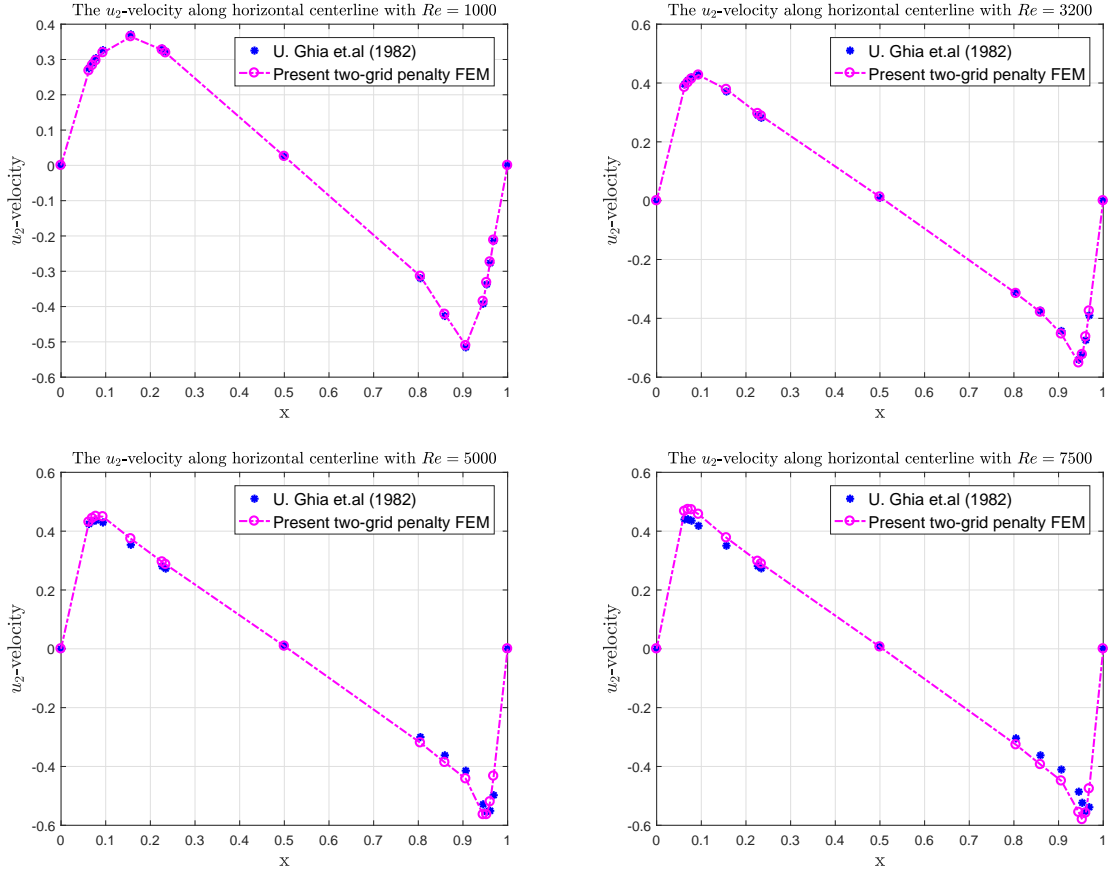


Fig. 4: A comparison of the  $u_2$ -velocity along horizontal centerline for  $Re = 1000, 3200, 5000$  and  $7500$ .

Table 2: Numerical results by using two-grid penalty FEM.

$1/h$	$1/H$	$\frac{\ \mathbf{u}-\mathbf{u}_h\ _1}{\ \mathbf{u}\ _1}$	$\mathbf{u}_{H^1}$ -Rate	$\frac{\ p-p_h\ }{\ p\ }$	$p_{L^2}$ -Rate	it	CPU(s)
4	2	0.16477		0.0159383		2	0.047
16	4	0.0113195	1.93178	0.000976975	2.01402	2	0.417
36	6	0.00225311	1.99058	0.000192908	2.00048	2	1.819
64	8	0.00071404	1.99722	6.10356e-005	2.00005	2	5.664
100	10	0.000292658	1.99857	2.50001e-005	2.00001	2	13.914
144	12	0.000141193	1.99887	1.20563e-005	2.00000	2	29.896

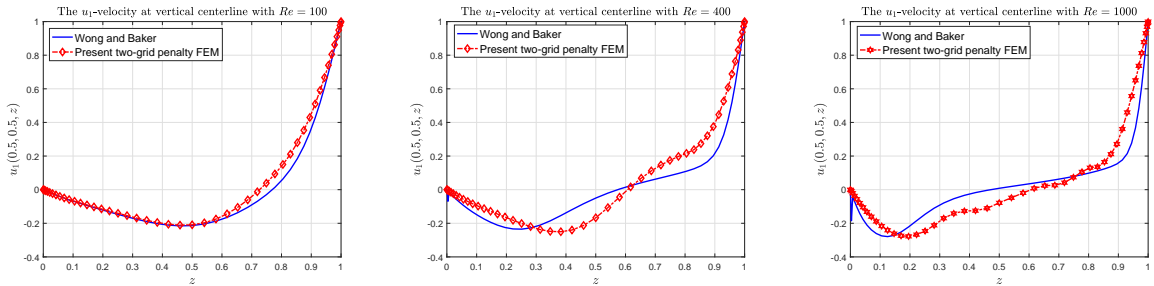
Table 3: Nonlinear iterations number and CPU time (s) of the 2D lid-driven cavity flow by using two-grid penalty FEM.

$\delta$	$4h$	$h$	$0.1h$	$0.1h^2$
$Re = 1000$	34(71.435)	30(65.03)	30(74.56)	30(72.095)
$Re = 3200$	37(77.4)	59(138.331)	264(595.596)	281(625.383)
$Re = 5000$	55(111.57)	126(284.613)	diverges	diverges
$Re = 7500$	165(311.477)	521(1024.09)	diverges	diverges

### 4.3 The 3D lid-driven cavity flow

Our final numerical example is the 3D lid-driven cavity flow problem, which is tested in [47]. The domain of this problem is the unit cube  $[0, 1]^3$ , equipped with horizontal velocity as boundary conditions for the top face ( $z = 1$ ) and homogeneous Dirichlet boundary conditions on the other faces. We implement the present two-grid penalty FEM with the mesh width  $h = 1/10$ ,  $H = 1/5$  and choose  $\delta = h$ .

In Fig. 5, we draw the centerline x-velocity at  $Re = 100, 400$  and  $1000$ , respectively, which shows that our results are comparable to the reference values given by Wone and Baker [47]. Figs. 6-8 plot the mid-plane velocity streamline pictures for  $Re = 100, 400$  and  $1000$ , respectively, which illustrate the effectiveness of our proposed method. All those numerical results are in good agreement with the reference solution in [47].


 Fig. 5: The centerline x-velocities of the 3D lid-driven cavity flow at  $Re = 100, 400$  and  $1000$ .

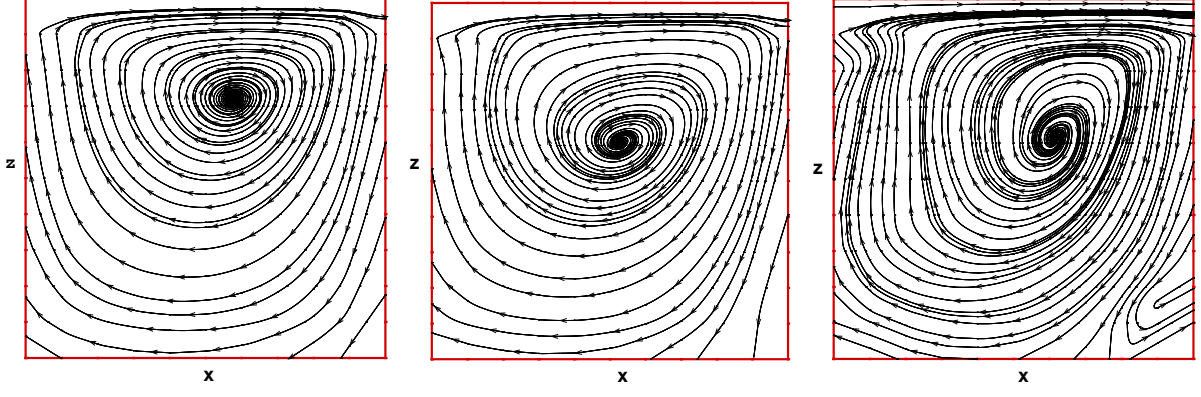


Fig. 6: The  $xz$ -plane velocity streamline pictures of the 3D lid-driven cavity flow at  $y = 0.5$ :  $Re = 100, 400$  and  $1000$  (from left to right).

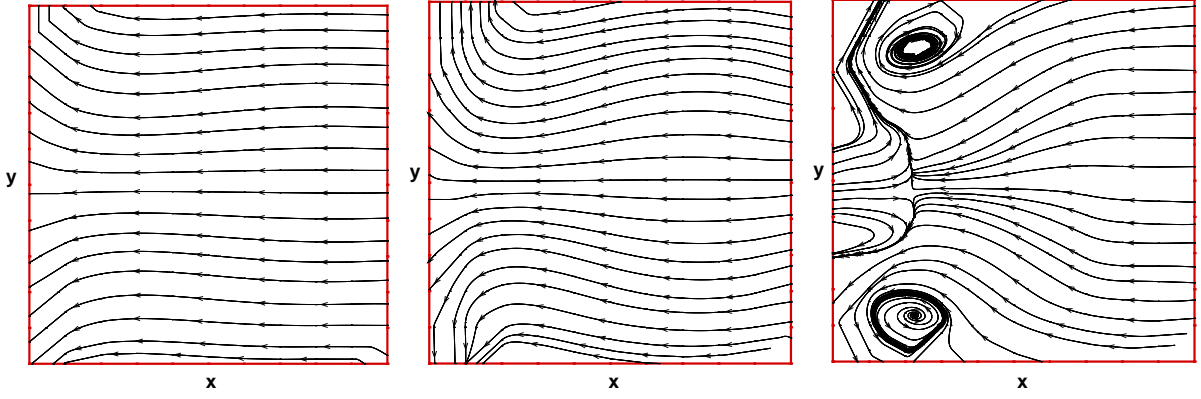


Fig. 7: The  $xy$ -plane velocity streamline pictures of the 3D lid-driven cavity flow at  $z = 0.5$ :  $Re = 100, 400$  and  $1000$  (from left to right).

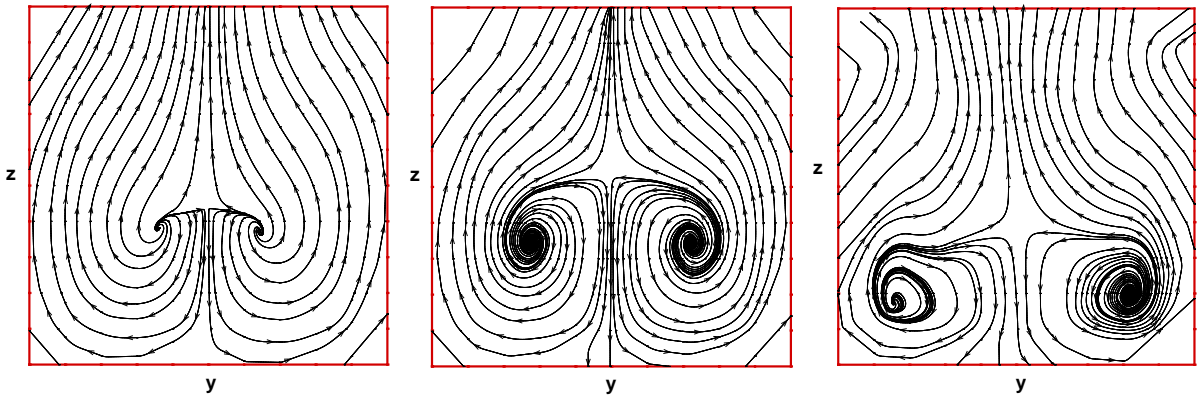


Fig. 8: The  $yz$ -plane velocity streamline pictures of the 3D lid-driven cavity flow at  $x = 0.5$ :  $Re = 100, 400$  and  $1000$  (from left to right).

## 5 Conclusions

In the paper, a two-grid penalty FEM has been developed and investigated for the Smagorinsky model. This method consist of solving a nonlinear Smagorinsky model by the one-grid penalty FEM with the proposed linearized iteration scheme on a coarse mesh, and then solving a linearized Smagorinsky model based on the Newton iteration on a fine mesh. Stability and error estimates of numerical solutions for two-grid penalty FEM are presented. Some numerical tests are provided to confirm the theoretical analysis and the effectiveness of the developed methods.

## References

- [1] Smagorinsky JS. General circulation experiments with the primitive equations. Mon Weather Rev. 1963; 91(3): 99-164.
- [2] Ladyžhenskaya OA. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. Proc Steklov Inst Math. 1967; 102: 95-118.
- [3] Ladyžhenskaya OA. Modifications of the Navier-Stokes equations for large gradients of the velocities. Zapiski Nauchnykh Seminarov POMI. 1968; 7: 126-154.
- [4] Du Q, Gunzburger MD. Analysis of a Ladyzhenskaya model for incompressible viscous flow. J Math Anal Appl. 1991; 155(1): 21-45.
- [5] Borggaard J, Iliescu T, Lee H, Roop JP, Son H. A two-level discretization method for the smagorinsky model. Multiscale Model Simul. 2008; 7(2): 599-621.
- [6] Borggaard J, Iliescu T. Approximate deconvolution boundary conditions for large eddy simulation. Appl Math Lett. 2006; 19:735-740.
- [7] Borggaard J, Iliescu T, Roop JP. A bounded artificial viscosity large eddy simulation model. SIAM J Numer Anal. 2009; 47:622-645.
- [8] Berselli LC, Iliescu T, Layton WJ. Mathematics of Large Eddy Simulation of Turbulent Flows. Berlin, Scientific Computation: Springer-Verlag; 2006.
- [9] Iliescu T, Fischer PF. Large eddy simulation of turbulent channel flows by the rational LES model. Phys Fluids. 2003; 15:3036-3047.
- [10] Sagaut P. Large Eddy Simulation for Incompressible Flows. Berlin, Scientific Computation: Springer-Verlag; 2006.
- [11] Von Neumann J, Richtmyer R. A method for the numerical calculation of hydrodynamic shocks. J Appl Phys. 1950; 21:232-237.
- [12] Huang P, Feng X, Liu D. Two-level stabilized method based on newton iteration for the steady Smagorinsky model. Nonlinear Anal Real World Appl. 2013; 14:1795-1805.
- [13] An R, Li Y, Zhang Y. Error estimates of two-level finite element method for Smagorinsky model. Appl Math Comput. 2016; 274:786-800.

- [14] Su H, Huang P, Wen J, Feng X. Three iterative finite element methods for the stationary Smagorinsky model. *E. Asian J Appl Math.* 2014; 4:132-151.
- [15] Shi D, Li M, Li Z. A nonconforming finite element method for the stationary Smagorinsky model. *Appl Math Comput.* 2019; 353:308-319.
- [16] Zheng B, Shang Y. A two-step stabilized finite element algorithm for the Smagorinsky model. *Appl Math Comput.* 2022; 422:126971.
- [17] Marion M, Xu JC. Error estimates on a new nonlinear Galerkin method based on two-grid finite elements. *SIAM J Numer Anal.* 1995; 32:1170-1184.
- [18] Xu J. A novel two-grid method for semilinear elliptic equations. *SIAM J Sci Comput.* 1994; 15:231-237.
- [19] Xu J. Two-grid discretization techniques for linear and nonlinear PDEs. *SIAM J Numer Anal.* 1996; 33:1759-1777.
- [20] Layton W. A two level discretization method for the Navier-Stokes equations. *Comput Math Appl.* 1993; 26:33-38.
- [21] Layton W, Lenferink W. Two-level Picard and modified Picard methods for the Navier-Stokes equations. *Appl. Math. Comput.* 1995; 69:263-274.
- [22] Layton W, Tobiska L. A two-level method with backtracking for the Navier-Stokes equations. *SIAM J Numer Anal.* 1998; 35:2035-2054.
- [23] Zhang Y, He YN. A two-level finite element method for the stationary Navier-Stokes equations based on a stabilized local projection. *Numer. Methods Partial Differential Equations.* 2011; 27:460-477.
- [24] Ervin V, Layton W, Maubach J. A posteriori error estimators for a two-level finite element method for the Navier-Stokes equations. *Numer Methods Partial Differential Equations.* 1996; 12:333-346.
- [25] Axelsson O, Layton W. A two-level method for the discretization of nonlinear boundary value problems. *SIAM J Numer Anal.* 1996; 33:2359-2374.
- [26] He YN, Wang AW. A simplified two-level method for the steady Navier-Stokes equations. *Comput Methods Appl Mech Engrg.* 2008; 197:1568-1576.
- [27] Dai X. Finite element approximation of the pure Neumann problem using the iterative penalty method. *Appl Math Comput.* 2007; 186:1367-1373.
- [28] Cheng X, Abdul WS. Analysis of the iterative penalty method for the Stokes equations. *Appl Math Lett.* 2006; 19:1024-1028.
- [29] Gunzburger MD. Iterated penalty methods for the Stokes and Navier-Stokes equations, in: *Finite Element Analysis in Fluids*. Huntsville, AL: University of Alabama in Huntsville Press; 1989.
- [30] An R, Shi F. Two-level iteration penalty methods for the incompressible flows. *Appl Math Model.* 2015; 39:630-641.
- [31] Carey GF, R Krishnan. Convergence of iterative methods in penalty finite element approximation of the Navier-Stokes equations. *Comput Methods Appl Mech Engrg.* 1987; 60:1-29.
- [32] Codina R. An iterative penalty method for the finite element solution of the stationary Navier-Stokes equations. *Comput Methods Appl Mech Engrg.* 1993; 110:237-262.

- [33] He YN. Optimal error estimate of the penalty finite element method for the time-dependent Navier-Stokes equations. *Math Comp.* 2005; 74:1201-1216.
- [34] He YN, Li J, Yang X. Two-level penalized finite element methods for the stationary Navier-Stokes equations. *Int J Inf Syst Sci.* 2006; 2:131-143.
- [35] Huang PZ. Iterative methods in penalty finite element discretizations for the steady Navier-Stokes equations. *Numer Methods Partial Differential Equations.* 2014; 30:74-94.
- [36] Shen J. On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations. *Numer Math.* 1992; 62:49-73.
- [37] Shen J. On error estimates of the penalty method for unsteady Navier-Stokes equations. *SIAM J Numer Anal.* 1995; 32:386-403.
- [38] Adams RA, Fournier JJ. *Sobolev Spaces*, New York: Academic press; 2003.
- [39] Minty G. Monotone (nonlinear) operators in Hilbert space. *Duke Math J.* 1961; 29:341-346.
- [40] Brenner SC, Scott LR. *The Mathematical Theory of Finite Element Methods*, Volume 15 of *Texts in Applied Mathematics*. New York, third edition: Springer; 2008.
- [41] Girault V, Raviart P. *Finite Element Method for Navier-Stokes Equations: Theory and algorithms*. Berlin, Springer; 1986.
- [42] Temam R. *Navie-Stokes equations, Theory and Numerical analysis*. Amsterdam, North-Holland; 1984.
- [43] Gunzburger M. *Finite Element Methods for Incompressible Viscous Flows: A Guide to Theory, Practice and Algorithms*. Boston, Academic Press; 1989.
- [44] Hood P, C Taylor. A numerical solution of the Navier-Stokes equations using the finite element technique. *Comput Fluids*; 1973 1:73-100.
- [45] Hecht F. New development in Freefem++. *J Numer Math.* 2012; 20:251-265.
- [46] Ghia U, Ghia KN, Shin CT. High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method. *J Comput Phys.* 1982; 48:387-441.
- [47] Wong K, Baker A. A 3D incompressible Navier-Stokes velocity-vorticity weak form finite element algorithm. *Int J Numer Methods Fluids.* 2002; 38:99-123.