Maximizing reliability of multi-stage uncertain random systems by maintenance strategy

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Abstract

The existing researches have shown that internal degradation processes and external shocks may simultaneously interfere with the reliability of dynamic systems in uncertain random environments. Assume that failure processes are dependent, that is, shocks may accelerate degradation processes by additional degradations. Wear and additional degradations are uncertain, while shocks are considered to be random. As a natural consideration, it is necessary to maximize the reliability of multi-stage uncertain random systems. In this paper, a maximizing reliability problem is presented, and recurrence equations are provided by Bellman's principle. These are successfully applied to maximize reliability index in two special cases with linear and quadratic state equations. In addition, two effective algorithms are developed to achieve optimal solutions. Finally, a numerical example of a metallized film pulse capacitor is proposed which aims to indicate that optimization method is beneficial to maximize the reliability of multi-stage systems.

Keywords: reliability; uncertainty; chance theory; multi-stage system

1 Introduction

The capability of a system to perform its required functions is one of its essential characteristics, which represents the reliability of a system. Degradation processes and shocks may affect a system's reliability due to internal factors (wear, corrosion, fatigue, etc.) as well as external abrupt factors (impact, stress, etc.). In the 1990s, degradation process modeling was found to be an effective method for analyzing the reliability and statistical characteristics of a system [1–3]. When external factors are taken into consideration, a great deal of researches [4–6] have also been done on modeling shocks for a system. In early reliability evaluation, experts believed that degradation processes and shocks on a system are independent of each other. However, most systems exposed to complex situations may be subject to degradation processes and shocks simultaneously. A system suffered from competing failures of extreme shock and degradations was presented by Ye et al. in [7]. Wang et al. [8] addressed that the degradation rate is accelerated by shocks. Jiang et al. [9] proposed a multiple s-dependent competing failure model with a shifting failure threshold. A new dependent competing failure process reliability model was developed by Fan et al. [10] where random shocks are accelerated by degradation processes.

The reliability evaluation method based on probability theory holds that the reliability of a system can be measured only when sufficient failure data are obtained. However, it is difficult to acquire a large number of failure data within a short time. Liu [11] made a pioneering work on the booming field of uncertain reliability analysis of a dynamic system with few failure data. Uncertainty theory based on the expert's belief degrees was developed by Liu [16], and refined by Liu [17] when there is no enough data to get a frequency distribution. The method of uncertainty theory to solve practical problems has been constantly penetrated into many application fields, such as manpower planning [12], risk analysis [13], reliability analysis [14], uncertain fractional differential equation [15] etc. On the basis of uncertainty theory, researchers from Zeng et al. [18] developed a set of uncertain reliability indexes for evaluating the reliability of a product. Zeng et al. [19] also carried out a numerical evaluation method based on minimal cut sets in order to compute the reliability index by uncertain measure. With the advent of uncertainty and randomness in complicated systems, Liu [20] introduced a new field on chance theory. Liu [21] also pointed out that an uncertain random optimization problem can be converted into its deterministic form when it is not easy to be solved. After that, chance distribution was presented by [22] to measure the system reliability. Zhang et al. [23] defined belief reliability index as reliability distribution, mean time to failure, and belief reliable life, respectively.

Optimal control problem is the process of finding a solution that is optimal from among all possible solutions, which is an essential pattern of modern control theory. Bellman [24] proposed a dynamic programming method for solving optimal control problems and created the theory of dynamic programming. The operation of a system is susceptible to disturbances of indeterministic factors, which affect the states of a system. There are currently two types of optimal control theories: stochastic optimal control theory based on probability theory, and uncertain optimal control theory based on uncertainty theory. The study of stochastic optimal control problems started in the 1950s, Bellman [25] applied dynamic programming methods to stochastic optimal control problems in 1958. With the deepening of uncertainty theory and applications, there have been some progress in applying uncertain optimal control theory. In 2011, Zhu [26] applied Bellman's optimality principle in dynamic programming to make the researches on the optimal control problem of multi-stage fuzzy systems, and obtained a set of recursive formulas. Kang and Zhu [27] also provided the optimal bang-bang control problem of multi-stage uncertain systems to make the objective function reach the maximum value. Subsequently, scholars have conducted some research on the topics of discrete uncertain optimal control [28–30].

The difference equation (recursive relation) is an effective mathematical model that is usually applied to describe practical issues, including queuing problems, economic issues, biological genetics, and some other application fields. Traditional reliability theory holds that soft failures caused by internal degradations are random based on probability theory. However, the lack of degradation data is common and the data may not be accurate even though they are sufficient. Due to the complexity within a system, it may be easy to make errors in reliability assessments by existing data. In order to describe wear degradation accurately, experienced experts in reliability fields are invited to modify observation data by means of their own experience and knowledge, that is, an uncertain difference equation may be employed in this paper to describe degradation failure processes of a complex system at each stage. In general, it is accepted that shock processes that a system suffers are considered to random when there are sufficient shock data. It is external shocks that not only contribute to hard failures, but also accelerate wear degradations and result in further uncertain additional degradations. The system is thus distinguished from a random system, but a complex uncertain random system. Currently, the study of maximizing reliability index for multi-stage uncertain uncertain systems is considered to be of great relevance. Thus, this paper proposes a optimal control method by applying Bellman's principle of optimality to maximize system reliability.

The paper is structured as follows. Section 2 reviews some concepts involved in uncertainty theory and chance theory. Firstly, section 3 defines belief reliability index of a multi-stage uncertain random model. Subsequently, two types of uncertain random reliability optimal control models are established in section 4 where the state equations are linear and quadratic, respectively. The corresponding optimal controls and values at each stage are obtained by recursive equations. Section 5 provides a numerical example to demonstrate the model. This paper is summarized in a brief conclusion in section 6.

2 Preliminary

Let Γ be a nonempty set, $\mathcal{L}$ be a σ-algebra over Γ, $(\Gamma,\mathcal{L})$ be a measurable space and each element Λ in $\mathcal{L}$ be a measurable set. A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ is called an uncertain measure to indicate the belief degree with which we believe the event Λ will happen. The uncertain measure $\mathcal{M}$ satisfies the following axioms:

(Normality Axiom) $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ.

(Duality Axiom) $\mathcal{M}\{\Gamma\}+\mathcal{M}\{\Gamma^c\}=1$ for any event Λ.

(Subadditivity Axiom) For every countable sequence of events $\Lambda_1,\Lambda_2,\cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\}\leq\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

(Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M})$ be uncertainty spaces for $k=1,2,\cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$, respectively. An uncertain variable is a function ξ from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers such that the set $\{\xi\in B\}=\{\gamma\in\Gamma\mid\xi(\gamma)\in B\}$ is an event for any Borel set of real numbers.

Definition 1 *(Liu [16]) The uncertainty distribution $\Phi(x):R\to[0,1]$ of an uncertain variable ξ is defined as $\Phi(x)=\mathcal{M}\{\xi\leq x\}$ for any $x\in R$.*

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x)=(1+\exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{-1},\ x\in R, \tag{1}$$

denoted by $\mathcal{N}_u(e,\sigma)$ where e and σ are real numbers with $\sigma>0$. $\mathcal{N}_r(e,\sigma)$ means normal distribution of a random variable η with probability density function

$$\phi(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-e)^2}{2\cdot\sigma^2}\right),\ x\in R.$$

Theorem 1 *(Liu [16]) Let ξ_1 and ξ_2 be independent normal uncertain variables $\mathcal{N}_u(e_1,\sigma_1)$ and $\mathcal{N}_u(e_2,\sigma_2)$, respectively. Then the sum $\xi_1+\xi_2$ is also a normal uncertain variable $\mathcal{N}_u(e_1+e_2,\sigma_1+\sigma_2)$, i.e.,*

$$\mathcal{N}_u(e_1,\sigma_1)+\mathcal{N}_u(e_2,\sigma_2)=\mathcal{N}_u(e_1+e_2,\sigma_1+\sigma_2).$$

The product of a normal uncertain variable $\mathcal{N}_u(e,\sigma)$ and a scalar number $k>0$ is also a normal uncertain variable $\mathcal{N}_u(ke,k\sigma)$, i.e.,

$$k\cdot\mathcal{N}_u(e,\sigma)=\mathcal{N}_u(ke,k\sigma).$$

Theorem 2 *(Liu [16]) Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If f is a strictly decreasing function, then*

$$\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(1-\alpha),\Phi_2^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

Theorem 3 *(Liu [16]) Assume $\xi_1,\xi_2,\cdots,\xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If f is a strictly increasing function with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, then*

$$\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$$

has an expected value

$$E[\xi]=\int_0^1 f(\Phi_1(\alpha),\Phi_2(\alpha),\cdots,\Phi_m(\alpha),\Phi_{m+1}(1-\alpha),\cdots,\Phi_n(1-\alpha))d\alpha.$$

Let $(\Gamma,\mathcal{L},\mathcal{M})$ be an uncertainty space and $(\Omega,\mathcal{A},\Pr)$ be a probability space. Then the product $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ is called a chance space. An uncertain random variable is a function ξ from a chance space $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ to the set of real numbers such that $\{\xi\in B\}$ is an event for any Borel set B of real numbers. The chance measure of the uncertain random event $\{\xi\in B\}$ is

$$\mathrm{Ch}\{\xi\in B\}=\int_0^1\Pr\left\{\omega\in\Omega\,\middle|\,\mathcal{M}\{\gamma\in\Gamma\,|\,\xi(\gamma,\omega)\in B\}\geq r\right\}dr.$$

Theorem 4 *(Liu [20]) Let $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ be a chance space. Then*

$$\mathrm{Ch}\{\Lambda\times A\}=\mathcal{M}\{\Lambda\}\times\Pr\{A\}$$

for any $\Lambda\in\mathcal{L}$ and any $A\in\mathcal{A}$.

3 System Reliability

It is considered that a system suffers from dual failure processes of degradation processes and shocks under uncertain random situations. In this section, we are to focus on how to describe competing failure processes and define reliability index of a system based on chance theory.

Extreme shock model

Shock model is an essential pattern of reliability statistical analysis which can be divided into extreme shock model, accumulative shock model, run shock model and δ shock model, etc. Suppose that a system considered in this paper is subjected to extreme random shocks, each of which would lead to damage to a certain extent. $z(j)$ represents the shock damage at stage j, N represents the total number of stages. Referring to the properties and characteristics of random shocks, $z(j),\ j=0,1,\cdots,N$, are considered as independent random variables. If the damage value of a shock exceeds failure threshold D, a system will fail immediately.

Definition 2 *The probability function of extreme shock model at a multi-stage system can be defined as follows,*

$$\Pr\left\{\bigcap_{j=0}^{N}\{z(j)<D\}\right\}. \tag{2}$$

Degradation process model

In practical applications, a system will be accompanied by degradation phenomena. Stochastic differential equation has often been applied in reliability modelling to propel a degradation process. It is worth considering that a discrete model can describe actual degradation states of a system more accurately where there is a certain relationship between each degradation states. We all know that the premise of stochastic equation is that we have sufficient data to obtain probability distribution. However, it is usually normal for a system to have less and insufficient internal observation data in reliability engineering. Therefore, we need to investigate another a suitable mathematical tool–uncertainty theory to simulate internal degradation of a system. We present a multi-stage degradation model with uncertain disturbances to drive wear degradation process. To improve the system reliability, a multi-stage maintenance strategy with optimal controls $u(j)\in U_j$ is presented to maximize system reliability. A wear degradation process of a multi-stage system is founded as

$$x(j+1)=\phi_1(x(j),u_j,j)+\varphi_1(x(j),u_j,j)\xi_{j+1},\quad j=0,1,\cdots,N-1, \tag{3}$$

where $x(j)$ is wear degradation state variable at stage j, ξ_j is an uncertain variable to indicate the degree of fluctuation of wear degradations, u_j is the control variable at stage j and ϕ_1,φ_1 are real functions.

Although we may have sufficient data sometimes, it may not be accurate enough to measure additional degradations caused by shocks on internal degradation due to the complexity of external environment and internal system. The uncertainty reliability theory suggests that when faced with inaccurate data, experts in the field of reliability can revise degradation data based on their own expertise and knowledge. The additional degradation can be driven by uncertain difference equation

$$y(j+1)=\phi_2(y(j),u_j,j)+\varphi_2(y(j),u_j,j)\eta_{j+1},\quad j=0,1,\cdots,N-1, \tag{4}$$

where $y(j)$ is additional degradation state variable at stage j, η_j is an uncertain variable to indicate the degree of fluctuation of additional degradations, u_j is the control variable at stage j and ϕ_2,φ_2 are real functions.

Definition 3 *It is believed that $x(j)$ is independent of $y(j)$ due to different degradation causes. Soft failure occurs when the sum of wear degradations and additional degradations exceed threshold value H. The reliability function of a system without soft failure is*

$$\mathcal{M}\left\{x(N)+\sum_{j=0}^{N}y(j)<H\right\}. \tag{5}$$

An evaluation of reliability requires the definition of a reliability index that measures reliability of an uncertain random system under degradation and shock processes simultaneously. The reliability index may be interpreted as chance measure when considering two dependent failures. It is hoped that total uncertain degradations do not exceed the threshold H and that random shocks do not cause system failures. Only when neither of failure processes occurs can a system operate normally.

Definition 4 *Assume that a system suffers from both dependent degradation processes and shocks. Chance measure of an extreme shock model*

$$\mathrm{Ch}\left\{x(N)+\sum_{j=0}^{N} y(j)<H, \bigcap_{j=0}^{N}\{z(j)<D\}\right\} \tag{6}$$

is defined as belief reliability index R with N stages.

Theorem 5 *The belief reliability index R of an extreme shock model with N stages which suffers from dependent failure processes is equal to*

$$\mathcal{M}\left\{x(N)+\sum_{j=0}^{N} y(j)<H\right\} \cdot \operatorname{Pr}\left\{\bigcap_{j=0}^{N}\{z(j)<D\}\right\}. \tag{7}$$

Proof: Obviously, $x(N)+\sum\limits_{j=0}^{N} y(j)<H$ is an uncertain event and $\bigcap\limits_{j=0}^{N}\{z(j)<D\}$ is an random event. According to Theorem 4 that the chance measure is the product of uncertain measure and probability measure. We have

$$\mathrm{Ch}\left\{x(N)+\sum_{j=0}^{N} y(j)<H, \bigcap_{j=0}^{N}\{z(j)<D\}\right\}=\mathcal{M}\left\{x(N)+\sum_{j=0}^{N} y(j)<H\right\} \cdot \operatorname{Pr}\left\{\bigcap_{j=0}^{N}\{z(j)<D\}\right\}.$$

The theorem is thus verified.

4 Uncertain Random Reliability Optimal Control Model With Dependent Failure

4.1 Optimal belief reliability index model for uncertain random systems

The reliability of a system subjected to a competitive failure process gradually decreases with time due to the combined effect of degradation and shock processes. In view of applications, a decision maker desires to maintain the reliability of a system at a high level by taking some maintenance methods. It is generally believed that maintenance funds are used to mitigate the effects of system degradations and to maximize system reliability. Assume that u_j is the maintenance fund at stage j. We are to present a maximizing reliability problem extreme shock satisfying a set of constraint,

$$\left\{\begin{array}{l}
R(x_0, y_0, 0)=\max\limits_{\substack{u_j \in U_j \\ 0 \leq j \leq N-1}} \mathrm{Ch}\left\{x(N)+\sum\limits_{j=0}^{N} y(j)<H, \bigcap\limits_{j=0}^{N}\{z(j)<D\}\right\} \\
\text{subject to} \\
\quad x(j+1)=\phi_1(x(j), u_j, j)+\varphi_1(x(j), u_j, j) \xi_{j+1}, \\
\quad y(j+1)=\phi_2(y(j), u_j, j)+\varphi_2(y(j), u_j, j) \eta_{j+1}, \\
\quad j=0, \cdots, N-1 \quad \text{and} \quad x(0)=x_0, y(0)=y_0,
\end{array}\right. \tag{8}$$

where $x(j)$ and $y(j)$ are state variables of degradation processes, $z(j),\ j=0,1,\cdots,N$, are independent state variables of shock processes, $\phi_1,\varphi_1,\phi_2,\varphi_2$ are real functions, $a_j,\sigma_{j+1},c_j,\chi_{j+1}$ are real numbers, ξ_j and η_j are independent uncertain variables. We define the new state variables by

$$\begin{aligned}
x(j+1) &= f_j(x(j),u_j,\xi_{j+1}),\\
&= \phi_1(x(j),u_j,j)+\varphi_1(x(j),u_j,j)\xi_{j+1}, \quad j=0,\cdots,N-1,\\
y(j+1) &= g_j(y(j),u_j,\eta_{j+1}),\\
&= \phi_2(y(j),u_j,j)+\varphi_2(y(j),u_j,j)\eta_{j+1}, \quad j=0,\cdots,N-1,\\
\hat{y}(j+1) &= \rho_j(\hat{y}(j),y(j),u_j),\\
&= \hat{y}(j)+y(j), \quad j=0,\cdots,N-2,\\
\hat{y}(N) &= \rho_{N-1}(\hat{y}(N-1),y(N-1),u_{N-1}),\\
&= \hat{y}(N-1)+y(N-1)+x(N)+y(N),\\
& x(0)=x_0, y(0)=y_0, \hat{y}(0)=0.
\end{aligned}$$

Then, problem (8) can be transformed into its equivalent form

$$\begin{cases}
R(x_0,y_0,\hat{y}_0,0) = \max\limits_{\substack{u_j\in U_j\\ 0\le j\le N-1}} \mathrm{Ch}\left\{\hat{y}(N)<H, \bigcap\limits_{j=0}^{N}\{z(j)<D\}\right\}\\
\text{subject to}\\
\quad x(j+1)=\phi_1(x(j),u_j,j)+\varphi_1(x(j),u_j,j)\xi_{j+1}, \quad j=0,\cdots,N-1,\\
\quad y(j+1)=\phi_2(y(j),u_j,j)+\varphi_2(y(j),u_j,j)\eta_{j+1}, \quad j=0,\cdots,N-1,\\
\quad \hat{y}(j+1)=\rho_j(\hat{y}(j),y(j),u_j), \quad j=0,\cdots,N-2,\\
\quad x(0)=x_0, \quad y(0)=y_0, \quad \hat{y}(0)=0.
\end{cases} \tag{9}$$

To solve problem (9), we propose a subproblem as follows,

$$\begin{cases}
J(x_0,y_0,\hat{y}_0,0) = \max\limits_{\substack{u_j\in U_j\\ 0\le j\le N-1}} E[I(\hat{y}(N)<H)]\\
\text{subject to}\\
\quad x(j+1)=f_j(x(j),u_j,\xi_{j+1}), \quad j=0,\cdots,N-1,\\
\quad y(j+1)=g_j(y(j),u_j,\eta_{j+1}), \quad j=0,\cdots,N-1,\\
\quad \hat{y}(j+1)=\rho_j(\hat{y}(j),y(j),u_j), \quad j=0,\cdots,N-2,\\
\quad x(0)=x_0, \quad y(0)=y_0, \quad \hat{y}(0)=0,
\end{cases} \tag{10}$$

where I is an indicator function of $\hat{y}(N)$.

Theorem 6 *Let $u'_j,\ j=0,1,\cdots,N-1$, and $R(x_0,y_0,\hat{y}_0,0)$ are optimal controls and optimal values of problem (9), and $u^*_j,\ j=0,1,\cdots,N-1$ and $J(x_0,y_0,\hat{y}_0,0)$ are optimal controls and optimal values of problem (10). Then, we have*

$$\begin{aligned}
& u'_j = u^*_j, \quad j=0,1,\cdots,N-1,\\
& R(x_0,y_0,\hat{y}_0,0) = J(x_0,y_0,\hat{y}_0,0)\cdot\prod_{j=0}^{N}\Pr\{z(j)<D\}.
\end{aligned} \tag{11}$$

Proof. It follows from Theorem 5 that

$$\begin{aligned}
\mathrm{Ch}\left\{\hat{y}(N)<H, \bigcap_{j=0}^{N}\{z(j)<D\}\right\} &= \mathcal{M}\{\hat{y}(N)<H\}\cdot\Pr\left\{\bigcap_{j=0}^{N}\{z(j)<D\}\right\}\\
&= \mathcal{M}\{\hat{y}(N)<H\}\cdot\prod_{j=0}^{N}\Pr\{z(j)<D\}.
\end{aligned}$$

Then, we would prefer to convert the uncertainty measure into an expectation over the indicator function $I(\cdot)$. Here, $I(\hat{y} < H) = 1$ if $\hat{y} < H$, and $I(\hat{y} < H) = 0$ if $\hat{y} \geq H$. We have

$$\mathcal{M}\{\hat{y}(N) < H\} = E[I(\hat{y}(N) < H)], \tag{12}$$

and

$$\mathrm{Ch}\left\{\hat{y}(N) < H, \bigcap_{j=0}^{N}(z(j) < D)\right\} = E[I(\hat{y}(N) < H)] \cdot \prod_{j=0}^{N} \Pr\{z(j) < D\}.$$

There are no controls for external shocks $z(j)$, and problem (9) has a different term $\bigcap_{j=0}^{N}\{z(j) < D\}$ in the objective function from problem (10). Thus, it is easy to derive $u'_j = u^*_j$, and

$$R(x_0, y_0, \hat{y}_0, 0) = J(x_0, y_0, \hat{y}_0, 0) \cdot \prod_{j=0}^{N} \Pr\{z(j) < D\}.$$

The theorem is proved.

For $0 \leq k < N$, let $J(x_k, y_k, \hat{y}_k, k)$ be the optimal value in $[k, N]$ with the condition that at stage k, we are in state $x(k) = x_k$, $y(k) = y_k$ and $\hat{y}(k) = 0$. That is, we have

$$\begin{cases} J(x_k, y_k, \hat{y}_k, k) = \max\limits_{\substack{u_j \in U_j \\ k \leq j \leq N}} E[I(\hat{y}(N) < H)] \\ \text{subject to} \\ \quad x(j+1) = f_j(x(j), u_j, \xi_{j+1}), \\ \quad y(j+1) = g_j(y(j), u_j, \eta_{j+1}), \\ \quad \hat{y}(j+1) = \rho_j(\hat{y}(j), y(j), u_j), \\ \quad j = k, k+1, \cdots, N-1 \quad \text{and} \quad x(k) = x_k, \quad y(k) = y_k, \quad \hat{y}(k) = 0. \end{cases} \tag{13}$$

Notice that problem (10) is called Mayer form, and conforms to a class of expected value models of uncertain optimal control problems which was studied by Kang and Zhu [27]. Applying Bellman' s principle of optimality, through a similar proof to [27], we can solve this specific problem (10) with the following recursion equations.

Theorem 7 *For problem (10), we have the following recurrence equations,*

$$\begin{aligned} J(x_N, y_N, \hat{y}_N, N) &= I(\hat{y}_N < H), \\ J(x_k, y_k, \hat{y}_k, k) &= \max_{u_k \in U_k} E[J(x(k+1), y(k+1), \hat{y}(k+1), k+1)] \\ &\quad \textit{for} \quad k = N-1, N-2, \cdots, 1, 0. \end{aligned} \tag{14}$$

Proof. Referring to recurrence equation of optimal control problem of multi-stage systems by Bellman's principle of optimality in [27], we have

$$J(x_N, y_N, \hat{y}_N, N) = I(\hat{y}(N) < H),$$

and for any $k = N-1, N-2, \cdots, 0$, we have

$$J(x_k, y_k, \hat{y}_k, k) = \max_{u_k \in U_k} E[J(x(k+1), y(k+1), \hat{y}(k+1), k+1)].$$

The theorem is thus proved.

4.2 Two special cases

Generally speaking, the approach to solve problems usually starts with some special cases and then expands to general cases, or from simple to complex problems. We might as well give the following examples to further investigate maximizing reliability problems of uncertain random systems via the method of recursion equations. A linear degradation model is quite common in reliability engineering, we desire to maximize belief reliability index of an uncertain random linear system

$$\begin{cases} R(x_0,y_0,0) = \max\limits_{\substack{u_j\in[M_1,M_2]\\ 0\le j\le N-1}} \mathrm{Ch}\left\{ x(N)+\sum\limits_{j=0}^{N} y(j)<H, \bigcap\limits_{j=0}^{N}\{z(j)<D\}\right\} \\ \text{subject to} \\ \quad x(j+1)=a_jx(j)-b_ju_j+\sigma_{j+1}\xi_{j+1}, \\ \quad y(j+1)=c_jy(j)-d_ju_j+\chi_{j+1}\eta_{j+1}, \\ \quad j=0,\cdots,N-1 \quad \text{and} \quad x(0)=x_0, y(0)=y_0, \end{cases} \tag{15}$$

where the coefficients $a_j,b_j,c_j,d_j,\sigma_{j+1},\chi_{j+1}$ are real numbers and $a_j,c_j,\sigma_{j+1},\chi_{j+1}>0$, and $z(j)$, $j=0,1,\cdots,N$ are independent random variables with probability function F_j. It follows from Theorem 6 that a subproblem of problem (15) with same control variables is presented,

$$\begin{cases} J(x_0,y_0,\hat{y}_0,0) = \max\limits_{\substack{u_j\in[M_1,M_2]\\ 0\le j\le N-1}} E[I(\hat{y}(N)<H)] \\ \text{subject to} \\ \quad x(j+1)=a_jx(j)-b_ju_j+\sigma_{j+1}\xi_{j+1}, \quad j=0,\cdots,N-1, \\ \quad y(j+1)=c_jy(j)-d_ju_j+\chi_{j+1}\eta_{j+1}, \quad j=0,\cdots,N-1, \\ \quad \hat{y}(j+1)=\hat{y}(j)+y(j), \quad j=0,\cdots,N-2, \\ \quad \hat{y}(N)=\hat{y}(N-1)+y(N-1)+x(N)+y(N), \\ \qquad x(0)=x_0, \quad y(0)=y_0, \quad \hat{y}(0)=0, \end{cases} \tag{16}$$

where the optimal controls u_j, $j=0,\cdots,N-1$, are equivalent to optimal controls of problem (15), and the optimal values of two problems have the relation

$$R(x_0,y_0,0)=J(x_0,y_0,\hat{y}_0,0)\cdot\prod_{j=0}^{N}F_j(D). \tag{17}$$

Theorem 8 *Assume that ξ_j, $j=1,2,\cdots,N$, are independent normal uncertain variables with uncertainty distribution $\xi_j\sim\mathcal{N}_u(e_j,v_j)$ and η_j, $j=1,2,\cdots,N$, are independent normal uncertain variables with uncertainty distribution $\eta_j\sim\mathcal{N}_u(e_j^{'},v_j^{'})$. The optimal controls of problem (16) are*

$$u_k^* = \begin{cases} M_2, & H_k>0, \\ M_1, & H_k<0, \\ undetermined, & H_k=0 \end{cases} \tag{18}$$

for $k = 0, 1, \cdots, N-1$ and the optimal values are

$$
\begin{aligned}
&J(x_N, y_N, \hat{y}_N, N) = I(\hat{y}_N < H),\\
&J(x_{N-1}, y_{N-1}, \hat{y}_{N-1}, N-1)\\
&= \left(1 + \exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N + \chi_N v_N^{'})}(\sigma_N e_N + \chi_N e_N^{'} - (H - \hat{y}_{N-1} - P_{N-1}x_{N-1} - Q_{N-1}y_{N-1} + H_{N-1}u_{N-1}^{*}))\right)\right)^{-1},\\
&J(x_k, y_k, \hat{y}_k, k)\\
&= \int_0^1 \cdots \int_0^1 \left(1 + \exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N + \chi_N v_N^{'})}\left(\sigma_N e_N + \chi_N e_N^{'} - \left(H - \hat{y}_k - P_k x_k - Q_k y_k + \sum_{i=k}^{N-1} H_i u_i^{*}\right.\right.\right.\right.\\
&\quad \left.\left.\left.\left. - \Phi_{k+1}^{-1}(1-\alpha_{k+1}) - \cdots - \Phi_{N-1}^{-1}(1-\alpha_{N-1})\right)\right)\right)\right)^{-1} d\alpha_{k+1} \cdots d\alpha_{N-1}, \quad k = N-2, \cdots, 1, 0,
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
&P_k = P_{k+1}a_k, \quad Q_k = 1 + Q_{k+1}c_k, \quad H_k = P_{k+1}b_k + Q_{k+1}d_k, \quad L_k = P_k\sigma_k, \quad S_k = Q_k\chi_k,\\
&\Phi_k^{-1}(1-\alpha_k) = L_k e_k + S_k e_k^{'} + \frac{(L_k v_k + S_k v_k^{'})\sqrt{3}}{\pi}\ln\frac{1-\alpha_k}{\alpha_k}
\end{aligned}
\tag{20}
$$

for $k = N-1, \cdots, 1, 0$ and $P_N = 1, Q_N = 1$.

Proof. Assume that optimal controls are $u_0^*, u_1^*, \cdots, u_{N-1}^*$, and applying the recursion equation (14). For $k = N$, we have

$$J(x_N, y_N, \hat{y}_N, N) = I(\hat{y}_N < H).$$

For $k = N-1$, using the recursion equation (14), we have the following:

$$
\begin{aligned}
J(x_{N-1}, y_{N-1}, \hat{y}_{N-1}, N-1) &= \max_{u_{N-1}\in[M_1,M_2]} E[J(x(N), y(N), \hat{y}(N), N]\\
&= \max_{u_{N-1}\in[M_1,M_2]} E[I(\hat{y}(N) < H)].
\end{aligned}
$$

According to the transformation (12), we have

$$E[I(\hat{y}(N) < H)] = \mathcal{M}\{\hat{y}_{N-1} + a_{N-1}x_{N-1} + (1 + c_{N-1})y_{N-1} - (b_{N-1} + d_{N-1})u_{N-1} + \sigma_N\xi_N + \chi_N\eta_N < H\}.$$

Since ξ_N and η_N are independent normal uncertain variables $\mathcal{N}_u(e_N, v_N)$ and $\mathcal{N}_u(e_N^{'}, v_N^{'})$, respectively. It follows from Theorem 1 that the sum $\sigma_N\xi_N + \chi_N\eta_N$ is also a normal uncertain variable $\mathcal{N}_u(\sigma_N e_N + \chi_N e_N^{'}, \sigma_N v_N + \chi_N v_N^{'})$, i.e.,

$$\sigma_N\mathcal{N}_u(e_N, v_N) + \chi_N\mathcal{N}_u(e_N^{'}, v_N^{'}) = \mathcal{N}_u(\sigma_N e_N + \chi_N e_N^{'}, \sigma_N v_N + \chi_N v_N^{'})$$

Set $P_{N-1} = a_{N-1}$, $Q_{N-1} = 1 + c_{N-1}$, $H_{N-1} = b_{N-1} + d_{N-1}$ and $P_N = Q_N = 1$. It is easy to obtain

$$
\begin{aligned}
&E[I(\hat{y}(N) < H)]\\
&= \left(1 + \exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N + \chi_N v_N^{'})}(\sigma_N e_N + \chi_N e_N^{'} - (H - \hat{y}_{N-1} - P_{N-1}x_{N-1} - Q_{N-1}y_{N-1} + H_{N-1}u_{N-1}))\right)\right)^{-1}
\end{aligned}
$$

and

$$\begin{aligned}
&J(x_{N-1},y_{N-1},\hat{y}_{N-1},N-1)\\
&=\max_{u(N-1)\in[M_1,M_2]}\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-1}-P_{N-1}x_{N-1}-Q_{N-1}y_{N-1}\right.\right.\\
&\quad\left.\left.+H_{N-1}u_{N-1}))\right)\right)^{-1}\\
&=\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-1}-P_{N-1}x_{N-1}-Q_{N-1}y_{N-1}\right.\right.\\
&\quad\left.\left.+\max_{u(N-1)\in[M_1,M_2]}H_{N-1}u_{N-1}))\right)\right)^{-1}.
\end{aligned}$$

Let u_{N-1}^{*} be the optimal solution, we have the following equation:

$$\max_{u_{N-1}\in[M_1,M_2]}H_{N-1}u_{N-1}=H_{N-1}u_{N-1}^{*}.$$

We can get the optimal control

$$u_{N-1}^{*}=\begin{cases} M_2, & H_{N-1}>0,\\ M_1, & H_{N-1}<0,\\ undetermined, & H_{N-1}=0,\end{cases}$$

and optimal value

$$\begin{aligned}
&J(x_{N-1},y_{N-1},\hat{y}_{N-1},N-1)\\
&=\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-1}-P_{N-1}x_{N-1}-Q_{N-1}y_{N-1}+H_{N-1}u_{N-1}^{*}))\right)\right)^{-1}.
\end{aligned}$$

For $k=N-2$, applying the recursion equation (14), we have

$$\begin{aligned}
&J(x_{N-2},y_{N-2},\hat{y}_{N-2},N-2)\\
&=\max_{u_{N-2}\in[M_1,M_2]}E[J(x(N-1),y(N-1),\hat{y}(N-1),N-1)],
\end{aligned}$$

where

$$\begin{aligned}
&J(x(N-1),y(N-1),\hat{y}(N-1),N-1)\\
&=\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-2}-P_{N-1}a_{N-2}x_{N-1}-(1+Q_{N-1}c_{N-2})y_{N-1}\right.\right.\\
&\quad\left.\left.+(P_{N-1}b_{N-2}+Q_{N-1}d_{N-2})u_{N-2}+H_{N-1}u_{N-1}^{*}-P_{N-1}\sigma_{N-1}\xi_{N-1}-Q_{N-1}\chi_{N-1}\eta_{N-1}))\right)\right)^{-1}.
\end{aligned}$$

Set $P_{N-2}=P_{N-1}a_{N-2}$, $Q_{N-2}=1+Q_{N-1}c_{N-2}$, $H_{N-2}=P_{N-1}b_{N-2}+Q_{N-1}d_{N-2}$, $L_{N-1}=P_{N-1}\sigma_{N-1}$ and $S_{N-1}=Q_{N-1}\chi_{N-1}$. Just as we assume that $a_{N-2},c_{N-2}>0$ and $\sigma_{N-2},\chi_{N-2}>0$, we have $P_{N-2},Q_{N-2}>,L_{N-1}$ and $S_{N-1}>0$. By using the independence of ξ_{N-1} and η_{N-1}, $L_{N-1}\xi_{N-1}+S_{N-1}\eta_{N-1}$ is also a normal uncertain variable, i.e.,

$$L_{N-1}\mathcal{N}_u(e_{N-1},v_{N-1})+S_{N-1}\mathcal{N}_u(e_{N-1}^{'},v_{N-1}^{'})=\mathcal{N}_u(L_{N-1}e_{N-1}+S_{N-1}e_{N-1}^{'},L_{N-1}v_{N-1}+S_{N-1}v_{N-1}^{'}),$$

We observe that $J(x(N-1),y(N-1),\hat{y}(N-1),N-1)$ is a strictly decreasing function with $L_{N-1}\xi_{N-1}+S_{N-1}\eta_{N-1}$. It follows from Theorem 2 that the inverse uncertainty distribution of $J(x(N-1),y(N-1),\hat{y}(N-1),N-1)$ is

$$\begin{aligned}
\Psi_{N-1}^{-1}(\alpha_{N-1})=&\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-2}-P_{N-2}x_{N-1}-Q_{N-2}y_{N-1}\right.\right.\\
&\left.\left.+H_{N-2}u_{N-2}+H_{N-1}u_{N-1}^{*}-\Phi_{N-1}^{-1}(1-\alpha_{N-1})))\right)\right)^{-1},
\end{aligned}$$

where

$$\Phi_{N-1}^{-1}(1-\alpha_{N-1}) = L_{N-1}e_{N-1} + S_{N-1}e_{N-1}^{'} + \frac{(L_{N-1}v_{N-1}+S_{N-1}v_{N-1}^{'})\sqrt{3}}{\pi}\ln\frac{1-\alpha_{N-1}}{\alpha_{N-1}}, \quad 0<\alpha_{N-1}<1.$$

By using Theorem 3, we have

$$\begin{aligned}
&J(x_{N-2},y_{N-2},N-2)\\
&= \max_{u_{N-2}\in[M_1,M_2]}\int_0^1\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-2}\right.\right.\\
&\quad \left.\left.-P_{N-2}x_{N-1}-Q_{N-2}y_{N-1}+H_{N-2}u_{N-2}+H_{N-1}u_{N-1}^{*}-\Phi_{N-1}^{-1}(1-\alpha_{N-1})))\right)\right)^{-1}d\alpha_{N-1}\\
&= \int_0^1\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-2}\right.\right.\\
&\quad \left.\left.-P_{N-2}x_{N-1}-Q_{N-2}y_{N-1}+\max_{u_{N-2}\in[M_1,M_2]}H_{N-2}u_{N-2}+H_{N-1}u_{N-1}^{*}-\Phi_{N-1}^{-1}(1-\alpha_{N-1})))\right)\right)^{-1}d\alpha_{N-1}.
\end{aligned}$$

Similarly, referring to the calculation of $j=N-1$ that $\max_{u_{N-2}\in[M_1,M_2]} H_{N-2}u_{N-2} = H_{N-2}u_{N-2}^{*}$, we derive the optimal controls

$$u_{N-2}^{*} = \begin{cases} M_2, & \text{if } H_{N-2}>0\\ M_1, & \text{if } H_{N-2}<0\\ undetermined, & \text{if } H_{N-2}=0,\end{cases}$$

and optimal value

$$\begin{aligned}
J(x_{N-2},y_{N-2},N-2) = \int_0^1&\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v_N^{'})}(\sigma_N e_N+\chi_N e_N^{'}-(H-\hat{y}_{N-2}\right.\right.\\
&\left.\left.-P_{N-2}x_{N-2}-Q_{N-2}y_{N-2}+H_{N-2}u_{N-2}^{*}+H_{N-1}u_{N-1}^{*}-\Phi_{N-1}^{-1}(1-\alpha_{N-1})))\right)\right)^{-1}d\alpha_{N-1}.
\end{aligned}$$

By induction, we can obtain the optimal controls (18) and optimal values (19). The theorem is thus verified.

It follows form Theorem 8 that an uncertain random reliability optimal control model whose objective function and state transition equations are linear at each stage with exact solutions. If the objective function is linear and the state transition equations are quadratic, we consider the following problem,

$$\begin{cases}
R(x_0,y_0,0) = \max\limits_{\substack{u_j\in[M_1,M_2]\\ 0\le j\le N-1}} \mathrm{Ch}\left\{x(N)+\sum\limits_{j=0}^{N}y(j)<H,\ \bigcap\limits_{j=0}^{N}\{z(j)<D\}\right\}\\
\text{subject to}\\
\quad x(j+1) = a_j x(j) - b_j u_j - l_j u_j^2 + \sigma_{j+1}\xi_{j+1},\\
\quad y(j+1) = c_j y(j) - d_j u_j - q_j u_j^2 + \chi_{j+1}\eta_{j+1},\\
\quad j=0,\cdots,N-1 \quad \text{and} \quad x(0)=x_0, y(0)=y_0,
\end{cases} \tag{21}$$

where the coefficients $a_j,b_j,c_j,d_j,l_j,q_j,\sigma_{j+1},\chi_{j+1}$ are real numbers with $a_j,c_j,\sigma_{j+1},\chi_{j+1}>0$, and $z(j)$, $j=0,1,\cdots,N$ are independent random variables with probability function F_j. It follows from Theorem 6 that a subproblem of problem (21) with same control variables is defined as

$$\begin{cases} J(x_0,y_0,\hat{y}_0,0)=\max\limits_{\substack{u_j\in[M_1,M_2]\\ 0\le j\le N-1}} E[I(\hat{y}(N)<H)] \\ \text{subject to} \\ \quad x(j+1)=a_jx(j)-b_ju_j-l_ju_j^2+\sigma_{j+1}\xi_{j+1}, \quad j=0,\cdots,N-1, \\ \quad y(j+1)=c_jy(j)-d_ju_j-q_ju_j^2+\chi_{j+1}\eta_{j+1}, \quad j=0,\cdots,N-1, \\ \quad \hat{y}(j+1)=\hat{y}(j)+y(j), \quad j=0,\cdots,N-2, \\ \quad \hat{y}(N)=\hat{y}(N-1)+y(N-1)+x(N)+y(N), \\ \quad x(0)=x_0, \quad y(0)=y_0, \quad \hat{y}(0)=0, \end{cases} \tag{22}$$

where the optimal controls u_j^*, $j=0,\cdots,N-1$, are equivalent to optimal controls of problem (21), and the optimal value of two problems have the relation

$$R(x_0,y_0,0)=J(x_0,y_0,\hat{y}_0,0)\cdot\prod_{j=0}^{N}F_j(D). \tag{23}$$

Theorem 9 *Assume that ξ_j, $j=1,2,\cdots,N$, are independent normal uncertain variables with uncertainty distribution $\xi_j\sim\mathcal{N}_u(e_j,v_j)$ and η_j, $j=1,2,\cdots,N$, are independent normal uncertain variables with uncertainty distribution $\eta_j\sim\mathcal{N}_u(e_j^{'},v_j^{'})$. The optimal controls of problem (22) are*

$$u_k^*=\begin{cases} M_1, & F_k=0 \text{ and } H_k<0, \text{ or } F_k<0 \text{ and } \frac{-H_k}{2F_k}<M_1, \\ & \text{or } F_k>0 \text{ and } |M_1+\frac{-H_k}{2F_k}|\ge|M_2+\frac{-H_k}{2F_k}|, \\ \frac{-H_k}{2F_k}, & F_k<0 \text{ and } M_1\le\frac{-H_k}{2F_k}\le M_2, \\ M_2, & F_k=0 \text{ and } H_k>0, \text{ or } F_k<0 \text{ and } \frac{-H_k}{2F_k}>M_2, \\ & \text{or } F_k>0 \text{ and } |M_1+\frac{H_k}{2F_k}|<|M_2+\frac{H_k}{2F_k}|, \\ undetermined, & F_k=0 \text{ and } H_k=0 \end{cases} \tag{24}$$

for $k=0,1,\cdots,N-1$ and the optimal values are

$$\begin{aligned} &J(x_N,y_N,\hat{y}_N,N)=I(\hat{y}_N<H), \\ &J(x_{N-1},y_{N-1},\hat{y}_{N-1},N-1) \\ &=\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_Nv_N+\chi_Nv_N^{'})}(\sigma_Ne_N+\chi_Ne_N^{'}-(H-\hat{y}_{N-1}-P_{N-1}x_{N-1}-Q_{N-1}y_{N-1}+G_{N-1}))\right)\right)^{-1}, \\ &J(x_k,y_k,\hat{y}_k,k), \\ &=\int_0^1\cdots\int_0^1\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_Nv_N+\chi_Nv_N^{'})}\left(\sigma_Ne_N+\chi_Ne_N^{'}-\left(H-\hat{y}_k-P_kx_k-Q_ky_k+\sum_{i=k}^{N-1}G_i\right.\right.\right.\right. \\ &\quad\left.\left.\left.\left.-\Phi_{k+1}^{-1}(1-\alpha_{k+1})-\cdots-\Phi_{N-1}^{-1}(1-\alpha_{N-1})\right)\right)\right)\right)^{-1}d\alpha_{k+1}\cdots d\alpha_{N-1}, \quad k=N-2,\cdots,1,0, \end{aligned} \tag{25}$$

where

$$\begin{aligned} &P_k=P_{k+1}a_k, \quad Q_k=1+Q_{k+1}c_k, \quad H_k=P_{k+1}b_k+Q_{k+1}d_k, \quad F_k=P_{k+1}l_k+Q_{k+1}q_k, \quad L_k=P_k\sigma_k, \quad S_k=Q_k\chi_k, \\ &\Phi_k^{-1}(1-\alpha_k)=L_ke_k+S_ke_k^{'}+\frac{(L_kv_k+S_kv_k^{'})\sqrt{3}}{\pi}\ln\frac{1-\alpha_k}{\alpha_k}, \end{aligned} \tag{26}$$

and

$$G_k = \begin{cases} M_1H_k + M_1^2F_k, & F_k = 0 \textit{ and } H_k < 0, \textit{ or } F_k < 0 \textit{ and } \frac{-H_k}{2F_k} < M_1, \\ & \textit{or } F_k > 0 \textit{ and } |M_1 + \frac{H_k}{2F_k}| \geq |M_2 + \frac{H_k}{2F_k}|, \\ \frac{-H_k^2}{4F_k}, & F_k < 0 \textit{ and } M_1 \leq \frac{-H_k}{2F_k} \leq M_2, \\ M_2H_k + M_2^2F_k, & F_k = 0 \textit{ and } H_k > 0, \textit{ or } F_k < 0 \textit{ and } \frac{-H_k}{2F_k} > M_2, \\ & \textit{or } F_k > 0 \textit{ and } |M_1 + \frac{-H_k}{2F_k}| < |M_2 + \frac{-H_k}{2F_k}|, \\ 0, & F_k = 0 \textit{ and } H_k = 0 \end{cases} \tag{27}$$

for $k = N-1, \cdots, 1, 0$ *and* $P_N = 1, Q_N = 1$.

Proof. Denote that optimal control variables are $u_0^*, u_1^*, \cdots, u_{N-1}^*$. For $k = N$, we applying the recursion equation (14), we have

$$J(x_N, y_N, \hat{y}_N, N) = I(\hat{y}_N < H).$$

For $k = N-1$, using the recursion equation (14), we have the following:

$$\begin{aligned} J(x_{N-1}, y_{N-1}, \hat{y}_{N-1}, N-1) &= \max_{u_{N-1} \in [M_1, M_2]} E[J(x(N), y(N), \hat{y}(N), N] \\ &= \max_{u_{N-1} \in [M_1, M_2]} E[I(\hat{y}(N) < H)]. \end{aligned}$$

According to the transformation (12), we have

$$\begin{aligned} & E[I(\hat{y}(N) < H)] \\ & = \mathcal{M}\{\hat{y}_{N-1} + a_{N-1}x_{N-1} + (1 + c_{N-1})y_{N-1} - (b_{N-1} + d_{N-1})u_{N-1} - (l_{N-1} - q_{N-1})u_{N-1}^2 + \sigma_N \xi_N + \chi_N \eta_N < H\}, \end{aligned}$$

where the parameters $\sigma_N, \chi_N > 0$. As the proof in Theorem 8, $\sigma_N \xi_N + \chi_N \eta_N$ is a normal uncertain variable $\mathcal{N}_u(\sigma_N e_N + \chi_N e_N^{'}, \sigma_N v_N + \chi_N v_N^{'})$. Set $P_{N-1} = a_{N-1}$, $Q_{N-1} = 1 + c_{N-1}$, $H_{N-1} = b_{N-1} + d_{N-1}$ and $F_{N-1} = l_{N-1} + q_{N-1}$. Using the recursion equation (14), we can derive

$$\begin{aligned} & E[I(\hat{y}(N) < H)] \\ & = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N + \chi_N v_N^{'})}(\sigma_N e_N + \chi_N e_N^{'} - (H - \hat{y}_{N-1} - P_{N-1}x_{N-1} - Q_{N-1}y_{N-1} + H_{N-1}u_{N-1} \right.\right. \\ & \quad \left.\left. + F_{N-1}u_{N-1}^2))\right)\right)^{-1} \end{aligned}$$

and

$$\begin{aligned} & J(x_{N-1}, y_{N-1}, \hat{y}_{N-1}, N-1) \\ & = \max_{u(N-1) \in [M_1, M_2]} \left(1 + \exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N + \chi_N v_N^{'})}(\sigma_N e_N + \chi_N e_N^{'} - (H - \hat{y}_{N-1} - P_{N-1}x_{N-1} - Q_{N-1}y_{N-1} \right.\right. \\ & \quad \left.\left. + H_{N-1}u_{N-1} + F_{N-1}u_{N-1}^2))\right)\right)^{-1} \\ & = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}(\sigma_N v_N + \chi_N v_N^{'})}(\sigma_N e_N + \chi_N e_N^{'} - (H - \hat{y}_{N-1} - P_{N-1}x_{N-1} - Q_{N-1}y_{N-1} \right.\right. \\ & \quad \left.\left. + \max_{u(N-1) \in [M_1, M_2]} H_{N-1}u_{N-1} + F_{N-1}u_{N-1}^2))\right)\right)^{-1}. \end{aligned}$$

Denote the following equation:

$$\max_{u_{N-1} \in [M_1, M_2]} H_{N-1}u_{N-1} + F_{N-1}u_{N-1}^2 = H_{N-1}u_{N-1}^* + F_{N-1}(u_{N-1}^*)^2 = G_{N-1},$$

Case 1: If $F_{N-1}=0$ and $H_{N-1}=0$, then for any $u^*_{N-1}\in[M_1,M_2]$ as the maximum point, $G_{N-1}=0$.
Case 2: If $F_{N-1}=0$ and $H_{N-1}<0$, then $u^*_{N-1}=M_1$, $G_{N-1}=M_1H_{N-1}+M_1^2F_{N-1}$.
Case 3: If $F_{N-1}=0$ and $H_{N-1}>0$, then $u^*_{N-1}=M_2$, $G_{N-1}=M_2H_{N-1}+M_2^2F_{N-1}$.
Case 4: If $F_{N-1}<0$ and $M_1\le\frac{-H_{N-1}}{2F_{N-1}}\le M_2$, then $u^*_{N-1}=\frac{-H_{N-1}}{2F_{N-1}}, G_{N-1}=\frac{-H^2_{N-1}}{4F_{N-1}}$.
Case 5: If $F_{N-1}<0$ and $\frac{-H_{N-1}}{2F_{N-1}}<M_1$, then $u^*_{N-1}=M_1, G_{N-1}=M_2H_{N-1}+M_2^2F_{N-1}$.
Case 6: If $F_{N-1}<0$ and $\frac{-H_{N-1}}{2F_{N-1}}>M_2$, then $u^*_{N-1}=M_2, G_{N-1}=M_1H_{N-1}+M_1^2F_{N-1}$.
Case 7: If $F_{N-1}>0$ and $|M_1+\frac{H_{N-1}}{2F_{N-1}}|\ge|M_2+\frac{H_{N-1}}{2F_{N-1}}|$, then $u^*_{N-1}=M_1, G_{N-1}=M_1H_{N-1}+M_1^2F_{N-1}$.
Case 8: If $F_{N-1}>0$ and $|M_1+\frac{H_{N-1}}{2F_{N-1}}|<|M_2+\frac{H_{N-1}}{2F_{N-1}}|$, then $u^*_{N-1}=M_2, G_{N-1}=M_2H_{N-1}+M_2^2F_{N-1}$.
Thus, we can obtain the optimal control as follows:

$$u^*_{N-1}=\begin{cases} M_1, & F_{N-1}=0 \text{ and } H_{N-1}<0, \text{ or } F_{N-1}<0 \text{ and } \frac{-H_{N-1}}{2F_{N-1}}<M_1,\\ & \text{or } F_{N-1}>0 \text{ and } |M_1+\frac{H_{N-1}}{2F_{N-1}}|\ge|M_2+\frac{H_{N-1}}{2F_{N-1}}|,\\ \frac{-H_{N-1}}{2F_{N-1}}, & F_{N-1}<0 \text{ and } M_1\le\frac{-H_{N-1}}{2F_{N-1}}\le M_2,\\ M_2, & F_{N-1}=0 \text{ and } H_{N-1}>0, \text{ or } F_{N-1}<0 \text{ and } \frac{-H_{N-1}}{2F_{N-1}}>M_2,\\ & \text{or } F_{N-1}>0 \text{ and } |M_1+\frac{-H_{N-1}}{2F_{N-1}}|<|M_2+\frac{-H_{N-1}}{2F_{N-1}}|,\\ undetermined, & F_{N-1}=0 \text{ and } H_{N-1}=0, \end{cases}$$

and optimal value

$$\begin{aligned}&J(x_{N-1},y_{N-1},\hat{y}_{N-1},N-1)\\ &=\left(1+\exp\left(\frac{\pi}{\sqrt{3}(\sigma_Nv_N+\chi_Nv_N^{'})}(\sigma_Ne_N+\chi_Ne_N^{'}-(H-\hat{y}_{N-1}-P_{N-1}x_{N-1}-Q_{N-1}y_{N-1}+G_{N-1}))\right)\right)^{-1},\end{aligned}$$

where

$$G_{N-1}=\begin{cases} M_1H_{N-1}+M_1^2F_{N-1}, & F_{N-1}=0 \text{ and } H_{N-1}<0, \text{ or } F_{N-1}<0 \text{ and } \frac{-H_{N-1}}{2F_{N-1}}<M_1,\\ & \text{or } F_{N-1}>0 \text{ and } |M_1+\frac{H_{N-1}}{2F_{N-1}}|\ge|M_2+\frac{H_{N-1}}{2F_{N-1}}|,\\ \frac{-H^2_{N-1}}{4F_{N-1}}, & F_{N-1}<0 \text{ and } M_1\le\frac{-H_{N-1}}{2F_{N-1}}\le M_2,\\ M_2H_{N-1}+M_2^2F_{N-1}, & F_{N-1}=0 \text{ and } H_{N-1}>0, \text{ or } F_{N-1}<0 \text{ and } \frac{-H_{N-1}}{2F_{N-1}}>M_2,\\ & \text{or } F_{N-1}>0 \text{ and } |M_1+\frac{H_{N-1}}{2F_{N-1}}|<|M_2+\frac{H_{N-1}}{2F_{N-1}}|,\\ 0, & F_{N-1}=0 \text{ and } H_{N-1}=0. \end{cases}$$

For $k=N-2$, by using the recursion equation (14), we derive

$$J(x_{N-2},y_{N-2},\hat{y}_{N-2},N-2)=\max_{u_{N-2}\in[M_1,M_2]}E[J(x(N-1),y(N-1),\hat{y}(N-1),N-1)],$$

where

$$\begin{aligned}&J(x(N-1),y(N-1),\hat{y}(N-1),N-1)\\ &=\Bigg(1+\exp\Bigg(\frac{\pi}{\sqrt{3}(\sigma_Nv_N+\chi_Nv_N^{'})}(\sigma_Ne_N+\chi_Ne_N^{'}-(H-\hat{y}_{N-2}-P_{N-1}a_{N-2}x_{N-1}-(1+Q_{N-1}c_{N-2})y_{N-1}\\ &\quad+(P_{N-1}b_{N-2}+Q_{N-1}d_{N-2})u_{N-2}+(P_{N-1}l_{N-2}+Q_{N-1}q_{N-2})u^2_{N-2}+G_{N-1}-P_{N-1}\sigma_{N-1}\xi_{N-1}\\ &\quad-Q_{N-1}\chi_{N-1}\eta_{N-1}))\Bigg)\Bigg)^{-1}.\end{aligned}$$

Set $P_{N-2}=P_{N-1}a_{N-2}$, $Q_{N-2}=1+Q_{N-1}c_{N-2}$, $H_{N-2}=P_{N-1}b_{N-2}+Q_{N-1}d_{N-2}$, $F_{N-2}=P_{N-1}l_{N-2}+Q_{N-1}q_{N-2}$, $L_{N-1}=P_{N-1}\sigma_{N-1}$ and $S_{N-1}=Q_{N-1}\chi_{N-1}$. The proof of Theorem 8 shows that $L_{N-1}\xi_{N-1}+S_{N-1}\eta_{N-1}$ is also a normal uncertain variable $\mathcal{N}_u(L_{N-1}e_{N-1}+S_{N-1}e^{'}_{N-1}, L_{N-1}v_{N-1}+S_{N-1}v^{'}_{N-1})$. We observe that $J(x(N-1), y(N-1), \hat{y}(N-1), N-1)$ is a strictly decreasing function with $L_{N-1}\xi_{N-1}+S_{N-1}\eta_{N-1}$. It follows from Theorem 2 that the inverse uncertainty distribution of $J(x(N-1), y(N-1), \hat{y}(N-1), N-1)$ is

$$\Psi^{-1}_{N-1}(\alpha_{N-1}) = \Bigg(1+\exp\Bigg(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v^{'}_N)}(\sigma_N e_N +_N e^{'}_N - (H-\hat{y}_{N-2} - P_{N-2}x_{N-1} - Q_{N-2}y_{N-1} + H_{N-2}u_{N-2} + F_{N-2}u^2_{N-2} + G_{N-1} - \Phi^{-1}_{N-1}(1-\alpha_{N-1})))\Bigg)\Bigg)^{-1},$$

where

$$\Phi^{-1}_{N-1}(1-\alpha_{N-1}) = L_{N-1}e_{N-1}+S_{N-1}e^{'}_{N-1} + \frac{(L_{N-1}v_{N-1}+S_{N-1}v^{'}_{N-1})\sqrt{3}}{\pi}\ln\frac{1-\alpha_{N-1}}{\alpha_{N-1}}, \quad 0<\alpha_{N-1}<1.$$

Thus, we have

$$\begin{aligned}
&J(x_{N-2}, y_{N-2}, N-2)\\
&= \max_{u_{N-2}\in[M_1,M_2]}\int_0^1 \Bigg(1+\exp\Bigg(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v^{'}_N)}(\sigma_N e_N+\chi_N e^{'}_N - (H-\hat{y}_{N-2}\\
&\quad - P_{N-2}x_{N-1} - Q_{N-2}y_{N-1} + H_{N-2}u_{N-2} + F_{N-2}u^2_{N-2} + G_{N-1} - \Phi^{-1}_{N-1}(1-\alpha_{N-1})))\Bigg)\Bigg)^{-1} d\alpha_{N-1}\\
&= \int_0^1 \Bigg(1+\exp\Bigg(\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v^{'}_N)}(\sigma_N e_N+\chi_N e^{'}_N - (H-\hat{y}_{N-2}\\
&\quad - P_{N-2}x_{N-1} - Q_{N-2}y_{N-1} + \max_{u_{N-2}\in[M_1,M_2]} H_{N-2}u_{N-2} + F_{N-2}u^2_{N-2} + G_{N-1} - \Phi^{-1}_{N-1}(1-\alpha_{N-1})))\Bigg)\Bigg)^{-1} d\alpha_{N-1}
\end{aligned}$$

The optimal solution u^*_{N-2} can be derived similarly to that in the case when $k=N-1$. Denoting that

$$\max_{u(N-1)\in[M_1,M_2]} H_{N-2}u_{N-2} + F_{N-2}u^2_{N-2} = H_{N-2}u^*_{N-2} + F_{N-2}(u^*_{N-2})^2 = G_{N-2},$$

we can obtain the optimal controls

$$u^*_{N-2} = \begin{cases} M_1, & F_{N-2}=0 \text{ and } H_{N-2}<0, \text{ or } F_{N-2}<0 \text{ and } \frac{-H_{N-2}}{2F_{N-2}}<M_1,\\ & \text{or } F_{N-2}>0 \text{ and } |M_1+\frac{H_{N-2}}{2F_{N-2}}| \geq |M_2+\frac{H_{N-2}}{2F_{N-2}}|,\\ \frac{-H_{N-2}}{F_{N-2}}, & F_{N-2}<0 \text{ and } M_1 \leq \frac{-H_{N-2}}{2F_{N-2}} \leq M_2,\\ M_2, & F_{N-2}=0 \text{ and } H_{N-2}>0, \text{ or } F_{N-2}<0 \text{ and } \frac{-H_{N-2}}{2F_{N-2}}>M_2,\\ & \text{or } F_{N-2}>0 \text{ and } |M_1+\frac{-H_{N-2}}{2F_{N-2}}| < |M_2+\frac{-H_{N-2}}{2F_{N-2}}|,\\ undetermined, & F_{N-2}=0 \text{ and } H_{N-2}=0, \end{cases}$$

and optimal value

$$\begin{aligned}J(x_{N-2}, y_{N-2}, N-2) = \int_0^1 \Bigg(1+\exp\Bigg(&\frac{\pi}{\sqrt{3}(\sigma_N v_N+\chi_N v^{'}_N)}(\sigma_N e_N+\chi_N e^{'}_N - (H-\hat{y}_{N-2}\\
&- P_{N-2}x_{N-2} - Q_{N-2}y_{N-2} + G_{N-2} + G_{N-1} - \Phi^{-1}_{N-1}(1-\alpha_{N-1})))\Bigg)\Bigg)^{-1} d\alpha_{N-1},\end{aligned}$$

where

$$G_{N-2}=\begin{cases} M_1H_{N-2}+M_1^2F_{N-2}, & F_{N-2}=0 \text{ and } H_{N-2}<0, \text{ or } F_{N-2}<0 \text{ and } \frac{-H_{N-2}}{2F_{N-2}}<M_1, \\ & \text{or } F_{N-2}>0 \text{ and } |M_1+\frac{H_{N-2}}{2F_{N-2}}|\geq |M_2+\frac{H_{N-2}}{2F_{N-2}}|, \\ \frac{-H_{N-2}^2}{4F_{N-2}}, & F_{N-2}<0 \text{ and } M_1\leq \frac{-H_{N-2}}{2F_{N-2}}\leq M_2, \\ M_2H_{N-2}+M_2^2F_{N-2}, & F_{N-2}=0 \text{ and } H_{N-2}>0, \text{ or } F_{N-2}<0 \text{ and } \frac{-H_{N-2}}{2F_{N-2}}>M_2, \\ & \text{or } F_{N-2}>0 \text{ and } |M_1+\frac{H_{N-2}}{2F_{N-2}}|<|M_2+\frac{H_{N-2}}{2F_{N-2}}|, \\ 0, & F_{N-2}=0 \text{ and } H_{N-2}=0. \end{cases}$$

By induction, we can get exact expressions of u_k^* and $J(x_k,y_k,\hat{y}_k,k)$. The theorem is thus verified.

In Theorems 8 and 9, the multi-stage reliability optimal control problems with linear objective function and quadratic state transition equation are proposed, respectively. We can obtain the exact expression of optimal solutions at each stage by recursion equation. For all $j=0,1,2,\cdots,N$, the following two algorithms can be applied to calculate the specific optimal controls and optimal values of uncertain random optimal control problems (15) and (21), respectively.

Algorithm 1 For problem (15) with linear controls

Step 1. Transform (15) to its subproblem (16) with same control variables.
Step 2. Calculate related parameters P_k,Q_k,H_k,L_k,S_k for $k=N-1,\cdots,1,0$ using (20) with $P_N=1,Q_N=1$.
Step 3. Calculate optimal controls using (18).
Step 4. For the initial states x_0,y_0, obtain all states through the following state equations:

$$x(j+1)=a_jx(j)-b_ju_j^*+\sigma_{j+1}\tilde{\xi}_{j+1},$$
$$y(j+1)=c_jy(j)-d_ju_j^*+\chi_{j+1}\tilde{\eta}_{j+1},$$

where $\tilde{\xi}_{j+1},\tilde{\eta}_{j+1}$ are real numbers such that $\xi_j\sim\mathcal{N}_u(e_j,v_j),\eta_j\sim\mathcal{N}_u(e_j^{'},v_j^{'})$.
Step 5. Calculate the optimal values of subproblem (16) by (19).
Step 6. Calculate the optimal values of problem (15) by (17)

Algorithm 2 For problem (21) with quadratic controls)

Step 1. Transform (21) to its subproblem (22) with same control variables.
Step 2. Calculate related parameters P_k,Q_k,H_k,F_k,L_k,S_k and G_k for $k=N-1,\cdots,1,0$ using (26) with $P_N=1,Q_N=1$.
Step 3. Calculate optimal controls using (24).
Step 4. For the initial states x_0,y_0, obtain all states through the following state equations:

$$x(j+1)=a_jx(j)-b_ju_j^*-l_j(u_j^*)^2+\sigma_{j+1}\tilde{\xi}_{j+1},$$
$$y(j+1)=c_jy(j)-d_ju_j^*-q_j(u_j^*)^2+\chi_{j+1}\tilde{\eta}_{j+1},$$

where $\tilde{\xi}_{j+1},\tilde{\eta}_{j+1}$ are real numbers such that $\xi_j\sim\mathcal{N}_u(e_j,v_j),\eta_j\sim\mathcal{N}_u(e_j^{'},v_j^{'})$.
Step 5. Calculate the optimal values of subproblem (22) by (25).
Step 6. Calculate the optimal values of problem (21) by (23).

Remark 1 *It can be seen from algorithms 1 and 2 that we calculate the optimal controls of each stage in reverse order from the last stage to the initial stage. When the initial state is given, the optimal values of all stages can be calculated from the initial stage to the last stage.*

The algorithms 1 and 2 allow us to solve specific optimal control problems (15) and (21), respectively. and obtain the optimal controls of maintenance fund and optimal belief reliability index. As a result of derived results, decisions are made by a decision maker to allocate maintenance funds for maximizing a system's reliability. The analytic solutions of two special examples are derived by recursive equations, which may not exist when generalized to other forms. It is therefore possible for future studies to find another effective numerical solutions.

5 Numerical Experiment

Compared with ordinary foil capacitor, a metallized film pulse capacitor processes the characteristics of high reliability, long life and soft failure. The decrease of capacitance value of a metallized film capacitor is caused by natural dielectric loss and sudden dielectric loss. The natural loss occurs randomly at any time, and the range of variation is relatively minor. The sudden loss can be explained as loosening or falling off of the gold spray layer connection at the end of the capacitor element, or the large area self-healing due to the breakdown of an organic film.

This paper selects a metallized film pulse capacitor as the research object, and briefly proposes competing failure model based on uncertain random degradations and shocks. Suppose that $z(j)$ are i.i.d random sudden shocks at stage j, $j=0,1,\cdots,N$, in terms of the arriving moment of shocks whose probability distribution is shown in Table 1. To optimize belief reliability index of a multi-stage system, it is necessary to allocate maintenance fund $u_j \in [0,1]$, that is, one million dollars maintenance fund is available for allocation at each stage,

$$\begin{aligned} h_1(u_j) &= b_j u_j + l_j u_j^2, \quad j=0,1,\cdots,N-1, \\ h_2(u_j) &= d_j u_j + q_j u_j^2, \quad j=0,1,\cdots,N-1, \end{aligned}$$

which reflects the reduction of wear degradations and additional degradations after investing the maintenance control fund u_j (one million dollars). The effectiveness of preventative maintenance depends on technology and how much investment is made in maintenance control. The coefficients b_j, l_j, d_j and q_j are varied which are assumed and shown in Tables 2. We define a wear process $x(j)$ and additional degradation process $y(j)$ as follows,

$$\begin{aligned} x(j+1) &= x(j) - b_j u_j - l_j u_j^2 + \sigma_{j+1}\xi_{j+1}, \quad j=0,1,\cdots,N-1, \\ y(j+1) &= y(j) - d_j u_j - q_j u_j^2 + \chi_{j+1}\eta_{j+1}, \quad j=0,1,\cdots,N-1, \end{aligned} \tag{28}$$

where the coefficients σ_j, χ_j are shown in Table 2, ξ_j and η_j are uncertain disturbances whose uncertainty distributions are shown in Table 1 in order to indicate how much the wear and additional degradations have fluctuated, respectively. By controlling the maintenance funds, we expect to maximize belief reliability index of a capacitor under the condition that both degradations and sudden shocks do not exceed their respective thresholds. Based on system (28), a maximizing system reliability problem with N stages is established as follows:

$$\begin{cases} R(x_0, y_0, 0) = \max\limits_{\substack{u_j \in [0,1] \\ 0 \le j \le N-1}} \mathrm{Ch}\left\{ x(N) + \sum\limits_{j=0}^{N} y(j) < H, \bigcap\limits_{j=0}^{N} z(j) < D \right\} \\ \text{subject to} \\ \quad x(j+1) = x(j) - b_j u_j - l_j u_j^2 + \sigma_{j+1}\xi_{j+1}, \\ \quad y(j+1) = y(j) - d_j u_j - q_j u_j^2 + \chi_{j+1}\eta_{j+1}, \\ \quad j=0,1,\cdots,N-1 \quad \text{and} \quad x(0)=x_0, y(0)=y_0, \end{cases} \tag{29}$$

where H, D, N and other relevant parameters are shown in Table 1. Algorithm 2 show the method that problem

(29) can be transformed into an subproblem with same optimal controls,

$$\begin{cases} J(x_0,y_0,\hat{y}_0,0)=\max\limits_{\substack{u_j\in[0,1]\\0\le j\le N-1}} E[I(\hat{y}(N)<H)] \\ \text{subject to} \\ \quad x(j+1)=a_jx(j)-b_ju_j-l_ju_j^2+\sigma_{j+1}\xi_{j+1}, \quad j=0,\cdots,N-1, \\ \quad y(j+1)=c_jy(j)-d_ju_j-q_ju_j^2+\chi_{j+1}\eta_{j+1}, \quad j=0,\cdots,N-1, \\ \quad \hat{y}(j+1)=\hat{y}(j)+y(j), \quad j=0,\cdots,N-2, \\ \quad \hat{y}(N)=\hat{y}(N-1)+y(N-1)+x(N)+y(N), \\ \quad x(0)=x_0, \quad y(0)=y_0, \quad \hat{y}(0)=0. \end{cases} \tag{30}$$

Table 1: Parameter values.

Parameters	Values	Sources
H	$2.8\mu F$	[31]
D	$2\mu F$	[31]
N	10	Assumption
x_0	$0.1\mu F$	Assumption
y_0	$0.08\mu F$	Assumption
ξ_j	$\mathcal{N}_u(0.25,1.5)$	Assumption
η_j	$\mathcal{N}_u(0.2,2.5)$	Assumption
$z(j)$	$\mathcal{N}_r(0.5,0.1)$	[31]

Table 2: Parameter values.

Stage j	b_j	d_j	l_j	q_j	σ_j	χ_j
0	0.01	0.04	-0.11	-0.08	-	-
1	0.02	0.05	-0.10	-0.08	0.11	0.12
2	0.03	0.06	-0.09	-0.07	0.12	0.13
3	0.04	0.07	-0.08	-0.07	0.13	0.13
4	0.05	0.08	-0.07	-0.07	0.13	0.14
5	0.06	0.09	-0.06	-0.06	0.14	0.15
6	0.07	0.10	-0.05	-0.06	0.14	0.16
7	0.08	0.11	-0.04	-0.06	0.15	0.17
8	0.09	0.12	-0.03	-0.06	0.16	0.18
9	0.10	0.13	-0.02	-0.06	0.17	0.19
10	$-$	$-$	$-$	$-$	0.17	0.20

As shown in Table 3, J and R represent the optimal value of subproblem (30) and optimal reliability index of problem (29) with optimal controls u_j^*, respectively. J' and R' represent the original value and original reliability index without controls u_j, i.e. $u_j=0$, respectively. Theorem 9 provides recursive equations for calculating the optimal reliability index. We may determine that optimal reliability index of a capacitor is 0.4104 and original reliability index is 0.2317 within $N=9$ stages. It can be concluded that optimal reliability index R is greater than its original reliability index R', which confirms that our optimization method can maximize the reliability of a capacitor. Figure 1 illustrates that the fluctuation of u_j^* which may guide us to choose the optimal maintenance fund at each stage for system optimization. As the capacitor continues to work, we believe that improving maintenance funds can better maximize system reliability. If $u_j^*=1$, it implies that all of maintenance funds should be put into usage to ensure the reliability of a capacitor. Figure 2 shows that wear degradation $x(j)$ with optimal controls is less than its original wear degradation without controls when $j=5,6,\cdots,10$. Figure 3 also indicates that addition degradations $y(j)$ with optimal controls are always less than its original addition degradations without controls. Numerical results also demonstrate that optimal control methods can effectively maximize the system reliability index.

The reliability index is maximized by decreasing degradation processes through maintenance funds in this paper. Besides, it is clear from Definition 4 that reliability index R is highly correlated with failure threshold values H and D, Accordingly, the sensitivity analysis of optimal reliability index on failure thresholds is worth studying, respectively. Figure 4 shows the sensitivity analysis of optimal reliability index R on soft failure threshold H, and we may observe that reliability index R is positively correlated with H. Figure 5 shows the sensitivity analysis of optimal reliability index R on hard failure threshold D, and it is observed that reliability index R is also positively correlated with D. It is recommended that attention be paid to increasing failure thresholds H,W of a system thereby improving system reliability. Materials that maintain high temperature and pressure resistance or corrosion resistance may be applied to raise the threshold values of a system.

Table 3: Controls and reliability index.

Stage j	Optimal control u_j^*	J	R	Control u_j	J'	R'
0	0.2253	0.8832	0.4104	0	0.6328	0.2317
1	0.2866			0		
2	0.3923			0		
3	0.4649			0		
4	0.5408			0		
5	0.7083			0		
6	0.8103			0		
7	0.9318			0		
8	1			0		
9	1			0		

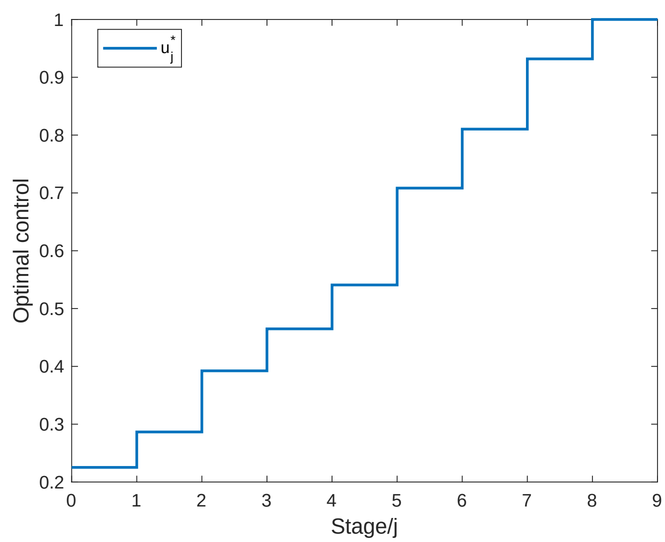

Figure 1: The optimal maintenance fund strategy.

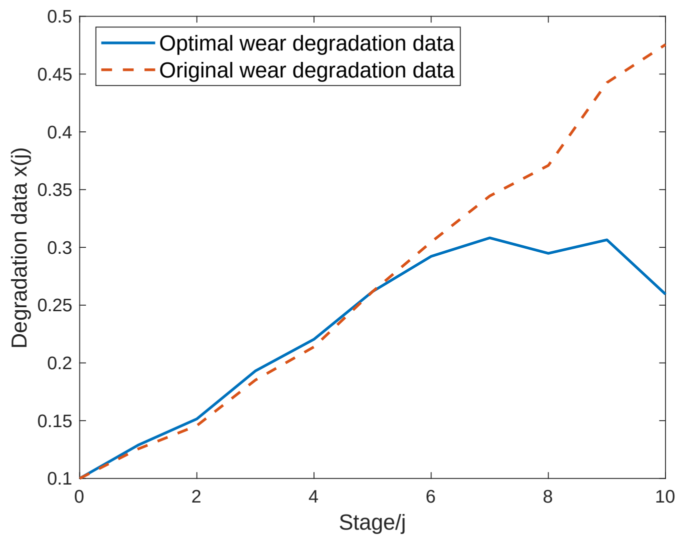

Figure 2: The optimal wear degradation data and original wear degradation data.

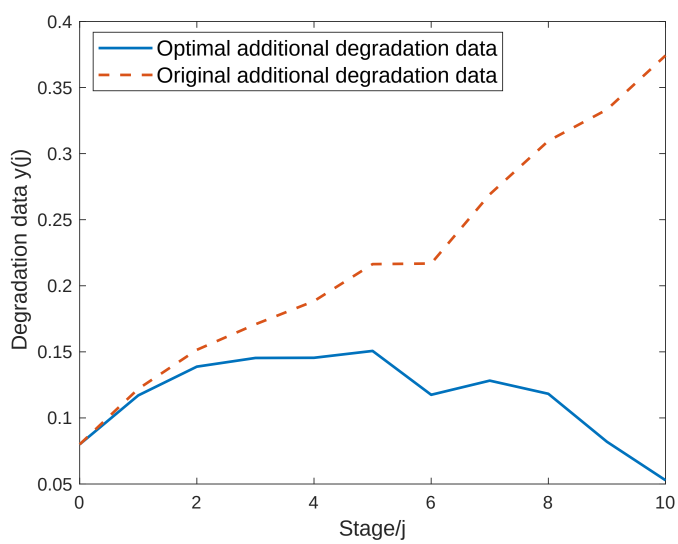

Figure 3: The optimal additional degradation data and original additional degradation data.

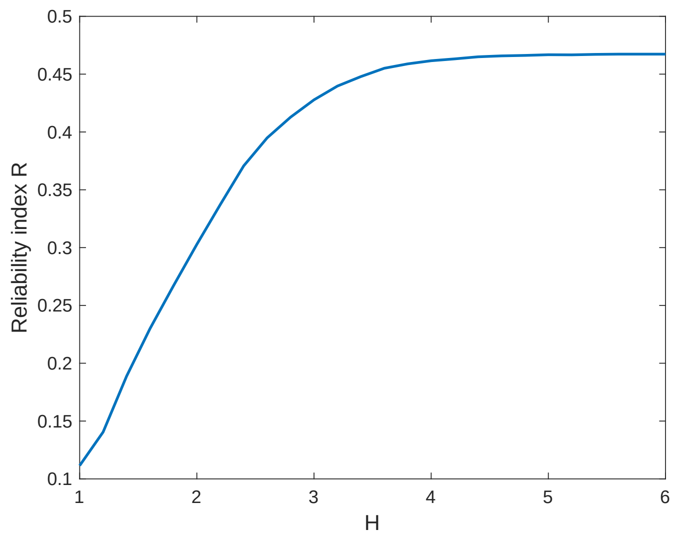

Figure 4: Sensitivity analysis of optimal reliability index R on H.

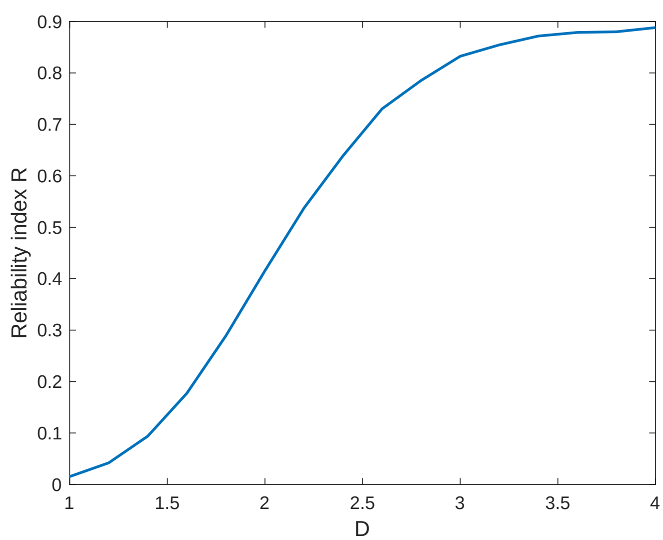

Figure 5: Sensitivity analysis of optimal reliability index R on D.

6 Conclusion

In this paper, we presented two types of maximizing reliability problems for multi-stage uncertain random systems when each stage is driven by competitive failure processes. The degradation processes were driven by uncertain difference equations, and shock processes were represented by random variables. Based on recursion equations, the exact solutions of maximizing reliability problems were presented in two special cases involving uncertain linear and quadratic systems. Finally, we gave an example of a metallized film pulse capacitor for the purpose of illustrating proposed methods. Analyzing the numerical results revealed that the optimal maintenance fund strategy is capable of maximizing system reliability. The sensitivity analysis of system reliability index with respect to H and H was conducted, and it was concluded that increasing failure thresholds may maximize the system reliability. Combining reliability problems with optimal control problems, we are able to come up with analytical solutions for linear and quadratic special systems. Taking advantage of the fact that there are analytical solutions for these two types of examples, we may illustrate the method of applying recursive equations to obtain the optimal reliability index. The analytical solution may not exist if it is generalized to other forms. Thus, it is possible to extend this work to other forms and investigate suitable numerical algorithms to obtain numerical solutions in the future.

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References

[1] J. Kharoufeh and S. Cos, Stochastic models for degradation-based reliability, *IIE Transactions*, **37** (6) (2005) 533–542.

[2] Z. Pan and N. Balakrishnan, Reliability modeling of degradation of products with multiple performance characteristics based on gamma processes, *Reliability Engineering & System Safety*, **96** (8) (2011) 949–957.

[3] G. Jin and D. Matthews, Reliability demonstration for long-Life products based on degradation testing and a wiener process model, *IEEE Transactions on Reliability*, **63** (3) (2014) 781–797.

[4] T. Nakagawa, *Shock and Damage Models in Reliability Theory*, Springer-Verlag: New York, (2007).

[5] Y. Chien, S. Sheu, Z. Zhang, et al., An extended optimal replacement model of systems subject to shocks, *European Journal of Operational Research*, **175** (1) (2006) 399–412.

[6] J. Bai, Z. Li and X. Kong, Generalized shock models based on a cluster point process, *IEEE Transactions on Reliability*, **55** (2006) 542–550.

[7] Z. Ye, L. Tang and H. Xu, A distribution-based systems reliability model under extreme shocks and natural degradation, *IEEE Transactions on Reliability*, **60** (1) (2011) 246–256.

[8] Z. Wang, H. Huang and Y. Li, An approach to reliability assessment under degradation and shock process, *IEEE Transactions on Reliability*, **60** (4) (2011) 852–863.

[9] L. Jiang, Q. Feng and D. Coit, Reliability and maintenance modeling for dependent competing failure processes with shifting failure thresholds, *IEEE Transactions on Reliability*, **61** (4) (2012) 932–948.

[10] M. Fan, Z. Zeng and E. Zio, et al, Modeling dependent competing failure processes with degradation-shock dependence, *Reliability Engineering & System Safety*, 2017, **61** (sep.) (2017) 422–430.

[11] B. Liu, Uncertain risk analysis and uncertain reliability analysis, *Journal of Uncertain System*, **4** (3) (2010) 163–170.

[12] B. Li, Y. Zhu, Y. Sun, et al, Deterministic conversion of uncertain manpower planning optimization problem, *IEEE Transactions on Fuzzy Systems*, **26** (5) (2018), 2748–2757.

[13] Y. Liu and D. Ralesc, Risk index in uncertain random risk analysis, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, **22** (4) (2014) 491–504.

[14] Q. Xu, Y. Zhu and Q. Lu, Optimizing reliability of linear fractional difference systems under uncertainty and randomness, *Fractals*, **29** (8) (2021) 2140031.

[15] Y. Zhu, Uncertain fractional differential equations and an interest rate model, *Mathematical Methods in the Applied Sciences*, **38** (2015) 3359–3368.

[16] B. Liu, *Uncertainty Theory*, 2nd edition, Berlin: Springer-Verlag, (2007).

[17] B. Liu, *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, (2010).

[18] Z. Zeng, M. Wen, and R. Kang, Belief reliability: a new metrics for products' reliability, *Fuzzy Optimization and Decision Making*, **12** (1) (2013) 15–27.

[19] Z. Zeng, R. Kang, M. Wen, et al., Uncertainty theory as a basis for belief reliability, *Information Sciences*, **429** (2018) 26–36.

[20] Y. Liu, Uncertain random variables: a mixture of uncertainty and randomness, *Soft Computing*, **17** (4) (2013) 625–634.

[21] Y. Liu, Uncertain random programming with applications, *Fuzzy Optimization and Decision Making*, **12** (2) (2013) 153–169.

[22] M. Wen and R. Kang, Reliability analysis in uncertain random system, *Fuzzy Optimization and Decision Making*, **15** (2016) 491–506.

[23] Q. Zhang, R. Kang and M. Wen, Belief reliability for uncertain random systems, *IEEE Transactions on Fuzzy Systems*, **26** (6) (2018) 3605–3614.

[24] R. Bellman, *Dynamic Programming*, New Jersey: Princeton University Press, (1957).

[25] R. Bellman, Dynamic programming and stochastic control process, *Information and Control*, **1** (3) (1958), 228–239.

[26] Y. Zhu, Fuzzy optimal control for multi-stage fuzzy systems, *IEEE Transactions on Systems, Man & Cybernetics: Part B,* **41** (4) (2011) 964–975.

[27] Y. Kang and Y. Zhu, Bang-Bang optimal control for multi-stage uncertain systems, *Information: An International Interdisciplinary Journal*, **15** (8) (2012) 3229–3237.

[28] H. Yan, Y. Sun and Y. Zhu, A linear-quadratic control problem of uncertain discrete-time switched systems, *Journal of Industrial and Management Optimization*, **13** (1) (2017) 267–282.

[29] Y, Chen and Y. Zhu, Indefinite LQ optimal control for discrete-time uncertain systems, *Soft Computing*, **24** (1), (2020) 267–279.

[30] L. Sheng, Y. Zhu and K. Wang, Analysis of a class of dynamiy programming models for multistage uncertain systems, *Applied Mathematical Modelling*, **86**, (2020) 446–459.

[31] Z. Sun, Y. Liang, et al., Reliability evaluation of related competitive failures based on stochastic shocks, *Fire Control & Command Control*, **4** (6) (2015) 132–135.