Stability and Finite/Fixed-Time Attractivity of Time-Delayed Filippov System: Application to Switched Neural Networks*

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Abstract

This paper studies different kinds of stability and finite/fixed-time attractivity problems for time-delayed Filippov system (TDFS) via time-delayed differential inclusions(TDDI). A series of novel criteria concerning basic stability and finite/fixed-time attractivity for TDFS are established by employing indefinite Lyapunov method. As an application, the problems of stabilization and attractivity in finite/fixed time are explored for delayed switched neural networks (DSNNs), where four control protocols are developed. Furthermore, several concrete examples are given to demonstrate the effectiveness and advantages of the main results.

Keywords: Stability; Finite-time attractivity (FNTA); Fixed-time attractivity (FXTA); Time-delayed Filippov system (TDFS); Differential inclusion (DI).

1 Introduction

Time-delayed Filippov system (TDFS) described by time-delayed differential equation possessing discontinuity is of practical significance. In 1964, based on set-valued maps approach, Filippov developed the differential inclusion theory to deal with the solution of discontinuous differential equation [1]. After that, Aubin and Cellina extended time-delayed differential inclusion (TDDI) to handle TDFS [2]. It should be pointed out that a suddenly change of system's state will occur because of uncertainties. Fortunately, TDDI can also deal with time-delayed dynamic system containing uncertainties very well. Nowadays, more and more scholars have carried out research on TDFS and TDDI [3-9]. Especially, the stability of TDFS/TDDI becomes one of the most fundamental and interesting research focuses. In [4], Surkov studied the stability of TDDI via Lyapunov function. In [5], the stability properties of TDDI were analyzed by Lyapunov functional method. In [6] and [7], the stability of TDDI/TDFS was investigated by utilizing Lyapunov-Krasovskii functional method. In [8], novel Lyapunov-Razumikhin approach was established to analyze the stability of TDDI. In [9], the asymptotic stability of TDDI was discussed via generalized Halanay's inequality.

On the other hand, finite-time attractivity (FNTA) and fixed-time attractivity (FXTA) have attracted extensive concern due to their potential applications in neural networks, aerospace technology, multiagent systems and so on. A main characteristic of FNTA is that the system's states converge to zero in finite-time and then stays there. In [10], the FNTA concept was defined for no-Lipschitz systems. In [11], the FNTA was extended to differential inclusion (DI). In [12], the FNTA was developed for time-delay system. In [13], the FNTA was discussed for impulsive systems. In [14], the concept of FNTA

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was introduced for stochastic systems. FXTA is a further extension of FNTA, and its settling-time is independent of any initial states. In 2012, Polyakov gave the definition of FXTA and presented its application to single input system [15]. Subsequently, based on FXTA, fixed-time stability and fixed-time control problems have been widely studied [16-20]. Whereas, there is still little research on stability and FNTA/FXTA analysis for TDFS based on TDDI. Moreover, most of the existing stability work for TDFS/TDDI is based on the Lyapunov method possessing negative definite derivatives. This makes the suitable Lyapunov function isn't easy to be constructed. For this reason, this paper uses indefinite derivative Lyapunov function method to investigate the issues of stability and FNTA/FXTA for TDFS via the framework of TDDI. At present, only a few papers have studied the fixed-time control problem by using lyapunov method possessing indefinite derivatives [21-24]. Thus, some more flexible FXTA criteria should be further established for TDFS, where the more diverse settling-time should also be estimated accurately.

Delayed switched neural networks (DSNNs) have many practical applications in image processing, communication secure, pattern tracking, etc. Because of time-delay and switching properties of neuron connection weights, DSNNs display very complex dynamics such as instability, oscillation, chaos, periodicity, bifurcation and sliding mode. Up to now, the research on various dynamics and control of DSNNs has been reported. In [25], lag synchronization of DSNNs was discussed via output controller. In [26], the DSNNs system was synchronized by event-triggered control. In [27], new ψ-type synchronization criteria were established for DSNNs. In [28], the multi-periodicity of DSNNs was studied. In [29], the finite/fixed-time synchronization problems of coupled DSNNs were analyzed. It should be noted that the research concerning the FNTA/FXTA of DSNNs is still rare and further studies are expected.

2 Preliminaries

A set including all nonnegative numbers is marked as $\mathbb{R}_+$. Any norm of $x \in \mathbb{R}^n$ is represented as $\|x\|$. Given function ψ, ψ^{-1} denotes the inverse function of ψ. Given $a, b \in \mathbb{R}$, $a \vee b$ means the maximum of a and b. Consider the non-autonomous TDFS:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x(t), x(t-\tau(t))), \tag{1}$$

where the state $x \in \mathbb{R}^n$, the time-varying delay $\tau(t) \leq \tau < +\infty$, the essentially locally bounded function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable. The initial state is provided as $x = \varphi \in C([t_0 - \tau, t_0], \mathbb{R}^n)$ and $t = t_0 \geq 0$ is initial time.

Definition 1 ([9]) The Filippov solution $x(t)$ is defined on $[0, T)$ for TDFS (1), if it has absolute continuity and

$$\frac{\mathrm{d}x}{\mathrm{d}t} \in F(t, x(t), x(t-\tau(t))), \tag{2}$$

where

$$F\big(t, x(t), x(t-\tau(t))\big) = \bigcap_{\rho_1>0,\rho_2>0} \ \bigcap_{\mu(\mathbb{N})=0,\mu(\mathbb{M})=0} \overline{co}\left[f\big(t, \mathcal{B}(x(t),\rho_1)\backslash\mathbb{N}, \mathcal{B}(x(t-\tau(t)),\rho_2)\backslash\mathbb{M}\big)\right]. \tag{3}$$

Here $\mu(\mathbb{N})$ and $\mu(\mathbb{M})$ denote Lebesgue measure sets; For $i = 1, 2$, $\mathcal{B}(x, \rho_i)$ denotes a ball with radius ρ_i at the center x; $\overline{co}$ means a closed convex hull is taken.

Definition 2 $\forall t \in \mathbb{R}$, if $0 \in F(t, 0, 0)$, then the origin $x = 0$ is defined for TDFS (1) via Filippov solution.

Definition 3 Let $C = C([t_0 - \tau, t_0], \mathbb{R}^n)$ be a Banach space possessing norm $\|\varphi\|_C = \sup\limits_{t_0-\tau\leq s\leq t_0} \|\varphi(s)\|$, which is composed of all continuous functions $\varphi : [t_0 - \tau, t_0] \to \mathbb{R}^n$. The origin of TDFS (1) is called

- stable if $\forall\varepsilon > 0$, $\forall t_0 \geq 0$, there exists $0 < \delta = \delta(\varepsilon, t_0)$ such that (s.t.) $\|x(t_0, \varphi)(t)\| < \varepsilon$ holds for all $\varphi \in \mathcal{B}(0, \delta) = \{\varphi \in C : \delta > \|\varphi\|_C\}$ and $t_0 \leq t$; if δ does not depend on t_0, then TDFS (1) is uniformly stable at origin;
- attractive if $\forall t_0 \geq 0$, there exists a $0 < \delta = \delta(t_0)$ s.t. $\lim\limits_{t\to+\infty} \|x(t_0, \varphi)(t)\| = 0$ holds for all $\varphi \in \mathcal{B}(0, \delta)$, i.e., $\forall\varepsilon > 0$, $\forall t_0 \in \mathbb{R}_+$ there exists $0 < \delta = \delta(t_0)$ and $\exists T = T(\varepsilon, t_0, \varphi) > 0$ s.t. $\|x(t_0, \varphi)(t)\| < \varepsilon$ holds for all $\varphi \in \mathcal{B}(0, \delta)$ and $t \geq t_0 + T$; if T does not depend on t_0 and φ, then TDFS (1) is uniformly attractive at origin; if δ can be arbitrarily large, then TDFS (1) is further called globally (uniformly) attractive at origin;
- (uniformly) asymptotically stable if it is (uniformly) stable and (uniformly) attractive;
- exponential stable if $\forall\varepsilon \in \mathbb{R}_+\backslash\{0\}$, $\exists\ell > 0$, $\exists\delta = \delta(\varepsilon, t_0) > 0$, s.t. $\|x(t_0, \varphi)(t)\| \leq \varepsilon \mathrm{e}^{-\ell(t-t_0)}$ holds for all $\varphi \in \mathcal{B}(0, \delta)$ and $t \geq t_0$;
- globally exponential stable if $\forall\delta > 0$, $\exists\ell > 0$, $\forall t_0 \in \mathbb{R}_+$, $\exists M(\delta) > 0$, s.t. $\|x(t_0, \varphi)(t)\| \leq M(\delta)\mathrm{e}^{-\ell(t-t_0)}$ holds for all $\varphi \in \mathcal{B}(0, \delta)$ and $t \geq t_0$.

Definition 4 The Filippov solution $x(t_0, \varphi)(t)$ of TDFS (1) is called bounded if $\forall(t_0, \varphi) \in \mathbb{R}_+ \times C$, there exists $M = M(t_0, \varphi) > 0$, s.t. $\|x(t_0, \varphi)(t)\| \leq M(t_0, \varphi)$, for all $t \geq t_0 - \tau$.

Definition 5 If $\forall\delta > 0$, there exists $M = M(\delta) > 0$, s.t. $\|x(t_0, \varphi)(t)\| \leq M$ holds for all $\|\varphi\|_C < \delta$ and $t \geq t_0$, then all Filippov solutions of TDFS (1) are said to be uniformly bounded.

Definition 6 TDFS (1) is called finite-time attractive (FNTA) at origin, if there exists $0 \leq T(t_0, \varphi) < +\infty$ s.t. $\lim\limits_{t\to T(t_0,\varphi)} x(t_0, \varphi)(t) = 0$ and $x(t_0, \varphi)(t) \equiv 0$ for all $t \geq T(t_0, \varphi)$. Here $T(t_0, \varphi)$ is named settling-time (S-T).

Definition 7 TDFS (1) is called fixed-time attractive (FXTA) at origin, if it is FNTA and its S-T $T(t_0, \varphi)$ is bounded with regard to φ, i.e., there exists a constant $T_{\max} > 0$ s.t. $T(t_0, \varphi) \leq t_0 + T_{\max}$, for any $\varphi \in C([t_0 - \tau, t_0], \mathbb{R}^n)$.

Definition 8 $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a KR-function (i.e., $\psi \in KR$), if it possesses strictly increasing continuity with $\psi(0) = 0$ and $\lim\limits_{r\to+\infty} \psi(r) = +\infty$.

Definition 9 ([30]) If $V(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous(LLC), then for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, a Clarke's generalized gradient of V is defined as

$$\partial V(t, x) = \overline{co}[\lim_{k\to\infty} \nabla V(t_k, x_k) : (t_k, x_k) \to (t, x), (t_k, x_k) \notin \mathbb{N} \cup \Omega].$$

Here the points' set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ means V fails to be differentiable and the arbitrary set $\mathbb{N} \subset \mathbb{R} \times \mathbb{R}^n$ has measure zero.

Definition 10 (C-regularity [31-33]) $V : \mathbb{R}^n \to \mathbb{R}$ is called C-regular $\Leftrightarrow$ V satisfies

(1) regularity;

(2) $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$;

(3) $V(x) \to +\infty$ when $\|x\| \to +\infty$.

Let $\partial_t V(t, x)$ be the Clarke generalized gradient of $V(t, x)$ at t and $\partial_x V(t, x)$ denote the Clarke generalized gradient of $V(t, x)$ at x. Similar to chain rule of [30], the following lemma can be obtained for TDFS (1).

Lemma 1 ([30]) If $x(t)$ is the Filippov solution of TDFS (1) for $t \in \mathbb{U}$ and $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is C-regular, then $x(t)$ and $V(t, x(t))$ are differentiable for a.e. $t \in \mathbb{U}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t, x(t))|_{(1)} = \eta + \zeta^{\mathrm{T}}\gamma(t),\ \forall \eta \in \partial_t V(t, x) \text{ and } \zeta \in \partial_x V(t, x),$$

where $\gamma(t) \in F(t, x(t), x(t-\tau(t)))$ is measurable and satisfies $\dot{x}(t) = \gamma(t)$, for a.e. $t \in \mathbb{U}$.

Lemma 2 ([34]) Set $\phi_i \geq 0$ for $i = 1, 2, \ldots, n$ and $0 < \mathbb{p} < \mathbb{q}$, then

$$n^{\frac{1}{\mathbb{p}}-\frac{1}{\mathbb{q}}}\left(\sum_{i=1}^{n}\phi_i^{\mathbb{q}}\right)^{1/\mathbb{q}} \geq \left(\sum_{i=1}^{n}\phi_i^{\mathbb{p}}\right)^{1/\mathbb{p}} \geq \left(\sum_{i=1}^{n}\phi_i^{\mathbb{q}}\right)^{1/\mathbb{q}}.$$

3 Main Results

Before giving the main results, we assume that the Filippov solution $x(t_0, \varphi)(t)$ of TDFS (1) exists on$[t_0 - \tau, +\infty)$ under initial-value $(t_0, \varphi) \in \mathbb{R}_+ \times C([t_0 - \tau, t_0], \mathbb{R}^n)$. Sometimes we use the abbreviation $x(t) = x(t_0, \varphi)(t)$. If there is no particular emphasis, we always make the following fundamental assumption:

(A0) $0 \in F(t, 0, 0)$, $\forall t \in \mathbb{R}$.

The following assumptions are needed.

(A1) There exists a continuous function $\Phi(t) : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\int_0^{+\infty} \Phi^+(s)\mathrm{d}s < +\infty$, where $\Phi^+(s) = \Phi(s) \vee 0$.

(A2) There exists a continuous function $\Phi(t) : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\Phi^* \overset{\text{def}}{=} \int_0^{+\infty} \Phi^+(s)\mathrm{d}s < +\infty$, and there exist constants $\ell > 0$ and $N \geq 0$ s.t. for any $t \geq t_0$,

$$\int_{t_0}^{t} \Phi^-(s)\mathrm{d}s \geq \ell(t - t_0) - N, \tag{4}$$

where $\Phi^+(s) = \Phi(s) \vee 0$ and $\Phi^-(s) = [-\Phi(s)] \vee 0$.

Theorem 1 Suppose that $\psi_1 \in KR$ and the assumption (A1) holds. If there exists a LLC function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t, 0) = 0$ for any $t \in \mathbb{R}$, s.t.

(A3) $V(t, x) \geq \psi_1(\|x\|)$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$;

(A4) for any $\eta \in \partial_t V(t, x)$ and $\zeta \in \partial_x V(t, x)$,

$$\eta + \zeta^{\mathrm{T}}\gamma(t) \leq \Phi(t)V(t, x),\ \text{for a.e. } t \in [t_0, +\infty),$$

where $\gamma(t) \in F(t, x(t), x(t - \tau(t)))$ is a measurable function and satisfies

$$\dot{x}(t) = \gamma(t),\ \text{for a.e. } t \in [t_0, +\infty),$$

then the origin of TDFS (1) is stable.

Proof. By virtue of Lemma 1, it yields from Condition (A4) that

$$\frac{\mathrm{d}V(t, x(t_0, \varphi)(t))}{\mathrm{d}t} = \eta + \zeta^{\mathrm{T}}\gamma(t) \leq \Phi(t)V(t, x(t_0, \varphi)(t)), \text{for a.e. } t \in [t_0, +\infty). \tag{5}$$

Multiplying both sides of (5) by $\mathrm{e}^{-\int_{t_0}^{t}\Phi(s)\mathrm{d}s}$, it has

$$\mathrm{e}^{-\int_{t_0}^{t}\Phi(s)\mathrm{d}s}\frac{\mathrm{d}V(t, x(t_0, \varphi)(t))}{\mathrm{d}t} \leq \Phi(t)V(t, x(t_0, \varphi)(t))\mathrm{e}^{-\int_{t_0}^{t}\Phi(s)\mathrm{d}s},\ \text{for a.e. } t \geq t_0.$$

By integrating between t_0 to t, it leads to

$$V(t, x(t_0, \varphi)(t)) \le V(t_0, x(t_0, \varphi)(t_0))\mathrm{e}^{\int_{t_0}^{t} \Phi(s)\mathrm{d}s}, \text{ for any } t \ge t_0.$$

Because $\Phi(s) \le \Phi^+(s)$, we can derive from above inequality that

$$\begin{aligned} V(t, x(t_0, \varphi)(t)) &\le V(t_0, x(t_0, \varphi)(t_0))\mathrm{e}^{\int_{t_0}^{t} \Phi(s)\mathrm{d}s} = V(t_0, \varphi(t_0))\mathrm{e}^{\int_{t_0}^{t} \Phi(s)\mathrm{d}s} \\ &\le V(t_0, \varphi(t_0))\mathrm{e}^{\int_0^{+\infty} \Phi^+(s)\mathrm{d}s}. \end{aligned} \tag{6}$$

Because $\int_0^{+\infty} \Phi^+(s)\mathrm{d}s < +\infty$, we denote $\mathscr{C} = \mathrm{e}^{\int_0^{+\infty} \Phi^+(s)\mathrm{d}s}$. Obviously, $\mathscr{C} > 0$ is a constant. Notice that $V(t_0, x)$ is continuous at x and $V(t_0, 0) = 0$. Then, $\forall \varepsilon > 0$, $\forall t_0 \in \mathbb{R}_+$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ s.t. $\forall \varphi \in \mathcal{B}(0, \delta) = \{\varphi \in C : \|\varphi\|_C < \delta\}$, it implies that $V(t_0, \varphi(t_0)) < \frac{\psi_1(\varepsilon)}{\mathscr{C}}$. Using Condition (A3), it implies from (6) that

$$\|x(t_0, \varphi)(t)\| \le \psi_1^{-1}\left(V(t_0, \varphi(t_0))\mathscr{C}\right) < \psi_1^{-1}\left(\frac{\psi_1(\varepsilon)}{\mathscr{C}}\mathscr{C}\right) = \varepsilon. \tag{7}$$

This tells us that TDFS (1) is stable at origin.

Remark 1 In Theorem 1, if we replace Condition (A3) with following condition (A5) where the Lyapunov function has an infinitesimal upper limit, then TDFS (1) is uniformly stable at origin.

(A5) $\psi_2(\|x\|) \ge V(t, x) \ge \psi_1(\|x\|)$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$, here ψ_1, $\psi_2 \in KR$.

Actually, we can derive from (6) and the condition (A5) that

$$\begin{aligned} V(t, x(t_0, \varphi)(t)) &\le V(t_0, \varphi(t_0))\mathrm{e}^{\int_0^{+\infty} \Phi^+(s)\mathrm{d}s} \le \psi_2(\|\varphi(t_0)\|)\mathrm{e}^{\int_0^{+\infty} \Phi^+(s)\mathrm{d}s} \\ &\le \psi_2(\|\varphi\|_C)\mathscr{C}. \end{aligned} \tag{8}$$

This leads to

$$\|x(t_0, \varphi)(t)\| \le \psi_1^{-1}\left(\psi_2(\|\varphi\|_C)\mathscr{C}\right). \tag{9}$$

Thus, $\forall \varepsilon > 0$, there exists $\delta = \psi_2^{-1}\left(\frac{\psi_1(\varepsilon)}{\mathscr{C}}\right) > 0$ which is independent of t_0, s.t. $\|x(t_0, \varphi)(t)\| < \varepsilon$ holds for all $\varphi \in \mathcal{B}(0, \delta) = \{\varphi \in C : \|\varphi\|_C < \delta\}$. This shows that TDFS (1) is uniformly stable at origin.

Remark 2 In Theorem 1, we can see from the inequality (5) that the derivative of $V(t, x(t))$ is relaxed to be positive definite for a.e. $t \ge t_0$. However, the existing Lyapunov function must have a negative/semi-negative definite derivative. So our results are improved and more practical.

Theorem 2 Suppose that the assumption (A2) holds. If there exists a LLC function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t, 0) = 0$ for any $t \in \mathbb{R}$, s.t. Condition (A5) holds and

(A6) for any $\eta \in \partial_t V(t, x)$ and $\zeta \in \partial_x V(t, x)$,

$$\eta + \zeta^{\mathrm{T}}\gamma(t) \le \Phi(t)V(t, x), \text{ for a.e. } t \in [t_0, +\infty),$$

where $\gamma(t) \in F(t, x(t), x(t - \tau(t)))$ is a measurable function and satisfies

$$\dot{x}(t) = \gamma(t), \text{ for a.e. } t \in [t_0, +\infty),$$

then TDFS (1) is uniformly asymptotically stable at origin.

Proof. According to Remark 1, we have proven that TDFS (1) is uniformly stable at origin. In the following, we need only to show TDFS (1) is uniformly attractive at origin. Similar to inequality (6), we can obtain that

$$V(t,x(t_0,\varphi)(t)) \le V(t_0,\varphi(t_0))\mathrm{e}^{\int_{t_0}^{t}\Phi(s)\mathrm{d}s},\ \forall t\ge t_0. \tag{10}$$

Because

$$\int_{t_0}^{t}\Phi(s)\mathrm{d}s=\int_{t_0}^{t}\Phi^{+}(s)\mathrm{d}s-\int_{t_0}^{t}\Phi^{-}(s)\mathrm{d}s, \tag{11}$$

we can deduce from (10), (11) and the condition (A5) that

$$\begin{aligned} V(t,x(t_0,\varphi)(t)) &\le V(t_0,\varphi(t_0))\mathrm{e}^{\int_{t_0}^{t}\Phi^{+}(s)\mathrm{d}s}\mathrm{e}^{-\int_{t_0}^{t}\Phi^{-}(s)\mathrm{d}s}\\ &\le V(t_0,\varphi(t_0))\mathrm{e}^{\int_{0}^{+\infty}\Phi^{+}(s)\mathrm{d}s}\mathrm{e}^{-\ell(t-t_0)+N}\\ &\le \psi_2(\|\varphi(t_0)\|)\mathrm{e}^{N+\int_{0}^{+\infty}\Phi^{+}(s)\mathrm{d}s}\mathrm{e}^{-\ell(t-t_0)}\\ &\le \psi_2(\|\varphi\|_C)\mathscr{I}\mathrm{e}^{-\ell(t-t_0)},\ \forall t\ge t_0. \end{aligned} \tag{12}$$

Here $\mathscr{I}=\mathrm{e}^{N+\int_{0}^{+\infty}\Phi^{+}(s)\mathrm{d}s}$. Again from the condition (A5) and (12), we can obtain

$$\|x(t_0,\varphi)(t)\| \le \psi_1^{-1}\left(\psi_2(\|\varphi\|_C)\mathscr{I}\mathrm{e}^{-\ell(t-t_0)}\right). \tag{13}$$

The above inequality yields that TDFS (1) is uniformly asymptotically stable at origin.

Remark 3 Especially, if we take $\psi_1(\|x\|)=\|x\|^r$ (r is a positive constant), then we can further prove that TDFS (1) is globally exponential stable at origin. If the condition (A5) in Theorem 2 is replaced with the condition (A3), then we can only derive that TDFS (1) is asymptotically stable at origin. Comparing with previous stability results in [5,30], the conditions obtained in Theorem 1 and Theorem 2 are relaxed. That is to say, the Lyapunov function $V(t,x(t))$ is non-smooth. Moreover, the Lyapunov function $V(t,x(t))$ along the trajectories of TDFS is allowed to have indefinite derivative.

Remark 4 In Theorem 2, if Condition (4) is replaced with $\int_{t_0}^{+\infty}\Phi^{-}(s)\mathrm{d}s=+\infty$, then we can only deduce that the origin of TDFS (1) is asymptotically stable. Obviously, condition (4) yields $\int_{t_0}^{+\infty}\Phi^{-}(s)\mathrm{d}s=+\infty$.

Theorem 3 Suppose the LLC function $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_+$ satisfying C-regularity, the functions $\Phi(t):\mathbb{R}_+\to\mathbb{R}_+$ and $\mathcal{O}(t):\mathbb{R}_+\to\mathbb{R}$ are continuous. If the condition (A5) holds and

(A7) for any $\eta\in\partial_t V(t,x)$ and $\zeta\in\partial_x V(t,x)$,

$$\eta+\zeta^{\mathrm{T}}\gamma(t)\le\Phi(t)V(t,x)+\mathcal{O}(t),\ \text{for a.e. } t\in[t_0,+\infty),$$

where $\gamma(t)\in F(t,x(t),x(t-\tau(t)))$ is a measurable function and satisfies

$$\dot{x}(t)=\gamma(t),\ \text{for a.e. } t\in[t_0,+\infty);$$

(A8) $\int_0^{+\infty}\Phi(s)\mathrm{d}s<+\infty$, and $\int_0^{+\infty}\mathcal{O}^{+}(s)\mathrm{d}s<+\infty$, where $\mathcal{O}^{+}(s)=\mathcal{O}(s)\vee 0$,

then all Filippov solutions of TDFS (1) are uniformly bounded.

Proof. By integrating between t_0 to t, it leads to

$$V(t,x(t_0,\varphi)(t)) = V(t_0,x(t_0,\varphi)(t_0)) + \int_{t_0}^{t} \dot{V}(s,x(t_0,\varphi)(s))\mathrm{d}s,\ \forall t \geq t_0. \tag{14}$$

By virtue of Lemma 1, it follows from condition (A7) that

$$\frac{\mathrm{d}V(t,x(t_0,\varphi)(t))}{\mathrm{d}t} = \eta + \zeta^{\mathrm{T}}\gamma(t) \leq \Phi(t)V(t,x(t_0,\varphi)(t)) + \mathcal{O}(t),\quad \text{a.e. } t \geq t_0. \tag{15}$$

This, together with (14), leads to

$$\begin{aligned} V(t,x(t_0,\varphi)(t)) &\leq V(t_0,x(t_0,\varphi)(t_0)) + \int_{t_0}^{t} \left[\Phi(s)V(s,x(t_0,\varphi)(s)) + \mathcal{O}(s)\right] \mathrm{d}s \\ &\leq \left(V(t_0,x(t_0,\varphi)(t_0)) + \int_{t_0}^{t} \mathcal{O}^{+}(s)\mathrm{d}s\right) + \int_{t_0}^{t} \Phi(s)V(s,x(t_0,\varphi)(s))\mathrm{d}s. \end{aligned} \tag{16}$$

Using Gronwall inequality, it obtains from (16) that

$$\begin{aligned} V(t,x(t_0,\varphi)(t)) &\leq \left(V(t_0,x(t_0,\varphi)(t_0)) + \int_{t_0}^{t} \mathcal{O}^{+}(s)\mathrm{d}s\right) \mathrm{e}^{\int_{t_0}^{t}\Phi(s)\mathrm{d}s} \\ &= \left(V(t_0,\varphi(t_0)) + \int_{t_0}^{t} \mathcal{O}^{+}(s)\mathrm{d}s\right) \mathrm{e}^{\int_{t_0}^{t}\Phi(s)\mathrm{d}s} \\ &\leq \left(V(t_0,\varphi(t_0)) + \int_{0}^{+\infty} \mathcal{O}^{+}(s)\mathrm{d}s\right) \mathrm{e}^{\int_{0}^{+\infty}\Phi(s)\mathrm{d}s} \\ &= \left(V(t_0,\varphi(t_0)) + \mathcal{O}^{*}\right)\mathscr{C}^{*},\ \text{for all } t \geq t_0, \end{aligned} \tag{17}$$

where $\mathcal{O}^{*} = \int_0^{+\infty} \mathcal{O}^{+}(s)\mathrm{d}s$ and $\mathscr{C}^{*} = \mathrm{e}^{\int_0^{+\infty}\Phi(s)\mathrm{d}s}$ are nonnegative constants. Using Condition (A5), it implies from (17) that

$$\begin{aligned} \|x(t_0,\varphi)(t)\| &\leq \psi_1^{-1}\left(\left(V(t_0,\varphi(t_0)) + \mathcal{O}^{*}\right)\mathscr{C}^{*}\right) \\ &\leq \psi_1^{-1}\left(\left(\psi_2(\|\varphi(t_0)\|) + \mathcal{O}^{*}\right)\mathscr{C}^{*}\right) \\ &\leq \psi_1^{-1}\left(\left(\psi_2(\|\varphi\|_C) + \mathcal{O}^{*}\right)\mathscr{C}^{*}\right),\ \text{for all } t \geq t_0. \end{aligned} \tag{18}$$

Therefore, $\forall\delta > 0$, take $M = \psi_1^{-1}\left(\left(\psi_2(\delta) + \mathcal{O}^{*}\right)\mathscr{C}^{*}\right) > 0$, s.t. $\|x(t_0,\varphi)(t)\| \leq M$ for any $\|\varphi\|_C < \delta$ and $t \geq t_0$. This means that all Filippov solutions of TDFS (1) are uniformly bounded.

Remark 5 In Theorem 3, if Condition (A5) is replaced with Condition (A3), then we can only derive $x(t_0,\varphi)(t)$ of TDFS (1) is bounded. In addition, when dealing with the boundness and uniform boundness of the Filippov solutions for TDFS (1), we do not need to assume that the condition (A0) holds. That is, removing the assumption (A0), the result of Theorem 3 is still correct.

Theorem 4 If there exists a LLC function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t,0) = 0$ for any $t \in \mathbb{R}$, s.t. Condition (A2) holds and

(A9) for any $\eta \in \partial_t V(t,x)$ and $\zeta \in \partial_x V(t,x)$,

$$\eta + \zeta^{\mathrm{T}}\gamma(t) \leq \Phi(t)V^{\alpha}(t,x),\ \text{for a.e. } t \in [t_0,+\infty),$$

where the constant $0 < \alpha < 1$, the function $\gamma(t) \in F(t,x(t),x(t-\tau(t)))$ is measurable and satisfies

$$\dot{x}(t) = \gamma(t),\ \text{for a.e. } t \in [t_0,+\infty),$$

then TDFS (1) is FNTA at origin. Furthermore, the S-T is given as

$$T(t_0,\varphi)=t_0+\frac{V^{1-\alpha}(t_0,\varphi(t_0))+(1-\alpha)(\Phi^*+N)}{(1-\alpha)\ell}. \tag{19}$$

Proof. Similar to (5), by virtue of Lemma 1, it follows from the condition (A9) that

$$\frac{\mathrm{d}V(t,x(t_0,\varphi)(t))}{\mathrm{d}t}=\eta+\zeta^{\mathrm{T}}\gamma(t)\leq\Phi(t)V^{\alpha}(t,x(t_0,\varphi)(t)),\text{for a.e. } t\geq t_0. \tag{20}$$

From (20), it gets

$$V^{-\alpha}(t,x(t_0,\varphi)(t))\mathrm{d}V(t,x(t_0,\varphi)(t))\leq\Phi(t)\mathrm{d}t. \tag{21}$$

By integrating (21) between t_0 to t, it has

$$\int_{t_0}^{t}V^{-\alpha}(t,x(t_0,\varphi)(t))\mathrm{d}V(t,x(t_0,\varphi)(t))\leq\int_{t_0}^{t}\Phi(s)\mathrm{d}s. \tag{22}$$

Recalling the inequality (11) and condition (A2), from (22), it has

$$\begin{aligned}V^{1-\alpha}(t,x(t_0,\varphi)(t))-V^{1-\alpha}(t_0,x(t_0,\varphi)(t_0))&\leq(1-\alpha)\int_{t_0}^{t}\Phi(s)\mathrm{d}s\\&=(1-\alpha)\int_{t_0}^{t}\Phi^{+}(s)\mathrm{d}s-(1-\alpha)\int_{t_0}^{t}\Phi^{-}(s)\mathrm{d}s\\&\leq(1-\alpha)\int_{0}^{+\infty}\Phi^{+}(s)\mathrm{d}s-(1-\alpha)[\ell(t-t_0)-N]\\&=(1-\alpha)(\Phi^*+N)-(1-\alpha)\ell(t-t_0),\end{aligned} \tag{23}$$

which yields

$$V^{1-\alpha}(t,x(t_0,\varphi)(t))\leq V^{1-\alpha}(t_0,\varphi(t_0))+(1-\alpha)(\Phi^*+N)-(1-\alpha)\ell(t-t_0). \tag{24}$$

In (24), letting $V^{1-\alpha}(t_0,\varphi(t_0))+(1-\alpha)(\Phi^*+N)-(1-\alpha)\ell(t-t_0)\leq 0$, then $V^{1-\alpha}(t,x(t_0,\varphi)(t))=0$ holds for any

$$t\geq t_0+\frac{V^{1-\alpha}(t_0,\varphi(t_0))+(1-\alpha)(\Phi^*+N)}{(1-\alpha)\ell}\triangleq T(t_0,\varphi). \tag{25}$$

This implies $x(t)=x(t_0,\varphi)(t)=0$ for all $t\geq T(t_0,\varphi)$.

In Theorem 4, if $\Phi=-\ell<0$, then $\Phi^*=\int_0^{+\infty}\Phi^{+}(s)\mathrm{d}s=0, N=0$ and the following result is obtained.

Corollary 1 Assume that there exists a LLC function $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_+$ satisfying C-regularity and $V(t,0)=0$ for any $t\in\mathbb{R}$, s.t. for any $\eta\in\partial_t V(t,x)$ and $\zeta\in\partial_x V(t,x)$,

$$\eta+\zeta^{\mathrm{T}}\gamma(t)\leq-\ell V^{\alpha}(t,x),\ \text{for a.e. } t\in[t_0,+\infty),$$

where $1>\alpha>0$ and $\ell>0$, the function $\gamma(t)\in F(t,x(t),x(t-\tau(t)))$ is measurable and satisfies

$$\dot{x}(t)=\gamma(t),\ \text{for a.e. } t\in[t_0,+\infty).$$

Then TDFS (1) is FNTA at origin. Furthermore, the S-T is given as

$$T(t_0,\varphi)=t_0+\frac{V^{1-\alpha}(t_0,\varphi(t_0))}{(1-\alpha)\ell}.$$

Theorem 5 If there exists a LLC function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t, 0) = 0$ for any $t \in \mathbb{R}$, s.t. Condition (A2) holds and

(A10) for any $\eta \in \partial_t V(t, x)$ and $\zeta \in \partial_x V(t, x)$,

$$\eta + \zeta^{\mathrm{T}} \gamma(t) \leq \Phi(t) V^{\alpha}(t, x) - bV^{\beta}(t, x), \text{ for a.e. } t \in [t_0, +\infty),$$

where $b > 0$, $1 > \alpha > 0$ and $\beta > 1$, the function $\gamma(t) \in F(t, x(t), x(t - \tau(t)))$ is measurable and satisfies

$$\dot{x}(t) = \gamma(t), \text{ for a.e. } t \in [t_0, +\infty),$$

then TDFS (1) is FXTA at origin. Furthermore, the S-T can be estimated as $T(t_0, \varphi) \leq t_0 + T^1_{\max}$

$$T^1_{\max} = \frac{1}{(1-\alpha)\ell} + \frac{1}{b(\beta-1)} + \Phi^* \left(\frac{1}{\ell} + \frac{1}{b}\right) + \frac{N}{\ell}. \tag{26}$$

Proof. Similar to (5), using Lemma 1, it yields from Condition (A10) that

$$\begin{aligned} \frac{\mathrm{d}V(t, x(t_0, \varphi)(t))}{\mathrm{d}t} &= \eta + \zeta^{\mathrm{T}} \gamma(t) \\ &\leq \Phi(t) V^{\alpha}(t, x(t_0, \varphi)(t)) - bV^{\beta}(t, x(t_0, \varphi)(t)), \text{for a.e. } t \in [t_0, +\infty). \end{aligned} \tag{27}$$

Firstly, let us prove there exists $T^* \geq t_0$ s.t. $V(T^*, x(t_0, \varphi)(T^*)) \leq 1$ and

$$T^* \leq t_0 + \frac{1}{b(\beta-1)} + \frac{\Phi^*}{b} \overset{\text{def}}{=} T_1. \tag{28}$$

If it is not true, then $1 < V(t, x(t_0, \varphi)(t))$ for any $t \in [t_0, T_1]$. Thus, under the condition $\alpha < \beta$, it implies

$$V^{\alpha}(t, x(t_0, \varphi)(t)) \leq V^{\beta}(t, x(t_0, \varphi)(t)). \tag{29}$$

It yields from (27) and (29) that

$$\begin{aligned} \frac{\mathrm{d}V(t, x(t_0, \varphi)(t))}{\mathrm{d}t} &\leq \Phi^+(t) V^{\alpha}(t, x(t_0, \varphi)(t)) - bV^{\beta}(t, x(t_0, \varphi)(t)) \\ &\leq \left(\Phi^+(t) - b\right) V^{\beta}(t, x(t_0, \varphi)(t)), \text{a.e. } t \geq t_0. \end{aligned} \tag{30}$$

From (30), it obtains

$$V^{-\beta}(t, x(t_0, \varphi)(t)) \mathrm{d}V(t, x(t_0, \varphi)(t)) \leq \left(\Phi^+(t) - b\right) \mathrm{d}t. \tag{31}$$

By integrating (31) from t_0 to T_1, it gets

$$\int_{t_0}^{T_1} V^{-\beta}(t, x(t_0, \varphi)(t)) \mathrm{d}V(t, x(t_0, \varphi)(t)) \leq \int_{t_0}^{T_1} \left(\Phi^+(t) - b\right) \mathrm{d}t, \tag{32}$$

which yields

$$\begin{aligned} \frac{1}{1-\beta} V^{1-\beta}(T_1, x(t_0, \varphi)(T_1)) &\leq \frac{1}{1-\beta} \left[V^{1-\beta}(T_1, x(t_0, \varphi)(T_1)) - V^{1-\beta}(t_0, x(t_0, \varphi)(t_0))\right] \\ &\leq \int_0^{+\infty} \Phi^+(t) \mathrm{d}t - \int_{t_0}^{T_1} b \mathrm{d}t = \Phi^* - b(T_1 - t_0). \end{aligned} \tag{33}$$

By substituting $T_1 = t_0 + \frac{1}{b(\beta-1)} + \frac{\Phi^*}{b}$ into (33), it has

$$\frac{1}{1-\beta}V^{1-\beta}(T_1, x(t_0,\varphi)(T_1)) \leq \Phi^* - b\left(t_0 + \frac{1}{b(\beta-1)} + \frac{\Phi^*}{b} - t_0\right) = \frac{1}{1-\beta}, \tag{34}$$

which yields $V^{1-\beta}(T_1, x(t_0,\varphi)(T_1)) \geq 1$. Consequently, $V(T_1, x(t_0,\varphi)(T_1)) \leq 1$ due to $0 > 1-\beta$. This is in contradiction with $V(T_1, x(t_0,\varphi)(T_1)) > 1$.

Secondly, because $b > 0$, it yields from (27) that

$$\frac{\mathrm{d}V(t, x(t_0,\varphi)(t))}{\mathrm{d}t} \leq \Phi(t)V^{\alpha}(t, x(t_0,\varphi)(t)), \text{for a.e. } t \geq T^*. \tag{35}$$

Similar to (21)-(24), by integrating between T^* to t, it has

$$V^{1-\alpha}(t, x(t_0,\varphi)(t)) \leq V^{1-\alpha}(T^*, x(t_0,\varphi)(T^*)) + (1-\alpha)\Phi^* - (1-\alpha)[\ell(t-T^*) - N]. \tag{36}$$

Since $V(T^*, x(t_0,\varphi)(T^*)) \leq 1$ and $0 < 1-\alpha$, it implies from (36) that

$$V(t, x(t_0,\varphi)(t)) \leq [1 + (1-\alpha)(\Phi^* + N) - (1-\alpha)\ell(t-T^*)]^{\frac{1}{1-\alpha}}. \tag{37}$$

In (37), let

$$1 + (1-\alpha)(\Phi^* + N) - (1-\alpha)\ell(t-T^*) \leq 0, \tag{38}$$

then $V(t, x(t_0,\varphi)(t)) = 0$ holds for

$$t \geq T^* + \frac{1}{(1-\alpha)\ell} + \frac{\Phi^* + N}{\ell}. \tag{39}$$

Recalling the formula $T^* \leq t_0 + \frac{1}{b(\beta-1)} + \frac{\Phi^*}{b}$, it implies that $V(t, x(t_0,\varphi)(t)) = 0$ for any

$$t \geq t_0 + \frac{1}{b(\beta-1)} + \frac{\Phi^*}{b} + \frac{1}{(1-\alpha)\ell} + \frac{\Phi^* + N}{\ell} \overset{\text{def}}{=} t_0 + T^1_{\max}. \tag{40}$$

Thus, $x(t) = x(t_0,\varphi)(t) \equiv 0$ for all $t \geq t_0 + T^1_{\max}$, where $t_0 + T^1_{\max}$ is the upper bound of the S-T $T(t_0,\varphi)$

In Theorem 5, if $\Phi = -\ell < 0$, then $\Phi^* = \int_0^{+\infty}\Phi^+(s)\mathrm{d}s = 0, N = 0$ and the following result is obtained.

Corollary 2 Assume that there exists a LLC function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t,0) = 0$ for any $t \in \mathbb{R}$, s.t. for any $\eta \in \partial_t V(t,x)$ and $\zeta \in \partial_x V(t,x)$,

$$\eta + \zeta^{\mathrm{T}}\gamma(t) \leq -\ell V^{\alpha}(t,x) - bV^{\beta}(t,x), \text{ for a.e. } t \in [t_0, +\infty),$$

where $\ell > 0$, $b > 0$, $1 > \alpha > 0$ and $\beta > 1$, the function $\gamma(t) \in F(t, x(t), x(t-\tau(t)))$ is measurable and satisfies

$$\dot{x}(t) = \gamma(t), \text{ for a.e. } t \in [t_0, +\infty).$$

Then TDFS (1) is FXTA at origin. Furthermore, the S-T can be estimated as $T(t_0,\varphi) \leq t_0 + \tilde{T}^1_{\max}$

$$\tilde{T}^1_{\max} = \frac{1}{(1-\alpha)\ell} + \frac{1}{b(\beta-1)}.$$

Theorem 6 If there exists a LLC function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t,0) = 0$ for any $t \in \mathbb{R}$, s.t. Condition (A2) holds and

(A11) for any $\eta \in \partial_t V(t,x)$ and $\zeta \in \partial_x V(t,x)$,

$$\eta + \zeta^{\mathrm{T}}\gamma(t) \le \Phi(t)e^{V^{\sigma}(t,x)}V^{1-\sigma}(t,x), \text{ for a.e. } t \in [t_0,+\infty),$$

where the constant $1 > \sigma > 0$, the function $\gamma(t) \in F(t, x(t), x(t-\tau(t)))$ is measurable and satisfies

$$\dot{x}(t) = \gamma(t), \text{ for a.e. } t \in [t_0,+\infty),$$

then TDFS (1) is FXTA at origin. Furthermore, the S-T can be estimated as $T(t_0,\varphi) \le t_0 + T^2_{\max}$

$$T^2_{\max} = \frac{1+\sigma(\Phi^*+N)}{\sigma\ell}. \tag{41}$$

Proof. Similar to (5), using Lemma 1, it yields from Condition (A11) that

$$\begin{aligned}\frac{\mathrm{d}V(t,x(t_0,\varphi)(t))}{\mathrm{d}t} &= \eta + \zeta^{\mathrm{T}}\gamma(t)\\ &\le \Phi(t)e^{V^{\sigma}(t,x(t_0,\varphi)(t))}V^{1-\sigma}(t,x(t_0,\varphi)(t)), \text{for a.e. } t \in [t_0,+\infty).\end{aligned} \tag{42}$$

Set $H(V) = 1 - e^{-V^{\sigma}}$, then $\mathrm{d}H = qe^{-V^{\sigma}}V^{\sigma-1}\mathrm{d}V$. Thus, the inequality (42) can be written as

$$\frac{\mathrm{d}H}{\mathrm{d}t} \le \sigma\Phi(t). \tag{43}$$

By integrating (43) between t_0 to t, it has

$$H(V(t,x(t_0,\varphi)(t))) - H(V(t_0,\varphi(t_0))) \le \sigma\int_{t_0}^{t}\Phi(s)\mathrm{d}s. \tag{44}$$

Using the condition (A2), it follows from (44) that

$$\begin{aligned}H(V(t,x(t_0,\varphi)(t))) &\le H(V(t_0,\varphi(t_0))) + \sigma\int_0^{+\infty}\Phi^+(s)\mathrm{d}s - \sigma[\ell(t-t_0)-N]\\ &= H(V(t_0,\varphi(t_0))) + \sigma(\Phi^*+N) - \sigma\ell(t-t_0).\end{aligned} \tag{45}$$

Obviously, $H(V(t,x(t_0,\varphi)(t))) \ge 0$ because of $V(t,x(t_0,\varphi)(t)) \ge 0$. In (45), let

$$H(V(t_0,\varphi(t_0))) + \sigma(\Phi^*+N) - \sigma\ell(t-t_0) \le 0, \tag{46}$$

then $H(V(t,x(t_0,\varphi)(t))) = 0$ for any

$$\begin{aligned}t &\ge t_0 + \frac{H(V(t_0,\varphi(t_0))) + \sigma(\Phi^*+N)}{\sigma\ell}\\ &= t_0 + \frac{1-e^{-V^{\sigma}(t_0,\varphi(t_0))} + \sigma(\Phi^*+N)}{\sigma\ell} \stackrel{\text{def}}{=} T(t_0,\varphi).\end{aligned} \tag{47}$$

Consequently, $V(t,x(t_0,\varphi)(t)) = H^{-1}(0) = 0$ for any $t \ge T(t_0,\varphi)$. This yields $x(t) = x(t_0,\varphi)(t) = 0$ for all $t \ge T(t_0,\varphi)$. Because, $0 < e^{-V^{\sigma}(t_0,\varphi(t_0))} \le 1$, the S-T $T(t_0,\varphi) \ge 0$ and it is bounded on φ, that is,

$$\begin{aligned}T(t_0,\varphi) &= t_0 + \frac{1-e^{-V^{\sigma}(t_0,\varphi(t_0))} + \sigma(\Phi^*+N)}{\sigma\ell}\\ &\le t_0 + \frac{1+\sigma(\Phi^*+N)}{\sigma\ell} \stackrel{\text{def}}{=} t_0 + T^2_{\max}.\end{aligned} \tag{48}$$

Therefore, $x(t) = x(t_0,\varphi)(t) = 0$ for all $t \ge t_0 + T^2_{\max}$.

In Theorem 6, if $\Phi = -\ell < 0$, then $\Phi^* = \int_0^{+\infty}\Phi^+(s)\mathrm{d}s = 0, N = 0$ and the following result is obtained.

Corollary 3 Assume that there exists a LLC function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying C-regularity and $V(t,0) = 0$ for any $t \in \mathbb{R}$, s.t. for any $\eta \in \partial_t V(t,x)$ and $\zeta \in \partial_x V(t,x)$,

$$\eta + \zeta^{\mathrm{T}}\gamma(t) \leq -\ell e^{V^{\sigma}(t,x)} V^{1-\sigma}(t,x), \text{ for a.e. } t \in [t_0, +\infty),$$

where $\ell > 0$ and $1 > \sigma > 0$, the function $\gamma(t) \in F(t, x(t), x(t-\tau(t)))$ is measurable and satisfies

$$\dot{x}(t) = \gamma(t), \text{ for a.e. } t \in [t_0, +\infty).$$

Then TDFS (1) is FXTA at origin. Furthermore, the S-T can be estimated as $T(t_0,\varphi) \leq t_0 + \tilde{T}^2_{\max}$

$$\tilde{T}^2_{\max} = \frac{1}{\sigma\ell}.$$

4 Application to DSNNs

Consider a delayed switched neural networks (DSNNs):

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} =& a_i(t)x_i(t) + \sum_{j=1}^{n} \mathscr{P}_{ij}(x_i(t)) f_j(x_j(t)) + \sum_{j=1}^{n} \mathscr{Q}_{ij}(x_i(t)) g_j(x_j(t-\tau(t))) + u_i(t), \\ & i \in \mathbb{N} = \{1, 2, \ldots, n\},\end{aligned} \tag{49}$$

where $x_i(t)$ denotes neuron state; $f_j(\cdot)$ and $g_j(\cdot)$ are neuron activations; $\tau(t)$ denotes time-varying delay; $u_i(t)$ denotes the external input; $a_i(t)$ denotes self-inhibition; $\mathscr{P}_{ij}(x_i(t))$ and $\mathscr{Q}_{ij}(x_i(t))$ denotes the connection weights which are measurable and possess following discontinuous property:

$$\mathscr{P}_{ij}(x_i(t)) = \begin{cases} \hat{\mathscr{P}}_{ij}, & \text{if } |x_i(t)| \leq J_i, \\ \check{\mathscr{P}}_{ij}, & \text{if } |x_i(t)| > J_i, \end{cases}$$

$$\mathscr{Q}_{ij}(x_i(t)) = \begin{cases} \hat{\mathscr{Q}}_{ij}, & \text{if } |x_i(t)| \leq J_i, \\ \check{\mathscr{Q}}_{ij}, & \text{if } |x_i(t)| > J_i, \end{cases}$$

where $i, j \in \mathbb{N}$, $\hat{\mathscr{P}}_{ij}$, $\check{\mathscr{P}}_{ij}$, $\hat{\mathscr{Q}}_{ij}$ and $\check{\mathscr{Q}}_{ij}$ are constants, the switching jumps $J_i > 0$.

The following two assumptions are needed.

(H1) The activation f_j satisfies $f_j(0) = 0$. Moreover, for any $z, z^* \in \mathbb{R}$ and $j \in \mathbb{N}$, there exist $K_j > 0$ s.t.

$$|f_j(z) - f_j(z^*)| \leq K_j |z - z^*|.$$

(H2) The activation g_j satisfies $g_j(0) = 0$ and $|g_j(\cdot)| \leq M_j$, where the constants $M_j > 0$.

By applying the DI theory with set-valued maps, if $x(t) = x(t_0,\varphi)(t)$ is a Filippov solution of DSNNs (49) under initial-value $(t_0,\varphi) \in \mathbb{R}_+ \times C([t_0 - \tau, t_0], \mathbb{R}^n)$, then it satisfies

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \in a_i(t)x_i(t) + \sum_{j=1}^{n} \overline{co}[\mathscr{P}_{ij}(x_i(t))] f_j(x_j(t)) + \sum_{j=1}^{n} \overline{co}[\mathscr{Q}_{ij}(x_i(t))] g_j(x_j(t-\tau(t))) + u_i(t), \tag{50}$$

for a.e. $t \geq t_0$, where

$$\overline{co}[\mathscr{P}_{ij}(x_i(t))] = \begin{cases} \hat{\mathscr{P}}_{ij}, & |x_i(t)| < J_i, \\ [\underline{\mathscr{P}}_{ij}, \overline{\mathscr{P}}_{ij}], & |x_i(t)| = J_i, \\ \check{\mathscr{P}}_{ij}, & |x_i(t)| > J_i, \end{cases}$$

$$\overline{co}[\mathscr{Q}_{ij}(x_i(t))] = \begin{cases} \hat{\mathscr{Q}}_{ij}, & |x_i(t)| < J_i, \\ [\underline{\mathscr{Q}}_{ij}, \overline{\mathscr{Q}}_{ij}], & |x_i(t)| = J_i, \\ \check{\mathscr{Q}}_{ij}, & |x_i(t)| > J_i, \end{cases}$$

$\overline{\mathscr{P}}_{ij} = \max\{\hat{\mathscr{P}}_{ij}, \check{\mathscr{P}}_{ij}\}$, $\underline{\mathscr{P}}_{ij} = \min\{\hat{\mathscr{P}}_{ij}, \check{\mathscr{P}}_{ij}\}$, $\overline{\mathscr{Q}}_{ij} = \max\{\hat{\mathscr{Q}}_{ij}, \check{\mathscr{Q}}_{ij}\}$ and $\underline{\mathscr{Q}}_{ij} = \min\{\hat{\mathscr{Q}}_{ij}, \check{\mathscr{Q}}_{ij}\}$, $i, j \in \mathbb{N}$. Thus, there exist measurable selections $p_{ij}(t) \in \overline{co}[\mathscr{P}_{ij}(x_i(t))]$ and $q_{ij}(t) \in \overline{co}[\mathscr{Q}_{ij}(x_i(t))]$ such that for a.e. $t \geq t_0$,

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = a_i(t)x_i(t) + \sum_{j=1}^{n} p_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} q_{ij}(t)g_j(x_j(t-\tau(t))) + u_i(t). \tag{51}$$

To ensure the stabilization and attractivity of DSNNs (49), several control protocols are developed:

$$u_i(t) = -c_i x_i(t) - h_i \mathrm{sign}(x_i(t)), \tag{52}$$

where c_i and h_i are control gains.

$$u_i(t) = -c_i x_i(t) + \frac{1}{2}\left(n^{\frac{\vartheta-1}{2}}\Phi^+(t) - \Phi^-(t)\right)|x_i(t)|^{\vartheta}\mathrm{sign}(x_i(t)) - h_i\mathrm{sign}(x_i(t)), \tag{53}$$

where c_i and h_i are control gains, the time-varying control gains $\Phi^+(t) = 0 \vee \Phi(t)$, $\Phi^-(t) = 0 \vee [-\Phi(t)]$ and $\Phi(t)$ is a continuous function, the constant $0 < \vartheta < 1$.

$$\begin{aligned} u_i(t) = &- c_i x_i(t) + \frac{1}{2}\left(n^{\frac{\vartheta-1}{2}}\Phi^+(t) - \Phi^-(t)\right)|x_i(t)|^{\vartheta}\mathrm{sign}(x_i(t)) \\ &- \frac{1}{2}n^{\frac{\omega-1}{2}} b|x_i(t)|^{\omega}\mathrm{sign}(x_i(t)) - h_i\mathrm{sign}(x_i(t)), \end{aligned} \tag{54}$$

where c_i, h_i and $b > 0$ are control gains, the time-varying control gains $\Phi^+(t) = \Phi(t) \vee 0$, $\Phi^-(t) = [-\Phi(t)] \vee 0$ and $\Phi(t)$ is a continuous function, the constants $0 < \vartheta < 1$ and $\omega > 1$.

$$u_i(t) = -c_i x_i(t) + \frac{1}{2}\left(n^{-\frac{\varrho}{2}}\Phi^+(t) - \Phi^-(t)\right)e^{\left(\sum_{i=1}^{n} x_i^2(t)\right)^{\frac{\varrho}{2}}} \cdot |x_i(t)|^{1-\varrho}\mathrm{sign}(x_i(t)) - h_i\mathrm{sign}(x_i(t)). \tag{55}$$

where c_i and h_i are control gains, the time-varying control gains $\Phi^+(t) = \Phi(t) \vee 0$, $\Phi^-(t) = [-\Phi(t)] \vee 0$ and $\Phi(t)$ is a continuous function, the constant $0 < \varrho < 1$.

Theorem 7 Let the hypotheses (H1) and (H2) hold. The function $\Phi(t) = 2\max_{i\in\mathbb{N}}\{a_i(t)\}$ is continuous and satisfies the condition (A1). Assume further that

(H3) for each $i \in \mathbb{N}$, $2c_i \geq \sum_{j=1}^{n}\left(\mathscr{P}_{ij}^D K_j + \mathscr{P}_{ji}^D K_i\right)$ and $h_i \geq \sum_{j=1}^{n} \mathscr{Q}_{ij}^D M_j$, where $\mathscr{P}_{ij}^D = \max\{|\underline{\mathscr{P}}_{ij}|, |\overline{\mathscr{P}}_{ij}|\}$ and $\mathscr{Q}_{ij}^D = \max\{|\underline{\mathscr{Q}}_{ij}|, |\overline{\mathscr{Q}}_{ij}|\}$.

Then the origin of DSNNs (49) is stable under the control protocol (52).

Proof. Take $V(t,x) = \sum_{i=1}^{n} x_i^2(t)$ as the Lyapunov function and compute the derivative of $V(t,x)$

along (49), it has

$$
\begin{aligned}
\dot{V}(t,x) &= 2\sum_{i=1}^{n} x_i(t)\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \\
&= 2\sum_{i=1}^{n} x_i(t)\left[a_i(t)x_i(t) + \sum_{j=1}^{n} p_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} q_{ij}(t)g_j(x_j(t-\tau(t))) + u_i(t)\right] \\
&\leq 2\sum_{i=1}^{n} a_i(t)x_i^2(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} |p_{ij}(t)||f_j(x_j(t))||x_i(t)| \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} |q_{ij}(t)||g_j(x_j(t-\tau(t)))||x_i(t)| + 2\sum_{i=1}^{n} x_i(t)u_i(t),
\end{aligned} \tag{56}
$$

for a.e. $t \geq t_0 \geq 0$. Using the hypotheses (H1) and (H2), it implies from (56) that

$$
\begin{aligned}
\dot{V}(t,x) &\leq 2\sum_{i=1}^{n} a_i(t)x_i^2(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{P}_{ij}^{D} K_j |x_j(t)||x_i(t)| \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{Q}_{ij}^{D} M_j |x_i(t)| + 2\sum_{i=1}^{n} x_i(t)u_i(t).
\end{aligned} \tag{57}
$$

Since $2|x_j(t)||x_i(t)| \leq x_j^2(t) + x_i^2(t)$, one has

$$
\begin{aligned}
2\sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{P}_{ij}^{D} K_j |x_j(t)||x_i(t)| &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{P}_{ij}^{D} K_j x_j^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{P}_{ij}^{D} K_j x_i^2(t) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{P}_{ji}^{D} K_i x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{P}_{ij}^{D} K_j x_i^2(t) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\mathscr{P}_{ji}^{D} K_i + \mathscr{P}_{ij}^{D} K_j\right) x_i^2(t).
\end{aligned} \tag{58}
$$

It yields from (57) and (58) that

$$
\begin{aligned}
\dot{V}(t,x) &\leq 2\sum_{i=1}^{n} a_i(t)x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\mathscr{P}_{ji}^{D} K_i + \mathscr{P}_{ij}^{D} K_j\right) x_i^2(t) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \mathscr{Q}_{ij}^{D} M_j |x_i(t)| + 2\sum_{i=1}^{n} x_i(t)u_i(t).
\end{aligned} \tag{59}
$$

Substituting the control protocol (52) into (59) and using the condition (H3), it gets

$$
\begin{aligned}
\dot{V}(t,x) &\leq 2\max_{i\in\mathbb{N}}\{a_i(t)\}\sum_{i=1}^{n} x_i^2(t) - \sum_{i=1}^{n}\left(2c_i - \sum_{j=1}^{n}\left(\mathscr{P}_{ji}^{D} K_i + \mathscr{P}_{ij}^{D} K_j\right)\right) x_i^2(t) \\
&\quad - 2\sum_{i=1}^{n}\left(h_i - \sum_{j=1}^{n} \mathscr{Q}_{ij}^{D} M_j\right)|x_i(t)| \\
&\leq \Phi(t)V(t,x),
\end{aligned} \tag{60}
$$

where $\Phi(t) = 2\max_{i\in\mathbb{N}}\{a_i(t)\}$ satisfies the condition (A1). By applying Theorem 1, DSNNs (49) is stable at origin under the control protocol (52).

Remark 6 In Theorem 7, if the condition (A1) is replaced by (A2), then DSNNs (49) is uniformly asymptotically stable at origin under the control protocol (52).

Theorem 8 Let the hypotheses (H1) and (H2) hold. The continuous function $\Phi(t)$ of control protocol (53) satisfies the condition (A2). Assume further that

(H4) for any $t \in \mathbb{R}$ and $i \in \mathbb{N}$, $2c_i \geq 2a_i(t) + \sum_{j=1}^{n} \left(\mathscr{P}_{ij}^D K_j + \mathscr{P}_{ji}^D K_i\right)$ and $h_i \geq \sum_{j=1}^{n} \mathscr{Q}_{ij}^D M_j$, where $\mathscr{P}_{ij}^D = \max\{|\underline{\mathscr{P}}_{ij}|, |\overline{\mathscr{P}}_{ij}|\}$ and $\mathscr{Q}_{ij}^D = \max\{|\underline{\mathscr{Q}}_{ij}|, |\overline{\mathscr{Q}}_{ij}|\}$.

Then DSNNs (49) is FNTA at origin under the protocol (53). Furthermore, the S-T is given as

$$\begin{aligned} T(t_0, \varphi) &= t_0 + \frac{V^{1-\frac{\vartheta+1}{2}}(t_0, \varphi(t_0)) + (1 - \frac{\vartheta+1}{2})(\Phi^* + N)}{(1 - \frac{\vartheta+1}{2})\ell} \\ &= t_0 + \frac{2\left(\sum\limits_{i=1}^{n} \varphi_i^2(t_0)\right)^{\frac{1-\vartheta}{2}} + (1-\vartheta)(\Phi^* + N)}{(1-\vartheta)\ell}. \end{aligned} \tag{61}$$

Proof. Similar to Theorem 7 with the same Lyapunov function, substituting the control protocol (53) into (59) and using the condition (H4), it gets

$$\begin{aligned} \dot{V}(t,x) \leq & -\sum_{i=1}^{n} \left(2c_i - 2a_i(t) - \sum_{j=1}^{n} \left(\mathscr{P}_{ji}^D K_i + \mathscr{P}_{ij}^D K_j\right)\right) x_i^2(t) \\ & - 2\sum_{i=1}^{n} \left(h_i - \sum_{j=1}^{n} \mathscr{Q}_{ij}^D M_j\right) |x_i(t)| + n^{\frac{\vartheta-1}{2}} \Phi^+(t) \sum_{i=1}^{n} |x_i(t)|^{\vartheta+1} - \Phi^-(t) \sum_{i=1}^{n} |x_i(t)|^{\vartheta+1} \\ \leq & \; n^{\frac{\vartheta-1}{2}} \Phi^+(t) \sum_{i=1}^{n} |x_i(t)|^{\vartheta+1} - \Phi^-(t) \sum_{i=1}^{n} |x_i(t)|^{\vartheta+1}. \end{aligned} \tag{62}$$

Since $0 < \vartheta < 1$, by Lemma 2, it yields

$$\left(\sum_{i=1}^{n} |x_i(t)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} |x_i(t)|^{\vartheta+1}\right)^{\frac{1}{\vartheta+1}} \leq n^{\frac{1}{\vartheta+1} - \frac{1}{2}} \left(\sum_{i=1}^{n} |x_i(t)|^2\right)^{\frac{1}{2}}. \tag{63}$$

Because $0 \leq \Phi^+(t)$, $0 \leq \Phi^-(t)$ and $\Phi(t) = \Phi^+(t) - \Phi^-(t)$, it implies from (62) and (63) that

$$\begin{aligned} \dot{V}(t,x) &\leq n^{\frac{\vartheta-1}{2}} \Phi^+(t) \left(n^{\frac{1}{\vartheta+1} - \frac{1}{2}}\right)^{\vartheta+1} \left(\sum_{i=1}^{n} |x_i(t)|^2\right)^{\frac{\vartheta+1}{2}} - \Phi^-(t) \left(\sum_{i=1}^{n} |x_i(t)|^2\right)^{\frac{\vartheta+1}{2}} \\ &= (\Phi^+(t) - \Phi^-(t)) \left(\sum_{i=1}^{n} x_i^2(t)\right)^{\frac{\vartheta+1}{2}} \\ &= \Phi(t) V^{\frac{\vartheta+1}{2}}(t,x). \end{aligned} \tag{64}$$

Obviously, $0 < \frac{\vartheta+1}{2} < 1$. By applying Theorem 4, DSNNs (49) is FNTA at origin under the protocol (53). Furthermore, the S-T can be given by (61).

Theorem 9 Under Hypotheses (H1), (H2) and (H4), if the continuous function $\Phi(t)$ of control protocol (54) satisfies the condition (A2), then DSNNs (49) is FXTA at origin under the protocol (54). Furthermore, the S-T can be estimated as $T(t_0, \varphi) \leq t_0 + T_{\max}^3$

$$T_{\max}^3 = \frac{2}{(1-\vartheta)\ell} + \frac{2}{b(\omega - 1)} + \Phi^* \left(\frac{1}{\ell} + \frac{1}{b}\right) + \frac{N}{\ell}. \tag{65}$$

Proof. Similar to Theorem 7 with the same Lyapunov function, substituting the control protocol (54) into (59) and using the condition (H4), it gets

$$\begin{aligned}\dot{V}(t,x) \leq & -\sum_{i=1}^{n}\left(2c_i - 2a_i(t) - \sum_{j=1}^{n}\left(\mathscr{P}_{ji}^{D}K_i + \mathscr{P}_{ij}^{D}K_j\right)\right)x_i^2(t) \\ & - 2\sum_{i=1}^{n}\left(h_i - \sum_{j=1}^{n}\mathscr{Q}_{ij}^{D}M_j\right)|x_i(t)| + n^{\frac{\vartheta-1}{2}}\Phi^{+}(t)\sum_{i=1}^{n}|x_i(t)|^{\vartheta+1} \\ & - \Phi^{-}(t)\sum_{i=1}^{n}|x_i(t)|^{\vartheta+1} - n^{\frac{\omega-1}{2}}b\sum_{i=1}^{n}|x_i(t)|^{\omega+1} \\ \leq & \, n^{\frac{\vartheta-1}{2}}\Phi^{+}(t)\sum_{i=1}^{n}|x_i(t)|^{\vartheta+1} - \Phi^{-}(t)\sum_{i=1}^{n}|x_i(t)|^{\vartheta+1} - n^{\frac{\omega-1}{2}}b\sum_{i=1}^{n}|x_i(t)|^{\omega+1}. \end{aligned} \tag{66}$$

Since $\omega > 1$, applying Lemma 2, one has

$$\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{1}{2}} \leq n^{\frac{1}{2}-\frac{1}{\omega+1}}\left(\sum_{i=1}^{n}|x_i(t)|^{\omega+1}\right)^{\frac{1}{\omega+1}}. \tag{67}$$

which implies

$$n^{\frac{1-\omega}{2}}\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{\omega+1}{2}} \leq \sum_{i=1}^{n}|x_i(t)|^{\omega+1}. \tag{68}$$

It can deduce from (63), (66) and (68) that

$$\begin{aligned}\dot{V}(t,x) \leq & \, n^{\frac{\vartheta-1}{2}}\Phi^{+}(t)\left(n^{\frac{1}{\vartheta+1}-\frac{1}{2}}\right)^{\vartheta+1}\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{\vartheta+1}{2}} - \Phi^{-}(t)\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{\vartheta+1}{2}} \\ & - n^{\frac{\omega-1}{2}}bn^{\frac{1-\omega}{2}}\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{\omega+1}{2}} \\ = & \left(\Phi^{+}(t) - \Phi^{-}(t)\right)\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\vartheta+1}{2}} - b\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\omega+1}{2}} \\ = & \, \Phi(t)V^{\frac{\vartheta+1}{2}}(t,x) - bV^{\frac{\omega+1}{2}}(t,x). \end{aligned} \tag{69}$$

Obviously, $0 < \frac{\vartheta+1}{2} < 1$ and $\frac{\omega+1}{2} > 1$. By applying Theorem 5, DSNNs (49) is FXTA at origin under the protocol (54). Furthermore, the S-T can be estimated as $T(t_0,\varphi) \leq t_0 + T_{\max}^3$, where $T_{\max}^3$ has been given by (65).

Theorem 10 Under Hypotheses (H1), (H2) and (H4), if the continuous function $\Phi(t)$ of protocol (55) satisfies the condition (A2), then DSNNs (49) is FXTA at origin under the protocol (55). Furthermore, the S-T can be estimated as $T(t_0,\varphi) \leq t_0 + T_{\max}^4$

$$T_{\max}^4 = \frac{2 + \varrho(\Phi^* + N)}{\varrho\ell}. \tag{70}$$

Proof. Similar to Theorem 7 with the same Lyapunov function, substituting the control protocol (55) into (59) and using the condition (H4), it gets

$$\begin{aligned}
\dot{V}(t,x) \leq & -\sum_{i=1}^{n}\left(2c_i - 2a_i(t) - \sum_{j=1}^{n}\left(\mathscr{P}_{ji}^{D}K_i + \mathscr{P}_{ij}^{D}K_j\right)\right)x_i^2(t) \\
& - 2\sum_{i=1}^{n}\left(h_i - \sum_{j=1}^{n}\mathscr{Q}_{ij}^{D}M_j\right)|x_i(t)| + n^{-\frac{\varrho}{2}}\Phi^{+}(t)e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\sum_{i=1}^{n}|x_i(t)|^{2-\varrho} \\
& - \Phi^{-}(t)e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\sum_{i=1}^{n}|x_i(t)|^{2-\varrho} \\
\leq & \, n^{-\frac{\varrho}{2}}\Phi^{+}(t)e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\sum_{i=1}^{n}|x_i(t)|^{2-\varrho} - \Phi^{-}(t)e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\sum_{i=1}^{n}|x_i(t)|^{2-\varrho}. \qquad (71)
\end{aligned}$$

Since $1 > \varrho > 0$, applying Lemma 2, it yields

$$\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n}|x_i(t)|^{2-\varrho}\right)^{\frac{1}{2-\varrho}} \leq n^{\frac{1}{2-\varrho}-\frac{1}{2}}\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{1}{2}}. \qquad (72)$$

Because $0 \leq \Phi^{+}(t)$, $0 \leq \Phi^{-}(t)$ and $\Phi(t) = \Phi^{+}(t) - \Phi^{-}(t)$, it yields from (71) and (72) that

$$\begin{aligned}
\dot{V}(t,x) \leq & \, n^{-\frac{\varrho}{2}}\Phi^{+}(t)e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\left(n^{\frac{1}{2-\varrho}-\frac{1}{2}}\right)^{2-\varrho}\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{2-\varrho}{2}} \\
& - \Phi^{-}(t)e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\left(\sum_{i=1}^{n}|x_i(t)|^2\right)^{\frac{2-\varrho}{2}} \\
= & \, (\Phi^{+}(t) - \Phi^{-}(t))e^{\left(\sum_{i=1}^{n}x_i^2(t)\right)^{\frac{\varrho}{2}}}\left(\sum_{i=1}^{n}x_i^2(t)\right)^{1-\frac{\varrho}{2}} \\
= & \, \Phi(t)e^{V^{\frac{\varrho}{2}}(t,x)}V^{1-\frac{\varrho}{2}}(t,x). \qquad (73)
\end{aligned}$$

Obviously, $0 < \frac{\varrho}{2} < 1$. By applying Theorem 6, DSNNs (49) is FXTA at origin under the protocol (55). Furthermore, the S-T can be estimated as $T(t_0,\varphi) \leq t_0 + T_{\max}^4$, where $T_{\max}^4$ has been given by (70).

5 Illustrative examples

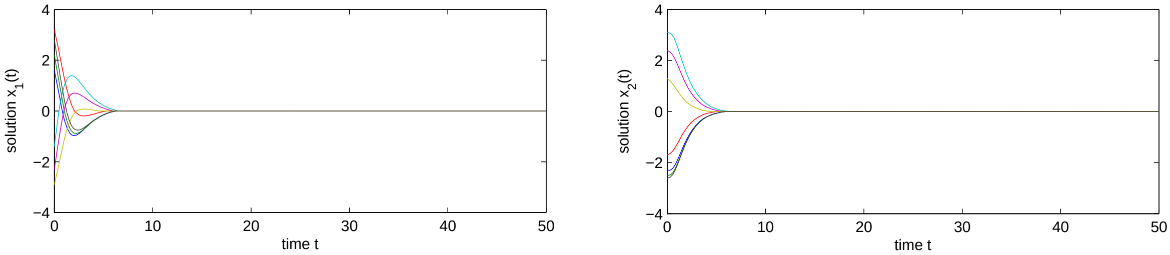
Lemma 3 ([35]) Set $G(t) = \frac{2}{1+t} - t|\cos t|$, $t \in [0,+\infty)$, then there exist $\ell > 0$ and $N > 0$ s.t.

$$\int_{t_0}^{t} G(s)\mathrm{d}s \leq -\ell(t-t_0) + N, \ \forall t \geq t_0 \geq 0,$$

where $\ell = \frac{4}{3\pi}$ and $N = 2\ln\left(1+\frac{3\pi}{2}\right) + 2$.

Example 1 Consider the 2-D DSNNs (49) possessing the following switched connection weights

$$\mathscr{P}_{11}(x_1) = \begin{cases} -0.4, & |x_1| \leq 0.45, \\ -0.3, & |x_1| > 0.45, \end{cases} \quad \mathscr{P}_{12}(x_1) = \begin{cases} 2.2, & |x_1| \leq 0.45, \\ 1.5, & |x_1| > 0.45, \end{cases}$$



(a) (b)

Figure 1: State variables of DSNNs (49) under the protocol (52) in Example 1.

$$\mathscr{P}_{21}(x_2)=\begin{cases}0.2, & |x_2|\le 0.45,\\ 0.3, & |x_2|>0.45,\end{cases}\quad \mathscr{P}_{22}(x_2)=\begin{cases}1.5, & |x_2|\le 0.45,\\ 1.4, & |x_2|>0.45.\end{cases}$$

$$\mathscr{Q}_{11}(x_1)=\begin{cases}-1.3, & |x_1|\le 0.45,\\ -1.1, & |x_1|>0.45,\end{cases}\quad \mathscr{Q}_{12}(x_1)=\begin{cases}1.2, & |x_1|\le 0.45,\\ 0.8, & |x_1|>0.45,\end{cases}$$

$$\mathscr{Q}_{21}(x_2)=\begin{cases}0.1, & |x_2|\le 0.45,\\ 0.2, & |x_2|>0.45,\end{cases}\quad \mathscr{Q}_{22}(x_2)=\begin{cases}-1.6, & |x_2|\le 0.45,\\ -1.2, & |x_2|>0.45.\end{cases}$$

The time delay $\tau(t)=2$. The self-inhibitions are taken as $a_i(t)=\frac{1}{1+t^2}(i=1,2)$, then $\Phi(t)=2\max_{i\in\mathbb{N}}\{a_i(t)\}=\frac{2}{1+t^2}$ and $\int_0^{+\infty}\Phi^+(s)\mathrm{d}s=\int_0^{+\infty}\frac{2}{1+s^2}\mathrm{d}s=\pi<+\infty$. This means the condition (A1) holds. The neuron activations are taken as

$$f_j(z)=0.4z,\ j=1,2,$$

$$g_j(z)=0.02\tanh(z),\ j=1,2.$$

Clearly, f_j satisfies the condition (H1) with $K_j=0.4$ $(j=1,2)$ and g_j satisfies the condition (H2) with $M_j=0.02$ $(j=1,2)$. Let us select the control protocol (52) with $c_1=1$, $c_2=1.5$, $h_1=0.06$ and $h_2=0.04$. Clearly, $2=2c_1\ge\sum\limits_{j=1}^{n}\left(\mathscr{P}_{1j}^DK_j+\mathscr{P}_{j1}^DK_1\right)=1.32$, $3=2c_2\ge\sum\limits_{j=1}^{n}\left(\mathscr{P}_{ij}^DK_j+\mathscr{P}_{j2}^DK_2\right)=2.2$, $0.06=h_1\ge\sum\limits_{j=1}^{n}\mathscr{Q}_{1j}^DM_j=0.05$ and $0.04=h_2\ge\sum\limits_{j=1}^{n}\mathscr{Q}_{2j}^DM_j=0.036$. Thus, the condition (H3) is satisfied. By Theorem 7, the origin of DSNNs (49) is stable under the control protocol (52). Take 7 random initial states and the initial time $t_0=0$, the simulation is given in Fig. 1.

Example 2 Consider the 2-D DSNNs (49) possessing the same switched connection weights as Example 1. The time delay $\tau(t)=1$. The self-inhibitions are taken as $a_i(t)=1(i=1,2)$. The activations are taken as

$$f_j(z)=0.8z,\ j=1,2,$$

$$g_j(z)=0.5\sin(z),\ j=1,2.$$

Clearly, f_j satisfies the condition (H1) with $K_j=0.8$ $(j=1,2)$ and g_j satisfies the condition (H2) with $M_j=0.5$ $(j=1,2)$. In the control protocol (53), take $c_1=3$, $c_2=3.5$, $h_1=2$, $h_2=1$, $\vartheta=0.5$ and $\Phi(t)=\frac{2}{1+t+t^2}-t|\cos t|$. Obviously,

$$\Phi^*=\int_0^{+\infty}\Phi^+(s)\mathrm{d}s\le\int_0^{+\infty}\frac{2}{1+s^2}\mathrm{d}s=\pi<+\infty. \tag{74}$$

Because $\Phi(t) = \Phi^+(t) - \Phi^-(t)$, applying Lemma 3, it yields

$$\begin{aligned}\int_{t_0}^{t} \Phi^-(s)\mathrm{d}s &= \int_{t_0}^{t} \Phi^+(s)\mathrm{d}s - \int_{t_0}^{t} \Phi(s)\mathrm{d}s \\ &\geq -\int_{t_0}^{t} \Phi(s)\mathrm{d}s = -\int_{t_0}^{t} \left(\frac{2}{1+s+s^2} - s|\cos s|\right)\mathrm{d}s \\ &\geq -\int_{t_0}^{t} \left(\frac{2}{1+s} - s|\cos s|\right)\mathrm{d}s \\ &\geq \ell(t-t_0) - N, \end{aligned} \tag{75}$$

where $\ell = \frac{4}{3\pi}$ and $N = 2\ln\left(1+\frac{3\pi}{2}\right) + 2$. This means the condition (A2) holds. Clearly, $6 = 2c_1 \geq 2a_1(t) + \sum_{j=1}^{n}\left(\mathscr{P}_{1j}^{D}K_j + \mathscr{P}_{j1}^{D}K_1\right) = 4.64$, $7 = 2c_2 \geq 2a_2(t) + \sum_{j=1}^{n}\left(\mathscr{P}_{2j}^{D}K_j + \mathscr{P}_{j2}^{D}K_2\right) = 6.4$ and $2 = h_1 \geq \sum_{j=1}^{n}\mathscr{Q}_{1j}^{D}M_j = 1.25$ and $1 = h_2 \geq \sum_{j=1}^{n}\mathscr{Q}_{2j}^{D}M_j = 0.9$. Thus, the condition (H4) is satisfied. By Theorem 8, the origin of DSNNs (49) is FNTA under the control protocol (53). Take 7 random initial states and the initial time $t_0 = 0$, the simulation is given in Fig. 2.

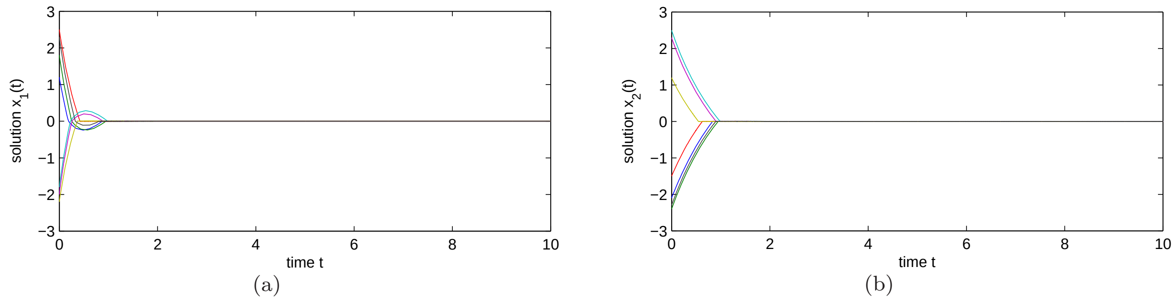


Figure 2: State variables of DSNNs (49) under the protocol (53) in Example 2.

Example 3 Consider the 3-D DSNNs (49) possessing the following switched connection weights

$$\mathscr{P}_{11}(x_1) = \begin{cases} 1.5, & |x_1| \leq 0.8, \\ 0.6, & |x_1| > 0.8, \end{cases} \quad \mathscr{P}_{12}(x_1) = \begin{cases} -0.8, & |x_1| \leq 0.8, \\ -0.7, & |x_1| > 0.8, \end{cases}$$

$$\mathscr{P}_{13}(x_1) = \begin{cases} -0.3, & |x_1| \leq 0.8, \\ -0.2, & |x_1| > 0.8, \end{cases} \quad \mathscr{P}_{21}(x_2) = \begin{cases} 0.6, & |x_2| \leq 0.8, \\ 0.4, & |x_2| > 0.8, \end{cases}$$

$$\mathscr{P}_{22}(x_2) = \begin{cases} 1.2, & |x_2| \leq 0.8, \\ 0.5, & |x_2| > 0.8, \end{cases} \quad \mathscr{P}_{23}(x_2) = \begin{cases} 0.9, & |x_2| \leq 0.8, \\ 1.3, & |x_2| > 0.8. \end{cases}$$

$$\mathscr{P}_{31}(x_3) = \begin{cases} 0.7, & |x_3| \leq 0.8, \\ 1.2, & |x_3| > 0.8, \end{cases} \quad \mathscr{P}_{32}(x_3) = \begin{cases} -0.4, & |x_3| \leq 0.8, \\ -0.2, & |x_3| > 0.8. \end{cases}$$

$$\mathscr{P}_{33}(x_3) = \begin{cases} 0.3, & |x_3| \leq 0.8, \\ 0.5, & |x_3| > 0.8, \end{cases} \quad \mathscr{Q}_{11}(x_1) = \begin{cases} 2, & |x_1| \leq 0.8, \\ 1.5, & |x_1| > 0.8. \end{cases}$$

$$\mathscr{Q}_{12}(x_1) = \begin{cases} -1.8, & |x_1| \leq 0.8, \\ -1.6, & |x_1| > 0.8, \end{cases} \quad \mathscr{Q}_{13}(x_1) = \begin{cases} -1.4, & |x_1| \leq 0.8, \\ -1.2, & |x_1| > 0.8, \end{cases}$$

$$\mathscr{Q}_{21}(x_2) = \begin{cases} 1.6, & |x_2| \le 0.8, \\ 1.3, & |x_2| > 0.8, \end{cases} \quad \mathscr{Q}_{22}(x_2) = \begin{cases} 2.2, & |x_2| \le 0.8, \\ 1.7, & |x_2| > 0.8. \end{cases}$$

$$\mathscr{Q}_{23}(x_2) = \begin{cases} 1.8, & |x_2| \le 0.8, \\ 2.1, & |x_2| > 0.8, \end{cases} \quad \mathscr{Q}_{31}(x_3) = \begin{cases} 1.4, & |x_3| \le 0.8, \\ 1.6, & |x_3| > 0.8. \end{cases}$$

$$\mathscr{Q}_{32}(x_3) = \begin{cases} -1.9, & |x_3| \le 0.8, \\ -1.8, & |x_3| > 0.8, \end{cases} \quad \mathscr{Q}_{33}(x_3) = \begin{cases} 1.5, & |x_3| \le 0.8, \\ 2.1, & |x_3| > 0.8. \end{cases}$$

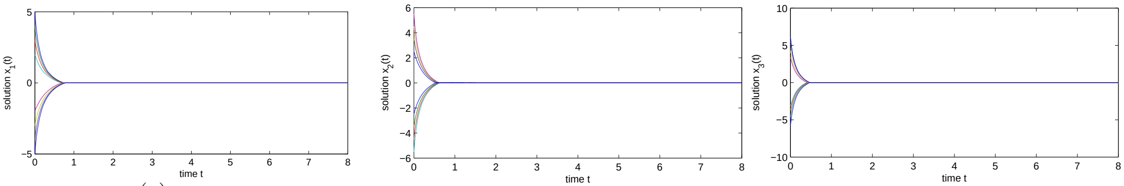
The time delay $\tau(t) = 1$. The activations are taken as

$$f_j(z) = 0.5z, \ j = 1, 2, 3,$$

$$g_j(z) = 0.2 \tanh(z), \ j = 1, 2, 3.$$

Clearly, f_j satisfies the condition (H1) with $K_j = 0.5$ $(j = 1, 2, 3)$ and g_j satisfies the condition (H2) with $M_j = 0.2$ $(j = 1, 2, 3)$.

Let us select the control protocol (54) with $c_1 = c_2 = c_3 = 3$, $h_1 = h_2 = h_3 = 2$, $\vartheta = \frac{1}{2}$, $\omega = 2$ and $b = 1$. Take $\Phi(t) = \frac{2}{1+t+t^2} - t|\cos t|$ which satisfies the condition (A2). The self-inhibitions are taken as $a_i(t) = 1 (i = 1, 2, 3)$. Clearly, $6 = 2c_1 \ge 2a_1(t) + \sum_{j=1}^{n} \left(\mathscr{P}_{1j}^D K_j + \mathscr{P}_{j1}^D K_1\right) = 4.95$, $6 = 2c_2 \ge 2a_2(t) + \sum_{j=1}^{n} \left(\mathscr{P}_{2j}^D K_j + \mathscr{P}_{j2}^D K_2\right) = 4.75$, $6 = 2c_3 \ge 2a_3(t) + \sum_{j=1}^{n} \left(\mathscr{P}_{3j}^D K_j + \mathscr{P}_{j3}^D K_3\right) = 4.1$ and $2 = h_1 \ge \sum_{j=1}^{n} \mathscr{Q}_{1j}^D M_j = 1.04$, $2 = h_2 \ge \sum_{j=1}^{n} \mathscr{Q}_{2j}^D M_j = 1.18$ and $2 = h_3 \ge \sum_{j=1}^{n} \mathscr{Q}_{3j}^D M_j = 1.12$. Thus, the condition (H4) is satisfied. By Theorem 9, DSNNs (49) is FXTA at origin under the protocol (54). Take 8 random initial states and the initial time $t_0 = 0$, the simulation is given in Fig. 3.



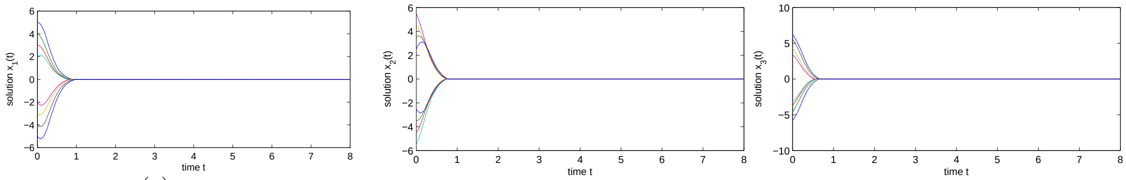
(a) (b) (c)

Figure 3: State variables of DSNNs (49) under the protocol (54) in Example 3.

Let us select the control protocol (55) with $c_1 = c_2 = c_3 = 4$, $h_1 = h_2 = h_3 = 2$ and $\varrho = \frac{1}{2}$. Take $\Phi(t) = \frac{2}{1+t+t^2} - t|\cos t|$ which satisfies the condition (A2). The self-inhibitions are taken as $a_i(t) = 2 (i = 1, 2, 3)$. Clearly, the condition (H4) is satisfied. By Theorem 10, DSNNs (49) is FXTA at origin under the protocol (55). Take 8 random initial states and the initial time $t_0 = 0$, the simulation is given in Fig. 4.

6 Conclusions

This paper explored the stability and finite/fixed-time attractivity problems for TDFS. By means of time-delayed differential inclusions and indefinite derivative Lyapunov function method, several novel criteria were provided to ascertain the stability, uniform stability, uniform asymptotic stability and uniform boundedness of zero for TDFS. Moreover, several more relaxed FNTA and FXTA criteria were developed for TDFS, in which some more flexible and diverse of settling time estimates were provided.



(a) (b) (c)

Figure 4: State variables of DSNNs (49) under the protocol (55) in Example 3.

Different from the traditional neural networks model possessing discontinuous activation, the DSNNs model with switching neuron connection weights was considered. By designing four control protocols with indefinite time-varying control gains, the stabilization and FNTA/FXTA of DSNNs have been analyzed. Simulation examples were provided to substantiate the effectiveness of theoretical results. In future studies, the stochastic/ impulsive neurodynamics and feedback control problems are expected to be explored further [36-41].

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