Hybrid projective method for solving split-null inclusion, variational inequality and hierarchical fixed point problems

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Abstract: In this paper, we solve a common problem of split null inclusion problem, variational inequality and hierarchical fixed point problem for nonexpansive and quasi-nonexpansive mappings using hybrid projective method. Under suitable conditions in a real Hilbert space, strong convergent theorems are proved. Further, we give an example for supporting our main result in infinitely dimensional spaces and show the efficiency of the algorithm by applying to solve the signal recovery problem.

Keywords: Split null inclusion problem, Variational inequality, Hierarchical fixed point problem, Signal recovery, Maximal monotone operators, Nonexpansive mapping, Quasi-nonexpansive mappings, Strong convergence.

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1 Introduction

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two real Hilbert spaces with inner products $\langle\cdot,\cdot\rangle$ and induced norms $\|\cdot\|$, respectively. Let $\mathcal{C}\subseteq\mathbf{H}_1$ and $\mathcal{Q}\subseteq\mathbf{H}_2$ be nonempty, closed and convex sets. A mapping $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ is called nonexpansive if $\|\mathcal{T}x-\mathcal{T}y\|\leq\|x-y\|$, for all $x,y\in\mathcal{C}$. Fix$(\mathcal{T})$ is denoted for the set of fixed points of $\mathcal{T}$, i.e., Fix$(\mathcal{T})$:= $\{x\in\mathcal{C}:\mathcal{T}x=x\}$. In this paper, we focus our attention on the following split null inclusion problem (in short, $\mathrm{S_P}$NIP) which was introduced Byrne *et al.* [8]: Find $x^*\in\mathbf{H}_1$ such that

$$0\in\mathcal{B}_1(x^*), \tag{1.1}$$

such that

$$y^*=\mathcal{A}x^*\in\mathbf{H}_2 \;\text{ solves } 0\in\mathcal{B}_2(y^*). \tag{1.2}$$

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where $\mathcal{B}_1 : \mathbf{H}_1 \to 2^{\mathbf{H}_1}$, $\mathcal{B}_2 : \mathbf{H}_2 \to 2^{\mathbf{H}_2}$ are multi-valued maximal monotone operators and $\mathcal{A} : \mathbf{H}_1 \to \mathbf{H}_2$ is a bounded linear operator. The solution set of $\mathrm{S_P}$NIP(1.1)-(1.2) is denoted by $\Omega = \{x^* \in \mathbf{H}_1 : x^* \in \mathrm{Sol(NIP(1.1))}$ and $\mathcal{A}x^* \in \mathrm{Sol(NIP(1.2))}\}$. Byrne *et al.* [8] studied the weak convergence theorems of iterative method for $\mathrm{S_P}$NPP(1.1)-(1.2). For a given $x_0 \in \mathbf{H}_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme: for $n \geq 1$,

$$x_{n+1} = \mathcal{J}_\lambda^{\mathcal{B}_1}(x_n + \gamma \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}x_n), \quad \text{for } \lambda > 0, \tag{1.3}$$

where $\mathcal{I}$ is an identity mapping, $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ and $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator $\mathcal{A}^*\mathcal{A}$. For obtaining strong convergence, Kazmi and Rizvi [17] modified viscosity method to solve the problem $\mathrm{S_P}$NIP(1.1)-(1.2) and fixed point problem for a nonexpansive mapping. For further related work, see [23].

It's well known that fixed point problems have been used to solve a powerful and effective method for solving many issues that emerge from real-world applications, for example see in [14, 15, 19, 21, 22, 26]. In 2006, Moudafi and Mainge [21] introduced and studied the following hierarchical fixed point problem (in short, HFPP) for a nonexpansive mapping $\mathcal{T}$ with respect to another nonexpansive mapping $\mathcal{S}$ on $\mathcal{C}$: Find $x^* \in \mathrm{Fix}(\mathcal{T})$ such that

$$\langle x^* - \mathcal{S}x^*, x^* - x \rangle \leq 0, \ \forall x \in \mathrm{Fix}(\mathcal{T}), \tag{1.4}$$

where $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ is a nonexpansive mapping. We know that HFPP(1.4) is equivalent to the following fixed point problem of the mapping $\mathcal{P}_{\mathrm{Fix}(\mathcal{T})} \circ \mathcal{S}$ where $P_{\mathrm{Fix}(\mathcal{T})}$ is the metric projection of $\mathbf{H}_1$ onto $\mathrm{Fix}(\mathcal{T})$. The solution set of HFPP(1.4) is denoted by $\mathcal{H} := \{x^* \in \mathcal{C} : x^* = (P_{\mathrm{Fix}(\mathcal{T})} \circ \mathcal{S})x^*\}$. Note that if $\mathcal{H} \neq \emptyset$, $\mathcal{H}$ is closed and convex. HFPP(1.4) generalizes many branches of optimization such as monotone variational inequality on fixed point sets, minimization problems over equilibrium constraints, hierarchical minimization problems, *etc.*, see for instance [9, 24].

On the other hand, we consider the classical variational inequality(VI) which is to find $x^* \in \mathcal{C}$ such that

$$\langle \mathcal{D}x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{C}, \tag{1.5}$$

introduced in [13] where $\mathcal{D} : \mathbf{H}_1 \to \mathbf{H}_1$. The set of solutions of VI(1.5) is denoted by Sol(VI). Note that the projected gradient scheme for solving VI(1.5) is:

$$x_{n+1} = P_\mathcal{C}(\mathcal{I} - \mu\mathcal{D})x_n, \ \forall n \geq 1, \tag{1.6}$$

where $\mu > 0$ and $P_\mathcal{C}$ is the metric projection of $\mathbf{H}_1$ onto $\mathcal{C}$. In order to converge, the algorithm (1.6) requires the Lipschitz condition on the operator $\mathcal{D}$. Indeed, if $\mathcal{D}$ is L-Lipschitz continuous with $0 < \mu < \frac{2}{L}$, then there exists a unique point in Sol(VI) and the sequence $\{x_n\}$ generated by (1.6) converge strongly to this point. There is no analytic expression for the metric projection operator in most cases. So the algorithm (1.6) is not very convenient in the practical calculation. Further, it was found that if $\mathcal{C}$ is a fixed point set of a nonexpansive mapping, then the metric projection is not be used. In 2001, Yamada [25] introduced the following hybrid steepest descent method:

$$x_{n+1} = P_{\mathcal{C}}(\mathcal{I} - \mu\beta_n\mathcal{D})\mathcal{T}x_n, \ \forall n \geq 1. \tag{1.7}$$

Under certain conditions, the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique point in Sol(VI) over the fixed point of $\mathcal{T}$.

In this paper, we modify a projective iterative method to approximate a common solution of split null inclusion problem, variational inequality and hierarchical fixed point problem for nonexpansive and quasi-nonexpansive mappings in real Hilbert spaces. Further, we prove that sequences generated by the proposed hybrid projective iterative method converge strongly to a common solution of these problems. As applications, signal recovery is considered.

2 Preliminaries

We recall some concepts and results needed in the sequel. Let the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively, and $\omega_w(x_n)$ denote the set of all weak limits of the sequence $\{x_n\}$.

Definition 2.1. *A mapping $\mathcal{D} : \mathbf{H}_1 \to \mathbf{H}_1$ is said to be:*

(i) *monotone, if*

$$\langle \mathcal{D}x - \mathcal{D}y, x - y\rangle \geq 0, \quad \forall x, y \in \mathbf{H}_1;$$

(ii) *k-strongly monotone, if there exists a constant $k \in \mathbb{R}$ with $k > 0$ such that*

$$\langle \mathcal{D}x - \mathcal{D}y, x - y\rangle \ \geq k\|x - y\|^2, \quad \forall x, y \in \mathbf{H}_1;$$

(ii) *k-inverse strongly monotone, if there exists a constant $k \in \mathbb{R}$ with $k > 0$ such that*

$$\langle \mathcal{D}x - \mathcal{D}y, x - y\rangle \ \geq k\|\mathcal{D}x - \mathcal{D}y\|^2, \quad \forall x, y \in \mathbf{H}_1;$$

(iii) *L-Lipschitz continuous, if there exists a constant $L > 0$ such that*

$$\|\mathcal{D}x - \mathcal{D}y\| \leq L\|x - y\|, \quad \forall x, y \in \mathbf{H}_1;$$

(iv) *firmly nonexpansive, if it is k-inverse strongly monotone with $k = 1$.*

We note that if $\mathcal{D}$ is an k-inverse strongly monotone mapping, then $\mathcal{D}$ is monotone and $\frac{1}{L}$-Lipschitz continuous.

Definition 2.2. [7]. *A multi-valued mapping $\mathcal{D} : \mathbf{H}_1 \to 2^{\mathbf{H}_1}$ is said to be:*

(i) *monotone if*

$$\langle u - v, x - y \rangle \geq 0, \text{ whenever } u \in \mathcal{D}(x),\ v \in \mathcal{D}(y);$$

(ii) *maximal monotone if* $\mathcal{D}$ *is monotone and the graph,* $\mathrm{graph}(\mathcal{D}) := \{(x, y) \in \mathbf{H}_1 \times \mathbf{H}_1 : y \in \mathcal{D}(x)\}$, *is not properly contained in the graph of any other monotone mapping.*

It is well known that for each $x \in \mathbf{H}_1$ and $\lambda > 0$ there is a unique $z \in \mathbf{H}_1$ such that $x \in (\mathcal{I} + \lambda\mathcal{D})z$. The mapping $\mathcal{J}_\lambda^{\mathcal{D}} := (\mathcal{I} + \lambda\mathcal{D})^{-1}$ is called the resolvent of $\mathcal{D}$. It is a single-valued and firmly nonexpansive mapping defined on $\mathbf{H}_1$.

Lemma 2.1. [12] *Let* $\mathcal{T}$ *is a nonexpansive mapping on* $\mathbf{H}_1$ *then* $\mathcal{T}$ *is demiclosed on* $\mathbf{H}_1$ *in the sense that, if* $\{x_n\}$ *converges weakly to* $x \in \mathbf{H}_1$ *and* $\{x_n - \mathcal{T}x_n\}$ *converges strongly to* 0*, then* $x \in \mathrm{Fix}(\mathcal{T})$.

Lemma 2.2. [2] *Let* $\mathcal{C} \subset \mathbf{H}_1$ *be a nonempty, closed and convex set and let* $\mathcal{T} : \mathcal{C} \to \mathbf{H}_1$ *be a nonexpansive mapping. Then* $\mathrm{Fix}(\mathcal{T})$ *is closed and convex.*

3 Strong convergence theorem

In this section, we prove a strong convergence theorem to approximate a common solution of SpNIP(1.1)-(1.2), VI(1.5) and HFPP(1.4) for a nonexpansive mapping $\mathcal{T}$ and $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ be a continuous quasi-nonexpansive mapping.

Algorithm 3.1.

Initialization: *Choose* $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$ *be real sequences in* $(0, 1)$ *and* $\{\mu\beta_n\} \subset (0, 2k)$ *and select an arbitrary starting point* x_0*: Set* $n = 0$.

Iterative Steps: *Given the current iterate* x_n*, for* $\lambda > 0$*:*

Step 1. *Compute*

$$\left.\begin{array}{l}
x_0 \in \mathcal{C},\quad \mathcal{C}_0 = \mathcal{C};\\
w_n = (1 - \delta_n)x_n + \delta_n \mathcal{P}_{\mathcal{C}}(x_n - \mu\beta_n \mathcal{D}x_n);\\
y_n = (1 - \sigma_n)w_n + \sigma_n \mathcal{S}x_n;\\
u_n = (1 - \alpha_n)x_n + \alpha_n \mathcal{T}y_n;\\
z_n = \mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n);\\
C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\};\\
x_{n+1} = \mathcal{P}_{\mathcal{C}_{n+1}} x_0,\ n \geq 0,
\end{array}\right\} \tag{3.1}$$

where $\gamma \in \left(0, \dfrac{1}{\|A\|^2}\right)$.

Step 2. *Set* $n := n + 1$ *and go to* ***Step 1****.*

Theorem 3.1. *Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be real Hilbert spaces and $\mathcal{C} \subseteq \mathbf{H}_1$ and $\mathcal{Q} \subseteq \mathbf{H}_2$ be nonempty, closed and convex sets. Let $\mathcal{A} : \mathbf{H}_1 \to \mathbf{H}_2$ be a bounded linear operator with its adjoint operator $\mathcal{A}^*$; let $\mathcal{B}_1 : \mathbf{H}_1 \to 2^{\mathbf{H}_1}$, $\mathcal{B}_2 : \mathbf{H}_2 \to 2^{\mathbf{H}_2}$ be multi-valued maximal monotone operators. Let $\mathcal{D} : \mathcal{C} \to \mathbf{H}_1$ be k-inverse strongly monotone mappings, let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a nonexpansive mapping and $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ be a continuous quasi-nonexpansive mapping such that $\mathcal{I}-\mathcal{S}$ is monotone and $\Gamma = \Omega \cap \mathcal{H} \cap \mathrm{Fix}(\mathcal{S}) \cap Sol(VI) \neq \emptyset$. Assume that the following conditions hold:*
(C1) $\lim_{n\to\infty} \inf \alpha_n > 0$*;*
(C2) $0 < \lim_{n\to\infty} \inf \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$*;*
(C3) $0 < \lim_{n\to\infty} \inf \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$*;*
(C4) $0 < \lim_{n\to\infty} \inf \mu\beta_n \leq \limsup_{n\to\infty} \mu\beta_n < 2k$.
Then the iterative sequences $\{z_n\}$ and $\{x_n\}$ be generated by Algorithm (3.1) converges strongly to $z \in \Gamma$, where $z = \mathcal{P}_\Gamma x_0$.

Proof. We divide the proof into several steps.

Step I. First, we show that Γ and $\mathcal{C}_n$ for all $n \geq 0$ both are closed and convex. Since $\Gamma \neq \emptyset$, it follows from Lemma 2.2 that Sol(NIP(1.1)) $= \mathrm{Fix}(\mathcal{J}_\lambda^{\mathcal{B}_1})$ and Sol(NIP(1.2)) $= \mathrm{Fix}(\mathcal{J}_\lambda^{\mathcal{B}_2})$ are closed and convex sets. Clearly $\mathcal{H}$ is closed and convex, since $\mathcal{H} = \mathrm{Fix}(P_{\mathrm{Fix}(\mathcal{T})} \circ \mathcal{S}) \neq \emptyset$. Further, it is easy to observe that $\mathrm{Fix}(\mathcal{S})$ and Sol(VI)are closed and convex. Thus, Γ is nonempty, closed and convex and $\mathcal{P}_\Gamma x_0$ is then well defined.

Next, we show that $\mathcal{C}_{n+1}$ is closed and convex. From the assumption, we see that $\mathcal{C}_0 = \mathcal{C}$ is closed and convex. Suppose that $\mathcal{C}_k$ is closed and convex for some $k \geq 1$. Next, we show that $\mathcal{C}_{k+1}$ is closed and convex for some k. For any $z \in \mathcal{C}_k$, we have

$$\begin{aligned} & & \|z_k - z\|^2 &\leq \|x_k - z\|^2 \\ \Leftrightarrow & & \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z\rangle &\leq 0. \end{aligned} \tag{3.2}$$

We easily observe from (3.2) that $\mathcal{C}_{k+1}$ is closed and convex for all $k \geq 1$. Therefore, $\mathcal{C}_n$ is closed and convex for all $n \geq 0$.

Step II. $\Gamma \subset \mathcal{C}_n$ for each $n \geq 0$, $\{x_n\}$ is well defined and the sequences $\{x_n\}$, $\{u_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. Let $p \in \Gamma$ then $p \in \mathcal{C}$. Since $\mathcal{P}_\mathcal{C}$ is firmly nonexpansive, we estimate

$$\begin{aligned} \|w_n - p\|^2 &\leq (1-\delta_n)\|x_n - p\|^2 + \delta_n\|\mathcal{P}_\mathcal{C}(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - \mathcal{P}_\mathcal{C}(\mathcal{I} - \mu\beta_n\mathcal{D})p\|^2 \\ &\leq (1-\delta_n)\|x_n - p\|^2 + \delta_n\|(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - (\mathcal{I} - \mu\beta_n\mathcal{D})p\|^2 \\ &\leq \|x_n - p\|^2 + \delta_n\big(\mu^2\beta_n^2\|\mathcal{D}x_n - \mathcal{D}p\|^2 - 2\mu\beta_n\langle x_n - p, \mathcal{D}x_n - \mathcal{D}p\rangle\big) \\ &\leq \|x_n - p\|^2 + \delta_n\big(\mu^2\beta_n^2\|\mathcal{D}x_n - \mathcal{D}p\|^2 - 2\mu\beta_n k\|\mathcal{D}x_n - \mathcal{D}p\|^2\big) \\ &\leq \|x_n - p\|^2 - \delta_n\mu\beta_n(2k - \mu\beta_n)\|\mathcal{D}x_n - \mathcal{D}p\|^2 \end{aligned}$$

$$\leq \quad \|x_n - p\|^2, \tag{3.3}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \|\sigma_n \mathcal{S}x_n + (1-\sigma_n)w_n - p\|^2 \\
&\leq (1-\sigma_n)\|w_n - p\|^2 + \sigma_n\|\mathcal{S}x_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}x_n - w_n\|^2 \\
&\leq \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|x_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}x_n - w_n\|^2 \\
&\leq \|x_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}x_n - w_n\|^2 \qquad (3.4) \\
&\leq \|x_n - p\|^2, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\|u_n - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n \mathcal{T}y_n - p\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{T}y_n - p\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p)\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[\|x_n - p)\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}x_n - w_n\|^2] \\
&\leq \|x_n - p\|^2 - \sigma_n(1-\sigma_n)\|\mathcal{S}x_n - w_n\|^2 \qquad (3.6) \\
&\leq \|x_n - p\|^2. \qquad (3.7)
\end{aligned}$$

Since $p \in \Gamma$ then $p \in \Omega$ and hence $\mathcal{J}_\lambda^{\mathcal{B}_1} p = p$ and $\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}p = \mathcal{A}p$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|\mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - p\|^2 \\
&= \|\mathcal{J}_\lambda^{\mathcal{B}_2}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - \mathcal{J}_\lambda^{\mathcal{B}_1}(p)\|^2 \\
&\leq \|u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n - p\|^2 \\
&= \|u_n - p\|^2 + \gamma^2\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2 + 2\gamma\langle u_n - p, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= \|u_n - p\|^2 + \gamma^2\|\mathcal{A}\|^2\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2 + 2\gamma\langle u_n - p, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle. \qquad (3.8)
\end{aligned}$$

Further, we have

$$\begin{aligned}
2\gamma\langle u_n &- p, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= 2\gamma\langle \mathcal{A}u_n - \mathcal{A}p, (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= 2\gamma\langle \mathcal{A}u_n - \mathcal{A}p + (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n - (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n, (\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle \\
&= 2\gamma\Big\{\langle \mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}p, \mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}u_n\rangle - \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\Big\} \\
&= \gamma\{\|\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}p\|^2 + \|\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{A}u_n\|^2 - \|\mathcal{A}u_n - \mathcal{A}p\|^2 - 2\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\} \\
&= \gamma\{\|\mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}u_n - \mathcal{J}_\lambda^{\mathcal{B}_2}\mathcal{A}p\|^2 + \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2 - \|\mathcal{A}u_n - \mathcal{A}p\|^2 - 2\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\} \\
&\leq \gamma\{\|\mathcal{A}u_n - \mathcal{A}p\|^2 - \|\mathcal{A}u_n - \mathcal{A}p\|^2 - \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\} \\
&= -\gamma\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2. \qquad (3.9)
\end{aligned}$$

Now, using (3.9) in (3.8), we obtain

$$\|z_n - p\|^2 \quad \le \quad \|u_n - p\|^2 - \gamma(1 - \gamma\|\mathcal{A}\|^2)\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2. \tag{3.10}$$

Next, using (3.7) and (3.10), we estimate

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \gamma(1 - \gamma\|\mathcal{A}\|^2)\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2. \tag{3.11}$$

Since $\gamma \in \left(0, \dfrac{1}{\|\mathcal{A}\|^2}\right)$, (3.11) implies

$$\|z_n - p\| \le \|x_n - p\|. \tag{3.12}$$

This implies that $p \in \mathcal{C}_{n+1}$ and hence $\Gamma \subset \mathcal{C}_{n+1}$ for all $n \ge 0$. Consequently, $\mathcal{C}_{n+1}$ is nonempty, closed and convex and hence $x_{n+1} = \mathcal{P}_{\mathcal{C}_{n+1}}x_0$ is well defined for all $n \ge 0$. Thus the sequence $\{x_n\}$ is well defined.

Let $l = \mathcal{P}_\Gamma x_0$. From $x_{n+1} = \mathcal{P}_{\mathcal{C}_{n+1}}x_0$ and $l \in \Gamma \subset \mathcal{C}_{n+1}$, we have

$$\|x_{n+1} - x_0\| \le \|l - x_0\|, \ \forall n \ge 0. \tag{3.13}$$

Therefore $\{x_n\}$ is bounded. It also follows from (3.3), (3.5), (3.7) and (3.12) that the sequences $\{w_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded.

Step III. $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$; $\lim\limits_{n\to\infty} \|z_n - x_n\| = 0$; $\lim\limits_{n\to\infty} \|z_n - u_n\| = 0$; $\lim\limits_{n\to\infty} \|\mathcal{P}_\mathcal{C}(x_n - \mu\beta_n\mathcal{D}x_n) - x_n\| = 0$; $\lim\limits_{n\to\infty} \|w_n - y_n\| = 0$ and $\lim\limits_{n\to\infty} \|x_n - u_n\| = 0$.
Since $x_n = \mathcal{P}_{\mathcal{C}_n}x_0$, $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ and $x_{n+1} \in \mathcal{C}_n$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \ \forall n \ge 0. \tag{3.14}$$

It follows from (3.13) and (3.14) that the sequence $\{\|x_n - x_0\|\}$ is monotonically increasing and bounded, and hence convergent. Therefore $\lim\limits_{n\to\infty} \|x_n - x_0\|$ exists.

By the properties of the metric projection $\mathcal{P}_{\mathcal{C}_n}$ that $x_n = \mathcal{P}_{\mathcal{C}_n}x_0$ and $x_{n+1} \in \mathcal{C}_{n+1}$, we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \ \forall n \ge 0,$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

Since $x_{n+1} = \mathcal{P}_{\mathcal{C}_{n+1}}x_0 \in \mathcal{C}_{n+1}$, it follows that

$$\|z_n - x_{n+1}\| \le \|x_n - x_{n+1}\|. \tag{3.16}$$

Hence, it follows from (3.15) and (3.16) that

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{3.17}$$

Since

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|, \tag{3.18}$$

it follows from (3.15), (3.17) and (3.18) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

Next, from the assumption $\liminf_{n\to\infty} \sigma_n > 0$, (3.11) and (3.19), we have

$$\lim_{n\to\infty} \|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| = 0. \tag{3.20}$$

Since $\mathcal{J}_\lambda^{\mathcal{B}_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|\mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - p\|^2 \\
&= \|\mathcal{J}_\lambda^{\mathcal{B}_1}(u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n) - \mathcal{J}_\lambda^{\mathcal{B}_1}p\|^2 \\
&\leq \langle z_n - p, u_n + \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n - p\rangle \\
&= \frac{1}{2}\left[\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - u_n - \gamma\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|^2\right] \\
&\leq \frac{1}{2}\big[\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - u_n\|^2 - \gamma^2\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\quad +2\gamma\langle z_n - u_n, \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\rangle\big] \\
&\leq \frac{1}{2}\big[\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - u_n\|^2 - \gamma^2\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\quad +2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|\big],
\end{aligned}$$

which in turn yields

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \|z_n - u_n\|^2 + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|, \tag{3.21}$$

and this together with (3.7) implies that

$$\begin{aligned}
\|z_n - u_n\|^2 &\leq \|u_n - p\|^2 - \|z_n - p\|^2 + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\| \\
&\leq L_1\|x_n - z_n\| + 2\gamma\|z_n - u_n\|\|\mathcal{A}^*\|\|(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n\|. \qquad (3.22)
\end{aligned}$$

Hence, it follows from (3.19), (3.20) and (3.22) that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{3.23}$$

Since

$$\|x_n - u_n\| \leq \|x_n - z_n\| + \|z_n - u_n\|. \tag{3.24}$$

It follows from (3.19), (3.23) and (3.24) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.25}$$

It follows from (3.1) that

$$\alpha_n\|\mathcal{T}y_n - x_n\| = \|u_n - x_n\|. \tag{3.26}$$

It follows from (3.25), (3.26) and $\lim_{n\to\infty} \inf \alpha_n > 0$ that

$$\lim_{n\to\infty} \|\mathcal{T}y_n - x_n\| = 0. \tag{3.27}$$

From the assumption $\lim_{n\to\infty} \inf \sigma_n > 0$, (3.6) and (3.27), we have

$$\lim_{n\to\infty} \|\mathcal{S}x_n - w_n\| = 0. \tag{3.28}$$

It follows from (3.1) that

$$\|y_n - w_n\| = \sigma_n\|\mathcal{S}x_n - w_n\| \tag{3.29}$$

It follows from (3.28), (3.29) that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.30}$$

Since $\{\mu\beta_n\} \subset (0, 2k)$, $\mathcal{P}_\mathcal{C}(\mathcal{I} - \mu\beta_n\mathcal{D})$ is nonexpansive. It follows from $\mathcal{T}$ and $\mathcal{S}$ are nonexpansive that

$$\begin{aligned}
\|u_n - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{T}y_n - p\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\big((1-\sigma_n)\|w_n - p\|^2 + \sigma_n\|\mathcal{S}x_n - p\|^2\big) \\
&\leq \|x_n - p\|^2 - \alpha_n(1-\sigma_n)(1-\delta_n)\delta_n\|\mathcal{P}_\mathcal{C}(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - x_n\|^2.
\end{aligned} \tag{3.31}$$

It follows the assumptions (C1)-(C3), (3.25) and (3.31) that

$$\lim_{n\to\infty} \|\mathcal{P}_\mathcal{C}(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - x_n\| = 0. \tag{3.32}$$

It follows from the assumption (C3) and (3.32) that

$$\|w_n - x_n\| \leq \delta_n\|\mathcal{P}_\mathcal{C}(\mathcal{I} - \mu\beta_n\mathcal{D})x_n - x_n\| \to 0, \tag{3.33}$$

as $n \to \infty$. Since

$$\|\mathcal{T}y_n - w_n\| \leq \|\mathcal{T}y_n - x_n\| + \|x_n - w_n\|,$$

it follows from (3.27) and (3.33) that

$$\lim_{n\to\infty} \|\mathcal{T}y_n - w_n\| = 0. \tag{3.34}$$

Since

$$\|\mathcal{T}y_n - y_n\| \leq \|\mathcal{T}y_n - w_n\| + \|w_n - y_n\|,$$

it follows from (3.30) and (3.34) that

$$\lim_{n\to\infty} \|\mathcal{T}y_n - y_n\| = 0. \tag{3.35}$$

Since

$$\|\mathcal{S}x_n - x_n\| \leq \|\mathcal{S}x_n - w_n\| + \|w_n - x_n\|,$$

it follows from (3.28) and (3.33) that

$$\lim_{n\to\infty} \|\mathcal{S}x_n - x_n\| = 0. \tag{3.36}$$

Step IV: $x^* \in \Gamma$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$. Further, it follows from (3.19), (3.25), (3.30) and (3.32) that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{w_n\}$ all have the same asymptotic behavior and hence there exist subsequences $\{y_{n_i}\}$ of $\{y_n\}$, $\{z_{n_i}\}$ of $\{z_n\}$, $\{u_{n_i}\}$ of $\{u_n\}$ and $\{w_{n_i}\}$ of $\{w_n\}$ such that $\{y_{n_i}\}$, $\{z_{n_i}\}$, $\{u_{n_i}\}$ and $\{w_{n_i}\}$ converge weakly to x^*. It follows from Lemma 2.1(ii), (3.35) and(3.36) that $x^* \in \text{Fix}(\mathcal{T})$ and $x^* \in \text{Fix}(\mathcal{S})$.

Now, we show that $x^* \in \mathcal{H}$. It follows from (3.1)

$$\frac{1}{\sigma_n}(\mathcal{T}y_n - y_n) = (\mathcal{I} - \mathcal{S})x_n + \frac{1}{\sigma_n}(\mathcal{T}y_n - w_n) + (w_n - x_n), \tag{3.37}$$

and hence for all $p \in \text{Fix}(\mathcal{T})$ and using monotonicity of $\mathcal{I} - \mathcal{S}$, we have

$$\begin{aligned} \langle \frac{\mathcal{T}y_n - y_n}{\sigma_n}, x_n - p\rangle &= \langle (\mathcal{I} - \mathcal{S})x_n - (\mathcal{I} - \mathcal{S})p, x_n - p\rangle + \langle (\mathcal{I} - \mathcal{S})p, x_n - p\rangle \\ &\quad + \frac{1}{\sigma_n}\langle \mathcal{T}y_n - w_n, x_n - p\rangle + \langle w_n - x_n, x_n - p\rangle \\ &\geq \langle (\mathcal{I} - \mathcal{S})p, x_n - p\rangle + \frac{1}{\sigma_n}\langle \mathcal{T}y_n - w_n, x_n - p\rangle \\ &\quad + \langle w_n - x_n, x_n - p\rangle \end{aligned} \tag{3.38}$$

Using (3.32), (3.34), (3.35), conditions on parameters α_n and σ_n in (3.38), we have

$$\overline{\lim_{n\to\infty}} \langle p - \mathcal{S}p, x_n - p\rangle \leq 0, \quad \forall p \in \text{Fix}(\mathcal{T}). \tag{3.39}$$

Due to the fact that x_n weakly converges to x^*, we have

$$\langle (\mathcal{I}-\mathcal{S})p, x^* - p\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(\mathcal{T}). \tag{3.40}$$

Since $\mathrm{Fix}(\mathcal{T})$ is convex, $\lambda p + (1-\lambda)x^* \in \mathrm{Fix}(\mathcal{T})$ for $\lambda \in (0,1)$ and hence

$$\begin{aligned} \langle (\mathcal{I}-\mathcal{S})(\lambda p + (1-\lambda)x^*), x^* - (\lambda p + (1-\lambda)x^*)\rangle &= \lambda\langle (\mathcal{I}-\mathcal{S})(\lambda p + (1-\lambda)x^*), x^* - p\rangle \\ &\leq 0, \quad \forall p \in \mathrm{Fix}(\mathcal{T}), \end{aligned} \tag{3.41}$$

which implies

$$\langle (\mathcal{I}-\mathcal{S})(\lambda p + (1-\lambda)x^*), x^* - p\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(\mathcal{T}).$$

On taking limits $\lambda \to 0_+$, we have

$$\langle (\mathcal{I}-\mathcal{S})x^*, x^* - p\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(\mathcal{T}). \tag{3.42}$$

That is $x^* \in \mathcal{H}$. Since $\{x_{n_i}\}$ converges weakly to x^*, it follows from Lemma 2.1 and (3.32) that $x^* \in \mathrm{Sol(VI)}$. Next, we show that $x^* \in \Omega$. Indeed, since $\{x_n\}$ weakly converges to x^* and the sequences $\{x_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ have the same asymptotic behavior then $u_n \rightharpoonup x^*$, $z_n \rightharpoonup x^*$ and $y_n \rightharpoonup x^*$. Since Algorithm 3.1 can be rewritten as

$$\frac{(u_n - z_n) + \mathcal{A}^*(\mathcal{J}_\lambda^{\mathcal{B}_2} - \mathcal{I})\mathcal{A}u_n}{\lambda} \in \mathcal{B}_1(z_n). \tag{3.43}$$

By passing to the limit $n \to \infty$ in (3.43) and taking account (3.20), (3.23) and the fact that graph of maximal monotone mapping is weakly-strongly closed, we obtain $0 \in \mathcal{B}_1(x^*)$. Since $\mathcal{A}$ is continuous then $\mathcal{A}u_n \rightharpoonup \mathcal{A}x^*$. It follows from the nonexpansivity of $\mathcal{J}_\lambda^{\mathcal{B}_2}$, (3.20) and Lemma 2.1 that $0 \in \mathcal{B}_2(\mathcal{A}x^*)$. This shows that $x^* \in \Omega$ and thus $x^* \in \Gamma$.

Step V. $x_n \to x^*$, where $x^* = \mathcal{P}_\Gamma x_0$. Since $x_n = \mathcal{P}_{\mathcal{C}_n} x_0$ and $x^* \in \Gamma$, we have

$$\|x_n - x_0\| \leq \|x^* - x_0\|.$$

It follows from $l = \mathcal{P}_\Gamma x_0$ and the lower semicontinuity of the norm that

$$\|l - x_0\| \leq \|x^* - x_0\| \leq \liminf_{n\to\infty} \|x_n - x_0\| \leq \limsup_{n\to\infty} \|x_n - x_0\| \leq \|l - x_0\|.$$

This implies that $\lim_{n\to\infty} \|x_n - x_0\| = \|l - x_0\| = \|x^* - x_0\|$. Since $x_n - x_0 \rightharpoonup x^* - x_0$ and $\|x_n - x_0\| \to \|x^* - x_0\|$ then from the Kadec-Klee property [12] of H_1, we have $\lim_{n\to\infty} x_n = x^* = l$. Thus, we conclude that $\{x_n\}$ converges strongly to x^*, where $x^* = \mathcal{P}_\Gamma x_0$. $\square$

4 Numerical illustrations

4.1 Function space

For supporting our main theorem, we now give an example in infinitely dimensional spaces $L_2[0,1]$ such that $\|\cdot\|$ is L_2-norm defined by $\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}$ where $x(t) \in L_2[0,1]$.

Example 4.1. *Let $\mathbf{H}_1 = \mathbf{H}_2 = L_2[0,1]$ and $\mathcal{C} = \{x(t) \in L_2[0,1] : \int_0^t x(s)ds < \infty\}$. Define mappings as follow:*

(i) bounded linear operator $\mathcal{A} : L_2[0,1] \to L_2[0,1]$ by $\mathcal{A}x(t) = 2x(t)$, $\forall x(t) \in L_2[0,1]$;

(ii) maximal monotone operators $\mathcal{B}_1, \mathcal{B}_2 : L_2[0,1] \to L_2[0,1]$ by $\mathcal{B}_1 x(t) = 3x(t)$ and $\mathcal{B}_2 x(t) = \frac{x(t)}{5}$, $\forall x(t) \in L_2[0,1]$;

(iii) nonexpansive mapping $\mathcal{T} : L_2[0,1] \to L_2[0,1]$ by $\mathcal{T}x(t) = \frac{x(t)}{2}$, $\forall x(t) \in L_2[0,1]$;

(iv) continuous quasi-nonexpansive mapping $\mathcal{S} : L_2[0,1] \to L_2[0,1]$ by $\mathcal{S}x(t) = \frac{x(t)}{2}$, $\forall x(t) \in L_2[0,1]$;

(v) $\frac{\pi}{2}$-inverse strongly monotone mapping $\mathcal{D} : \mathcal{C} \to L_2[0,1]$ by $\mathcal{D}x(t) = \int_0^t x(s)ds$.

For each $\lambda > 0$, we see that $\mathcal{J}_\lambda^{\mathcal{B}_1}(x) = \frac{x}{1+3\lambda}$ and $\mathcal{J}_\lambda^{\mathcal{B}_2}(x) = \frac{x}{1+\frac{1}{5}\lambda}$. For the experiments in this section, we use the Cauchy error $\|x_{n+1} - x_n\|^2 < 10^{-10}$ for the stopping criterion. We split considering all of the performances of our algorithm in five cases.

Case I: Comparison of the proposed algorithm with different parameters δ_n are shown when we choose $\mu\beta_n = \frac{n}{n+1}$, $\gamma = 0.1$, $\lambda = 0.1$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

δ_n	0.1	0.3	0.5	0.9	0.999
No. of Iter.	18	18	17	17	17
CPU time(s)	3.204624	3.219476	3.104514	3.132228	3.212694

Table1: Numerical results of different parameters δ_n.

Case II: Comparison of the proposed algorithm with different parameters $\mu\beta_n$ are shown when we choose $\delta_n = 0.5$, $\gamma = 0.1$, $\lambda = 0.1$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

$\mu\beta_n$	$\frac{n}{n+1}$	$\frac{n}{10n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^3n+1}$	$\frac{n}{10^4n+1}$
No. of Iter.	17	18	18	18	18
CPU time(s)	3.302351	3.310532	3.369519	3.301081	3.176498

Table2: Numerical results of different parameters $\mu\beta_n$.

Case III: The performance of the algorithm with different parameters γ are shown by choosing $\mu\beta_n = \frac{n}{10^4n+1}$, $\delta_n = 0.5$, $\lambda = 0.1$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

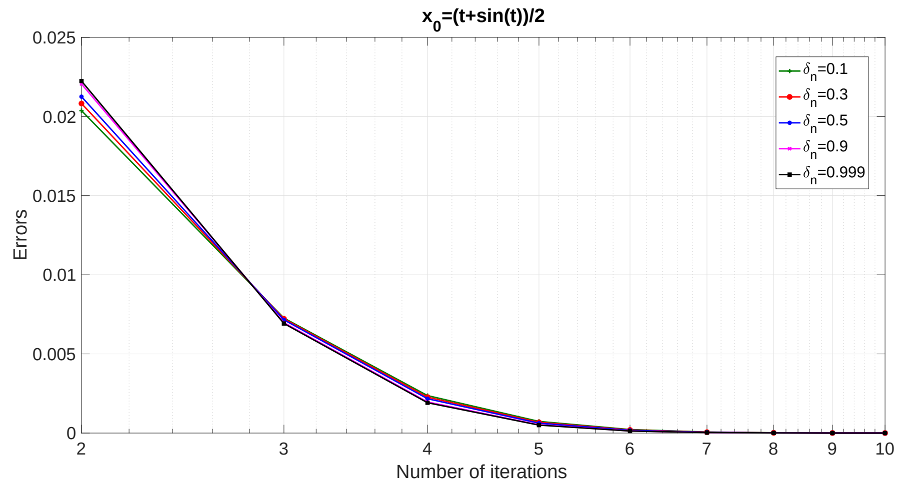

Figure 1: The error plotting of our proposed algorithm (3.1) for different parameters δ_n .

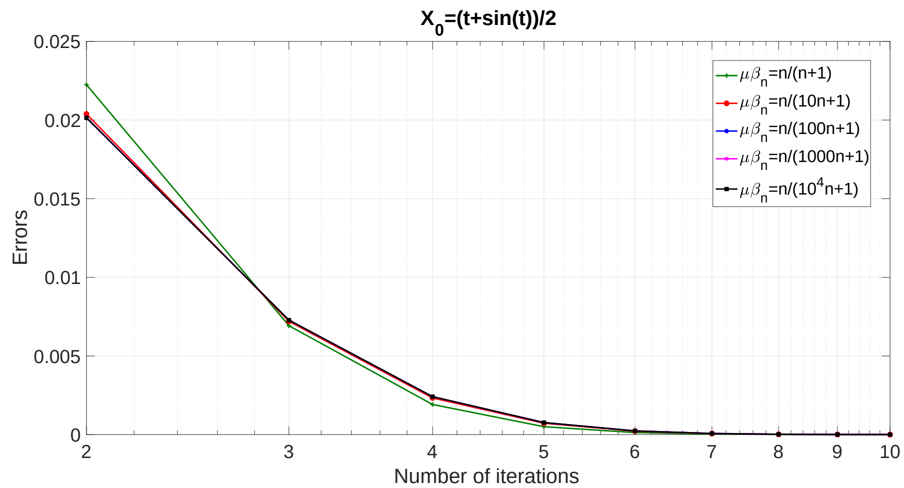

Figure 2: The error plotting of our proposed algorithm (3.1) for different parameters $\mu\beta_n$.

γ	0.2	0.1	0.01	0.001	0.0001
No. of Iter.	18	18	19	19	19
CPU time(s)	3.062286	3.016581	3.620591	3.267629	3.195616

Table3: Numerical results of different parameters γ.

Case IV: The performance of the algorithm with different parameters λ are shown by choosing $\gamma = 0.1$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\delta_n = 0.5$, $\sigma_n = \alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

λ	0.1	1	10	100	10^3
No. of Iter.	18	8	4	3	3
CPU time(s)	3.189750	1.538551	0.9133888	0.752417	0.758241

Table4: Numerical results of different parameters λ.

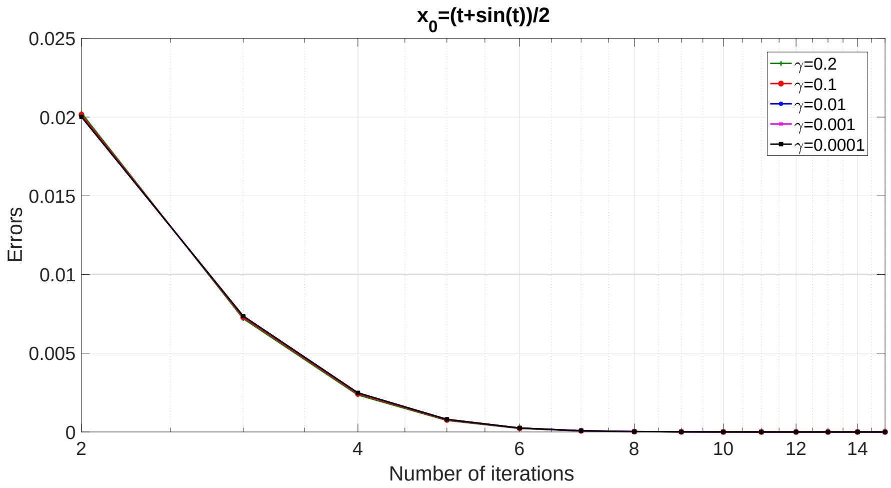

Figure 3: The error plotting of our proposed algorithm (3.1) for different parameters γ .

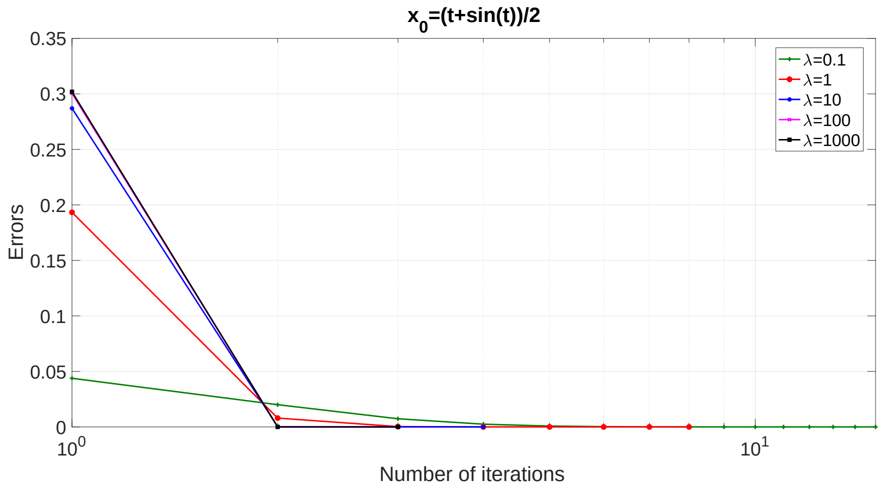

Figure 4: The error plotting of our proposed algorithm (3.1) for different parameters λ .

Case V: The performance of the algorithm with different parameters σ_n are shown by choosing $\lambda = 100$, $\gamma = 0.1$, $\delta_n = 0.5$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\alpha_n = \frac{n}{2n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

σ_n	$\frac{n}{2n+1}$	$\frac{n}{10n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^3 n+1}$	$\frac{n}{10^4 n+1}$
No. of Iter.	3	3	3	3	3
CPU time(s)	0.743157	0.723417	0.744866	0.760672	0.675156

Table5: Numerical results of different parameters σ_n.

Case VI: The performance of the algorithm with different parameters α_n are shown by choosing $\lambda = 100$, $\gamma = 0.1$, $\delta_n = 0.5$, $\mu\beta_n = \frac{n}{10^4 n+1}$, $\sigma_n = \frac{n}{10^4 n+1}$ and initializations $x_0 = \frac{\sin(t)+t}{2}$. Then the results are presented as follows:

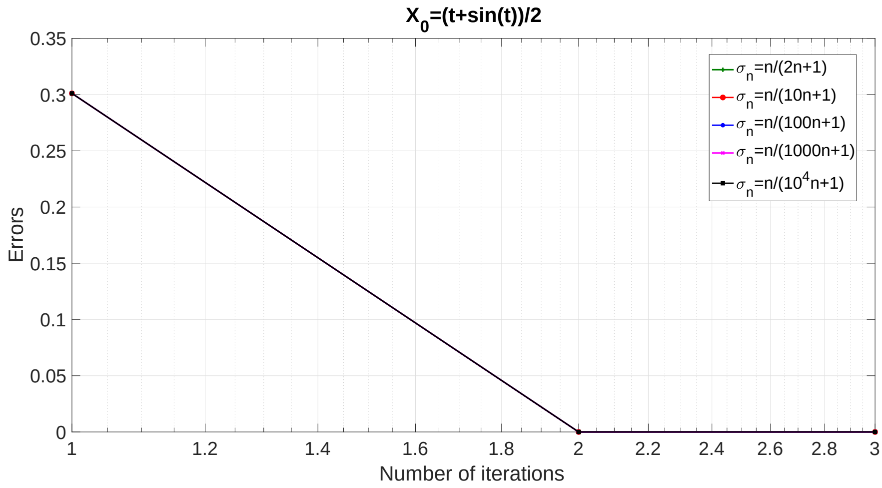

Figure 5: The error plotting of our proposed algorithm (3.1) for different parameters σ_n .

α_n	$\frac{n}{2n+1}$	$\frac{n}{10n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^3n+1}$	$\frac{n}{10^4n+1}$
No. of Iter.	3	3	3	3	3
CPU time(s)	0.746312	0.748333	0.741478	0.735886	0.769183

Table6: Numerical results of different parameters α_n.

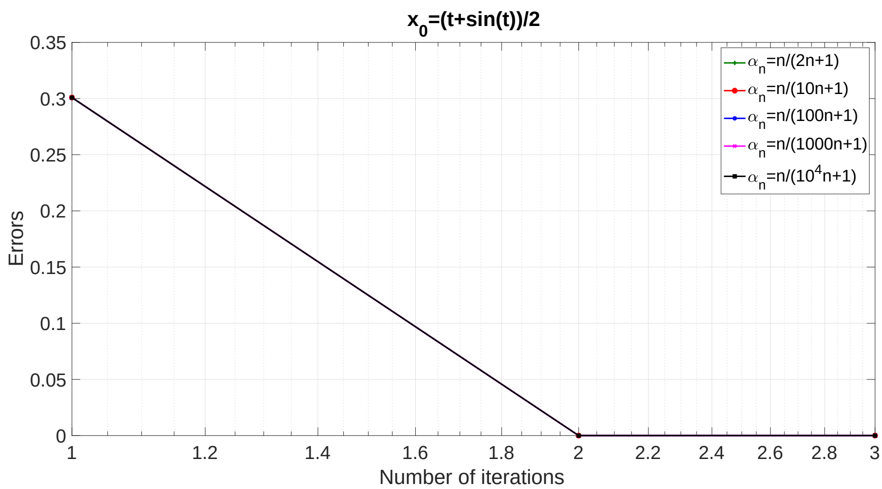

Figure 6: The error plotting of our proposed algorithm (3.1) for different parameters α_n .

From Tables 1-6 and Figures 1-6, we noticed that in all the above 6 cases, selecting $\lambda = 100$, $\gamma = 0.1$, $\delta_n = 0.5$, $\mu\beta_n = \frac{n}{10^4n+1}$, $\sigma_n = \frac{n}{10^4n+1}$ and $\alpha_n = \frac{n}{10^3n+1}$ for initialization $x_0 = \frac{\sin(t)+t}{2}$ yield the best results.

4.2 Signal recovery

In this section, a signal recovery problem in compressed sensing is considered for giving an example of our algorithm application in real world problem. A signal recovery problem can be modeled as the following underdeterminated linear equation system:

$$b = \mathcal{A}x + \varepsilon, \tag{4.1}$$

where $x \in \mathbb{R}^N$ is the original signal, $b \in \mathbb{R}^M$ is the observed signal with noise ε, and $\mathcal{A} \in \mathbb{R}^{M\times N}$ $(M < N)$ is filter matrix. It is well known that the problem (4.1) can be solved by the LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - \mathcal{A}x\|_2^2 + \lambda\|x\|_1, \tag{4.2}$$

where $\lambda > 0$ is a given constant. In this case, we set $\mathcal{D}(x) = \nabla f(x)$, $\mathcal{S}x = \mathcal{T}x = \mathcal{J}_\lambda^{\partial g}(x - \lambda\nabla f(x))$ where $f(x) = \frac{1}{2}\|b-\mathcal{A}x\|_2^2$ and $g(x) = \lambda\|x\|_1$. We know that if $\lambda \in (0, 2/\|\mathcal{A}\|^2)$, then $\mathcal{S}, \mathcal{T}$ are nonexpansive, then our algorithm (3.1) can be applied. And we set $\mathcal{B}_1(x) = \partial g(x)$ and $\mathcal{B}_2(x) = x$. For the experiments in this section, we choose the signal size to be $N = 1024$ and $M = 512$, and the original signal x is generated by the uniform distribution in $[-2, 2]$ with $m = 100$ nonzero elements. We use the mean-squared error to measure the restoration accuracy defined as follows: $MSE_n = \frac{1}{N}\|x_n - x\|_2^2 < 5 \times 10^{-5}$. Let $\mathcal{A}$ be the Gaussian matrix generated by the MATLAB routine $randn(M, N)$, the observation b be generated by white Gaussian noise with signal-to-noise ratio SNR=40. The original signal and the measurement by using $\mathcal{A}$ with $m = 100$. Given that the initial points x_0 is generated by command $randn(N, 1)$. We split considering all of the performances of our algorithm in five cases.

Case I: Comparison of the proposed algorithm with different parameters δ_n are shown when we choose $\mu\beta_n = \frac{1.5}{\|\mathcal{A}\|^2}$, $\gamma = \frac{0.5}{\|\mathcal{A}\|^2}$, $\lambda = \frac{1.5}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

δ_n	0.1	0.3	0.5	0.9	0.999
No. of Iter.	9092	9088	9083	9075	9074
CPU time(s)	13.9853	15.5584	16.4712	17.8711	17.9848

Table7: Numerical results of different parameters δ_n.

Case II: Comparison of the proposed algorithm with different parameters $\mu\beta_n$ are shown when we choose $\delta_n = 0.999$, $\gamma = \frac{0.5}{\|\mathcal{A}\|^2}$, $\lambda = \frac{1.5}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

$\mu\beta_n$	$\frac{1.5}{\|\mathcal{A}\|^2}$	$\frac{1.7}{\|\mathcal{A}\|^2}$	$\frac{1.9}{\|\mathcal{A}\|^2}$	$\frac{1.99}{\|\mathcal{A}\|^2}$	$\frac{1.999}{\|\mathcal{A}\|^2}$
No. of Iter.	9074	9073	9072	9072	9072
CPU time(s)	16.8849	17.0220	18.0354	17.9963	18.9953

Table8: Numerical results of different parameters $\mu\beta_n$.

Case III: The performance of the algorithm with different parameters γ are shown by choosing $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$, $\delta_n = 0.999$, $\lambda = \frac{1.5}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

γ	$\frac{0.5}{\|\mathcal{A}\|^2}$	$\frac{0.7}{\|\mathcal{A}\|^2}$	$\frac{0.9}{\|\mathcal{A}\|^2}$	$\frac{0.99}{\|\mathcal{A}\|^2}$	$\frac{0.999}{\|\mathcal{A}\|^2}$
No. of Iter.	9072	8995	8965	8959	8959
CPU time(s)	17.1674	15.7297	15.9147	15.4883	15.5108

Table9: Numerical results of different parameters γ.

Case IV: The performance of the algorithm with different parameters λ_n are shown by choosing $\gamma = \frac{0.99}{\|A\|^2}$, $\delta_n = 0.999$, $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$, $\sigma_n = \frac{n}{100n+1}$ and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

λ_n	$\frac{1.5}{\|\mathcal{A}\|^2}$	$\frac{1.7}{\|\mathcal{A}\|^2}$	$\frac{1.9}{\|\mathcal{A}\|^2}$	$\frac{1.99}{\|\mathcal{A}\|^2}$	$\frac{1.999}{\|\mathcal{A}\|^2}$
No. of Iter.	9050	7996	7167	6848	6818
CPU time(s)	14.5482	12.5826	11.1581	11.1900	11.1400

Table10: Numerical results of different parameters λ_n.

Case V: The performance of the algorithm with different parameters σ_n are shown by choosing $\lambda_n = \frac{1.999}{\|\mathcal{A}\|^2}$, $\delta_n = 0.999$, $\gamma = \frac{0.99}{\|\mathcal{A}\|^2}$, $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$, and $\alpha_n = \frac{1}{n^2+1}$. Then the results are presented as follows:

σ_n	0.1	0.9	$\frac{n}{n+1}$	$\frac{n}{100n+1}$	$\frac{n}{10^4n+1}$
No. of Iter.	6611	6608	6587	6583	6611
CPU time(s)	10.4733	10.7185	8.2354	7.7671	11.1091

Table11: Numerical results of different parameters σ_n.

Case VI: The performance of the algorithm with different parameters α_n are shown by choosing $\sigma_n = \frac{n}{100n+1}$, $\delta_n = 0.999$, $\lambda_n = \frac{1.999}{\|\mathcal{A}\|^2}$, $\gamma = \frac{0.99}{\|\mathcal{A}\|^2}$, and $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$. Then the results are presented as follows:

α_n	$\frac{1}{n^2+1}$	$\frac{1}{100n^2+1}$	$\frac{1}{10^4n^2+1}$	$\frac{1}{n^3+1}$	$\frac{1}{100n^3+1}$
No. of Iter.	6732	6764	6732	6758	6765
CPU time(s)	10.5964	10.4635	10.4464	10.8398	5.5015

Table12: Numerical results of different parameters α_n.

From Table7- Table12, we see that in all the above 6 cases, selecting $\alpha_n = \frac{1}{10^4n^2+1}$, $\delta_n = 0.999$, $\sigma_n = \frac{n}{100n+1}$, $\lambda_n = \frac{1.999}{\|\mathcal{A}\|^2}$, $\gamma = \frac{0.99}{\|\mathcal{A}\|^2}$, and $\mu\beta_n = \frac{1.99}{\|\mathcal{A}\|^2}$ yield the best results, we denote that choosing the best parameters is depended on number of iterations. We next show the original signal, the measurement by using $\mathcal{A}$ with $m = 100$, and the reconstructed signals in Figure 7.

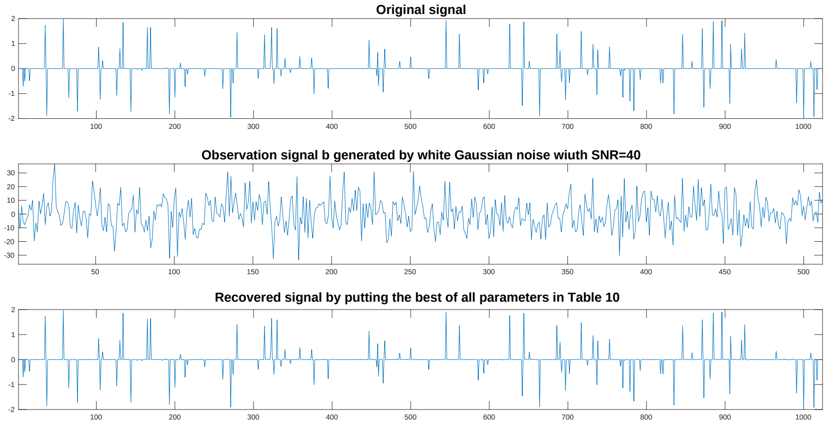

Figure 7: From top to bottom: the original signal, the measurement by using $\mathcal{A}$ with m = 100, and the reconstructed signals by using the best of all parameters in Table 12.

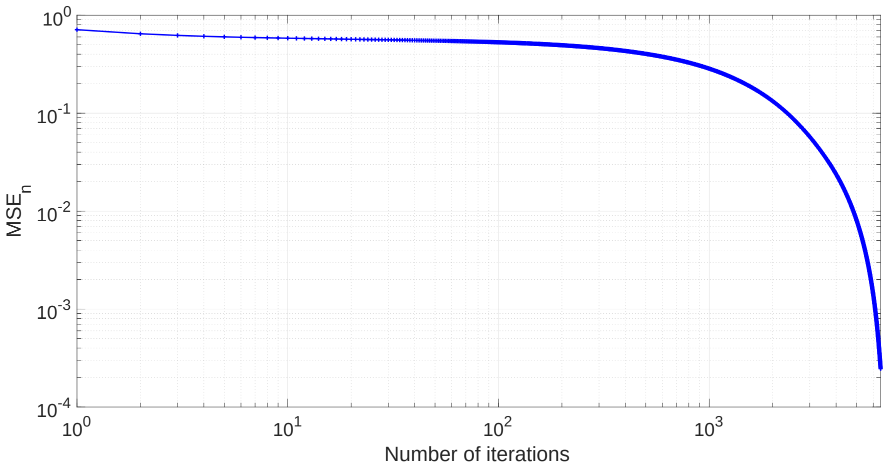

Figure 8: The mean-squared error versus number of iterations.

From Figure 8, we see that our proposed algorithm (3.1) converges to the original signal.

5 Conclusion

In this paper, we modify a hybrid projective method for approximating a common solution of split null inclusion problem, variational inequality and hierarchical fixed point problem for nonexpansive and quasi-nonexpansive mappings of nonexpansive and quasi-nonexpansive mappings. We also prove strong convergent theorems under some mind conditions in Hilbert spaces and give an example for supporting our main result in infinitely dimensional spaces. Finally, we show the efficiency of the algorithm by applying to solve the signal recovery problem.

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