Existence of solutions for anisotropic parabolic Ni-Serrin type equations originated from a capillary phenomena with nonstandard growth nonlinearity

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Abstract. We consider an initial boundary value problem for a class of anisotropic parabolic Ni-Serrin type equations with nonstandard nonlinearity in a bounded smooth domain with homogeneous Dirichlet boundary condition. Because the nonlinear perturbation leads to difficulties (it does not have a definite sign) in obtaining a priori estimates in the energy method, we had to modify the Tartar method significantly. Under suitable assumptions, we obtain the global existence, decay, and extinction of solutions.

Keyword. Anisotropic parabolic Ni-Serrin type equations; variable exponents; global existence; decay estimates; extinction of solutions

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1 Introduction

In this paper, we discuss the global, extinction and decay of solutions for the following parabolic equations with the Ni-Serrin type problems a nonstandard growth nonlinearity:

$$\begin{cases} u' - Au + f(x,u) = 0,\ x \in \Omega, t > 0, \\ u(x,t) = 0,\ x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain with a smooth boundary $\partial\Omega$, $u' = \partial u/\partial t$, $f(x,u) = \lambda |u|^{q(x)-1}$ and the Ni-Serrin type operator A given by

$$Au := -\operatorname{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{a + |\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right),$$

where $a \geq 0$ is a constant. Moreover, q is continuous and p is log-Hölder continuous (see [12]), that is, there exists a constant $C > 0$ such that, for any $x, y \in \overline{\Omega}$, we have

$$|p(x) - p(y)| \leq \frac{C}{\ln\left(e + |x-y|^{-1}\right)}. \tag{2}$$

Assume further that

$$2 \leq p^- := \min_{\overline{\Omega}} p(\cdot) \leq p(x) \leq \max_{\overline{\Omega}} p(\cdot) := p^+ < +\infty, \tag{3}$$

and

$$2 < q^- := \min_{\overline{\Omega}} q(\cdot) \leq q(x) \leq \max_{\overline{\Omega}} q(\cdot) := q^+ < +\infty. \tag{4}$$

The main feature of equation (1) is the variable character of nonlinearity f which causes a gap between the monotonicity and coercivity conditions. Because of this gap, equations of the type (1) are usually termed equations with nonstandard growth conditions.

PDEs with nonstandard growth nonlinearities (or variable exponent type) have been very interested from the purely mathematical point of view. On the other hand, their study is motivated by various applications where such equations appear in the most natural way. Recently, parabolic equations involving nonstandard growth nonlinearities were studied in the mathematical descriptions of motions of the non-newtonian fluids, electrorheological fluids in [1, 2, 4, 15, 27].

Let's note that, in the stationary case with $a = 1$ and $f(x,u) = -g(x,u)$ problem (1) turns into the following problem:

$$\begin{cases} -\operatorname{div}\left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = g(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Recently, problem (5) has begun studied more and more (see [5, 7, 11, 14, 26, 28, 32, 33]). Let us recall some known results of the problem (5). When the primitive G of g oscillates at infinity, Shokooh and Neirameh [28] obtained the existence of infinitely many weak solutions for this problem by using Ricceri's variational principle. For the case of $g(x,u) = \lambda h(x,u)$ is p^+-superlinear at infinity, Zhou [33] and Ge [14] both obtained the existence of a nontrivial solution of the problem (5) for every parameter $\lambda > 0$, under suitable conditions on the function h. Rodrigues in [26], by using Fountain Theorem, established the existence of sequence of high energy solutions for problem (5) under appropriate conditions on the function f.

Problem (5) can be viewed as a generalization of the equation

$$-\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}=f(u), \text{ in } \mathbb{R}^N, \tag{6}$$

with very general right hand side f, where

$$G(u)=\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$$

is the Kirchhoff stress term and the source term f was very general, was initiated by Ni and Serrin [23, 24]. Some authors studied the radial solutions of the problem (6) in the context of the analysis of capillary surfaces for a function f of the form $f(u)=ku$, for $k>0$ (for more details see [10, 13, 17]). Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e. the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e. the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally-occurring phenomena such as the motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical, and pharmaceutical to microfluidic systems.

In the case $a=0$ problem (1) turns into the following $p(\cdot)$-Laplacian problem:

$$\begin{cases} u'-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+f(x,u)=0,\ x\in\Omega, t>0,\\ u(x,t)=0,\ x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x),\ x\in\Omega.\end{cases} \tag{7}$$

When $f(x,u)=|u|^{\sigma(x)}$, Lourêdo et. al concerned with the study of the global existence and the decay of solutions of an evolution problem driven by an anisotropic operator and a nonlinear perturbation, both of them having a variable exponent in [20]. Because the nonlinear perturbation leads to difficulties in obtaining a priori estimates in the energy method, the authors had to significantly modify the Tartar method. As a result, they proved the existence of global solutions at least for small initial data. The decay of the energy was derived by using a differential inequality and applying a non-standard approach.

There are many works regarding parabolic problems with nonlinearities of nonstandard growth nonlinearities (see [3, 6, 8, 9, 16, 18, 21, 25].

In this paper, we consider (1) and establish the global, extinction and decay of weak solutions. We cannot apply the energy method to obtain the existence of solutions of problem (1) because the term $\lambda\int_\Omega |u|^{q(x)-1}udx, \lambda>0$ does not have a definite sign. To overcome this difficulty we apply the method which has its motivation in the work of Tartar [31] (see also [20, 22]).

In developing our study, we consider both the Lebesgue space with variable exponent $L^{q(\cdot)}(\Omega)$ and the Sobolev spaces with variable exponents $W^{1,p(\cdot)}(\Omega)$.

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. Let $q \in C_+(\overline{\Omega}) := \left\{q \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} q(x) > 1\right\}$. We define the Lebesgue space with variable exponent as

$$L^{q(\cdot)}(\Omega) := \left\{u : u \in S(\Omega), \int_\Omega |u(x)|^{q(x)}\, dx < \infty\right\}.$$

The set $L^{q(\cdot)}(\Omega)$, equipped with the Luxemburg norm

$$\|u\|_{L^{q(\cdot)}(\Omega)} := \|u\|_{q(\cdot)} = \inf\left\{\gamma > 0 : \int_\Omega \left|\frac{u(x)}{\gamma}\right|^{q(x)} dx \leq 1\right\},$$

is a Banach space. The modular of $L^{q(\cdot)}(\Omega)$, which is the mapping $\rho_{q(\cdot)}(u) : L^{q(\cdot)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{q(\cdot)}(u) := \int_\Omega |u(x)|^{q(x)}\, dx$$

is a modular on $L^{q(\cdot)}(\Omega)$. For $p \in C_+(\overline{\Omega})$, we define the Sobolev space with variable exponent, $W^{1,p(\cdot)}(\Omega)$, as the space of functions $u \in L^{p(\cdot)}(\Omega)$, such that $\frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\Omega)$, $i = 1, ..., N$, equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},\ u \in W^{1,p(\cdot)}(\Omega).$$

We denote $W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$. Furthermore, for all $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ such that

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)},$$

since Ω is bounded. Let us first observe that, for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and $|\nabla u|^{p(x)-2}\nabla u \in L^{p'(\cdot)}(\Omega)$, the dual space of $W_0^{1,p(\cdot)}(\Omega)$ is denoted by $\left(W_0^{1,p(\cdot)}(\Omega)\right)^* := W_0^{-1,p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$, $\forall x \in \overline{\Omega}$.

Set

$$B(u) = \int_\Omega \sum_{i=1}^N \frac{1}{p(x)} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-1} + \sqrt{a + \left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}\right) dx, \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

We have the following proposition.

Proposition 1 *(see [26]). The following assertions hold:*

(i) $B \in C^1(W_0^{1,p(\cdot)}(\Omega), \mathbb{R})$ *is convex and sequentially weakly lower semi-continuous and*

$$\langle B'(u), \upsilon\rangle = \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2} \frac{\partial u}{\partial x_i} + \frac{\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)-2} \frac{\partial u}{\partial x_i}}{\sqrt{a + \left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}}\right) \frac{\partial \upsilon}{\partial x_i} dx$$

for all $u, v \in W_0^{1,p(\cdot)}(\Omega)$*;*

(ii) $B' : W_0^{1,p(\cdot)}(\Omega) \to W_0^{-1,p'(\cdot)}(\Omega)$ *is a mapping of type* (S_+)*, i.e.,* $u_n \rightharpoonup n$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *and*

$$\limsup_{n\to\infty} \langle B'(u_n), u_n - u\rangle \leq 0$$

imply $u_n \to u$ *in* $W_0^{1,p(\cdot)}(\Omega)$*;*

(iii) $B' : W_0^{1,p(\cdot)}(\Omega) \to W_0^{-1,p'(\cdot)}(\Omega)$ *is a strictly monotone, bounded homeomorphism.*

Moreover, it is well known that if $1 < q^- \leq q^+ < \infty$, $1 < p^- \leq p^+ < \infty$, then the spaces $\left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)$, $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ and $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{0,1,p(\cdot)}\right)$ are separable and reflexive Banach spaces. We refer to [12, 19] for further properties of variable exponent Lebesgue-Sobolev spaces.

Proposition 2 *(see [12, 19]). If* $1 < h^- \leq h^+ < +\infty$ *is satisfied, then for any* $u \in L^{h(\cdot)}(\Omega)$ *the following inequalities are provided.*

(i) $\|u\|_{h(\cdot)}^{h^-} \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h^+}$ *if* $\|u\|_{h(\cdot)} > 1$*;*

(ii) $\|u\|_{h(\cdot)}^{h^+} \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h^-}$ *if* $\|u\|_{h(\cdot)} \leq 1$*.*

Proposition 3 *(Hölder-type inequality, see [12, 19]). Let* $p \in L_+^\infty(\Omega)$*. The conjugate space to* $L^{p(\cdot)}(\Omega)$ *is* $L^{p'(\cdot)}(\Omega)$*, where* $1/p(x)+1/p'(x) = 1$ *for almost every (a.e.)* $x \in \Omega$*. Moreover, the following inequality hold*

$$\left|\int_\Omega u(x)v(x)dx\right| \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

for all $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*.*

Proposition 4 *(see [12, 19]). (i) Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with smooth boundary* $\partial\Omega$ *and* $p \in P_{\log}(\Omega)$*. Let* $q : \Omega \to [1, +\infty)$ *be a measurable and bounded function and suppose that* $q(x) \leq p^*(x) = Np(x)/(N - p(x))_+$ *for a.e.* $x \in \Omega$*. Then* $W^{1,p(\cdot)}(\Omega)$ *is continuously embedded in* $L^{q(\cdot)}(\Omega)$*. In addition, assume that* $\operatorname*{ess\,inf}_{x\in\Omega}\{p^*(x) - q(x)\} > 0$*. Then the embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

In particular, if $p^- \geq \frac{2N}{N+2}$*, then there exists a positive constant* S *such that*

$$\|u\|_2 \leq S\|u\|_{W_0^{1,p(\cdot)}(\Omega)}, \ \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{8}$$

(ii) If p_1, $p_2 \in L^\infty(\Omega)$, $1 < p_1(x) \leq p_2(x)$ *for any* $x \in \overline{\Omega}$*, then* $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$*, and the imbedding is continuous.*

Under the $(3), (4)$ and Proposition 4, we obtain

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q^+}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \hookrightarrow L^{q^-}(\Omega) \hookrightarrow L^2(\Omega). \tag{9}$$

For the sake of simplicity, we denote the norm $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ $(1 \leq p < +\infty)$ and $W_0^{1,p(\cdot)}(\Omega)$ by X_0.

Then for weak solution of problem (1), we have the following definition.

Definition 5 *We define a function $u \in L^\infty(0,T;X_0)$ with $u' \in L^2(0,T;L^2(\Omega))$ to be a weak solution of problem* (1)*, if it satisfies the initial condition $u(.,0) := u^0 \in X_0$, and*

$$(u', \upsilon) + \langle B'(u), \upsilon\rangle + \lambda \int_\Omega |u|^{q(x)-1} \upsilon dx = 0$$

for all $\upsilon \in X_0$, and for a.e. $t \in (0,T)$.

2 Main Results

In this section, we study the global solutions, decay estimates, and extinction of solutions of the problem (1) under suitable assumptions.

By (9) there exists a constant $K > 0$ such that

$$\|u\|_{q(\cdot)} \leq K \|u\|_{X_0}, \forall u \in X_0. \tag{10}$$

Let $K_0 = \max\{1, K\}$ where K is the embedding constant given in inequality (10).

Now we can state our main results as follows.

Theorem 6 *(Global solutions). Assume that hypothesis* (2) *holds. If p^-, p^+, q^+ satisfy*

$$2 \leq p^- \leq p^+ < q^- \leq q^+ < \frac{Np^-}{N-p^-}, \tag{11}$$

and $u^0 \in X_0$ satisfies

$$\left\|u^0\right\|_{X_0} < \beta \leq 1, \tag{12}$$

with

$$\frac{2}{p^-}\left(\left\|u^0\right\|_{X_0}^2 + \left\|u^0\right\|_{X_0}^{p^-}\right) + \lambda K_0^{q^+}\left\|u^0\right\|_{X_0}^{q^-} + K_1 < \left(\frac{2}{p^+N^{q^--1}} - \lambda K_0^{q^+}\right)\beta^{q^-}, \tag{13}$$

where

$$K_1 = \frac{2p^- + 2p^+ + |\Omega|\sqrt{a}p^+}{p^-p^+} + \frac{q^- + q^+}{q^-q^+}, a \geq 0,$$

$$\lambda \in \left(0, \min\left\{\frac{2}{p^+N^{q^--1}K_0^{q^+}}, q^+\right\}\right),$$

then there exists a function $u : \Omega \times [0, +\infty)$, in the class:

$$u \in L^\infty(0,\infty;X_0), u' \in L^2(0,\infty;L^2(\Omega)) \tag{14}$$

which is a weak solution of (1) *in $L^2_{loc}(0,\infty;X_0^*)$ and $u(0) := u^0$ in Ω.*

Let u be the solution given by Theorem 6. Define the energy $F(t)$ by

$$F(t)=\int_{\Omega}u^{2}dx. \tag{15}$$

From (14), we have that $F\in C([0,+\infty);L^{2}(\Omega))$.

Theorem 7 *(Decay estimates)***.** *Let u be the solution given by Theorem 6. Suppose that (2) and (11) hold. Then*

(i) if $p^{-}=2$, that is, $p(x)=2$, $\forall x\in\overline{\Omega}$, we have

$$F(t)\leq F(0)e^{-\eta t}, \tag{16}$$

where $F(0)=\int_{\Omega}u^{2}(0)dx$, $\eta=\frac{1+2b}{S^{2}}$, $b=1-\frac{a}{a+D^{4}}>0$ with $a\geq 0$, $D>0$ are constants and S is the embedding constant of the (8).

(ii) if $p^{-}>2$, we have

$$\begin{cases}\|u\|_{2}\leq\left[\left\|u^{0}\right\|_{2}^{2-p^{-}}+\eta_{1}\frac{p^{-}-2}{2}t\right]^{\frac{1}{2-p^{-}}}, \\ \left\|u^{0}\right\|\geq\left(\frac{b_{1}}{\eta_{1}}\right)^{\frac{1}{p^{-}}},\end{cases} \tag{17}$$

where , $\eta_{1}=\frac{1+2b}{S^{p^{+}}}$, $b=1-\frac{a}{a+\min\left\{D^{2p^{-}},D^{2p^{+}}\right\}}>0$.

Theorem 8 *(Extinction of solutions)***.** *Assume that hypotheses (3), $0<u^{0}\in L^{\infty}(\Omega)\cap X_{0}$ and*

$$\frac{2N}{N+2}<p^{-}\leq p^{+}<q^{-}<2$$

hold. Then the weak solution of problem (1) vanishes in finite time for non-negative initial data satisfying following estimate:

$$\frac{\min\left\{1,\left\|u^{0}\right\|_{2}^{2-p^{-}}\right\}}{\max\left\{\left\|u^{0}\right\|_{2}^{q^{-}-p^{-}},\left\|u^{0}\right\|_{2}^{q^{+}-p^{-}}\right\}}>\frac{2\left(|\Omega|+1\right)^{\left(2-q^{-}\right)/2}}{\left(b+1\right)S_{0}^{-p^{-}}},$$

where $S_{0}=\max\{1,S\}$ and S is embedding constant given in (8). More precisely speaking, we have the following estimates

$$\begin{aligned}&\left\|u\left(t\right)\right\|_{2}^{2-p^{-}}\\ \leq\;&\left\|u^{0}\right\|_{2}^{2-p^{-}}-\left(2-p^{-}\right)\left[\alpha_{1}\min\left\{1,\left\|u^{0}\right\|_{2}^{p^{+}-p^{-}}\right\}-\beta_{1}\max\left\{\left\|u^{0}\right\|_{2}^{q^{-}-p^{-}},\left\|u^{0}\right\|_{2}^{q^{+}-p^{-}}\right\}\right]t\end{aligned}$$

for $t\in(0,T^{*})$, and

$$\left\|u\left(t\right)\right\|_{2}\equiv 0$$

for $t\in[T^{*},+\infty)$, with

$$T^{*}=\frac{\left\|u^{0}\right\|_{2}^{2-p^{-}}}{\left(2-p^{-}\right)\left[\alpha_{1}\min\left\{1,\left\|u^{0}\right\|_{2}^{p^{+}-p^{-}}\right\}-\beta_{1}\max\left\{\left\|u^{0}\right\|_{2}^{q^{-}-p^{-}},\left\|u^{0}\right\|_{2}^{q^{+}-p^{-}}\right\}\right]},$$

where $\alpha_{1}=2\left(1+b\right)S_{0}^{-p^{-}}$, $\beta_{1}=4\left(|\Omega|+1\right)^{\left(2-q^{-}\right)/2}$.

2.1 Global Existence

In this section, we first introduce the following auxiliary lemma, which we will use to obtain existence of global weak solution of problem (1).

Lemma 9 *Assume that* $2 \leq p^- \leq p^+ < q^- \leq q^+$ *and* $\xi_1, ..., \xi_N \geq 0$ *with* $\sum_{i=1}^N \xi_i \leq 1$ *hold. Then, we have*

$$\left(\sum_{i=1}^N \xi_i^{p^-}\right)^{\frac{1}{2}} \leq \sum_{i=1}^N \xi_i \leq N^{\frac{q^- -1}{q^-}} \left(\sum_{i=1}^N \xi_i^{p^+}\right)^{\frac{1}{q^-}}, \tag{18}$$

and

$$\left(\sum_{i=1}^N \xi_i^{p^+}\right)^{\frac{1}{p^-}} \leq \sum_{i=1}^N \xi_i \leq N^{\frac{p^+ -1}{p^+}} \left(\sum_{i=1}^N \xi_i^{p^-}\right)^{\frac{1}{p^+}}. \tag{19}$$

Proof of Lemma 9. For $\xi_i < 1$ $(i = 1, ..., N)$, we have

$$\left(\xi_i^{p^-}\right)^{\frac{1}{2}} = \xi_i^{\frac{p^-}{2}} \leq \xi_i \leq \xi_i^{\frac{p^+}{q^-}} = \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}. \tag{20}$$

Summing (20) over i, we easily get

$$\sum_{i=1}^N \xi_i \geq \sum_{i=1}^N \left(\xi_i^{p^-}\right)^{\frac{1}{2}} \geq \left(\sum_{i=1}^N \xi_i^{p^-}\right)^{\frac{1}{2}},$$

and

$$\sum_{i=1}^N \xi_i \leq \sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}.$$

Since $\varphi(t) = t^{q^-}$ is convex, by applying Jensen's inequality we have

$$\left(\frac{\sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N}\right)^{q^-} = \varphi\left(\frac{\sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N}\right) \leq \frac{\sum_{i=1}^N \varphi\left(\left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}\right)}{N}.$$

Then

$$\frac{\sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N} \leq \left(\frac{\sum_{i=1}^N \xi_i^{p^+}}{N}\right)^{\frac{1}{q^-}}.$$

That is

$$\sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}} \leq N^{\frac{q^- -1}{q^-}} \left(\sum_{i=1}^N \xi_i^{p^+}\right)^{\frac{1}{q^-}}.$$

Thus

$$\sum_{i=1}^{N}\xi_i \leq \sum_{i=1}^{N}\left(\xi_i^{p^+}\right)^{\frac{1}{q^-}} \leq N^{\frac{q^--1}{q^-}}\left(\sum_{i=1}^{N}\xi_i^{p^+}\right)^{\frac{1}{q^-}}.$$

Similarly we obtain

$$\left(\xi_i^{p^+}\right)^{\frac{1}{p^-}} = \xi_i^{\frac{p^+}{p^-}} \leq \xi_i \leq \xi_i^{\frac{p^-}{p^+}} = \left(\xi_i^{p^-}\right)^{\frac{1}{p^+}}. \tag{21}$$

Summing (21) over i, we have

$$\sum_{i=1}^{N}\xi_i \geq \sum_{i=1}^{N}\left(\xi_i^{p^+}\right)^{\frac{1}{p^-}} \geq \left(\sum_{i=1}^{N}\xi_i^{p^+}\right)^{\frac{1}{p^-}},$$

and

$$\sum_{i=1}^{N}\xi_i \leq \sum_{i=1}^{N}\left(\xi_i^{p^-}\right)^{\frac{1}{p^+}} \leq N^{\frac{p^+-1}{p^+}}\left(\sum_{i=1}^{N}\xi_i^{p^-}\right)^{\frac{1}{p^+}}.$$

Thus the proof of Lemma 9 is complete. ∎

Now, we prove the existence of global weak solution for (1) with small initial data. We shall employ the Galerkin's method. Consider a Schauder basis $\{\omega_1, \omega_2, ..., \omega_N, ...\}$ of X_0. Let u_m be an approximate solution of problem (1) defined by

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)\,\omega_j(x),\ m = 1, 2, ...,$$

where the coefficients $g_{jm}(t) \in C^1[0,T]$ $(1 \leq j \leq m)$ satisfy the system of ordinary differential equations

$$(u_m', \omega_j) + \langle A'(u_m), \nabla\omega_j\rangle + \int_\Omega |u_m|^{q(x)-1}\omega_j dx = 0, \tag{22}$$

with

$$u_m(x,0) = \sum_{j=1}^{m} g_{jm}(0)\,\omega_j(x) \to u^0(x) \text{ strongly in } X_0, \tag{23}$$

for all $v \in V_m = span\{\omega_1, \omega_2, ..., \omega_m\}$ the subspace of dimension m of X_0 generated by $\omega_1, \omega_2, ..., \omega_m$ and $\omega_m(0) := \omega_m^0 \in V_m$,as $m \to +\infty$.

We denote by $[0, t_m)$ the maximal interval of existence of the solution u_m. By (11) and (12), we obtain

$$\left\|u_m^0\right\|_{X_0} < \beta \leq 1, \forall m \geq m_0,$$

and

$$\frac{2}{p^-}\left(\left\|u_m^0\right\|_{X_0}^2 + \left\|u_m^0\right\|_{X_0}^{p^-}\right) + \lambda K_0^{q^+}\left\|u_m^0\right\|_{X_0}^{q^-} + K_1 < \left(\frac{2}{p^+N^{q^--1}} - \lambda K_0^{q^+}\right)\beta^{q^-}.$$

Fixing m such that $m \geq m_0$, we have the following estimate:

Lemma 10 *Assume that hypotheses* $(3), (4)$ *and* (11) *hold. Suppose that* β *satisfy the conditions* (12) *and* (13) *of Theorem 6. Then we have* $\|u_m(t)\|_{X_0} < \beta, \forall t \geq 0, m \in \mathbb{N}$.

Proof of Lemma 10. We argue by contradiction. In fact, suppose that there exist $m \in \mathbb{N}$ and $t_1 \in (0, t_m)$ such that

$$\|u_m(t_1)\|_{X_0} \geq \beta.$$

Consider the subset σ of $(0, t_m)$ defined by:

$$\Re = \left\{\sigma \in (0, t_m) : \|u_m(\sigma)\|_{X_0} \geq \beta\right\}, \tag{24}$$

and $\inf_{\sigma \in \Re} \sigma = t_0$. Then we have $\|u_m(t_0)\|_{X_0} = \beta$ and $t_0 > 0$. $\Re$ is not empty, because of (24). It is a closed set because the function $\varphi(t) := \|u_m(t)\|_{X_0}$ is continuous on $[0, t_m)$. In fact, the function φ is continuous on $[0, t_m)$ then $\varphi(t_0) \geq \beta$. If $\varphi(t_0) > \beta$, the Intermediate Value Theorem and noting that $\varphi(0) < \beta$, imply that t_0 is not the infimum on $\Re$, which is a contradiction. Thus $\varphi(t_0) = \beta$. Also $t_0 > 0$ because $\varphi(0) < \beta$. Note that $\varphi(t) < \beta$ for all $0 \leq t < t_0$.

Consider $t \in [0, t_0)$ and multiplying (22) by $g'_{jm}(t)$ and summing over j give

$$\begin{aligned}
&\|u'_m(\tau)\|_2^2 \\
&+ \sum_{i=1}^{N} \int_\Omega \left(\left| \frac{\partial u_m(t)}{\partial x_i} \right|^{p(x)-2} + \frac{\left| \frac{\partial u_m(t)}{\partial x_i} \right|^{p(x)-2}}{\sqrt{a + \left| \frac{\partial u_m(t)}{\partial x_i} \right|^{2p(x)}}} \right) \frac{\partial u_m(t)}{\partial x_i} \frac{\partial u'_m(t)}{\partial x_i} dx \\
&\quad + \lambda \int_\Omega |u_m(t)|^{q(x)-1} u'_m(t) dx = 0.
\end{aligned}$$

It follows

$$\begin{aligned}
&\|u'_m(\tau)\|_2^2 + \frac{d}{dt} \sum_{i=1}^{N} \int_\Omega \frac{1}{p(x)} \left(\left| \frac{\partial u_m(t)}{\partial x_i} \right|^{p(x)} + \sqrt{a + \left| \frac{\partial u_m(t)}{\partial x_i} \right|^{2p(x)}} \right) dx \\
&+ \lambda \frac{d}{dt} \int_\Omega \frac{|u_m(t)|^{q(x)-1} u_m(t)}{q(x)} dx = 0.
\end{aligned}$$

Integrating both sides of this equality with respect to t,

$$\begin{aligned}
&\int_0^t \|u_m'(\tau)\|_2^2\, d\tau + \sum_{i=1}^{N}\int_\Omega \frac{1}{p(x)}\left(\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} + \sqrt{a+\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{2p(x)}}\right)dx \\
&+\lambda\int_\Omega \frac{|u_m(t)|^{q(x)-1}u_m(t)}{q(x)}dx \\
= &\sum_{i=1}^{N}\int_\Omega \frac{1}{p(x)}\left(\left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} + \sqrt{a+\left|\frac{\partial u_m^0}{\partial x_i}\right|^{2p(x)}}\right)dx \\
&+\lambda\int_\Omega \frac{|u_m^0|^{q(x)-1}u_m^0}{q(x)}dx, \qquad (25)
\end{aligned}$$

and then we have

$$\int_0^t \|u_m'(\tau)\|_2^2\, d\tau + I(u_m) = I(u_m^0),$$

where $I : X_0 \to \mathbb{R}$ is defined by:

$$\begin{aligned}
&I(u) \\
= &\sum_{i=1}^{N}\int_\Omega \frac{1}{p(x)}\left(\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} + \sqrt{a+\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{2p(x)}}\right)dx + \lambda\int_\Omega \frac{|u|^{q(x)-1}u}{q(x)}dx.
\end{aligned}$$

The main question, in this point of the proof, is to show that under the assumptions (11) and (12), we can control the sign of $I(u)$, for $u = u_m(\cdot,t)$ approximate solution of (22), $0 \le t < t_0$ and at $u = u_0$, in the inequality (25).

Since

$$\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} \le \sqrt{a+\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{2p(x)}}, a \ge 0, \qquad (26)$$

and

$$\sqrt{a+\left|\frac{\partial u_m^0}{\partial x_i}\right|^{2p(x)}} \le \sqrt{a} + \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)}, \qquad (27)$$

by $(25)\,,(26)$ and (27) we get

$$\begin{aligned}
&\int_0^t \|u_m'(\tau)\|_2^2\, d\tau + \frac{2}{p^+}\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)}dx + \frac{\lambda}{q^+}\int_\Omega |u_m(t)|^{q(x)-1}u_m(t)\,dx \\
\le\ &\frac{1}{p^-}\sum_{i=1}^{N}\int_\Omega\left(\sqrt{a}+2\int_\Omega\left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)}\right)dx + \frac{\lambda}{q^-}\int_\Omega |u_m^0|^{q(x)-1}u_m^0 dx. \qquad (28)
\end{aligned}$$

By using Proposition $2\,(ii)$, (10), and since $\|u_m\|_{X_0} < 1$, we have

$$\begin{aligned}
&\left|\frac{\lambda}{q^+}\int_\Omega |u_m(t)|^{q(x)-1}u_m(t)dx\right| \le \frac{\lambda}{q^+}\int_\Omega |u_m(t)|^{q(x)}dx \\
\le\ &\frac{\lambda}{q^+}\left(\|u_m(t)\|_{q(\cdot)}^{q^+} + \|u_m\|_{q(\cdot)}^{q^-}\right) \le K_0^{q^+}\ \|u_m(t)\|_{X_0}^{q^-}, \qquad (29)
\end{aligned}$$

Let us choise a λ such that $\frac{\lambda}{q^+} < 1$. Therefore it follows from Proposition 2 (ii) that

$$\frac{2}{p^+}\sum_{i=1}^{N}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{p(\cdot)}^{p^+} \leq \frac{2}{p^+}\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx. \tag{30}$$

By (18) and (30), we have

$$\begin{aligned}\frac{2}{p^+N^{q^--1}}\left\|u_m(t)\right\|_{X_0}^{q^-} &\leq \frac{2}{p^+}\sum_{i=1}^{N}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{p(\cdot)}^{p^+}\\ &\leq \frac{2}{p^+}\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx,\end{aligned} \tag{31}$$

By (18) and (19), we have

$$\begin{aligned}&\frac{2}{p^-}\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx\\ \leq\ & \frac{2}{p^-}\sum_{i=1}^{N}\left\|\frac{\partial u_m^0}{\partial x_i}\right\|_{p(\cdot)}^{p^-}+\frac{2}{p^-}\sum_{i=1}^{N}\left\|\frac{\partial u^0}{\partial x_i}\right\|_{p(\cdot)}^{p^+}\\ \leq\ & \frac{2}{p^-}\left\|u_m^0\right\|_{X_0}^{2}+\frac{2}{p^-}\left\|u_m^0\right\|_{X_0}^{p^-}.\end{aligned} \tag{32}$$

Similarly the inequality (29) and noting that $\left\|u_m^0\right\|_{X_0} < 1$, because $t \in [0, t_0)$ it follows that

$$\left|\frac{1}{q^-}\int_\Omega\left|u_m^0\right|^{q(x)-1}u_m^0 dx\right| \leq K_0^{q^+}\left\|u_m^0\right\|_{X_0}^{q^-},$$

on the other hand

$$\frac{1}{p^-}\left(2\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)}+\sqrt{a}\right)dx \leq \frac{2}{p^-}\left(\left\|u_m^0\right\|_{X_0}^{p^-}\right)+\frac{\sqrt{a}N\left|\Omega\right|}{p^-}. \tag{33}$$

Plugging $(29), (31), (32), (33)$ into (28), we obtain

$$\begin{aligned}&\int_0^t\left\|u_m'(\tau)\right\|_2^2 d\tau+\left(\frac{2}{p^+N^{q^--1}}-\lambda K_0^{q^+}\right)\left\|u_m(t)\right\|_{X_0}^{q^-}\\ \leq\ & \frac{2}{p^-}\left(\left\|u_m^0\right\|_{X_0}^{2}+\left\|u_m^0\right\|_{X_0}^{p^-}\right)+\lambda K_0^{q^+}\left\|u_m^0\right\|_{X_0}^{q^-}+K_1,\end{aligned}$$

for all $0 \leq t < t_0$, where

$$\lambda < \min\left\{\frac{2}{p^+N^{q^--1}K_0^{q^+}}, q^+\right\},$$

and

$$K_1 = \frac{\sqrt{a}\left|\Omega\right|N}{p^-}.$$

Therefore,

$$\frac{2}{p^-}\left(\left\|u_m^0\right\|_{X_0}^2+\left\|u_m^0\right\|_{X_0}^{p^-}\right)+\lambda K_0^{q^+}\left\|u_m^0\right\|_{X_0}^{q^-}+K_1<\left(\frac{2}{p^+N^{q^--1}}-\lambda K_0^{q^+}\right)\beta^{q^-},$$

and

$$\begin{aligned}\left(\frac{2}{p^+N^{q^--1}}-\lambda K_0^{q^+}\right)\|u_m(t)\|_{X_0}^{q^-} &< \frac{2}{p^-}\left(\left\|u_m^0\right\|_{X_0}^2+\left\|u_m^0\right\|_{X_0}^{p^-}\right)+\lambda K_0^{q^+}\left\|u_m^0\right\|_{X_0}^{q^-}+K_1\\ &< r<\left(\frac{2}{p^+N^{q^--1}}-\lambda K_0^{q^+}\right)\beta^{q^-}\end{aligned}$$

for some $r\in\mathbb{R}_+$. Taking the limit $t\to t_0$, $t<t_0$, in the above inequality,

$$0\leq\left(\frac{2}{p^+N^{q^--1}}-\lambda K_0^{q^+}\right)\|u_m(t)\|_{X_0}^{q^-}\leq r<\left(\frac{2}{p^+N^{q^--1}}-\lambda K_0^{q^+}\right)\beta^{q^-}$$

which is a contradiction because $\|u_m(t_0)\|_{X_0}=\beta$. Thus the Lemma 10 is proved. ■

Proof of Theorem 6. Using the convergence (23), Lemma 10 and the Gronwall's lemma, there exists a constant $C>0$ independent of t,m such that

$$\int_0^t\|u_m'(\tau)\|_2^2\,d\tau+\frac{2}{p^+}\|u_m(t)\|_{X_0}^{p^-}\leq C.$$

By properties of operator A, we obtain that there exist u, χ and a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$), such that, as $m\to\infty$,

$$u_m\overset{*}{\rightharpoonup}u\text{ in }L^\infty(0,\infty;X_0),\tag{34}$$

$$u_m'\rightharpoonup u'\text{ in }L^2(0,\infty;L^2(\Omega)).\tag{35}$$

Since B is monotone, bounded homeomorphism and (S_+) type (see Proposition 1) we get

$$B(u_m)\overset{*}{\rightharpoonup}\chi\text{ in }L^\infty(0,\infty;X_0^*).$$

The next step is to prove that $\chi=Bu$ and for that we need to show that

$$\int_0^T\int_\Omega|u_m|^{q(x)-1}u_m\,dxdt\to\int_0^T\int_\Omega|u|^{q(x)-1}u\,dxdt\tag{36}$$

for any $T>0$. By compactness $X_0\hookrightarrow\hookrightarrow L^{q^+}(\Omega)$, (34), (35) and Aubin–Lions–Simon Lemma (see [29], Corollary 4), we get

$$u_m\to u\text{ in }C([0,T];L^{q^+}(\Omega)).$$

So,

$$u_m\to u\text{ in }L^{q^+}(Q_T),\tag{37}$$

and

$$u_m(x,t) \to u(x,t) \text{ a.e. in } Q_T.$$

This implies

$$|u_m|^{q(x)-1}u_m \to |u|^{q(x)-1}u \text{ a.e. } (x,t) \in Q_T. \tag{38}$$

By (37), we derive

$$\begin{aligned}
& \int_{Q_T} \left(|u_m|^{q(x)-1} u_m\right)^{\frac{q^+}{q^+-1}} dxdt \\
\leq & \int_{Q_T \cap \{(x,t)\in Q_T : |u_m(x,t)| \leq 1\}} \left(|u_m|^{q(x)-1}\right)^{\frac{q^+}{q^+-1}} dxdt \\
& + \int_{Q_T \cap \{(x,t)\in Q_T : |u_m(x,t)| > 1\}} \left(|u_m|^{q(x)-1}\right)^{\frac{q^+}{q^+-1}} dxdt \\
\leq & \; T|\Omega| + \int_{Q_T} |u_m|^{q^+} dxdt \leq C, \forall m \in \mathbb{N},
\end{aligned}$$

that is,

$$\int_{Q_T} |u_m|^{q^+} dxdt \leq C, \forall m \in \mathbb{N}. \tag{39}$$

From (38), (39) and Lions' Lemma (see [30]) it follows that

$$|u_m|^{q(x)-1}u_m \rightharpoonup |u|^{q(x)-1}u \text{ in } L^{\frac{q^+}{q^+-1}}(Q_T). \tag{40}$$

This result and convergence (37) imply convergence (36).

From the monotonicity of the operator $s \longmapsto |s|^{p-2}s + \frac{|s|^{2p-1}s}{\sqrt{a+|s|^{2p}}}$ (see [26]), the convergences (36) and (38), we deduce (see [30])

$$\chi = Au. \tag{41}$$

Also, by applying the diagonalization process to the sequence of $\{u_m\}$, we find from (40)

$$|u_m|^{q(x)-1}u_m \rightharpoonup |u|^{q(x)-1}u \text{ in } L^{\frac{q^+}{q^+-1}}(Q_T), \forall T > 0. \tag{42}$$

Convergences (41) and (42) allows us to pass to the limit in the approximate equation (22) and so it holds that

$$\begin{aligned}
& \int_0^\infty (u'(t), \upsilon)\, dt \\
& + \int_0^\infty \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2} + \frac{\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)-2}}{\sqrt{a + \left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}} \right) \frac{\partial u}{\partial x_i} \cdot \frac{\partial \upsilon}{\partial x_i} dxdt + \lambda \int_0^\infty |u(t)|^{q(x)-1} \upsilon dt = 0,
\end{aligned}$$

for all $\upsilon \in L^2_{loc}(0,\infty; X_0)$ and supp υ compact in $(0,\infty)$. Taking $\upsilon \in C_0^\infty(\Omega \times (0,T))$ in the last equality, we find equation (1). The initial condition $u(0) = u^0$

in Ω follows by convergences (34) and (35). This concludes the proof of Theorem 6. ∎

2.2 Decay of Solutions

Now, we give the decay of solution to the problem (1).

Proof of Theorem 7. By Multiplying both sides of the first equation in 1 by u and integrating on Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_2^2+\sum_{i=1}^{N}\int_\Omega\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}+\frac{\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}{\sqrt{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}}\right)dx$$

$$+\lambda\int_\Omega\left|u(t)\right|^{q(x)-1}u(t)\,dx=0. \tag{43}$$

It is easy to see that

$$\begin{aligned}\frac{\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}{\sqrt{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}} &= \frac{\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}\sqrt{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}}{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}\\ &= \sqrt{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}\left(1-\frac{a}{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}\right)\\ &\geq b\sqrt{a+\left|\frac{\partial u}{\partial x_i}\right|^{2p(x)}}\geq b\left|\frac{\partial u}{\partial x_i}\right|^{p(x)},\, i=1,...,N \end{aligned}\tag{44}$$

where

$$\left|\frac{\partial u}{\partial x_i}\right|\geq D>0,\, b=1-\frac{a}{a+\min\left\{D^{2p^-},D^{2p^+}\right\}}>0.$$

From (43) and (44) we get

$$\frac{d}{dt}\left\|u(t)\right\|_2^2+2(b+1)\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p(x)}dx+2\int_\Omega\left|u(t)\right|^{q(x)-1}u(t)\,dx\leq 0. \tag{45}$$

As $\left\|u(t)\right\|_{X_0}<1$, from Proposition $2\,(ii)$ and (10), we find

$$\begin{aligned}\left|\lambda\int_\Omega\left|u(t)\right|^{q(x)-1}u(t)\,dx\right| &\leq \lambda\int_\Omega|u(t)|^{q(x)}dx\\ &\leq \lambda\left(\left\|u(t)\right\|_{q(.)}^{q^+}+\left\|u(t)\right\|_{q(.)}^{q^-}\right)\\ &\leq \lambda K_0^{q^+}\left\|u(t)\right\|_{X_0}^{q^+}, \end{aligned}\tag{46}$$

and

$$\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u\left(t\right)}{\partial x_{i}}\right|^{p(x)}dx\geq\sum_{i=1}^{N}\left\|\frac{\partial u\left(t\right)}{\partial x_{i}}\right\|_{p(.)}^{p^{-}}=\left\|u\left(t\right)\right\|_{X_{0}}^{p^{-}}. \tag{47}$$

Plugging $(46),(47)$ into (45) we get

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_{2}^{2}+2\left(b+1\right)\left\|u\left(t\right)\right\|_{X_{0}}^{p^{-}}-2K_{0}^{q^{+}}\left\|u\left(t\right)\right\|_{X_{0}}^{q^{+}}\leq 0. \tag{48}$$

Noting that by $\frac{2}{p^{+}}\leq 1$, we derive the inequality (48) as following

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_{2}^{2}+\left(1+2b\right)\left\|u\left(t\right)\right\|_{X_{0}}^{p^{-}}+\left(\frac{2}{p^{+}}\left\|u\left(t\right)\right\|_{X_{0}}^{p^{-}}-2K^{q^{+}}\left\|u\left(t\right)\right\|_{X_{0}}^{q^{+}}\right)\leq 0. \tag{49}$$

By (49) we get

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_{2}^{2}+\left(2b+1\right)\left\|u\left(t\right)\right\|_{X_{0}}^{p^{-}}\leq 0.$$

By using (15) and (8) we obtain

$$F'\left(t\right)+\frac{1+2b}{S^{p^{-}}}F(t)^{\frac{p^{-}}{2}}\leq 0. \tag{50}$$

(i) if $p^{-}=2$, that is, $p(x)=2$, $\forall x\in\overline{\Omega}$, we have

$$F'\left(t\right)+\eta F(t)\leq 0,$$

where $\eta=\frac{1+2b}{S^{2}}$, then we get

$$\left(\ln\left(\eta F(t)\right)\right)'\leq-\eta.$$

Integrating both sides of this inequality with respect to t,

$$\eta F(t)\leq\eta F\left(0\right)e^{-\eta t}.$$

Thus

$$F(t)\leq F(0)e^{-\eta t}.$$

that implies (16).

(ii) Let $p^{-}>2$. If $u^{0}=0$, we take $u^{0}\equiv 0$ as the solution of problem (1). Assume $u^{0}\neq 0$. If there exists $t_{1}\in(0,+\infty)$ such that $F(t_{1})=0$, we consider the set $\Im=\{\nu\in(0,+\infty):F(\nu)=0\}$ and $\inf_{\nu\in\Im}\nu=t_{0}$. Then $t_{0}>0$ because $F(0)>0$. Also $F\left(t_{0}\right)=0$. As $F'(t)\leq 0$ a.e. in $(0,+\infty)$, then $F(t)$ is decreasing, therefore $F(t)=0$ for all $t\geq t_{0}$. Thus either $F(t)=0$ for all $t\geq t_{0}$ or $F(t)>0$, for all $t>0$. We prove inequality (43) for the second case, that is, $F(t)>0$, for all $t\in[0,+\infty)$. The inequality (43) for $t\in[0,t_{0})$ is derived in a similar way. Recalling that $\frac{p^{-}}{2}=1+\gamma,\gamma>0$. By (50), we obtain

$$F'\left(t\right)+\eta_{1}F(t)^{\frac{p^{-}}{2}}\leq 0,$$

where $\eta_1 = \frac{2b+1}{S^{p^+}}$. Thus, we have

$$F'(t) \leq -\eta_1 F(t)^{\frac{p^-}{2}}. \tag{51}$$

and

$$\frac{\frac{2-p^-}{2} F'(t)}{F^{\frac{p^-}{2}}(t)} \geq \eta_1 \left(\frac{p^- - 2}{2} \right).$$

Therefore,

$$\left(F^{\frac{2-p^-}{2}}(t) \right)' \geq \frac{\eta_1 (p^- - 2)}{2}$$

Integrating both sides of this inequality with respect to t, we obtain

$$F^{\frac{2-p^-}{2}}(t) \leq F^{\frac{2-p^-}{2}}(0) + \eta_1 \frac{p^- - 2}{2} t.$$

This implies inequality (17). This concludes the proof of Theorem 7. ■

2.3 Extinction of Solutions

In this section, firstly, in order to obtain the extinction properties of weak solutions, we introduce an auxiliary lemma on the ordinary differential inequality as follows.

Lemma 11 *(see Lemma 3.2 in [6]) Assume* $0 < l_1 \leq l_2 < r_1 \leq r_2 \leq 1$ *and* $\alpha \geq 0$, $\beta \geq 0$ *and* φ *is a nonnegative and absolutely continuous function, which satisfies*

$$\begin{cases} \varphi'(t) + \alpha \min\left\{\varphi^{l_1}(t), \varphi^{l_2}(t)\right\} \leq \beta \max\left\{\varphi^{r_1}(t), \varphi^{r_2}(t)\right\}, t \geq 0, \\ \varphi(0) > 0, \ \beta \frac{\max\left\{\varphi^{r_1 - l_1}(0), \varphi^{r_2 - l_1}(0)\right\}}{\min\left\{1, \varphi^{l_2 - l_1}(0)\right\}} < \alpha, \end{cases}$$

then it holds

$$\begin{cases} \varphi(t) \leq \left[\varphi^{1-l_1}(0) - \alpha_0 (1 - l_1) t\right]^{\frac{1}{1-l_1}}, 0 < t < T_0, \\ \varphi(t) \equiv 0, \ t \geq T_0, \end{cases}$$

where

$$\alpha_0 = \alpha \min\left\{1, \varphi^{l_2 - l_1}(0)\right\} - \beta \max\left\{\varphi^{r_1 - l_1}(0), \varphi^{r_2 - l_1}(0)\right\} > 0,$$

and

$$T_0 = \alpha_0^{-1} (1 - l_1)^{-1} \varphi^{1-l_1}(0) > 0.$$

Now, we can give the proof of Theorem 8.

Proof of Theorem 8. From (43) and (44) we have

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_2^2+2\left(1+b\right)\int_\Omega\left|\nabla u\left(t\right)\right|^{p(x)}dx+2\lambda\int_\Omega\left|u\left(t\right)\right|^{q(x)-1}u\left(t\right)dx\leq 0. \quad (52)$$

Furthermore, by using (15), Proposition 2 (i) and (8), we obtain

$$\begin{aligned} & 2\left(1+b\right)\int_\Omega\left|\nabla u\right|^{p(x)}dx \\ \geq\ & 2\left(1+b\right)\min\left\{\left\|\nabla u\right\|_{p(\cdot)}^{p^-},\left\|\nabla u\right\|_{p(\cdot)}^{p+}\right\}\geq\alpha_1\min\left\{\left\|u\right\|_2^{p^-},\left\|u\right\|_2^{p^+}\right\} \\ =\ & \alpha_1\min\left\{F^{\frac{p^-}{2}}(t),F^{\frac{p^+}{2}}(t)\right\}, \end{aligned} \quad (53)$$

where $\alpha_1=2\left(1+b\right)S_0^{-p^-}>0$. By Proposition 3 we have

$$\begin{aligned} 2\left|\lambda\int_\Omega\left|u\left(t\right)\right|^{q(x)-1}u\left(t\right)dx\right| &\leq 2\lambda\int_\Omega\left|u\right|^{q(x)}dx\leq 4\lambda\left\|\left|u\right|^{q(\cdot)}\right\|_{\frac{2}{q(\cdot)}}\left\|1\right\|_{\frac{2}{2-q(\cdot)}} \\ &\leq \beta_1\max\left\{\left\|u\right\|_2^{q^-},\left\|u\right\|_2^{q^+}\right\} \\ &= \beta_1\max\left\{F^{\frac{q^-}{2}}\left(t\right),F^{\frac{q^+}{2}}\left(t\right)\right\}, \end{aligned} \quad (54)$$

where

$$\beta_1=4\lambda\left(\left|\Omega\right|+1\right)^{\left(2-q^-\right)/2}>0.$$

Plugging $(53),(54)$ into (52) we get

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_2^2+\alpha_1\left(1+b\right)\min\left\{F^{\frac{p^-}{2}}(t),F^{\frac{p^+}{2}}(t)\right\}\leq\beta_1\max\left\{F^{\frac{q^-}{2}}\left(t\right),F^{\frac{q^+}{2}}\left(t\right)\right\}.$$

with $0<\frac{p^-}{2}\leq\frac{p^+}{2}<\frac{q^-}{2}\leq\frac{q^+}{2}\leq 1$. By using Lemma 11, we obtain

$$F'(t)\leq-\alpha_0F^{\frac{p^-}{2}}(t),$$

where

$$\alpha_0=\alpha_1\min\left\{1,F^{\frac{p^+-p^-}{2}}(0)\right\}-\beta_1\max\left\{F^{\frac{q^--p^-}{2}}(0),F^{\frac{q^+-p^-}{2}}(0)\right\}>0,$$

that is

$$F(t)\leq\left(F^{\frac{2-p^-}{2}}(0)-\frac{\alpha_0\left(2-p^-\right)}{2}t\right)^{\frac{2}{2-p^-}},t\geq 0.$$

Thus, from $F(t)\geq 0$ with $F(0)>0$, we get

$$\begin{aligned} & F^{\frac{2-p^-}{2}}(t)\leq F^{\frac{2-p^-}{2}}(0) \\ & -\frac{2-p^-}{2}\left(\alpha_1\min\left\{1,F^{\frac{p^+-p^-}{2}}(0)\right\}+\beta_1\max\left\{F^{\frac{q^--p^-}{2}}(0),F^{\frac{q^+-p^-}{2}}(0)\right\}\right)t \end{aligned}$$

for $t \in (0, T^*)$, and

$$F(t) \equiv 0$$

for $t \in [T^*, +\infty)$, where

$$T^* = \frac{2F^{\frac{2-p^-}{2}}(0)}{(2-p^-)\left(\alpha_1 \min\left\{1, F^{\frac{p^+-p^-}{2}}(0)\right\} - \beta_1 \max\left\{F^{\frac{q^--p^-}{2}}(0), F^{\frac{q^+-p^-}{2}}(0)\right\}\right)}.$$

Thus the proof of Theorem 8 is complete. ■

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