Convergence and numerical treatment of Bratu's problem with initial conditions via advanced decomposition technique

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Abstract

In the present research, an advanced decomposition technique based on the Adomian decomposition method is proposed to achieve the highly accurate numerical solution of non-linear initial value problems of Bratu's-type without any linearization, perturbation and discretization. For the completeness of the proposed technique, convergence analysis is also addressed. The reliability, generality and validity of the proposed technique are examined by calculating the absolute errors of some initial value problems of Bratu's type. Moreover, the obtained solutions are compared graphically with the precise solution and also with some existing approaches solutions.

Keywords Bratu's type equation · Initial value problems · Advanced decomposition technique · Approximation · Convergence analysis

AMS Subject Classification 65L05 · 34A12 · 34B15

1 Introduction

The study of nonlinear differential equations attracts many researchers due to their wide applications in several fields of science and engineering such as chemistry, mathematical physics and mechanics etc. In this article, we consider the non-linear initial value problems (IVPs) of Bratu's type. First time, Bratu's problem was introduced and became highlight in 1914 by G. Bratu [1]. In the esteem of mathematicians Liouville and Gelfand, this problem is also known as the "Liouville-Bratu-Gelfand" problem [2,3]. The most reliable form of the Bratu's problem is as

$$\xi''(\theta) + \lambda\, \mathcal{N}(\xi(\theta)) = 0, \quad 0 < \theta < 1, \tag{1}$$

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with initial conditions (ICs)

$$\xi(0) = \sigma, \quad \xi'(0) = \gamma \tag{2}$$

where $\mathcal{N}(\xi(\theta)) = e^{\mu\xi(\theta)}$, μ is $+1$ or -1 and λ, σ, γ are constant.

In real-world applications, Bratu's problem (1)-(2) comes out in many physical models as radiative heat transfer, fuel ignition model in the thermal combustion theory, nanotechnology, Chandrasekhar model of the expansion of the universe [4–6].

Bratu's problem (1) has been used by many researchers to test the accuracy of their numerical methods as: G. Hariharan et al. [7] applied Chebyshev wavelets method, in [8] authors used the Legendre wavelet method. Recently, Adomian's decomposition method (ADM) [9–11], Collocation method [12], Runge-Kutta method [13, 14], Operational matrix method [15], Bernstein and Gegenbauer-wavelet methods [16], Perturbation method [17, 18], Optimal homotopy asymptotic method [19, 20], Bessel collocation method [21], Hybrid method [22], etc have been used to solve the Bratu's problem (1).

Here, we establish an efficient advanced decomposition technique, which is based on the Duan-Rach modified decomposition approach (DRA) [23, 24], to obtain the numerical solution of IVPs of Bratu's type (1). This technique is a powerful tool for the analytical and numerical solution of singualr and non-singular differential equations which arises in the modeling of real-world physical problems [25, 26]. According to Duan-Rach approach, solution ξ and the non-linear term $\mathcal{N}(\xi)$ decomposes into an infinite series as: If

$$\xi(\theta) = \sum_{n=0}^{\infty} C_n(\theta - \theta_0)^n, \tag{3}$$

then nonlinear term $\mathcal{N}(\xi)$ transformed into the following series

$$\mathcal{N}(\xi) = \sum_{n=0}^{\infty} A_n(\theta - \theta_0)^n, \tag{4}$$

where $A_n = A_n(\xi_0, \xi_1, \cdots, \xi_n)$ are the Adomian polynomials [27, 28] and given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[\mathcal{N}\left(\sum_{k=0}^{\infty}\mu^k\xi_k\right)\right]_{\mu=0}, \qquad n \geq 0 \tag{5}$$

where μ is a grouping constant.

Here, we can note that to handle the Adomian polynomials using formula (5) is a very lengthy and time-dominating process. So, to generate the Adomian polynomials rapidly and easily, in [29–31] authors presented an algorithm which works very well on the computer software like Mathematica, Matlab, Python etc.

Algorithm:

For $\quad n \geq 1,$

$$C_n^1 = \xi_n, \tag{6}$$

For $\quad 2 \leq k \leq n,$

$$C_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)\xi_{j+1}C_{n-1-j}^{k-1}. \tag{7}$$

Then A_n are given as

$$A_0 = \mathcal{N}(\xi_0), \quad and \quad A_n = \sum_{k=1}^{n} \mathcal{N}^k(\xi_0)\ C_n^k, \quad for \quad n \geq 1. \tag{8}$$

The rest of this paper is organised as: In the next Section 2, we discuss the decomposition technique to solve the IVPs of Bratu's type. Section 3, presents the convergence analysis of the proposed technique. In section 4, the decomposition technique is implemented on various IVPs of Bratu's type. In the end, conclusions are summarized in section 5.

2 Advanced decomposition technique for the IVPs of Bratu's type

Consider the Bratu's problem (1)

$$\xi''(\theta) + \lambda\, \mathcal{N}(\xi(\theta)) = 0, \tag{9}$$

subject to the initial conditions (IC)

$$\xi(0) = \sigma, \quad \xi'(0) = \gamma. \tag{10}$$

In this technique, solution $\xi(\theta)$ decomposed into an infinite series as

$$\xi(\theta) = \sum_{i=0}^{\infty} C_i \theta^i, \tag{11}$$

then non-linear term $\mathcal{N}(\xi(\theta))$ transformed into the series of Adomian polynomials A_i as,

$$\mathcal{N}(\xi(\theta)) = \mathcal{N}\left(\sum_{i=0}^{\infty} C_i \theta^i\right) = \sum_{i=0}^{\infty} A_i \theta^i. \tag{12}$$

Now our job is to determine the solution coefficients C_i of the series (11).
Applying the initial conditions (10) on (11), we obtain

$$C_0 = \sigma, \quad C_1 = \gamma. \tag{13}$$

On substituting the equations (11)-(13) into (9), we have

$$\sum_{i=0}^{\infty} C_{i+2}(i+2)(i+1)\theta^i + \lambda \sum_{i=0}^{\infty} A_i \theta^i = 0, \tag{14}$$

which gives the relation

$$C_{i+2} = \frac{-\lambda\ A_i(C_0, C_1, \cdots, C_i)}{(1+i)(2+i)}, \quad i \geq 0 \tag{15}$$

where A_i in terms of the C_i are calculated by using the algorithm (6-7).
To attain the n^{th}- order numerical solution of problem (1) truncate the series (11) as

$$\xi_n(\theta) = \sum_{i=0}^{n} C_i \theta^i. \tag{16}$$

From the relations (13) and (15), (16) becomes

$$\xi_n(\theta) = \sigma + \gamma\theta - \lambda \sum_{i=0}^{n-2} \frac{A_i}{(1+i)(2+i)} \theta^{i+2} \tag{17}$$

Equation (17) provides the desired numerical solution of problem (1).

3 Convergence analysis of decomposition technique

Convergence of Adomian decomposition method (ADM) and its modifications for the initial, boundary value problems are discussed by many authors [25, 26, 32–34].
Here, we discuss the convergence of advanced decomposition technique for IVPs of Bratu's type (1). For this, equation (17) can be written in operator from as

$$\xi = \sigma + \mathcal{M}(\xi) \tag{18}$$

where

$$\mathcal{M}(\xi) = \mathcal{M}\left(\sum_{i=0}^{\infty} C_i \theta^i\right) = \gamma\theta - \lambda \sum_{i=0}^{\infty} \frac{A_i}{(2+i)(1+i)} \theta^{i+2}. \tag{19}$$

Equation (17) can be rewritten as

$$\xi_n(\theta) = \sigma + \gamma\theta - \lambda \sum_{i=1}^{n-1} \frac{A_{i-1}}{i(i+1)} \theta^{i+1}. \tag{20}$$

Using (19) and (20), the operator form of (17) is as

$$\xi_n = \sigma + \mathcal{M}(\xi_{n-1}), \quad n \geq 1. \tag{21}$$

The convergence of approximate solution ξ_n defined by (21) is given by the ensuing theorem.

Theorem 1. Let $Z = C[0,1]$ be a Banach space with the norm $||\xi|| = \max_{0\leq\theta\leq 1} |\xi(\theta)|,\ \xi \in C[0,1]$.
Suppose the non-linear operator $\mathcal{M}(\xi)$ given by (19) satisfies the Lipschitz condition
$||\mathcal{M}(\xi) - \mathcal{M}(\psi)|| \leq r\ ||\xi - \psi||, \forall\ \xi,\ \psi \in C[0,1]$ with Lipschitz constant r, $0 \leq r < 1$. If $||\sigma|| < \infty$, then the sequence $\{\xi_n\}$ defined by (21) converges to ξ.

Proof. For the convergence of $\{\xi_n\}$, first we prove the relation

$$||\xi_{n+1} - \xi_n|| \leq r^n\ ||\sigma||, \tag{22}$$

to prove it, we use induction principle.
For $n = 1$, from (21) and by the Lipschitz condition for $\mathcal{M}(\xi)$, we have

$$||\xi_2 - \xi_1|| = ||\mathcal{M}(\xi_1) - \mathcal{M}(\xi_0)|| \leq r\ ||\xi_1 - \xi_0|| = r\ ||\sigma||, \tag{23}$$

so (22) is true for $n = 1$.
For $n = k$, by induction principle (22) is true and we have

$$||\xi_{k+1} - \xi_k|| = ||\mathcal{M}(\xi_k) - \mathcal{M}(\xi_{k-1})|| \leq r^k\ ||\xi_k - \xi_{k-1}|| = r^k\ ||\sigma||. \tag{24}$$

Finally, for $n = k + 1$,

$$||\xi_{k+2} - \xi_{k+1}|| = ||\mathcal{M}(\xi_{k+1}) - \mathcal{M}(\xi_k)|| \le r^{k+1} \ ||\xi_{k+1} - \xi_k|| = r^{k+1} \ ||\sigma||. \tag{25}$$

From the relations (23)-(25), we observe that (22) is hold $\forall \ n \in N$.
Now, to show that sequence $\{\xi_n\}$ is convergent, we prove that $\{\xi_n\}$ is a Cauchy sequence in the Banach space $C[0, 1]$.
For every $n, m \in N, \ \ n \ge m$, using (22), we have

$$\begin{aligned}||\xi_n - \xi_m|| &= ||(\xi_n - \xi_{n-1}) + (\xi_{n-1} - \xi_{n-2}) + \cdots + (\xi_{m+1} - \xi_m)|| \\ &\le \ ||(\xi_n - \xi_{n-1})|| + ||(\xi_{n-1} - \xi_{n-2})|| + \cdots + ||(\xi_{m+1} - \xi_m)|| \\ &\le r^{n-1} \ ||\sigma|| + r^{n-2} \ ||\sigma|| + \cdots + r^{m+1} \ ||\sigma|| + r^m \ ||\sigma|| \\ &\le r^m (1 + r + r^2 + \cdots + r^{n-m-1}) \ ||\sigma|| \\ &\le r^m \left(\frac{1 - r^{n-m}}{1 - r} \right) ||\sigma||\end{aligned} \tag{26}$$

since $0 \le r < 1$, so $1 - r^{n-m} < 1$ and $||\sigma|| < \infty$, then (26) reduces to

$$||\xi_n - \xi_m|| \le \frac{r^m}{1 - r} ||\sigma|| \tag{27}$$

as $m \to \infty$ in (27), we get $||\xi_n - \xi_m|| \to 0$.
Therefore, $\{\xi_n\}$ is a Cauchy sequence in the Banach space $C[0, 1]$. It implies that there exist a ξ such that

$$\lim_{n \to \infty} \xi_n = \xi,$$

Hence ξ_n converges to ξ.

4 Simulation and results

The proposed advanced decomposition technique discussed in Section 2 is implemented on some IVPs of Bratu's type and the obtained results are compared with their exact solution, and also with the existing methods in [7, 8, 19, 35]. To check the robustness and effectiveness of the proposed technique, we introduce the error function as: If $\xi(\theta)$ and $\xi_n(\theta)$ are the exact and *nth*-order approximate solution of given problem (1), then the absolute error $E_n(\theta)$ is given by

$$E_n(\theta) = |\xi(\theta) - \xi_n(\theta)|. \tag{28}$$

All numerical simulation related to these problems have been done on MATLAB R2018b.

Problem 1: Let $\lambda = -2$ and $\mathcal{N}(\xi(\theta)) = e^{\xi(\theta)}$ in equation (1), we obtain the IVP of Bratu's type as

$$\xi''(\theta) - 2e^{\xi(\theta)} = 0, \ 0 < \theta < 1 \tag{29}$$

with initial conditions

$$\xi(0) = 0, \ \ \xi'(0) = 0. \tag{30}$$

This problem has the exact solution

$$\xi(\theta) = -2\ ln(\cos(\theta)).$$

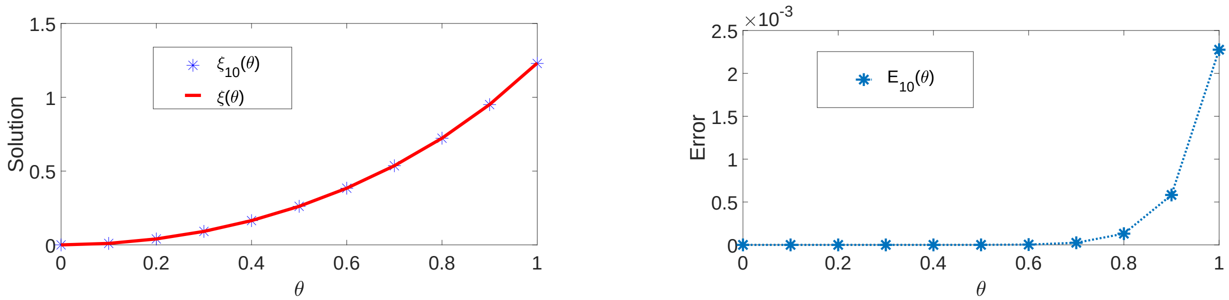
Adomian polynomials for the term $-2e^{\xi(\theta)}$ of (29) by using the algorithm (6)-(7) are estimated as

$$\begin{aligned}
A_0 &= -2\ e^{C_0} \\
A_1 &= -2\ C_1\ e^{C_0} \\
A_2 &= -C_1^2\ e^{C_0} - 2\ C_2\ e^{C_0} \\
A_3 &= -2\ C_3\ e^{C_0} - \frac{1}{3} C_1^3\ e^{C_0} - 2\ C_1 C_2\ e^{C_0} \\
&\ \vdots
\end{aligned}$$

Applying the decomposition technique discussed in Section 2, we obtain $\xi_n(\theta)$ for different n as

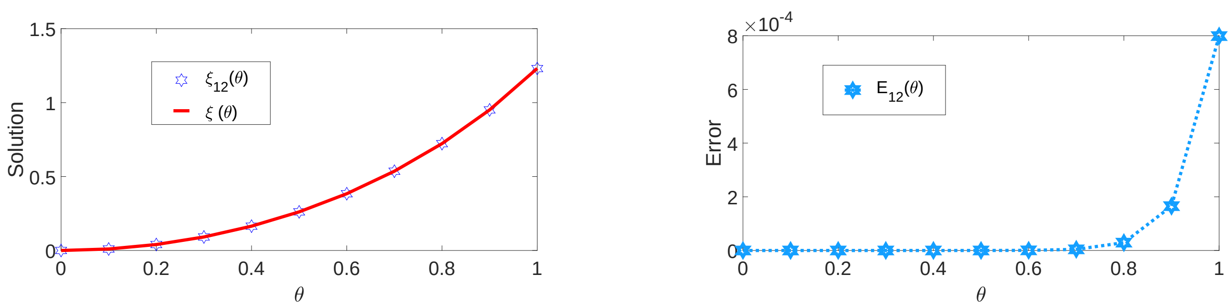
$$\xi_{10}(\theta) = (4.4e-03)\ \theta^{10} + (1.3e-02)\ \theta^8 + (4.4e-02)\ \theta^6 + (0.166)\ \theta^4 +\ \theta^2.$$

$$\xi_{12}(\theta) = (1.5e-03)\ \theta^{12} + (4.4e-03)\ \theta^{10} + (1.3e-02)\ \theta^8 + (4.4e-02)\ \theta^6 + (0.166)\ \theta^4 +\ \theta^2.$$



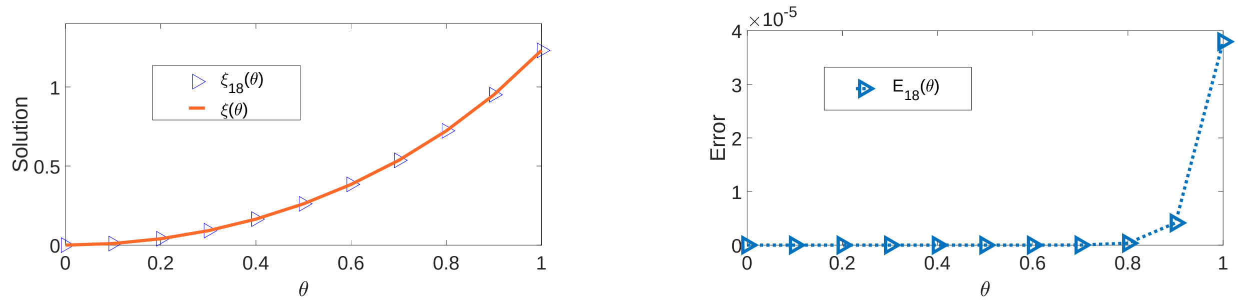
(a) (b)

Fig. 1: (a) Comparison between approximate solution $\xi_{10}(\theta)$ and exact solution $\xi(\theta)$ of Problem 1. (b) Absolute error $E_{10}(\theta)$ in the approximate solution $\xi_{10}(\theta)$ of Problem 1.



(a) (b)

Fig. 2: (a) Comparison between approximate solution $\xi_{12}(\theta)$ and exact solution $\xi(\theta)$ of Problem 1. (b) Absolute error $E_{12}(\theta)$ in the approximate solution $\xi_{12}(\theta)$ of Problem 1.



(a) (b)

Fig. 3: (a) Comparison between approximate solution $\xi_{18}(\theta)$ and exact solution $\xi(\theta)$ of Problem 1. (b) Absolute error $E_{18}(\theta)$ in the approximate solution $\xi_{18}(\theta)$ of Problem 1.

Table 1: Comparison of the Absolute Errors of Problem 1.

	Present method					
θ	$E_{10}(\theta)$	$E_{12}(\theta)$	$E_{18}(\theta)$	LWM [8]	OHAM [19]	TWM [35]
0	0	0	0	0	0	$8.563e{-}05$
0.1	$1.595e{-}15$	$1.196e{-}16$	$1.145e{-}16$	$9.023e{-}08$	$6.410e{-}07$	$2.696e{-}05$
0.2	$6.135e{-}12$	$8.529e{-}14$	$6.939e{-}18$	$1.506e{-}07$	$9.746e{-}06$	$2.389e{-}05$
0.3	$8.104e{-}10$	$2.535e{-}11$	$9.437e{-}16$	$6.140e{-}07$	$4.529e{-}05$	$1.013e{-}05$
0.4	$2.624e{-}08$	$1.460e{-}09$	$2.794e{-}13$	$8.880e{-}06$	$1.271e{-}04$	$2.124e{-}05$
0.5	$3.950e{-}07$	$3.437e{-}08$	$2.512e{-}11$	$5.671e{-}05$	$2.686e{-}04$	$1.153e{-}05$
0.6	$3.676e{-}06$	$4.611e{-}07$	$1.008e{-}09$	$2.557e{-}04$	$4.836e{-}04$	$1.851e{-}05$
0.7	$2.466e{-}05$	$4.215e{-}06$	$2.329e{-}08$	$9.246e{-}04$	$8.367e{-}04$	$1.154e{-}05$
0.8	$1.307e{-}04$	$2.923e{-}05$	$3.610e{-}07$	$2.861e{-}03$	$1.600e{-}03$	$2.264e{-}05$
0.9	$5.823e{-}04$	$1.651e{-}04$	$4.149e{-}06$	$7.912e{-}03$	$3.649e{-}03$	$1.139e{-}05$
1.0	$2.275e{-}03$	$7.986e{-}04$	$3.796e{-}05$	$2.014e{-}02$	$9.391e{-}03$	$8.555e{-}05$

Fig. 1(a), Fig. 2(a) and Fig. 3(a) depict the numerical result of the approximate solution $\xi_n(\theta)$ for $n = 10, 12, 18$ and the exact solution $\xi(\theta)$ of Problem 1. Here, we observe that the approximate solution and exact solution coincide nearly. Also, the absolute error between exact $\xi(\theta)$ and approximate solution $\xi_n(\theta)$ for $n = 10, 12, 18$ of Problem 1 are shown in Fig. 1(b), Fig. 2(b) and Fig. 3(b). From these Figures, it is important to note that when we increase the order of numerical solution errors are decreasing.

Table 1, represent the comparison of absolute errors $E_n(\theta)$ obtained by the proposed decomposition technique for $n = 10, 12, 18$ and the existing method as Legendre wavelet method (LWM) [8], optimal homotopy analysis method (OHAM) [19] and Taylor wavelet method (TWM) [35], for Problem 1. Table 1 shows that the Proposed technique provides the higher accuracy result with less error in comparison to the methods in [8, 19, 35].

Problem 2: Let $\lambda = -\pi^2$ and $\mathcal{N}(\xi(\theta)) = e^{\xi(\theta)}$ in equation (1), we obtain the IVP of Bratu's type as

$$\xi''(\theta) - \pi^2 e^{\xi(\theta)} = 0, \;\; 0 < \theta < 1 \tag{31}$$

with initial conditions

$$\xi(0) = 0, \quad \xi'(0) = \pi. \tag{32}$$

This problem has the exact solution

$$\xi(\theta) = -ln\left[1 + \cos\frac{(2\theta+1)}{2}\pi\right].$$

Adomian polynomials for the term $-\pi^2 e^{\xi(\theta)}$ of (31) are as

$$\begin{aligned}
A_0 &= -\pi^2 e^{C_0} \\
A_1 &= -\pi^2 C_1 \; e^{C_0} \\
A_2 &= -\pi^2 C_2 \; e^{C_0} - \frac{1}{2}\pi^2 C_1^2 e^{C_0} \\
A_3 &= -\pi^2 C_3 \; e^{C_0} - \frac{1}{6}\pi^2 c_1^3 e^{C_0} - \pi^2 C_1 C_2 e^{C_0} \\
&\;\;\vdots
\end{aligned}$$

Applying the decomposition technique described in section 2, we obtain the approximate solution,

$$\xi_{12}(\theta) = (682.668)\;\theta^{12} + (372.3615)\;\theta^{11} + (204.8035)\;\theta^{10} + (113.7721)\;\theta^{9} + (64.0099)\;\theta^{8} + (36.5551)\;\theta^{7}$$
$$+ (21.364)\;\theta^{6} + (12.7508)\;\theta^{5} + (8.1174)\;\theta^{4} + (5.1677)\;\theta^{3} + (4.9348)\;\theta^{2} + \pi\;\theta.$$

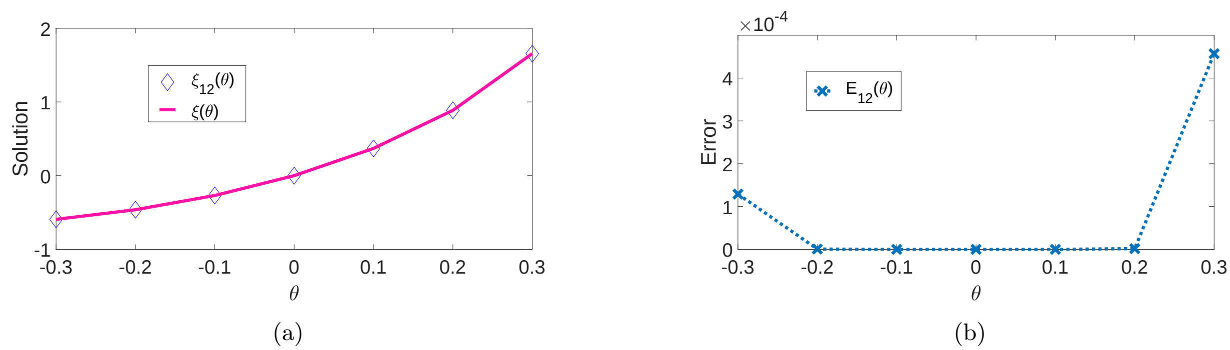
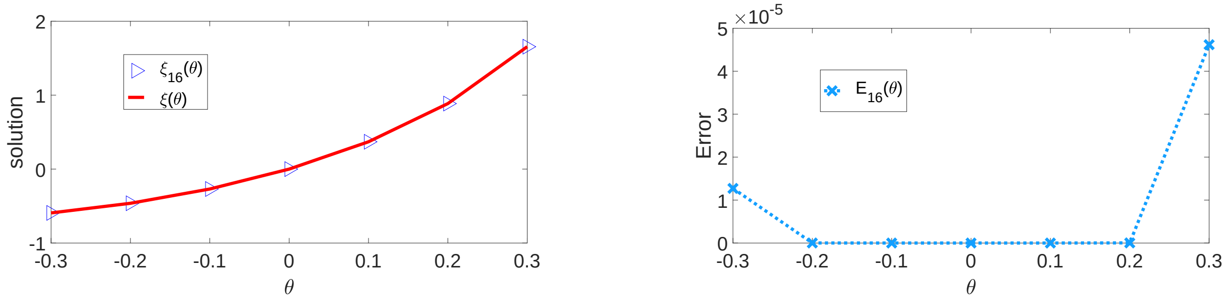
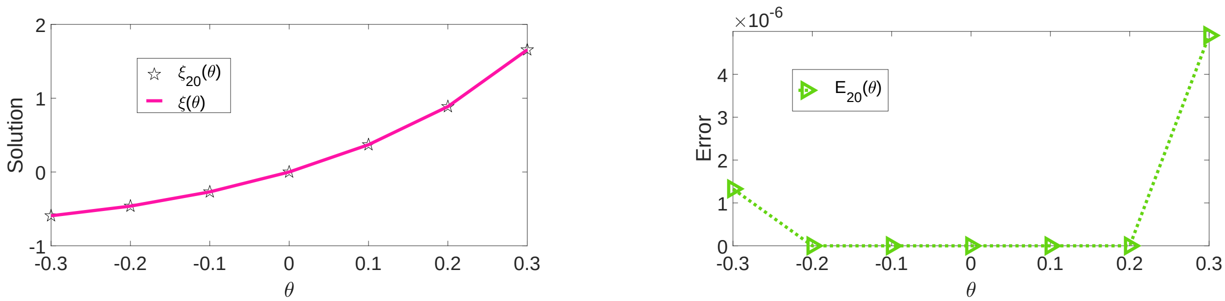


Fig. 4: (a) Comparison between approximate solution $\xi_{12}(\theta)$ and exact solution $\xi(\theta)$ of Problem 2. (b) Absolute error $E_{12}(\theta)$ in the approximate solution $\xi_{12}(\theta)$ of Problem 2.



(a) (b)

Fig. 5: (a) Comparison between approximate solution $\xi_{16}(\theta)$ and exact solution $\xi(\theta)$ of Problem 2. (b) Absolute error $E_{16}(\theta)$ in the approximate solution $\xi_{16}(\theta)$ of Problem 2.



(a) (b)

Fig. 6: (a) Comparison between approximate solution $\xi_{20}(\theta)$ and exact solution $\xi(\theta)$ of Problem 2. (b) Absolute error $E_{20}(\theta)$ in the approximate solution $\xi_{20}(\theta)$ of Problem 2.

Table 2: Comparison of the Absolute Errors of Problem 2.

	Present Method				
θ	$E_{12}(\theta)$	$E_{16}(\theta)$	$E_{20}(\theta)$	EWM [7]	OHAM [19]
-0.3	$1.291e{-}04$	$1.272e{-}05$	$1.329e{-}06$	$1.280e{-}04$	$2.277e{-}03$
-0.2	$7.531e{-}07$	$1.467e{-}08$	$3.032e{-}10$	$4.025e{-}05$	$3.892e{-}04$
-0.1	$1.063e{-}10$	$1.297e{-}13$	$1.665e{-}16$	$4.822e{-}07$	$1.061e{-}05$
0.0	0	0	0	0	0
0.1	$1.548e{-}10$	$1.9e{-}13$	$1.110e{-}16$	$1.068e{-}07$	$4.681e{-}05$
0.2	$1.645e{-}06$	$3.252e{-}08$	$6.781e{-}10$	$7.068e{-}04$	$1.938e{-}03$
0.3	$4.570e{-}04$	$4.617e{-}05$	$4.906e{-}06$	$1.010e{-}06$	$2.534e{-}02$

Again in Fig. 4(a), Fig. 5(a) and Fig. 6(a) we observe that the approximate solution $\xi_n(\theta)$ for $n = 12, 16, 20$ and the exact solution $\xi(\theta)$ of Problem 2 coincide nearly. Also, the absolute error between exact $\xi(\theta)$ and approximate solution $\xi_n(\theta)$ for $n = 12, 16, 20$ in Fig. 4(b), Fig. 5(b) and

Fig. 6(b) show that on increasing the order of the numerical solution errors are decreasing. In case of Problem 2 from Table 2, we can see that the proposed technique provides better result in comparison to the efficient wavelet method (EWM) [7] and optimal homotopy analysis method (OHAM) [19].

5 Conclusion

In this paper, an advanced decomposition technique is presented to obtain the numerical solution of non-linear IVPs of Bratu's type (1)-(2), with a fast algorithm to generate the Adomian polynomials. The proposed technique does not require the linearization of non-linear terms, discretization of the variables and any perturbed parameter to handle the non-linear problems of the type (1), so attained results are more physically realistic in comparison to the existing approaches. For the completeness of the proposed technique, in Theorem 1 we have proved that the decomposition technique is convergent.

In section 4, to test the robustness and effectiveness of the proposed technique, we considered two non-linear IVPs of Bratu's type. In Fig. 1(a), Fig. 2(a), Fig. 3(a) and similarly in Fig. 4(a), Fig. 5(a), Fig. 6(a) of Problem 1 and Problem 2, we observed that approximate solution $\xi_n(\theta)$ are indistinguishable to the exact solution $\xi(\theta)$, which demonstrate that the proposed technique works very well for Bratu's problem (1). In addition, in Fig. 1(b), Fig. 2(b), Fig. 3(b) of Problem 1 and similarly in Fig. 4(b), Fig. 5(b), Fig. 6(b) of Problem 2, it is to be noted that as we increase the order of numerical solution error are decreasing. Hence the accuracy of our obtained solutions can be improved by adding more terms in the approximate series solution. Also, Tables 1 and 2 reveal that the proposed technique provides more accurate results in comparison to the methods in [7, 8, 19, 35].

Compliance with ethical standards
Conflict of interest The authors declare that they have no Conflict of interest.

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