Existence and multiplicity solutions for critical Choquard-Kirchhoff type equations with variable exponents

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Abstract

In this paper we establish existence and multiplicity solutions for a class of Choquard-Kirchhoff type equations with variable exponents and critical reaction. Because of the critical reaction, we can use the concentration-compactness principle to deal with the lack of compactness. The results are also based on the combination of the mountain pass theorem and the Hardy-Littlewood-Sobolev inequality for variable exponents. And we discuss the results in non-degenerate and degenerate cases.

Keywords: p-Laplacian operator; Choquard equation; Hardy-Littlewood-Sobolev inequality; Critical nonlinearity; Concentration-compactness principles; Variational methods.

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1 Introduction

In this paper we deal with the following critical nonlocal Choquard equations with variable exponents of the form:

$$\begin{cases} M(\mathcal{T}_{p(x)}(u))((-\Delta)_{p(x)}u + V(x)|u|^{p(x)-2}u) = \lambda \left(\int_{\mathbb{R}^N} \frac{F(y,u(y))}{|x-y|^{\alpha(x,y)}}dy\right) f(x,u) + |u|^{p^*(x)-2}u \quad \text{in } \mathbb{R}^N, \\ u \in W_V^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

e1.1

where

$$\mathcal{T}_{p(x)}(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})dx,$$

M is the Kirchhoff function, $V \in C\left(\mathbb{R}^N, \mathbb{R}^+\right)$, $\alpha : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}$ and $\alpha > 0$, f is a continuous function, λ is a real parameter, p: $\mathbb{R}^N \mapsto \mathbb{R}$ is a function and $p^*(x) = \frac{Np(x)}{N-p(x)}$ is the critical Sobolev exponent.

In the sequel, if $h_1, h_2 \in C\left(\mathbb{R}^N\right)$, we say that $h_1 \ll h_2$ if $\inf\left\{h_2(x) - h_1(x) : x \in \mathbb{R}^N\right\} > 0$. And in the paper, C may denote a positive constant and the same C may represent different constants. Throughout this paper, we consider the following hypotheses:

($\mathcal{P}$) $p : \mathbb{R}^N \to \mathbb{R}$ *is continuous such that*

$$1 < p^- := \inf_{x\in\mathbb{R}^N} p(x) \leq p(x) \leqslant p^+ := \sup_{x\in\mathbb{R}^N} p(x) < N.$$

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($\mathcal{V}$) *$V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, with V_0 being a positive constant. Moreover, for any $\mathcal{D} > 0$, meas$\{x \in \mathbb{R}^N : V(x) \leq \mathcal{D}\} < \infty$, where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.*

($\mathcal{M}$) *(M_1) $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and there exists $m_0 > 0$ such that $\inf_{t\geq 0} M(t) = m_0$.*

(M_2) There exists $\sigma \in [1, p^*(x)/2p^+)$ satisfying $\sigma\mathscr{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\mathscr{M}(t) = \int_0^t M(s)ds$.

(M_3) There exists $m_1 > 0$ such that $M(t) \geq m_1 t^{\sigma-1}$ for all $t \in \mathbb{R}^+$ and $M(0) = 0$.

($\mathcal{F}$) *(f_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that f is odd with respect to the second variable.*

(f_2) There exist nonnegative functions r, a, with $r \in C(\mathbb{R}^N)$, $p \ll rq^- \leq rq^+ \ll p^*$ such that

$$|f(x,t)| \leq a(x)|t|^{r(x)-2}t \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

where

$$0 \leq a \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^+}{p^*(x)-r(x)q^+}}(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^-}{p^*(x)-r(x)q^-}}(\mathbb{R}^N)$$

and

$$\frac{1}{q(x)} + \frac{\alpha(x,y)}{N} + \frac{1}{q(y)} = 2 \text{ for all } x, y \in \mathbb{R}^N$$

and

$$0 < \alpha^- := \inf_{x,y\in\mathbb{R}^N} \alpha(x,y) \leq \lambda^+ := \sup_{x,y\in\mathbb{R}^N} \alpha(x,y) < N.$$

(f_3) there exists θ, with $p^+/\sigma < \theta < p^*(x)$ such that $0 < \theta F(x,t) \leq 2f(x,t)t$ for all $t \in \mathbb{R}^+$, where $F(x,t) = \int_0^t f(x,s)ds$.

rem1.1 **Remark 1.1.** *A typical example of M is given by $M(t) = a + b\,t^{\sigma-1}$ for $t \in \mathbb{R}_0^+$, where $a \in \mathbb{R}_0^+$, $b \in \mathbb{R}_0^+$ and $a + b > 0$. When M is of this type, problem (1.1) is said to be non–degenerate if $a > 0$, while it is called degenerate if $a = 0$. Clearly, assumptions (M_1)–(M_2) cover the non–degenerate case and (M_2)–(M_3) are automatic in the non–degenerate case.*

The paper was motivated by some works appeared in recent years. From the point of view of mathematical theory, the study of the $p(\cdot)$-Laplacian is a natural extension of the p-Laplacian, which itself is also a natural extension of the Laplacian ($p = 2$). Lebesgue spaces with variable exponents appeared in the literature in 1931 in the paper by Orlicz [39], since then Zhikov [50] started a new direction of investigation, which created the relationship between spaces with variable exponents and variational integrals with nonstandard growth conditions. From the point of application, variable exponents problem has many applications, for examples, in image processing [12] and electrorheological fluids [45]. For these reasons, many authors have begun to study the existence of solutions to variable exponents problem. For instance, various parametric boundary value problems with variable exponents can be found in the book of Rădulescu-Repovš [44] and also one can refer to the book by Diening et al. [14] for the properties of such operator and associated variable exponent Lebesgue spaces and variable exponent Sobolev spaces. For the critical problem, it was initially studied in the seminal paper of Brézis and Nirenberg [9], which treated Laplace equations, and then there have been extensions of [9] in many directions. Elliptic equations involving critical growth are delicate due to the lack of compactness arising in connection with the variational approach. To overcome this difficulty, the concentration-compactness principles introduced by P.L. Lions in [30, 31] and its variants at infinity [6, 7, 11] have played a decisive role in showing that a minimizing sequence or a Palais-Smale sequence is precompact. In the last decade, many authors have extended the results for p-Laplacian involving critical Sobolev exponents to variable exponents case. The variable exponents version of the second concentration-compactness principles for a bounded domain was independently obtained in [17] and [8]. Moreover, much more results regarding $p(x)$-Laplace equations and variable-order fractional $p(x)$-Laplace equations are in progress, for example, we refer to [1, 18, 25, 26] and [22, 33, 27] , respectively.

On the other hand, the study of the Choquard equation begin with Fröhlich [19] and Pekar [40], they dealt with the modeling of quantum polaron:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2 \right) u \quad \text{ in } \mathbb{R}^3. \tag{1.2}$$

e1.2

Then in the theory of Bose-Einstein condensation, there is a significant Choquard equation:

$$-\Delta u + V(x) u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^\lambda} \right) |u|^{p-2} u \quad \text{ in } \mathbb{R}^N, \tag{1.3}$$

e1.3

where $N \geq 3, 0 < \lambda < N$. Next, when $N = 3, p = 2$ and $\lambda = 1$, Lieb in [28] use problem (1.3) to get the approximation to Hartree-Fock theory of one-component plasma. As is known to all, the Choquard equation is famous as the Schrödinger-Newton equation. And Penrose in[35, 41] applied equation (1.3) as a model of self-gravitating matter. Recently, more and more works have studied the existence and multiplicity of solutions for problem (1.3). We cite [36, 37, 38] for the work of Choquard type equations over the whole domain $\mathbb{R}^N$. For critical case, Gao and Yang [20] studied the Brezis-Nirenberg type existence results for the following critical Choquard problem in bounded domains Ω:

$$-\Delta u = \lambda u + \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u|^{2^*_\mu - 2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\lambda > 0$, $0 < \mu < N$. Later in [21] author used variational methods to prove the existence and multiplicity of positive solutions for equations involving convex and convex-concave type nonlinearities. Once we turn our attention to the Choquard problem with variable exponents, we immediately see that the literature is relatively scarce. In this case, we call attention to [32], the authors firstly considered the nonhomongeneous Choquard equation with $p(x)$-Laplacian operator and obtained the existence of a weak solution by using variational methods. Secondly, by using truncation arguments and Krasnoselskii's genus, they also showed a multiplicity of solutions for the $p(x)$-Laplacian Choquard equation with non-degenerate Kirchhoff term. In [46], the authors investigated a class of nonhomogeneous Choquard equations involving p-Laplacian, the existence of at least two nontrivial solutions is obtained by using of Nehari manifold and minimax methods.

Recently, Alves and Tavares [2] in which work the authors considered the following quasilinear Choquard equations involving variable exponent:

$$\begin{cases} (-\Delta)_{p(x)} u + V(x)|u|^{p(x)-2} u = \displaystyle\int_{\mathbb{R}^N} \frac{F(y, u(y))}{|x-y|^{\alpha(x,y)}} dy f(x,u) \quad \text{ in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.4}$$

e1.4

where $V, p : \mathbb{R}^N \to \mathbb{R}$, $\alpha : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $F(x,t)$ is the primitive of $f(x,t)$. The existence of solution for this problem are obtained by using the Hardy-Littlewood-Sobolev type inequality for variable exponents together with variational methods.

Zhang etc. in [49] considered the following Choquard problem with variable exponents and critical reaction:

$$\begin{cases} -\Delta_{p(x)}u + \mu|u|^{p(x)-2}u = \displaystyle\int_{\mathbb{R}^N} \frac{F(y,u(y))}{|x-y|^{\alpha(x,y)}}dyf(x,u) + \beta(x)|u|^{p^*(x)-2}u \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.5}$$ e1.5

where $\alpha : \mathbb{R}^N \to \mathbb{R}$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R}^N \to \mathbb{R}$ are continuous functions, and $p : \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz radially symmetric function, $p^*(x) = Np(x)/(N - p(x))$ denote the critical Sobolev exponent and assume that $\mu > 0$. The existence of infinitely many solutions for problem (1.5) are obtained by variational and analytic methods, including the Hardy-Littlewood-Sobolev inequality for variable exponents and the concentration-compactness principle for problems with variable growth. To the best of our knowledge, the existence and multiplicity of solutions for the critical Choquard-Kirchhoff type equations with variable exponents (1.1) has not yielded any results, especially for the degenerate cases.

The goal of this research is to complete and improve the study of the critical Choquard-Kirchhoff type equations with variable exponents. And we discuss the results in non-degenerate and degenerate cases. It is worth mentioning that the degenerate and non-degenerate case are rather interesting and are treated in well-known papers in Kirchhoff theory [13]. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. Very recently, there are also some authors working on the degenerate Kirchhoff problem. For example, Wang et al. [48] dealt with the existence and multiplicity of solutions for critical Kirchhoff-type p-Laplacian problems in the degenerate case by using the mountain pass theorem and an abstract critical point theorem based on the cohomological index. Song and Shi [47] obtained the existence of infinitely many solutions for a class of degenerate p-fractional Kirchhoff equations with critical Hardy-Sobolev nonlinearities by means of the Kajikiya's new version of the symmetric mountain pass lemma. Subsequently, Liang et. al. [24] were concerned with the existence and multiplicity of solutions for Kirchhoff problem with the fractional Choquard-type in the degenerate Kirchhoff case. On some interesting results recovering the degenerate case of Kirchhoff–type problems, we refer to [5, 10, 34, 42] and the references therein for more details in bounded domains and in the whole space.

Then the main theorems are the following in this paper.

the1.1 **Theorem 1.1.** *Assume $(\mathcal{P})$ and $(\mathcal{V})$ hold. If M satisfies (M_1)–(M_2) and f verifies (f_1)–(f_3), then there exists $\lambda_1 > 0$ such that for any $\lambda \geq \lambda_1$ problem (1.1) has a nontrivial solution in $W_V^{1,p(x)}(\mathbb{R}^N)$.*

the1.2 **Theorem 1.2.** *Assume $(\mathcal{P})$ and$(\mathcal{V})$ hold. If M satisfies (M_1)–(M_2), f verifies (f_1)–(f_3) and suppose one of the following conditions holds:*
(i) there exists a constant $m^* > 0$ such that for each $m_0 > m^*$ and $\lambda > 0$;
(ii) there exists a constant $\lambda_2 > 0$ such that, for all $\lambda > \lambda_2$ and $m_0 > 0$.
Then, problem (1.1) has at least n pairs of nontrivial weak solutions in $W_V^{1,p(x)}(\mathbb{R}^N)$.

We also obtain the following existence results for equation(1.1) in the degenerate case.

the1.3 **Theorem 1.3.** *Assume $(\mathcal{P})$ and $(\mathcal{V})$ hold. If M satisfies (M_2)–(M_3) and f verifies (f_1)–(f_3), then there exists $\lambda_3 > 0$ such that for any $\lambda \geq \lambda_3$ problem (1.1) has a nontrivial solution in $W_V^{1,p(x)}(\mathbb{R}^N)$.*

the1.4 **Theorem 1.4.** *Assume $(\mathcal{P})$ and $(\mathcal{V})$ hold. If M satisfies (M_2)–(M_3), f verifies (f_1)–(f_3) and suppose one of the following conditions holds:*
(i) there exists a constant $m_* > 0$ such that for each $m_1 > m_*$ and $\lambda > 0$;

(ii) there exists a constant $\lambda_4 > 0$ such that, for all $\lambda > \lambda_4$ and $m_1 > 0$.
Then, problem (1.1) has at least n pairs of nontrivial weak solutions in $W_V^{1,p(x)}(\mathbb{R}^N)$.

The paper is organized as follows. Section 2 contains some properties of the Lebesgue spaces with variable exponents and the Sobolev spaces with variable exponents. In Section 3, we give the proof of the Palais-Smale condition at some special conditions by using the concentration-compactness principles for Sobolev spaces with variable exponents. Section 4 proves the existence and multiplicity result for problem (1.1) in the non–degenerate case. Finally in Section 5, we give the proof of Theorems 1.3 and 1.4, which present the proof of existence and multiplicity of solutions for problem (1.1) in the degenerate case.

2 Preliminaries

sec2

In this section, we review some basic definitions on the Lebesgue spaces with variable exponents and the Sobolev spaces with variable exponents. We refer to the [14, 15, 16] for more details.

Set Ω be a bounded domain of $\mathbb{R}^N$, and

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : h(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^- = \min_{x\in\bar{\Omega}} h(x), h^+ = \max_{x\in\bar{\Omega}} h(x).$$

For any $p \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}$$

which endowed with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The Lebesgue-Sobolev space with variable exponents $W^{1,p(x)}\left(\mathbb{R}^N\right)$ is defined by:

$$W^{1,p(x)}\left(\mathbb{R}^N\right) = \left\{u \in L^{p(x)}\left(\mathbb{R}^N\right); |\nabla u|^{p(x)} \in L^{p(x)}\left(\mathbb{R}^N\right)\right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

For problem (1.1) , the appropriate Sobolev space is $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$, which defined as the completion of $C_0^\infty\left(\mathbb{R}^N\right)$ with the norm

$$\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} = \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)} + \|u\|_{L_V^{p(x)}(\mathbb{R}^N)}$$

where

$$\|u\|_{L_V^{p(x)}(\mathbb{R}^N)} = \inf\left\{\eta > 0 : \int_{\mathbb{R}^N} V(x)\left|\frac{u}{\eta}\right|^{p(x)} dx \le 1\right\}.$$

pro2.1 **Proposition 2.1** (see [16]). (1) *Denote by* $L^{p'(x)}(\Omega)$ *the conjugate space of* $L^{p(x)}(\Omega)$ *with* $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$,

$$\int_\Omega |uv|dx \leq \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)},\ u\in L^{p(x)}(\Omega), v\in L^{p'(x)}(\Omega)$$

holds.

(2) *Define mapping* $\rho : L^{p(x)}(\Omega)\to\mathbb{R}$ *by* $\rho(u)=\int_\Omega |u|^{p(x)}dx$, *then the following relations hold*

$$\begin{aligned}
&|u|_{p(x)}<1(=1,>1)\Leftrightarrow \rho(u)<1(=1,>1),\\
&|u|_{p(x)}>1\Rightarrow |u|_{p(x)}^{p^-}\leq\rho(u)\leq|u|_{p(x)}^{p^+},\\
&|u|_{p(x)}<1\Rightarrow |u|_{p(x)}^{p^+}\leq\rho(u)\leq|u|_{p(x)}^{p^-}.
\end{aligned}$$

Then we give the Hardy-Littlewood-Sobolev inequality for variable exponents, see [2].

pro2.2 **Proposition 2.2.** *Let* $p,q\in C^+\left(\mathbb{R}^N\right), h\in L^{p^+}\left(\mathbb{R}^N\right)\cap L^{p^-}\left(\mathbb{R}^N\right), g\in L^{q^+}\left(\mathbb{R}^N\right)\cap L^{q^-}\left(\mathbb{R}^N\right)$ *, and* $\alpha:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ *be a continuous function such that* $0<\alpha^-:=\inf_{x\in\mathbb{R}^N}\alpha(x)\leq\alpha^+:=\sup_{x\in\mathbb{R}^N}\alpha(x)<N$ *and* $\frac{1}{p(x)}+\frac{\alpha(x,y)}{N}+\frac{1}{q(y)}=2$, *for* $\forall x,y\in\mathbb{R}^N$.
Then, we have

$$\left|\iint_{\mathbb{R}^{2N}}\frac{h(x)g(y)}{|x-y|^{\alpha(x,y)}}dxdy\right|\leq C\left(|h|_{L^{p^+}(\mathbb{R}^N)}|g|_{L^{q^+}(\mathbb{R}^N)}+|h|_{L^{p^-}(\mathbb{R}^N)}|g|_{L^{q^-}(\mathbb{R}^N)}\right)$$

where $C>0$ *is a constant that does not depend on* h *and* g.

cor2.1 **Corollary 2.1.** *In particular for* $h(x)=g(x)=|u(x)|^{\beta(x)}\in L^{r^+}(\mathbb{R}^N)\cap L^{r^-}(\mathbb{R}^N)$, *we have*

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)|^{\beta(x)}|u(y)|^{\beta(y)}}{|x-y|^{\alpha(x,y)}}dxdy\right|\leq C\left(\left||u|^{\beta(\cdot)}\right|^2_{L^{r^+}(\mathbb{R}^N)}+\left||u|^{\beta(\cdot)}\right|^2_{L^{r^-}(\mathbb{R}^N)}\right),$$

where β, $r\in C_+(\overline{\mathbb{R}^N})$ *such that* $1<\beta^-r^-\leq\beta(x)r^-\leq\beta(x)r^+<p^*(x)$ *for all* $x\in\overline{\mathbb{R}^N}$, $C>0$ *is a constant that does not depend on* r.

rem2.1 **Remark 2.1.** *If* $(\mathcal{P})$ *and* $(\mathcal{V})$ *hold, then the Sobolev embedding*

$$W_V^{1,p(x)}\left(\mathbb{R}^N\right)\hookrightarrow L^{s(x)}\left(\mathbb{R}^N\right)$$

is compact for all $s\in C^+\left(\mathbb{R}^N\right)$ *and* $p(x)\leq s(x)\leq p^*(x), \forall x\in\mathbb{R}^N$. *Hence, it is obvious that*

$$\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)}\leq S|u|_{L^{s(x)}(\mathbb{R}^N)},$$

where S *is the best Sobolev constant.*

rem2.2 **Remark 2.2.** *We can find that there exists a constant* $b>0$ *such that*

$$\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)dx\geq b\left(\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+|u|^{p(x)}\right)dx\right)$$

3 Verification of $(PS)_c$ condition

sec3

Let's first recall the definition of the $(PS)_c$ condition. Given $c \in \mathbb{R}$, we say the functional J_λ satisfies the Palais-Smale condition at level c if any sequence $(u_n)_n \subset W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ such that $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ has a convergent subsequence. In this paper, we will prove the functional J_λ satisfies the $(PS)_c$ condition at special levels c.

We say that $u \in W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ is a weak solution of problem (1.1) if

$$\begin{aligned} &M(\mathcal{T}_{p(x)}(u)) \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right) dx \\ &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N} |u|^{p^*(x)-2}uvdx \end{aligned} \tag{3.1}$$ e3.1

for all $v \in W_V^{1,p(x)}\left(\mathbb{R}^N\right)$.

The energy functional $J_\lambda : W_V^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (1.1) is given by

$$J_\lambda(u) = \mathscr{M}\left(\mathcal{T}_{p(x)}(u)\right) - \lambda\Phi(u) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)}|u|^{p^*(x)}dx, \tag{3.2}$$ e3.2

where

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\lambda(x,y)}}dxdy$$

and $\mathscr{M}(t) = \int_0^t M(s)ds$ and $F(x,t) = \int_0^t f(x,s)ds$. It is obvious that $J_\lambda \in C^1(W_V^{1,p(x)}(\mathbb{R}^N))$. Moreover, for all $u, v \in W_V^{1,p(x)}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned} \langle J'_\lambda(u), v\rangle =& M(\mathcal{T}_{p(x)}(u)) \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right) dx \\ &- \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\alpha(x,y)}}dxdy - \int_{\mathbb{R}^N} |u|^{p^*(x)-2}uvdx. \end{aligned} \tag{3.3}$$ e3.3

Hence, the weak solutions of problem (1.1) are the critical points of the functional J_λ.

lem3.1 **Lemma 3.1.** *Assume $(\mathcal{P})$, $(\mathcal{V})$, $(\mathcal{F})$ and (M_1)–(M_2) hold. Let $(u_n)_n \subset W_V^{1,p(x)}(\mathbb{R}^N)$ be a Palais-Smale sequence of functional J_λ, then*

$$J_\lambda(u_n) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{in} \quad (W_V^{1,p(x)}(\mathbb{R}^N))' \tag{3.4}$$ e3.4

as $n \to \infty$, where $(W_V^{1,p(x)}(\mathbb{R}^N))'$ is the dual of $W_V^{1,p(x)}(\mathbb{R}^N)$. If

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{\left(m_0 S^{p^+}\right)^{\tau^+}, \left(m_0 S^{p^+}\right)^{\tau^-}\right\}, \tag{3.5}$$

where $\tau(x) = \frac{p^*(x)}{p^*(x)-p^+}$ and S is defined as in Remark 2.1, then there exists a subsequence of $(u_n)_n$ strongly convergent in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Proof. First, We claim that $(u_n)_n$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Let $(u_n)_n$ be a $(PS)_{c_\lambda}$ sequence for J_λ, with c_λ satisfying (3.5). Then, from (f_3), we can deduce that

$$\begin{aligned}
c_\lambda + 1 + o(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{\theta}\langle J_\lambda'(u_n), u_n\rangle \\
&= \mathscr{M}\left(\mathcal{T}_{p(x)}(u_n)\right) - \frac{1}{\theta} M\left(\mathcal{T}_{p(x)}(u_n)\right)\left[\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} dx\right] \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)|u_n|^{p^*(x)} dx + \lambda \iint_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|x-y|^{\alpha(x,y)}}\left(\frac{f(x,u_n)u_n}{\theta} - \frac{F(x,u_n)}{2}\right) dxdy \\
&\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right)\frac{m_0}{p^+}\left[\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} dx\right] + \int_{\mathbb{R}^N}\left(\frac{1}{\theta} - \frac{1}{p^*}\right)|u_n|^{p^*} dx \\
&\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right)\frac{m_0}{p^+}\|u_n\|^{p^-}.
\end{aligned} \tag{3.6}$$

e3.6

This fact implies that sequence $(u_n)_n$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Next, we prove that

$$\left\langle \Phi'(u_n) - \Phi'(u), u_n - u\right\rangle \to 0 \quad \text{as } n \to \infty.$$

Since $u_n \to u$ weakly in $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ as $n \to \infty$, when $\Phi'(u) \in \left(W_V^{1,p(x)}\left(\mathbb{R}^N\right)\right)'$, we can yield that

$$\left\langle \Phi'(u), u_n - u\right\rangle \to 0 \quad \text{as } n \to \infty.$$

So we only need to prove that

$$\left\langle \Phi'(u_n), u_n - u\right\rangle \to 0 \text{ as } n \to \infty.$$

In view of Proposition 2.2, we deduce that

$$\begin{aligned}
\left|\left\langle \Phi'(u_n), u_n - u\right\rangle\right| \leq & C\|F(x,u_n)\|_{L^{p^+}(\mathbb{R}^N)}\|f(x,u_n)(u_n - u)\|_{L^{q^+}(\mathbb{R}^N)} \\
& + C\|F(x,u_n)\|_{L^{p^-}(\mathbb{R}^N)}\|f(x,u_n)(u_n - u)\|_{L^{q^-}(\mathbb{R}^N)}.
\end{aligned} \tag{3.7}$$

e3.7

Combining (f_2) and the boundedness of $(u_n)_n$ in $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$, we can get that

$$\begin{aligned}
\|F(x,u_n)\|_{L^{p^+}(\mathbb{R}^N)} \leq & C\left(\int_{\mathbb{R}^N}\left(|u_n|^{p^+ r(x)}\right) dx\right)^{\frac{1}{p^+}} \\
\leq & C \max\left\{\|u_n\|^{r^+}_{L^{p^+ r(x)}(\mathbb{R}^N)}, \|u_n\|^{r^-}_{L^{p^+ r(x)}(\mathbb{R}^N)}\right\} \\
\leq & C
\end{aligned} \tag{3.8}$$

e3.8

and

$$\begin{aligned}
\|F(x,u_n)\|_{L^{p^-}(\mathbb{R}^N)} \leq & C \max\left\{\|u_n\|^{r^+}_{L^{p^- r(x)}(\mathbb{R}^N)}, \|u_n\|^{r^-}_{L^{p^- r(x)}(\mathbb{R}^N)}\right\} \\
\leq & C.
\end{aligned} \tag{3.9}$$

e3.9

According to (f_2), Remark 2.1 and the boundedness of $(u_n)_n$ in $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$, we can yield that

$$\begin{aligned}
&\|f(x,u_n)(u_n-u)\|_{L^{q^+}(\mathbb{R}^N)}^{q^+}\\
\leq& C\left\||u_n|^{q^+(r(x)-1)}\right\|_{L^{\frac{r(x)}{r(x)-1}}(\mathbb{R}^N)}\left\||u_n-u|^{q^+}\right\|_{L^{r(x)}(\mathbb{R}^N)}\\
\leq& C\max\left\{\|u_n-u\|_{L^{q^+r(x)}(\mathbb{R}^N)}^{q^+},\|u_n-u\|_{L^{q^+r(x)}(\mathbb{R}^N)}^{\frac{q^+r^-}{r^+}}\right\}\\
&+C\max\left\{\|u_n-u\|_{L^{q^+r(x)}(\mathbb{R}^N)}^{\frac{q^+r^+}{r^-}},\|u_n-u\|_{L^{q^+r(x)}(\mathbb{R}^N)}^{q^+}\right\}\\
=&o_n(1)\quad\text{as } n\to\infty
\end{aligned} \tag{3.10}$$

e3.10

and

$$\begin{aligned}
&\|f(\cdot,u_n)(u_n-u)\|_{L^{q^-}(\mathbb{R}^N)}^{q^-}\\
\leq& C\left\||u_n|^{q^-(r(\cdot)-1)}\right\|_{L^{\frac{r(x)}{r(x)-1}}(\mathbb{R}^N)}\left\||u_n-u|^{q^-}\right\|_{L^{r(x)}(\mathbb{R}^N)}\\
\leq& C\max\left\{\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{q^-},\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{\frac{q^-r^-}{r^+}}\right\}\\
&+C\max\left\{\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{\frac{q^-r^+}{r^-}},\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{q^-}\right\}\\
=&o_n(1)\quad\text{as}\quad n\to\infty.
\end{aligned} \tag{3.11}$$

e3.11

Combining (3.7)-(3.11), we can obtain $\langle\Phi'(u_n),u_n-u\rangle\to 0$ as $n\to\infty$. So we deduce that

$$\left\langle\Phi'(u_n)-\Phi'(u),u_n-u\right\rangle\to 0,\ \text{as } n\to\infty.$$

Then, in view of the concentration-compactness principle for variable exponents in [23], we get

$$\begin{aligned}
&u_n\to u\quad\text{a.e. in }\mathbb{R}^N,\\
&u_n\rightharpoonup u\quad\text{in}\quad W_V^{1,p(x)}(\mathbb{R}^N),\\
&U_n(x)\overset{*}{\rightharpoonup}\mu\geq U(x)+\sum_{i\in I}\delta_{x_i}\mu_i,\\
&|u_n|^{p^*(x)}\overset{*}{\rightharpoonup}\nu=|u|^{p^*}+\sum_{i\in I}\delta_{x_i}\nu_i,\\
&S\nu_i^{\frac{1}{p^*(x)}}\leq\mu_i^{\frac{1}{p(x)}}\quad\text{for}\quad i\in I,
\end{aligned} \tag{3.12}$$

e3.12

where

$$U_n(x):=|\nabla u_n(x)|^{p(x)}+V(x)|u_n(x)|^{p(x)}$$

and

$$U(x):=|\nabla u(x)|^{p(x)}+V(x)|u(x)|^{p(x)}.$$

Furthermore, we yield that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N} U_n(x)dx=\mu(\mathbb{R}^N)+\mu_\infty,\\
&\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*(x)}dx=\nu(\mathbb{R}^N)+\nu_\infty,\\
&S\nu_\infty^{1/p_\infty^*}\le \mu_\infty^{1/p_\infty},
\end{aligned} \tag{3.13}$$

e3.13

where

$$\begin{aligned}
&\mu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{|x|>R\}}(|\nabla u_n(x)|^{p(x)}+V(x)|u_n(x)|^{p(x)})dx,\\
&\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{|x|>R\}}|u_n|^{p^*(x)}dx,\\
&p_\infty=\lim_{|x|\to\infty}p(x)\quad\text{and}\quad p_\infty^*=\lim_{|x|\to\infty}p^*(x).
\end{aligned}$$

Now we claim that

$$I=\emptyset\quad\text{and}\quad \nu_\infty=0.$$

We assume that $I\neq\emptyset$. For any $i\in I$ and any $\epsilon>0$ small, we define a smooth cut-off function $\phi_{\epsilon,i}$ centered at z_i such that

$$0\le\phi_{\epsilon,i}(x)\le 1,\ \phi_{\epsilon,i}(x)=1 \text{ in } B_{2\epsilon}(z_i),\ \phi_{\epsilon,i}(x)=0 \text{ in } B_\epsilon(z_i)^c,\ |\nabla\phi_{\epsilon,i}(x)|\le 2/\epsilon.$$

Combining the boundedness of $(u_n\phi_{\epsilon,i})_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ with $\langle J_\lambda'(u_n),u_n\phi_{\epsilon,i}\rangle\to 0$, we deduce that

$$\begin{aligned}
&M(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p(x)}\phi_{\epsilon,i}+V(x)|u_n|^{p(x)}\phi_{\epsilon,i}+|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_{\epsilon,i}u_n\Big)dx\\
&=\lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_{\epsilon,i}}{|x-y|^{\alpha(x,y)}}dxdy+\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\epsilon,i}dx+o_n(1).
\end{aligned} \tag{3.14}$$

e3.14

Since $u_n\to u$ in $L^{p(x)}\left(B_{2\epsilon}(z_i)\right)$, we get that

$$\|\nabla\phi_{\epsilon,i}u_n\|_{L^{p(x)}(\mathbb{R}^N)}\to\|\nabla\phi_{\epsilon,i}u\|_{L^{p(x)}(\mathbb{R}^N)}\quad\text{as } n\to\infty.$$

Then we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_{\epsilon,i}u_ndx\right|\\
&\le\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-1}|\nabla\phi_{\epsilon,i}u_n|dx\\
&\le\limsup_{n\to\infty}C\left\||\nabla u_n|^{p(x)-1}\right\|_{L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)}\|\nabla\phi_{\epsilon,i}u_n\|_{L^{p(x)}(\mathbb{R}^N)}\\
&\le C\|\nabla\phi_{\epsilon,i}u\|_{L^{p(x)}(\mathbb{R}^N)}
\end{aligned} \tag{3.15}$$

e3.15

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \phi_{\epsilon,i} u|^{p(x)} dx \\
&= \int_{B_{2\epsilon}(z_i)} |\nabla \phi_{\epsilon,i} u|^{p(x)} dx \leq C \left\| |\nabla \phi_{\epsilon,i}|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p^*(x)-p(x)}}(B_{2\epsilon}(z_i))} \left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\epsilon}(z_i))} \\
&\leq C \max \left\{ \left(\int_{B_{2\epsilon}(z_i)} |\nabla \phi_{\epsilon,i}|^N dx \right)^{\frac{p^+}{N}}, \left(\int_{B_{2\epsilon}(z_i)} |\nabla \phi_{\epsilon,i}|^N dx \right)^{\frac{p^-}{N}} \right\} \left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\epsilon}(z_i))} \qquad (3.16) \quad \boxed{\texttt{e3.16}} \\
&\leq C \max \left\{ \left(\frac{4^N w_N}{N} \right)^{\frac{p^+}{N}}, \left(\frac{4^N w_N}{N} \right)^{\frac{p^-}{N}} \right\} \left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\epsilon}(z_i))} \\
&= o_\epsilon(1) \quad \text{as } \epsilon \to 0,
\end{aligned}
$$

where w_N is the surface area of the unit sphere in $\mathbb{R}^N$.

Next, we prove that

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y)) f(x, u_n(x)) u_n \phi_{\epsilon,i}}{|x-y|^{\alpha(x,y)}} dxdy = \left\langle \Phi'(u_n), u_n \phi_{\epsilon,i} \right\rangle
$$

$$
\to \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u(y)) f(x, u(x)) u(x) \phi_{\epsilon,i}(x)}{|x-y|^{\alpha(x,y)}} dxdy = \left\langle \Phi'(u), u \phi_{\epsilon,i} \right\rangle \text{ as } n \to \infty.
$$

According to Proposition 2.2, the boundedness of $(u_n)_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ and the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
&\left| \left\langle \Phi'(u_n), u_n \phi_{\epsilon,i} \right\rangle - \left\langle \Phi'(u), u \phi_{\epsilon,i} \right\rangle \right| \\
&\leq \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n(y)) \left(f(x, u_n(x)) u_n(x) - f(x, u(x)) u(x) \right)}{|x-y|^{\alpha(x,y)}} dxdy \right| \\
&\quad + \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(F(y, u_n(y)) - F(y, u(y)) \right) f(x, u(x)) u(x)}{|x-y|^{\alpha(x,y)}} dxdy \right| \\
&\leq C \left\| F(x, u_n) \right\|_{L^{p^+}(\mathbb{R}^N)} \left\| f(x, u_n) u_n - f(x, u) u \right\|_{L^{q^+}(\mathbb{R}^N)} \\
&\quad + C \left\| F(x, u_n) \right\|_{L^{p^-}(\mathbb{R}^N)} \left\| f(x, u_n) u_n - f(x, u) u \right\|_{L^{q^-}(\mathbb{R}^N)} \qquad (3.17) \quad \boxed{\texttt{e3.17}} \\
&\quad + C \left\| F(x, u_n) - F(x, u) \right\|_{L^{p^+}(\mathbb{R}^N)} \left\| f(x, u) u \right\|_{L^{q^+}(\mathbb{R}^N)} \\
&\quad + C \left\| F(x, u_n) - F(x, u) \right\|_{L^{p^-}(\mathbb{R}^N)} \left\| f(x, u) u \right\|_{L^{q^-}(\mathbb{R}^N)} \\
&\leq C \left\| f(x, u_n) u - f(x, u) u \right\|_{L^{q^+}(\mathbb{R}^N)} + C \left\| f(x, u_n) u - f(x, u) u \right\|_{L^{q^-}(\mathbb{R}^N)} \\
&\quad + C_u \left\| F(x, u_n) - F(x, u) u \right\|_{L^{p^+}(\mathbb{R}^N)} + C_u \left\| F(x, u_n) - F(x, u) u \right\|_{L^{p^-}(\mathbb{R}^N)} \\
&= o_n(1) \quad \text{as } n \to \infty,
\end{aligned}
$$

where C_u is a positive constant. And

$$\begin{aligned}
\left|\left\langle \Phi'(u), u\phi_{\epsilon,i}\right\rangle\right| &\leq C\|F(x,u)\|_{L^{p^+}(\mathbb{R}^N)}\|f(x,u)u\phi_{\epsilon,i}\|_{L^{q^+}(\mathbb{R}^N)} \\
&\quad + C\|F(x,u)\|_{L^{p^-}(\mathbb{R}^N)}\|f(x,u)u\phi_{\epsilon,i}\|_{L^{q^-}(\mathbb{R}^N)} \\
&\leq C\|f(x,u)u\phi_{\epsilon,i}\|_{L^{q^+}(\mathbb{R}^N)} + C\|f(x,u)u\phi_{\epsilon,i}\|_{L^{q^-}(\mathbb{R}^N)} \\
&\leq C\left(\int_{B_{2\epsilon}(z_i)} a(x)^{q^+}|u|^{r(x)q^+}dx\right)^{\frac{1}{q^+}} + C\left(\int_{B_{2\epsilon}(z_i)} a(x)^{q^-}|u|^{r(x)q^-}dx\right)^{\frac{1}{q^-}} \qquad (3.18) \\
&\leq C\left(\int_{B_{2\epsilon}(z_i)} |u|^{r(x)q^+}dx\right)^{\frac{1}{q^+}} + C\left(\int_{B_{2\epsilon}(z_i)} |u|^{r(x)q^-}dx\right)^{\frac{1}{q^-}} \\
&= o_\epsilon(1) \quad \text{as } \epsilon \to 0.
\end{aligned}$$

e3.18

Combining (3.14)-(3.18), we deduce that

$$M(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_{\epsilon,i} + V(x)|u_n|^{p(x)}u_n\phi_{\epsilon,i}\right)dx = \int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\epsilon,i}dx + o_n(1). \qquad (3.19)$$

e3.19

Since $\phi_{\epsilon,i}$ has compact support and (M_1), choosing $n\to\infty$ and $\epsilon\to 0$ in (3.19), we get

$$m_0\mu_i \leq \nu_i.$$

In view of (3.12), we yield that

$$\nu_i \geq \left(m_0 S^{p^+}\right)^{\frac{p^*(z_i)}{p^*(z_i)-p^+}} \geq \min\left\{\left(m_0S^{p^+}\right)^{\tau^+}, \left(m_0S^{p^+}\right)^{\tau^-}\right\}, \qquad (3.20)$$

e3.20

where $\tau(x) = \frac{p^*}{p^*-p^+}$. According to $J_\lambda(u_n)\to c_\lambda$ and $J'_\lambda(u_n)\to 0$ as $n\to\infty$, we obtain from (3.20) and (3.5) that

$$\begin{aligned}
c_\lambda &= \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle\right) \\
&\geq \int_{\mathbb{R}^N}\left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)|u_n|^{p^*(x)}dx \\
&\geq \left(\frac{1}{\theta} - \frac{1}{p^+}\right)\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\epsilon,i}dx \geq \left(\frac{1}{\theta} - \frac{1}{p^+}\right)\nu_i \qquad (3.21) \\
&\geq \left(\frac{1}{\theta} - \frac{1}{p^+}\right)\min\left\{\left(m_0S^{p^+}\right)^{\tau^+}, \left(m_0S^{p^+}\right)^{\tau^-}\right\} > c_\lambda.
\end{aligned}$$

e3.21

We get a contradiction, so $I = \emptyset$.

Then we show that $\nu_\infty = 0$. We suppose that $\nu_\infty > 0$. Similarly, we define a cut off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ in B_R and $\phi_R(x) = 1$ in B^c_{R+1}. According to the boundedness of $(u_n\phi_R)_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ and $\langle J'_\lambda(u_n), u_n\phi_R\rangle \to 0$ as $n\to\infty$, we deduce that

$$\begin{aligned}
&M(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_R + V(x)|u_n|^{p(x)}u_n\phi_R + |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_R u_n\right)dx \\
&= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_R dx + o_n(1). \qquad (3.22)
\end{aligned}$$

e3.22

Similarly, we can prove that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_R u_n dx\right|=0$$

and

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}}dxdy=\left\langle\Phi'\left(u_n\right),u_n\phi_R\right\rangle=0.$$

So we get

$$M(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_R+V(x)|u_n|^{p(x)}u_n\phi_R\right)dx=\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_R dx+o_n(1). \tag{3.23}$$

e3.23

Letting $R\to\infty$ in (3.23), we deduce

$$m_0\mu_\infty\le\nu_\infty. \tag{3.24}$$

e3.24

According to (3.13) and (3.24), we can also infer that

$$\begin{aligned}c_\lambda&=\lim_{n\to\infty}J_\lambda(u_n)=\lim_{n\to\infty}\left(J_\lambda(u_n)-\frac{1}{\theta}\langle J'_\lambda(u_n),u_n\rangle\right)\\&\ge\int_{\mathbb{R}^N}\left(\frac{1}{\theta}-\frac{1}{p^*(x)}\right)|u_n|^{p^*(x)}dx\\&\ge\left(\frac{1}{\theta}-\frac{1}{p^+}\right)\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_R dx\ge\left(\frac{1}{\theta}-\frac{1}{p^+}\right)\nu_\infty\\&\ge\left(\frac{1}{\theta}-\frac{1}{p^+}\right)\min\left\{\left(m_0S^{p^+}\right)^{\tau^+},\left(m_0S^{p^+}\right)^{\tau^-}\right\}>c_\lambda.\end{aligned} \tag{3.25}$$

e3.25

Then we get a contradiction, so $\nu_\infty=0$.

Therefore, in view of $I=\emptyset$ and $\nu_\infty=0$, we obtain that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*(x)}dx=\int_{\mathbb{R}^N}|u|^{p^*(x)}dx.$$

According to the Brézis-Lieb type lemma, we get

$$\int_{\mathbb{R}^N}|u_n-u|^{p^*}dx\to 0,$$

thus $\|u_n-u\|_{L^{p^*(x)}(\mathbb{R}^N)}\to 0$. Consequently, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n|^{p^*(x)-2}u_n-|u|^{p^*(x)-2}u\right)(u_n-u)\,dx=0. \tag{3.26}$$

e3.26

Then we get

$$\lim_{n\to\infty}\left(M\left(T_{p(x)}(u_n)\right)-M\left(T_{p(x)}(u)\right)\right)\langle L(u),u_n-u\rangle=0, \tag{3.27}$$

e3.27

where the functional $L(v)$ on $W_V^{1,p(x)}(\mathbb{R}^N)$ defined by

$$\begin{aligned}\langle L(v),w\rangle&=\int_{\mathbb{R}^N}\left(|\nabla v|^{p(x)-2}\nabla v\nabla w+V(x)|v|^{p(x)-2}vw\right)dx\\&=\lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,v(y))f(x,v(x))w(x)}{|x-y|^{\alpha(x,y)}}dxdy+\int_{\mathbb{R}^N}|v|^{p^*(x)-2}vwdx,\end{aligned} \tag{3.28}$$

e3.28

for all $w \in W_V^{1,p(x)}(\mathbb{R}^N)$.

Combining the weak convergence of $(u_n)_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ with the boundedness of $(M(T_{p(x)}(u_n)) - M(T_{p(x)}(u)))_n$ in $\mathbb{R}^N$, we obtain that

$$\lim_{n\to\infty} \left(M\left(\mathcal{T}_{p(x)}(u_n)\right) - M\left(\mathcal{T}_{p(x)}(u)\right)\right)\langle L(u), u_n - u\rangle = 0. \tag{3.29}$$ e3.29

It follows from $\langle J_\lambda(u_n), u_n - u\rangle \to 0$ $(n\to\infty)$ that

$$\begin{aligned} o(1) &= \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\ &= M\left(\mathcal{T}_{p(x)}(u_n)\right)[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle] + [M\left(\mathcal{T}_{p(x)}(u_n)\right) - M\left(\mathcal{T}_{p(x)}(u)\right)]\langle L(u), u_n - u\rangle \\ &\quad - \lambda\left\langle \Phi'(u_n) - \Phi'(u), u_n - u\right\rangle - \int_{\mathbb{R}^N}\left(|u_n|^{p^*(x)-2}u_n - |u|^{p^*(x)-2}u\right)(u_n - u)\,dx \\ &= M\left(\mathcal{T}_{p(x)}(u_n)\right)[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle] + o(1). \end{aligned} \tag{3.30}$$ e3.30

Hence, we yield that

$$\lim_{n\to\infty}[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle] = 0,$$

that is,

$$\int_{\mathbb{R}^N}\left(|\nabla(u_n - u)|^{p(x)} + V(x)|u_n - u|^{p(x)}\right)dx = 0.$$

Hence, we deduce that $(u_n)_n$ strongly converges to u in $W_V^{1,p(x)}(\mathbb{R}^N)$. This completes the proof. $\square$

4 Non–degenerate case for problem (1.1)

sec4

In this part, we use the mountain pass theorem (see [3]) and the Krasnoselskii genus (see [43]) to prove Theorem 1.1 and Theorem 1.2, respectively.

4.1 Proof of Theorem 1.1

First, we try to prove that the energy functional J_λ has mountain pass structure.

lem4.1 **Lemma 4.1.** *Let $J_\lambda \in C^1(E)$, with $J_\lambda(0) = 0$, where E is a real Banach space. Assume that*

(1) *there exists $\rho, \alpha > 0$ satisfying $J_\lambda(u) \geq \alpha$ for all $u \in E$ with $\|u\|_E = \rho$;*

(2) *there exists $e \in E$ such that $J_\lambda(e) < 0$ with $\|e\|_E > \rho$.*

Then we can define $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 1, \gamma(1) = e\}$. Hence,

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J_\lambda(\gamma(t)) \geq \alpha$$

and there exists a $(PS)_c$ sequence $(u_n)_n \subset E$.

Proof. In view of assumption (M_1) and Remark 2.1, with $\|u\|\leq 1$, we get

$$
\begin{aligned}
J_\lambda(u) &= \mathscr{M}\left(\mathcal{T}_{p(x)}(u)\right) - \lambda\Phi(u) - \int_{\mathbb{R}^N}\frac{1}{p^*(x)}|u|^{p^*(x)}dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\|F(x,u)\|^2_{L^{q^+}(\mathbb{R}^N)} - C\|F(x,u)\|^2_{L^{q^-}(\mathbb{R}^N)} \\
&\geq \frac{m_0}{p^+}\|u\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\max\left\{\|u\|^{2r^+}_{L^{q^+r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\} \\
&\quad - C\max\left\{\|u\|^{2r^+}_{L^{q^-r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\} \\
&\geq \frac{m_0}{p^+}\|u\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{2C}{S^{2r^-}}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \text{ for any } u\in W_V^{1,p(x)}(\mathbb{R}^N).
\end{aligned}
$$

Hence, let ρ, $\alpha>0$ and the fact $p^-\leq q^-r(x)\leq q^+r(x)\ll p^*$ such that $J_\lambda(u)\geq\alpha$ for $\|u\|=\rho$. So we prove (1) of Lemma 4.1.

In order to prove the conclusion (2) of Lemma 4.1, we choose $\psi\in C_0^\infty(\mathbb{R}^N)$ and $\psi>0$, we can deduce from (M_2) that

$$
\mathscr{M}(t)\leq\mathscr{M}(1)t^\sigma \quad \text{for all } t\geq 1. \tag{4.1}
$$

e4.2

According to assumption $(\mathcal{F})$ and for any $t>1$, we obtain that

$$
\begin{aligned}
J_\lambda(t\psi) &= \mathscr{M}\left(\mathcal{T}_{p(x)}(t\psi)\right) - \lambda\Phi(t\psi) - \int_{\mathbb{R}^N}\frac{1}{p^*(x)}|t\psi|^{p^*(x)}dx \\
&\leq \mathscr{M}\left(\mathcal{T}_{p(x)}(t\psi)\right) - \int_{\mathbb{R}^N}\frac{1}{p^*(x)}|t\psi|^{p^*(x)}dx \\
&\leq \mathscr{M}(1)t^{\sigma p^+}T_{p(x)}(\psi) - \frac{t^{p^*(x)}}{p^*(x)}\int_{\mathbb{R}^N}|\psi|^{p^*(x)}dx.
\end{aligned} \tag{4.2}
$$

e4.3

In view of $\sigma p^+<p^*$ and for t_0 large enough, we obtain $J_\lambda(t_0\psi)<0$ and $t_0\|\psi\|>\rho$. Set $e=t_0\psi$. Hence e is the required function and the conclusion (2) in Lemma 4.1 is true. This proof is complete. □

Proof of Theorem 1.1. Next, for any λ large enough, we prove that

$$
c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J_\lambda(\gamma(t)) < \left(\frac{1}{\theta}-\frac{1}{p^*}\right)\min\left\{\left(m_0S^{p^+}\right)^{\tau^+}, \left(m_0S^{p^+}\right)^{\tau^-}\right\}. \tag{4.3}
$$

e4.4

Combining (4.3), Lemma 3.1 and Lemma 4.1, it is obvious that we can deduce the existence of nontrivial critical points of J_λ. So we need to prove (4.3). Let $v_0\in W_V^{1,p(x)}(\mathbb{R}^N)$ such that

$$
\mathcal{T}_{p(x)}(v_0)=1 \quad \text{and} \quad \lim_{t\to\infty}J_\lambda(tv_0)=-\infty.
$$

Hence, for some $t_\lambda>0$, we get $\sup_{t\geq 0}J_\lambda(tv_0)=J_\lambda(t_\lambda v_0)$. And

$$
\begin{aligned}
&M(\mathcal{T}_{p(x)}(tv_0))\int_{\mathbb{R}^N}\left(|\nabla tv_0|^{p(x)}+V(x)|tv_0|^{p(x)}\right)dx \\
&= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,tv_0(y))f(x,tv_0(x))tv_0}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|tv_0|^{p^*(x)}dx.
\end{aligned} \tag{4.4}
$$

e4.5

Next, we claim that $\{t_\lambda\}_{\lambda>0}$ is bounded. First, we assume that $t_\lambda \geq 1$ for all $\lambda > 0$. According to (4.4), we obtain that

$$\begin{aligned}
p^+\sigma\mathscr{M}(1)t_\lambda^{2p^+\sigma} &\geq p^+\sigma\mathscr{M}(1)\left(\mathcal{T}_{p(x)}(t_\lambda v_0)\right)^\sigma \\
&\geq p^+M\left(\mathcal{T}_{p(x)}(t_\lambda v_0)\right)T_{p(x)}(t_\lambda v_0) \\
&\geq M\left(\mathcal{T}_{p(x)}(t_\lambda v_0)\right)\left[|\nabla t_\lambda v_0|^{p(x)} + V(x)|t_\lambda v_0|^{p(x)}\right] \\
&= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,t_\lambda v_0(y))f(x,t_\lambda v_0(x))t_\lambda v_0}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|t_\lambda v_0|^{p^*(x)}dx \\
&\geq t_\lambda^{p^*(x)}\int_{\mathbb{R}^N}|v_0|^{p^*(x)}dx. \qquad (4.5)
\end{aligned}$$

Since $\sigma \in [1, p^*(x)/2p^+)$, so $2p^+\sigma < p^*(x)$ and (4.5), we get the boundedness of $\{t_\lambda\}_\lambda$.

Then we prove that $t_\lambda \to 0$ as $\lambda \to \infty$. We can assume that there exist $t_0 > 0$ and a sequence $(\lambda_n)_n$, with $\lambda_n \to \infty$ as $n \to \infty$, such that $t_{\lambda_n} \to t_0$ as $n \to \infty$. In view of the Lebesgue dominated convergence theorem, we yield

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,t_{\lambda_n}v_0(y))f(x,t_{\lambda_n}v_0(x))t_{\lambda_n}v_0}{|x-y|^{\alpha(x,y)}}dxdy \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,t_\lambda v_0(y))f(x,t_\lambda v_0(x))t_\lambda v_0}{|x-y|^{\alpha(x,y)}}dxdy$$

as $n \to \infty$. And

$$\lambda_n\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,t_\lambda v_0(y))f(x,t_\lambda v_0(x))t_\lambda v_0}{|x-y|^{\alpha(x,y)}}dxdy \to \infty \quad \text{as} \quad n \to \infty.$$

So it contracts $M\left(T_{p(x)}(t_0v_0)\right) = \infty$ as $n \to \infty$ which implied in (4.4). Hence, $t_\lambda \to 0$ as $\lambda \to \infty$. Then we have

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^N}\frac{F(y,t_\lambda v_0(y))f(x,t_\lambda v_0(x))t_\lambda v_0}{|x-y|^{\alpha(x,y)}}dxdy = 0$$

and

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^N}|t_\lambda v_0|^{p^*(x)}dx = 0.$$

Moreover, by easy computation we deduce that

$$\lim_{\lambda\to\infty}\left(\sup_{t\geq 0}J_\lambda(tv_0)\right) = \lim_{\lambda\to\infty}J_\lambda(t_\lambda v_0) = 0.$$

Then there exists $\lambda_1 > 0$ such that

$$\sup_{t\geq 0}J_\lambda(tv_0) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\min\left\{\left(m_0S^{p^+}\right)^{\tau^+}, \left(m_0S^{p^+}\right)^{\tau^-}\right\},$$

for any $\lambda \geq \lambda_1$. Letting $e = \tau v_0$, with τ large enough such that $J_\lambda(e) < 0$, we have

$$c_\lambda \leq \max_{t\in[0,1]}J_\lambda(\gamma(t)) \quad \text{by taking} \quad \gamma(t) = t\tau v_0.$$

Finally, if λ large enough, we obtain

$$c_\lambda \leq \sup_{t\geq 0}J_\lambda(tv_0) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\min\left\{\left(m_0S^{p^+}\right)^{\tau^+}, \left(m_0S^{p^+}\right)^{\tau^-}\right\}.$$

4.2 Proof of Theorem 1.2

To prove Theorem 1.2, we need to use the Krasnoselskii genus introduced by Krasnoselskii himself in [43]. Let X be a Banach space and Λ be the class of all closed subsets $A \subset X \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

lem4.2 **Lemma 4.2** (see [43]). *Let X be an infinite dimensional Banach space and let $J_\lambda \in C^1(X)$ be an even functional, with $J_\lambda(0) = 0$. Suppose that $X = Y \oplus Z$, where Y is finite dimensional, and that J_λ satisfies*

- (I_1) *There exists constant ρ, $\alpha > 0$ such that $J_\lambda(u) \geq \alpha$ for all $u \in \partial B_\rho \bigcap Z$;*
- (I_2) *There exists $\Theta > 0$ such that J_λ satisfies the $(PS)_c$ condition for all c, with $c \in (0, \Theta)$;*
- (I_3) *For any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $J_\lambda(u) \leq 0$ on $\widetilde{X} \setminus B_R$.*

Assume furthermore that Y is k dimensional and $Y = span\{v_1, \cdots, v_k\}$. For $n \geq k$, inductively choose $v_{n+1} \notin E_n = span\{v_1, \cdots, v_n\}$. Let $R_n = R(E_n)$ and $\Omega_n = B_{R_n} \bigcap E_n$. Define

$$G_n = \{\psi \in C(\Omega_n, X) \, : \, \psi|_{\partial B_{R_n} \bigcap E_n} = id \text{ and } \psi \text{ is odd}\}$$

and

$$\Gamma_j = \Big\{\psi\big(\overline{\Omega_n \setminus V}\big) \, : \, \psi \in G_n, \; n \geq j, \; V \in \Lambda, \; \gamma(V) \leq n - j\Big\},$$

where $\gamma(V)$ is the Krasnoselskii genus of V. For $j \in \mathbb{N}$, set

$$c_j = \inf_{E \in \Gamma_j} \max_{u \in E} J_\lambda(u).$$

Thus, $0 \leq c_j \leq c_{j+1}$ and $c_j < \Theta$ for $j > k$, then we get c_j is a critical value of J_λ. Furthermore, if $c_j = c_{j+1} = \cdots = c_{j+m} = c < \Theta$ for $j > k$, then $\gamma(K_c) \geq m + 1$, where

$$K_c = \{u \in X : J_\lambda(u) = c \text{ and } J'_\lambda(u) = 0\}.$$

Proof of Theorem 1.2. Then we can apply Lemma 4.2 to J_λ. Since $W_V^{1,p(x)}(\mathbb{R}^N)$ is a Banach space and $J_\lambda \in C^1\big(W_V^{1,p(x)}(\mathbb{R}^N)\big)$. According to (3.2), the functional J_λ satisfies $J_\lambda(0) = 0$. The proof is similar to the proof of (1) and (2) in Lemma 4.1. Since J_λ satisfies (I_1) and (I_3) of Lemma 4.2. First, we claim that there exists a sequence $(\Upsilon_n)_n \subset \mathbb{R}^+$, with $\Upsilon_n \leq \Upsilon_{n+1}$, such that

$$c_n^\lambda = \inf_{E \in \Gamma_n} \max_{u \in E} J_\lambda(u) < \Upsilon_n.$$

According to the definition of c_n^λ, we deduce that

$$c_n^\lambda = \inf_{E \in \Gamma_n} \max_{u \in E} J_\lambda(u) \leq \inf_{E \in \Gamma_n} \max_{u \in E} \left\{\mathscr{M}\left(\mathcal{T}_{p(x)}(u)\right) - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx\right\}.$$

Set

$$\Upsilon_n = \inf_{E \in \Gamma_n} \max_{u \in E} \left\{\mathscr{M}\left(\mathcal{T}_{p(x)}(u)\right) - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx\right\},$$

so that $\Upsilon_n < \infty$ and $\Upsilon_n \leq \Upsilon_{n+1}$. We show that equation (1.1) has at least k pairs of weak solutions. And we can analyze the following possibilities:
I. Fix $\lambda > 0$. Let m_0 so large that

$$\sup_n \Upsilon_n < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{\left(m_0 S^{p^+}\right)^{\tau^+}, \left(m_0 S^{p^+}\right)^{\tau^-}\right\}.$$

II. Similarly, in (4.3), for any $\lambda > \lambda_2$, there exists $\lambda_2 > 0$ such that

$$c_n^\lambda \leq \Upsilon_n < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{\left(m_0 S^{p^+}\right)^{\tau^+}, \left(m_0 S^{p^+}\right)^{\tau^-}\right\}.$$

Hence, we yield

$$0 < c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_n^\lambda < \Upsilon_n < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{\left(m_0 S^{p^+}\right)^{\tau^+}, \left(m_0 S^{p^+}\right)^{\tau^-}\right\}.$$

In view of Proposition 9.30 in [43], we get that $c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_n^\lambda$ are critical values of J_λ.

If $c_j^\lambda = c_{j+1}^\lambda$ for some $j = 1, 2, \cdots, k-1$, we can get the set $K_{c_j^\lambda}$ contains infinitely many distinct points according to [4]. Hence, equation (1.1) has infinitely many weak solutions and we deduce the result that equation (1.1) has at least k pairs of solutions.

5 Degenerate case for problem (1.1)

sec5

In this section, we discuss the degenerate case of problem (1.1). We assume that M satisfies (M_2) and (M_3), and f verifies (f_1)–(f_3). We first give a crucial lemma.

lem5.1 **Lemma 5.1.** *Let $(u_n)_n \subset W_V^{1,p(x)}(\mathbb{R}^N)$ be a Palais-Smale sequence of functional J_λ, then*

$$J_\lambda(u_n) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{in} \quad (W_V^{1,p(x)}(\mathbb{R}^N))'$$

as $n \to \infty$, where $\left(W_V^{1,p(x)}(\mathbb{R}^N))\right)'$ is the dual of $W_V^{1,p(x)}(\mathbb{R}^N)$. If

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{\left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}, \tag{5.1}$$ e5.1

where $\tau_\sigma(x) := \frac{p^*}{p^* - p^+\sigma}$, then there exists a subsequence of $(u_n)_n$ strongly convergent in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Proof. If $\inf_{n\geq 1} \|u_n\| = 0$, then there exists a subsequence of $(u_n)_n$ such that $u_n \to 0$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ as $n \to \infty$. Hence, we assume that $d := \inf_{n\geq 1} \|u_n\| > 0$ and $\|u_n\| > 1$ for any integer n in the following proof. In view of $J_\lambda(u_n) \to c_\lambda$ and $J'_\lambda(u_n) \to 0$ as $n \to \infty$,

$$\begin{aligned}
c_\lambda + 1 + o(1)\|u_n\| =& J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\
=& \mathscr{M}\left(\mathcal{T}_{p(x)}(u_n)\right) - \frac{1}{\theta} M\left(\mathcal{T}_{p(x)}(u_n)\right)\left[\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} dx\right] \\
&+ \int_{\mathbb{R}^N} \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)|u_n|^{p^*(x)} dx + \lambda \iint_{\mathbb{R}^N} \frac{F(y, u_n(y))}{|x-y|^{\alpha(x,y)}}\left(\frac{f(x, u_n)u_n}{\theta} - \frac{F(x, u_n)}{2}\right) dxdy.
\end{aligned}$$

Combining (M_2), (M_3) and (f_3), we have

$$\begin{aligned} c_\lambda + 1 + o(1)\|u_n\| &= J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\ &\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right) M\left(\mathcal{T}_{p(x)}(u_n)\right)\mathcal{T}_{p(x)}(u_n) \\ &\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right) m_1 \left(\mathcal{T}_{p(x)}(u_n)\right)^\sigma \\ &\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right) m_1 \|u_n\|^{p^+\sigma}. \end{aligned} \tag{5.2}$$ e5.2

Since $p^+\sigma > 1$, we deduce that $(u_n)_n$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Similar to the proof of Lemma 3.1, we can assume $I \neq \emptyset$. For any $i \in I$ and any $\epsilon > 0$ small, we define a smooth cut-off function $\phi_{\epsilon,i}$ as Lemma 3.1. In view of the boundedness of $(u_n)_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ and $\langle J'_\lambda(u_n), u_n\phi_{\epsilon,i}\rangle \to 0$ as $n \to \infty$. Thus we have

$$\begin{aligned} &M(\mathcal{T}_{p(x)}(u_n)) \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)}\phi_{\epsilon,i} + V(x)|u_n|^{p(x)}u_n\phi_{\epsilon,i} + |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_{\epsilon,i} u_n\right) dx \\ &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_{\epsilon,i}}{|x-y|^{\alpha(x,y)}} dxdy + \int_{\mathbb{R}^N} |u_n|^{p^*(x)}\phi_{\epsilon,i} dx + o_n(1). \end{aligned} \tag{5.3}$$ e5.3

Similarly in Lemma 3.1, we deduce that

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty} \left|\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_{\epsilon,i} u_n dx\right| = 0$$

and

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_{\epsilon,i}}{|x-y|^{\alpha(x,y)}} dxdy = \left\langle \Phi'(u_n), u_n\phi_{\epsilon,i}\right\rangle = 0.$$

By (M_3) and (5.3), we have

$$\begin{aligned} &M(\mathcal{T}_{p(x)}(u_n)) \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)}\phi_{\epsilon,i} + V(x)|u_n|^{p(x)}u_n\phi_{\epsilon,i}\right) dx \\ &= M\big(\mathcal{T}_{p(x)}(u_n)\big) \int_{\mathbb{R}^N} U_n(x)\phi_{\epsilon,i} dx \geq M\big(\mathcal{T}_{p(x)}(u_n\phi_{\epsilon,i})\big) \int_{\mathbb{R}^N} U_n(x)\phi_{\epsilon,i} dx \\ &\geq p^-\left(\mathcal{T}_{p(x)}(u_n\phi_{\epsilon,i})\right)\mathcal{T}_{p(x)}(u_n\phi_{\epsilon,i}) \geq p^- m_1 \left(T_{p(x)}(u_n\phi_{\epsilon,i})\right)^\sigma \\ &\geq (p^-)^{\sigma+1} m_1 \left(\int_{\mathbb{R}^N} U_n(x)\phi_{\epsilon,i} dx\right)^\sigma \\ &\geq (p^-)^{\sigma+1} m_1 \mu_i^\sigma. \end{aligned} \tag{5.4}$$ e5.4

According to (5.3), we get

$$(p^-)^{\sigma+1} m_1 \mu_i^\sigma \leq \nu_i.$$

Then we find that either $\nu_i = 0$ or

$$\nu_i \geq \left(p^- m_1 S^{p^+\sigma}\right)^{\frac{p^*(z_i)}{p^*(z_i)-p^+\sigma}} \geq \min\left\{\left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}. \tag{5.5}$$ e5.5

Assume that (5.5) holds true. In view of Lemma 3.1, (M_2) and (f_3), we obtain

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} |u_n|^{p^*} \phi_{\epsilon,i} dx. \tag{5.6}$$ e5.6

So we get

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{ \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^-} \right\}.$$

Thus we get a contradiction with (5.1). Hence $\nu_i = 0$.

Next, we claim that $\nu_\infty = 0$. Similarly, we define a smooth cut-off function ϕ_R as in Lemma 3.1. Since $\langle J'_\lambda(u_n), u_n\phi_R\rangle \to 0$ as $n \to \infty$,

$$\begin{aligned} &M(\mathcal{T}_{p(x)}(u_n)) \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)}\phi_R + V(x)|u_n|^{p(x)}u_n\phi_R + |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_R u_n \right) dx \\ &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}} dxdy + \int_{\mathbb{R}^N} |u_n|^{p^*(x)}\phi_R dx + o_n(1). \end{aligned} \tag{5.7}$$ e5.7

Similarly in Lemma 3.1, we deduce

$$\lim_{R\to\infty} \limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_R u_n dx \right| = 0$$

and

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(y,u_n(y))f(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}} dxdy = \left\langle \Phi'\left(u_n\right), u_n\phi_R \right\rangle = 0.$$

According to (5.5) and choosing $R \to \infty$ and $n \to \infty$, we obtain that

$$M\left(\mathcal{T}_{p(\cdot,\cdot)}(u_n)\right) \int_{\mathbb{R}^N} U_n(x)\phi_R dx \geq (p^-)^{\sigma+1} m_1 \mu_\infty^\sigma \tag{5.8}$$ e5.8

and

$$(p^-)^{\sigma+1} m_1 \mu_\infty^\sigma \leq \nu_\infty.$$

Combining with (3.12), we find that either $\nu_\infty = 0$ or

$$\nu_\infty \geq \left(p^- m_1 S^{p^+\sigma}\right)^{\frac{p^*_\infty}{p^*_\infty - p^+\sigma}} \geq \min\left\{ \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^-} \right\}. \tag{5.9}$$ e5.9

Then we prove (5.9). Similarly, in view of (3.21), we deduce that

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \min\left\{ \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^-} \right\}. \tag{5.10}$$ e5.10

By (5.1), it is an obvious contradiction. Thus $\nu_\infty = 0$.

Hence, we have $I = \emptyset$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx = \int_{\mathbb{R}^N} |u|^{p^*(x)} dx.$$

According to a Brézis-Lieb lemma for the Lebesgue spaces with variable exponents (see [23], Lemma 3.9) and last equality, we yield

$$\int_{\mathbb{R}^N} |u_n - u|^{p^*(x)} dx \to 0,$$

that is, $u_n \to u$ in $L^{p^*(x)}(\mathbb{R}^N)$. Moreover, we have

$$\int_{\mathbb{R}^N} |u_n|^{p^*(x)-2} u_n (u_n - u) dx \to 0.$$

Set $L(v)$ be defined as in (3.28). Then it follows from (3.30) and $\langle J_\lambda(u_n), u_n - u\rangle \to 0$ that

$$\lim_{n\to\infty} M\left(\mathcal{T}_{p(x)}(u_n)\right)\left[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle\right] = 0.$$

We get

$$\lim_{n\to\infty} \left[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle\right] = 0.$$

that is,

$$\int_{\mathbb{R}^N} \left(|\nabla(u_n - u)|^{p(x)} + V(x)|u_n - u|^{p(x)}\right) dx = 0.$$

So we can find a u in $W_V^{1,p(x)}(\mathbb{R}^N)$ such that $(u_n)_n$ strongly converges to u. The proof is now complete. $\square$

lem5.2 **Lemma 5.2.** *The functional J_λ satisfies the conditions* (1) *and* (2) *of Lemma* 4.1.

Proof. According to (M_3), (f_2) and Remark 2.1, for any $\lambda > 0$, $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ and $\|u\| < 1$, we deduce that

$$\begin{aligned}
J_\lambda(u) =& \mathscr{M}\left(\mathcal{T}_{p(x)}(u)\right) - \lambda\Phi(u) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)}|u|^{p^*(x)} dx \\
\geq& \frac{1}{\sigma} M\left(\mathcal{T}_{p(x)}(u)\right)\mathcal{T}_{p(x)}(u) - -\frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\|F(\cdot,u)\|^2_{L^{q^+}(\mathbb{R}^N)} - C\|F(\cdot,u)\|^2_{L^{q^-}(\mathbb{R}^N)} \\
\geq& \frac{m_1}{\sigma(p^+)^\sigma}\|u\|^{p^+\sigma}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\max\left\{\|u\|^{2r^+}_{L^{q^+r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\} \\
& - C\max\left\{\|u\|^{2r^+}_{L^{q^-r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\} \qquad (5.11) \\
\geq& \frac{m_1}{\sigma(p^+)^\sigma}\|u\|^{p^+\sigma}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{2C}{S^{2r^-}}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)}.
\end{aligned}$$ e5.11

Since $p^+\sigma < p^*$ and $p \ll rq^- \leq rq^+ \ll p^*$, choosing $\rho, \alpha > 0$, we have $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$. Thus we get the proof of (1) in Lemma 4.1. Similarly, we can prove that (2) of Lemma 4.1 is true. $\square$

Proof of Theorem 1.3 Similar to the proof of Lemma 4.1, we obtain that

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\min\left\{\left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- m_1 S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}.$$

The rest of the proof is the same as Theorem1.1.

Proof of Theorem 1.4 The proof of Theorem 1.4 is the same as that of Theorem 1.2.

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Competing interests

The authors declare that they have no competing interests.

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