A Q-ERKUS- SRIVASTAVA POLYNOMIALS OPERATOR

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Abstract. We construct a sequence of linear positive operators by means of the Erkus- Srivastava multivariable polynomials which include q- Lagrange polynomial operators discussed in [5] and the Lagrange Hermite polynomial operators considered in [1]. We study the Korovkin type theorems for the constructed operators by using summability techniques of statistical convergence and the power series method. We also define a $k-th$ order Taylor generalization of the multivariable polynomials operator and investigate the approximation of $k-th$ times continuously differentiable Lipschitz class elements.

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1. Introduction

Erkus and Srivastava [11] introduced the multivariable polynomials defined as:

$$h_n^{(\xi_1,\cdots\ \xi_r)}(z_1,z_2,\cdots\ z_r)=\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=n}\left\{\prod_{i=1}^{r}(\xi_i)_{l_i}\frac{(z_i)^{l_i}}{l_i!}\right\},$$

where $(\lambda)_l=\lambda(\lambda+1)\cdots(\lambda+l-1)$ and $(\lambda)_0=1$. These polynomials have the generating function of the form

$$\prod_{i=1}^{r}(1-z_it^{m_i})^{-\xi_i}=\sum_{n=0}^{\infty}h_n^{(\xi_1,\xi_2,\cdots\ \xi_r)}(z_1,z_2,\cdots\ z_r)t^n,$$

where $\xi_i\in\mathbb{C},(i=1,...,r)$ and $|t|<\min\{|z_1|^{-1/m_1},\cdots,|z_r|^{-1/m_r}\}$. Duman [9] constructed a q-analogue of these polynomials as follows:

$$h_{n,q}^{(\xi_1,\cdots\ \xi_r)}(z_1,z_2,\cdots\ z_r)=\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=n}\left\{\prod_{i=1}^{r}(q^{\xi_i},q)_{l_i}\frac{(z_i)^{l_i}}{(q,q)_{k_i}}\right\},\tag{1.1}$$

where the generating function is given by

$$\prod_{i=1}^{r}(z_it^{m_i};q))^{-\xi_i}=\sum_{n=0}^{\infty}h_{n,q}^{(\xi_1,\xi_2,\cdots\ \xi_r)}(z_1,z_2,\cdots\ z_r)t^n,\tag{1.2}$$

the q-integer of any $m\in\mathbb{N}$, is given in [4] as

$$[m]_q:=\begin{cases}\frac{1-q^m}{1-q}, & \text{if } q\neq1,\\ m, & \text{if } q=1,\end{cases}$$

and

$$(\lambda;q)_\mu=\begin{cases}1, & \text{if}\quad \mu=0,\\ (1-\lambda)(1-\lambda q)...(1-\lambda q^{\mu-1}), & \text{if}\quad \mu\in\mathbb{N}.\end{cases}$$

Inspired by the above research, for $f \in \mathcal{C}(I)$, the space of continuous functions on $I=[0,1]$, endowed with the sup norm $\|.\|$, we propose the following sequence of Erkus-Srivastava type positive linear operators:

$$\begin{aligned}\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x) &= \left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=0}^{\infty}\Bigg\{\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=s}\left\{\prod_{k=1}^{r}(q^n,q)_{l_k}\frac{(\beta_n^{(k)})^{l_k}}{(q,q)_{l_k}}\right\}\\ &\qquad f\left(\frac{[l_1]_q}{[n+l_1-1]_q}\right)\Bigg\}x^s, \qquad (1.3)\end{aligned}$$

where $\langle \beta_n^{(j)}\rangle_{n\in\mathbb{N}}$, $(j=1,2,\cdots r)$ are the sequences of real numbers in $(0,1)$, $x \in I_0^1$, and $r,n \in \mathbb{N}$. In particular, if $m_i=1, \forall\, i \in \mathbb{N}$ then these operators reduce to the Lagrange polynomial operators studied in [5] and in the case $m_i=i, \forall i \in \mathbb{N}$, they include Lagrange Hermite polynomial operators discussed in [1]. The approximation properties of q-analogue of an integral type operator based on multivariate q-Lagrange polynomials via summability method is studied by Agrawal et. al. [2]. The purpose of this paper is to study various approximation properties of the operators (1.3) with respect to the statistical convergence and the power series method.

In the following lemmas, we obtain the estimates for some raw moments of the operators defined by (1.3).

Lemma 1. *The operators* $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$ *satisfy*

$$\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(1;x)=1, \quad \textit{for all } x \in I_0^1.$$

Proof. Using (1.1) and (1.2) , proof of the lemma is straight-forward hence the details are omitted. □

Lemma 2. *For the operators* $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$*, we have*

$$\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u;x)=x^{m_1}\beta_n^{(1)}.$$

Proof. From the definition (1.3) of the Erkus- Srivastava q-operator, we can write

$$\begin{aligned}\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u;x) &= \left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}\Bigg\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_1\geq 1}}\left\{\prod_{k=1}^{r}(q^n,q)_{l_k}\frac{(\beta_n^{(k)})^{l_k}}{(q,q)_{l_k}}\right\}\\ &\qquad \left(\frac{[l_1]_q}{[n+l_1-1]_q}\right)\Bigg\}x^s.\end{aligned}$$

Using the elementary identities, $\frac{[l_1]_q}{(q;q)_{l_1}}=\frac{1}{(1-q)(q;q)_{l_1-1}}$, $\frac{(q^n;q)_{l_1}}{[n+l_1-1]_q}=(1-q)(q^n;q)_{l_1-1}$, and replacing l_1 *by* (l_1+1), we have

$$\begin{aligned}\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u;x) &= x^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}\Bigg\{\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=s-m_1}(q^n;q)_{l_1}\cdots(q^n;q)_{l_r}\\ &\qquad \frac{(\beta_n^{(1)})^{l_1}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1}\cdots(q;q)_{l_r}}\Bigg\}x^{s-m_1}\\ &= x^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}h_{s-m_1,q}^{(n,\cdots,n)}(\beta_n^{(1)},\beta_n^{(2)},\cdots,\beta_n^{(r)})x^{s-m_1}\\ &= x^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=0}^{\infty}h_{s,q}^{(n,\cdots,n)}(\beta_n^{(1)},\beta_n^{(2)},\cdots,\beta_n^{(r)})x^{s}\\ &= x^{m_1}\beta_n^{(1)}, \quad \text{in view of Lemma 1.}\end{aligned}$$

□

Lemma 3. *The operators* $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$ *satisfy*

$$\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x) \le q(x^{m_1}\beta_n^{(1)})^2 + \frac{x^{m_1}\beta_n^{(1)}}{[n]_q};$$

and

$$\left|\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2-x^2;x)\right| \le 2x^2(1-\beta_n^{(1)}x^{m_1-1}) + \frac{x^{m_1}\beta_n^{(1)}}{[n]_q}.$$

Proof. From the definition of the operator $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$, we have

$$\begin{aligned}
\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x) &= \left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}\left\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_1\ge 1}}\left\{\prod_{k=1}^{r}(q^n,q)_{l_k}\frac{(\beta_n^{(k)})^{l_k}}{(q,q)_{l_k}}\right\}\left(\frac{[l_1]_q}{[n+l_1-1]_q}\right)^2\right\}x^s\\
&= x^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}\left\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_1\ge 1}}(q^n;q)_{l_1-1}\cdots(q^n;q)_{l_r}\right.\\
&\quad\left.\frac{[l_1]_q}{[n+l_1-1]_q}\frac{(\beta_n^{(1)})^{l_1-1}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1-1}\cdots(q;q)_{l_r}}\right\}x^{s-m_1}\\
&= x^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}\left\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_r\ge 1}}(q^n;q)_{l_1-1}\cdots(q^n;q)_{l_r}\right.\\
&\quad\left.\left(\frac{1+q[l_1-1]_q}{[n+l_1-1]_q}\right)\frac{(\beta_n^{(1)})^{l_1-1}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1-1}\cdots(q;q)_{l_r}}\right\}x^{s-m_1} = \sum_1 + \sum_2. \qquad (1.4)
\end{aligned}$$

Here,

$$\begin{aligned}
\sum_1 &= x^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=m_1}^{\infty}\left\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_1\ge 1}}(q^n;q)_{l_1-1}\cdots(q^n;q)_{l_r}\right.\\
&\quad\left.\left(\frac{1}{[n+l_1-1]_q}\right)\frac{(\beta_n^{(1)})^{l_1-1}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1-1}\cdots(q;q)_{l_r}}\right\}x^{s-m_1},
\end{aligned}$$

since $\frac{1}{[n+l_1-1]_q} \le \frac{1}{[n]_q}$, then using Lemma 1, we have

$$\sum_1 \le \frac{x^{m_1}\beta_n^{(1)}}{[n]_q}.$$

Now,

$$\begin{aligned}
\sum_2 &= qx^{m_1}\beta_n^{(1)}\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=1}^{\infty}\Bigg\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r-1=p-1\\ l_1\geq 1}}(q^n;q)_{l_1-1}\cdots(q^n;q)_{l_r}\\
&\quad\left(\frac{[l_1-1]_q}{[n+l_1-1]_q}\right)\frac{(\beta_n^{(1)})^{l_1-1}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1-1}\cdots(q;q)_{l_r}}\Bigg\}x^{s-1}\\
&= q(x^{m_1}\beta_n^{(1)})^2\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=2m_1}^{\infty}\Bigg\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_1\geq 2}}(q^n;q)_{l_1-2}\cdots(q^n;q)_{l_r}\\
&\quad\left(\frac{(1-q^{l_1-1})(1-q^{n+l_1-2})}{(1-q^{l_1-1})(1-q)[n+l_1-1]_q}\right)\frac{(\beta_n^{(1)})^{l_1-2}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1-2}\cdots(q;q)_{l_r}}\Bigg\}x^{s-2m_1}\\
&= q(x^{m_1}\beta_n^{(1)})^2\left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=2m_1}^{\infty}\Bigg\{\sum_{\substack{m_1l_1+m_2l_2+\cdots+m_rl_r=s\\ l_1\geq 2}}(q^n;q)_{l_1-2}\cdots(q^n;q)_{l_r}\\
&\quad\left(\frac{[n+l_1-2]_q}{[n+l_1-1]_q}\right)\frac{(\beta_n^{(1)})^{l_1-2}\cdots(\beta_n^{(r)})^{l_r}}{(q;q)_{l_1-2}\cdots(q;q)_{l_r}}\Bigg\}x^{s-2}.
\end{aligned}$$

Since $\frac{[n+l_1-2]_q}{[n+l_1-1]_q}<1$, in view of Lemma 1 we get

$$\sum_2 \leq q(x^{m_1}\beta_n^{(1)})^2.$$

Finally, using the estimates of $\sum_1$ and $\sum_2$ in (1.4), we obtain

$$\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x) \leq q(x^{m_1}\beta_n^{(1)})^2+\frac{x^{m_1}\beta_n^{(1)}}{[n]_q}. \tag{1.5}$$

Hence, we can write

$$\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x)-x^2 \leq (q(x^{m_1}\beta_n^{(1)})^2-x^2)+\frac{x^{m_1}\beta_n^{(1)}}{[n]_q}=-x^2(1-q(\beta_n^{(1)})^2x^{2m_1-2})+\frac{x^{m_1}\beta_n^{(1)}}{[n]_q}.$$

As $q,\beta_n^{(1)}\in(0,1)$, for all $x\in\ I$ we have

$$\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x)-x^2 \leq \frac{x^{m_1}\beta_n^{(1)}}{[n]_q}. \tag{1.6}$$

Since the operator $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$ is linear and positive, using Lemmas 1 and 2, we may write

$$\begin{aligned}
0\leq\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}((u-x)^2;x) &= \mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x)-2x\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u;x)+x^2;\\
\Longrightarrow\quad -2x^2(1-\beta_n^{(1)}x^{m_1-1}) &\leq \mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x)-x^2\\
\text{or,}\quad -2x^2(1-\beta_n^{(1)}x^{m_1-1})-\frac{x^{m_1}\beta_n^{(1)}}{[n]_q} &\leq \mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(s^2;x)-x^2.
\end{aligned}$$

Hence, in view of (1.6), we obtain

$$\left|\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u^2;x)-x^2\right| \leq 2x^2(1-\beta_n^{(1)}x^{m_1-1})+\frac{x^{m_1}\beta_n^{(1)}}{[n]_q}. \tag{1.7}$$

This completes the proof of the lemma. □

Throughout this paper, let $q_n\in(0,1)$ be a sequence such that $\lim_{n\to\infty}q_n=1$ and $\lim_{n\to\infty}q_n^n=\beta,\ 0\leq\beta<1$.

From the Lemmas 1-3, it is clear that the sequence of operators $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$ satisfies the Korovkin conditions

$$\lim_{n\to\infty}||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)-e_i|| \quad = \quad 0, \quad \text{for } i=0,1,2,$$

where $e_i(x)=x^i$, provided $m_1=1$. Hence, let us assume from now onwards that $m_1=1$. Then, by Korovkin theorem,

$$\lim_{n\to\infty}||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f|| \quad = \quad 0, \quad \text{for all } f\in\mathcal{C}(I). \tag{1.8}$$

Lemma 4. *For each $n\in\mathbb{N}$, the operators $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(.;x)$ satisfy the following*

(i) $\left|\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(u-x;x)\right|=(1-\beta_n^{(1)})x;$

(ii) $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}((u-x)^2;x) \ \leq \ \{1-2(\beta_n^{(1)})+q(\beta_n^{(1)})^2\}x^2+\frac{x\beta_n^{(1)}}{[n]_q}.$

Proof. Using the Lemmas 1- 3, the proof is straight-forward. Hence, we omit the details. □

Now, we recall the definitions of the usual modulus of continuity and the Lipschitz class. Let $f\in\mathcal{C}(I)$ and $\lambda>0$ then the modulus of continuity $\omega(f;\lambda)$ given in[29] is defined as

$$\omega(f;\lambda)=\sup_{|u-x|\leq\lambda}|f(u)-f(x)|.$$

From [29], for any $f\in\mathcal{C}(I)$ and $u,x\in I$, we have

$$|f(u)-f(x)|\leq\left(\frac{|u-x|}{\lambda}+1\right)\omega(f;\lambda). \tag{1.9}$$

For any $f\in\mathcal{C}(I)$, the Lipschitz class is defined as follows:

$$Lip_{\mathcal{M}}\nu=\{f\in\mathcal{C}(I):|f(x)-f(u)|\leq\mathcal{M}|x-u|^\nu,\ 0<\nu\leq1\},$$

where $\mathcal{M}>0$, is a constant depending on f.

The following result yields us an estimate of the approximation degree for the operators (1.3) in terms of the usual modulus of continuity.

Theorem 1. *Let $f\in\mathcal{C}(I)$ then for each $x\in I$, we have*

$$\left||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f\right||\leq 2\omega(f;\sqrt{\gamma_{n,q_n}}),$$

where $\gamma_{n,q_n}=\{1-2(\beta_n^{(1)})+q_n(\beta_n^{(1)})^2\}+\frac{\beta_n^{(1)}}{[n]_{q_n}}.$

Proof. Considering the property (1.9) of the usual modulus of continuity, the proof of the theorem follows from (1.3), Lemma 1 and the Cauchy-Schwarz inequality. □

Corollary 1. *Let $f\in\mathcal{C}(I)$ such that $f\in Lip_{\mathcal{M}}\upsilon$, then we have*

$$\left||\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f\right||\leq 2\mathcal{M}\gamma_{n,q}^{\upsilon/2}.$$

2. Convergence via power series method

This section is devoted to the study of convergence of the operators $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ in the sense of power series method. Using this approach, Taş and Altihan [38] studied the Korovkin type theorems for the sequences of positive linear operators defined on $C[a,b]$ and $L_p[a,b]$. We verify the Korovkin type theorem for the operators $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ via this method. We also prove with the help of an example that our theorem is a non-trivial generalization of the classical Korovkin theorem. We construct a sequence of positive linear operators which satisfies the Korovkin type theorem via power series method but does not work in the classical sense.

Let $R>0$, be the radius of convergence of the power series

$$\rho(t)=\sum_{n=1}^{\infty}a_nt^{n-1},\ t\in(0,R)$$

where (a_n) is a sequence of non-negative real numbers such that $a_1 > 0$. A sequence of real numbers (β_n) is said to be convergent to the number l in the sense of power series method[24][36], if for all $t \in (0, R)$

$$\lim_{t \to R^-} \frac{1}{\rho(t)} \sum_{n=1}^{\infty} a_n t^{n-1} \beta_n = l.$$

From[6], it is important to note that the power series method is regular if and only if for each $n \in \mathbb{N}$

$$\lim_{t \to R^-} \frac{a_n t^{n-1}}{\rho(t)} = 0.$$

Note that , if $a_n = 1$ for $n \geq 1$ then $\rho(t) = \frac{1}{1-t}$ and $R = 1$, in this case power series method reduces to Abel method which is a sequence to function transformation and if $a_n = \frac{1}{n!}$ for $n \geq 1$ then $\rho(t) = e^t$ and $R = \infty$, then the power series method reduces to Borel method. Both Abel's and Borel methods are defined via power series not by using matrix as statistical method is defined(see[6][39]). More details about the power series method can be found in the papers[30], [33] and [40].
For any $f \in \mathcal{C}(I)$, let us define

$$\mathcal{G}_t(f; x) = \frac{1}{\rho(t)} \sum_{n=1}^{\infty} \mathcal{L}_{n,q_n}^{\beta_n^{(1)}, \cdots, \beta_n^{(r)}}(f; x) a_n t^{n-1}, \ t \in (0, R) \tag{2.1}$$

then $\mathcal{G}_t(f; x)$ is a bounded positive linear operator.

Now, we verify the Korovkin type theorem[38] for the operators $\mathcal{G}_t$ in the sense of power series method.

Theorem 2. *For all $f \in \mathcal{C}(I)$, the positive linear operators defined by (2.1) satisfies*

$$\lim_{t \to R^-} ||\mathcal{G}_t(f) - f|| = 0.$$

Proof. From equation (2.1), we may write

$$\mathcal{G}_t(f; x) - f(x) = \frac{1}{\rho(t)} \sum_{n=1}^{\infty} [\mathcal{L}_{n,q_n}^{\beta_n^{(1)}, \cdots, \beta_n^{(r)}}(f; x) - f(x)] a_n t^{n-1}.$$

From Lemma 1, we get

$$\mathcal{L}_{n,q_n}^{\beta_n^{(1)}, \cdots, \beta_n^{(r)}}(e_0; x) - e_0(x) = 0, \text{ which implies that } ||\mathcal{G}_t(e_0) - e_0|| = 0,$$

and hence

$$\sup_{t \to R^-} ||\mathcal{G}_t(e_0) - e_0|| = 0$$

Now, from Lemma 4, we may write

$$\begin{aligned} |\mathcal{G}_t(e_1; x) - e_1(x)| &\leq \frac{1}{\rho(t)} \sum_{n=1}^{\infty} |\mathcal{L}_{n,q_n}^{\beta_n^{(1)}, \cdots, \beta_n^{(r)}}(e_1; x) - e_1(x)| a_n t^{n-1} \\ &\leq \frac{1}{\rho(t)} \sum_{n=1}^{\infty} (1 - \beta_n^{(1)}) x a_n t^{n-1}. \end{aligned}$$

Therefore,

$$||\mathcal{G}_t(e_1) - e_1|| \leq \frac{1}{\rho(t)} \sum_{n=1}^{\infty} (1 - \beta_n^{(1)}) a_n t^{n-1}. \tag{2.2}$$

Since the sequence $(1 - \beta_n^{(1)}) \to 0$, as $n \to \infty$, for a given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$(1 - \beta_n^{(1)}) < \frac{\epsilon}{2}, \ for \ all \ n > n_0.$$

Hence equation (2.2) becomes

$$||\mathcal{G}_t(e_1)-e_1|| < \frac{\epsilon}{2} + \frac{1}{\rho(t)}\sum_{n=1}^{n_0}(1-\beta_n^{(1)})a_n t^{n-1}. \tag{2.3}$$

Let $\mathcal{M} = \max_{1\le n\le n_0}\{1-\beta_n^{(1)}\}$, then

$$||\mathcal{G}_t(e_1)-e_1|| < \frac{\epsilon}{2} + \mathcal{M}\frac{1}{\rho(t)}\sum_{n=1}^{n_0} a_n t^{n-1}. \tag{2.4}$$

From the regularity condition of the power series method, we have $\lim_{t\to R^-}\frac{a_n t^{n-1}}{\rho(t)} = 0$. Hence For a given $\epsilon>0$, for every $1\le n\le n_0$ there exists ν_n such that

$$\frac{a_n t^n}{\rho(t)} < \frac{\epsilon}{2\mathcal{M}n_0},\ \forall\ R-\nu_n<t<R.$$

Let $\nu = \min_{1\le n\le n_0}\nu_n$, we have

$$\frac{1}{\rho(t)}\sum_{n=1}^{n_0} a_n t^{n-1} < \frac{\epsilon}{2\mathcal{M}},\ \forall\ R-\nu<t<R.$$

Thus from equation (2.4), we get

$$\lim_{t\to R^-}||\mathcal{G}_t(e_1)-e_1|| = 0.$$

Finally,

$$\begin{aligned}
|\mathcal{G}_t(e_2;x)-e_2(x)| &\le \frac{1}{\rho(t)}\sum_{n=1}^{\infty}|\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_2;x)-e_2(x)|a_n t^{n-1}\\
&\le \frac{1}{\rho(t)}\sum_{n=1}^{\infty}\left\{2x^2(1-\beta_n^{(1)}) + \frac{x\beta_n^{(1)}}{[n]_{q_n}}\right\}a_n t^{n-1},
\end{aligned}$$

which implies that

$$||\mathcal{G}_t(e_2)-e_2|| \le \frac{1}{\rho(t)}\sum_{n=1}^{\infty}\left\{2(1-\beta_n^{(1)}) + \frac{\beta_n^{(1)}}{[n]_{q_n}}\right\}a_n t^{n-1}.$$

Now proceeding in a manner similar to the proof of $\lim_{t\to R^-}||\mathcal{G}_t(e_1)-e_1|| = 0$, we obtain

$$\lim_{t\to R^-}||\mathcal{G}_t(e_2)-e_2|| = 0.$$

Hence applying (Theorem 1,[38]), for all $f\in\mathcal{C}(I)$ we have

$$\lim_{t\to R^-}||\mathcal{G}_t(f)-f|| = 0.$$

□

Next result concerns with the rate of convergence of the operators $\mathcal{G}_t$ by means of the usual modulus of continuity $\omega(f;\lambda)$.

Theorem 3. *Let $\omega(f;\lambda)$ be the usual modulus of continuity then for any $f\in\mathcal{C}(I)$, we have*

$$||\mathcal{G}_t(f)-f|| \le \mathcal{M}\omega(f;\gamma(t)),$$

where $\gamma(t) = \sqrt{\frac{1}{\rho(t)}\sum_{n=1}^{\infty}\{1+(\frac{1}{[n]_{q_n}}-2)\beta_n^{(1)}+q_n(\beta_n^{(1)})^2\}a_n t^{n-1}}$ *and $\mathcal{M}>0$ is a constant.*

Proof. Following the proof of (Theorem 2, [38]), we have

$$||\mathcal{G}_t(f)-f|| \le \mathcal{M}\{\omega(f;\gamma(t)) + ||\mathcal{G}_t(e_0)-e_0||\},$$

where $\gamma(t)=\sqrt{||\mathcal{G}_t((u-x)^2;x)||}$ and $\mathcal{M}>0$ is a constant. Since $||\mathcal{G}_t(e_0)-e_0||=0$ and

$$\begin{aligned}\mathcal{G}_t((u-x)^2;x) &= \frac{1}{\rho(t)}\sum_{n=1}^{\infty}\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}((u-x)^2;x)a_n t^{n-1}\\ &\leq \frac{1}{\rho(t)}\sum_{n=1}^{\infty}\left\{\{1-2\beta_n^{(1)}+q_n(\beta_n^{(1)})^2\}x^2+\frac{x\beta_n^{(1)}}{[n]_{q_n}}\right\}a_n t^{n-1},\end{aligned}$$

we have

$$||\mathcal{G}_t((u-x)^2;x)||\leq \frac{1}{\rho(t)}\sum_{n=1}^{\infty}\left\{1+(\frac{1}{[n]_{q_n}}-2)\beta_n^{(1)}+q_n(\beta_n^{(1)})^2\right\}a_n t^{n-1}.$$

Hence the theorem is proved. □

We conclude this section by giving an example of a sequence of positive linear operators that converges in the sense of power series method, but it does not converge in the usual sense.
For each $n\in\mathbb{N}$, let us define a sequence of positive linear operators on $\mathcal{C}(I)$ as

$$\mathcal{T}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x)=(1+\alpha_n)\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x) \tag{2.5}$$

where

$$\alpha_n=\begin{cases}1, & n=m^4,\, m\in\mathbb{N}\\ 0, & otherwise.\end{cases}$$

It is evident that the sequence (α_n) is divergent, we show that it is Abel convergent to 0. For Abel convergence $a_n=1$ therefore $\rho(t)=\frac{1}{1-t}$ and $R=1$. By Cauchy root test and using the fact that $|t|<1$, we have

$$\lim_{m\to\infty}|t^{m^4-1}|^{1/m}=\lim_{m\to\infty}|t|^{m^3-(1/m)}=0,$$

therefore the series $\sum_{m=1}^{\infty}t^{m^4-1}$ is convergent for $|t|<1$ and hence

$$\lim_{t\to 1^-}(1-t)\sum_{m=1}^{\infty}t^{m^4-1}=0.$$

Thus the sequence (α_n) is Abel convergent to 0. From equation (2.5), we get

$$\mathcal{T}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i;x)=(1+\alpha_n)\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i;x),\ i=0,1,2.$$

For $i=0,1,2$ we have

$$\begin{aligned}||\mathcal{T}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)-e_i|| &= \sup_{0\leq x\leq 1}|(1+\alpha_n)\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i;x)-e_i(x)|\\ &\leq \sup_{0\leq x\leq 1}|\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i;x)-e_i(x)|+|\alpha_n|\sup_{0\leq x\leq 1}|\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i;x)|\\ &\leq ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)-e_i||+|\alpha_n|||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)|| \end{aligned}\tag{2.6}$$

To show the convergence of the operators $\mathcal{T}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ in the sense of power series method, let us define a linear operator on $\mathcal{C}(I_0^1)$ as

$$T_t(f;x)=\frac{1}{\rho(t)}\sum_{n=1}^{\infty}\mathcal{T}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x)t^{n-1}.$$

For $i=0,1,2$, from equation (2.6), we obtain

$$\begin{aligned}||T_t(e_i;x)-e_i(x)|| &\leq \frac{1}{\rho(t)}\sum_{n=1}^{\infty}||\mathcal{T}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i;x)-e_i(x)||t^{n-1}\\ &\leq \frac{1}{\rho(t)}\sum_{n=1}^{\infty}||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)-e_i||t^{n-1}+\frac{1}{\rho(t)}\sum_{n=1}^{\infty}|\alpha_n|||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)||t^{n-1}.\end{aligned}$$

Since, the sequence $||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i) - e_i|| \to 0$, as $n \to \infty$, it also converges to 0, in the sense of power series method and the sequence $||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)||$ is convergent for every $i = 0, 1, 2$, hence it is also bounded so there exists a constant $M > 0$ such that $||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i)|| < M$, for all $n \geq 1$ and $i = 0, 1, 2$. Therefore, we have

$$||T_t(e_i; x) - e_i(x)|| \leq (1-t)\sum_{n=1}^{\infty} ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i) - e_i||t^{n-1} + M(1-t)\sum_{n=1}^{\infty} \alpha_n t^{n-1}. \quad (2.7)$$

Since the sequence (α_n) converges to zero in the sense of power series method, it follows from equation (2.7) that

$$\lim_{t\to 1^-} ||T_t(e_i; x) - e_i(x)|| = 0, \text{ for } i = 0, 1, 2.$$

Thus, the operators T_t satisfy all the conditions of (Theorem 1, [38]), hence

$$\lim_{t\to 1^-} ||T_t(f) - f|| = 0$$

for all $f \in \mathcal{C}(I)$.

Since the sequence (α_n) is not convergent in the classical sense to 0, therefore the classical Korovkin type theorem of Gadjiev[15] does not work here. However, Korovkin type theorem holds in the sense of power series method.

3. A-Statistical weighted convergence

Fast[12] and Steinhaus[37] introduced independently the concept of statistical convergence of sequences of real numbers in the same year 1951. The application of the statistical convergence in the approximations theory needs Korovkin type theorem in the statistical sense which is given by Gadjiev and Orhan[16]. Karakaya and Chishti[19] introduced the concept of weighted statistical convergence. Later Mursaleen et al.[27] rectified the definition of weighted statistical convergence given in [19]. Mohiuddine[26] introduced the concept of statistical weighted A-summability of a sequence and its convergence.

Let $\mathcal{K}$ be a subset of the set of natural numbers $\mathbb{N}$. The natural density[13] of the set $\mathcal{K}$ is defined as

$$\delta(\mathcal{K}) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \chi_{\mathcal{K}}(i)$$

where $\chi_{\mathcal{K}}$ is the characteristic function on $\mathcal{K}$, provided the above limit exists. A sequence of real number (α_n) is to be converge statistically[12] to a limit l if for each $\epsilon > 0$, $\delta\{k \in \mathbb{N} : |\alpha_k - l| \geq \epsilon\} = 0$. In this case, we write $st - \lim_{n\to\infty} \alpha_n = l$. For more details about the statistical convergence, the reader may refer to ([7],[14],[22],[28],[34]).

Let (x_κ) be a sequence of real numbers and $A = (a_{\kappa n})$ be an infinite summability matrix, then the A-transform of the sequence (x_κ) is defined as

$$(Ax)_n = \sum_{\kappa=1}^{\infty} a_{\kappa n} x_\kappa$$

whenever the series converges for each $n \in \mathbb{N}$. From [18], we know that a summability matrix A is said to be regular if $\lim_{n\to\infty}(Ax)_n = l$ whenever $\lim_{\kappa\to\infty} x_\kappa = l$. In particular, if we take $A = C_1$, the Cesáro matrix of order one, then the A-statistical convergence reduces to the statistical convergence. The ordinary convergence is obtained if we consider $A = I$.

Let (ρ_κ) be a sequence of non-negative real numbers such that $\rho_1 > 0$ and $\mathcal{P}_n = \sum_{\kappa=1}^{n} \rho_\kappa \to \infty$, as $n \to \infty$, then an infinite matrix $A = (a_{\kappa n})$ is said to be a weighted regular matrix if

$$\lim_{n\to\infty} \frac{1}{\mathcal{P}_n} \sum_{m=1}^{n}\sum_{\kappa=1}^{\infty} a_{\kappa m}\rho_m x_\kappa = l,$$

whenever $\lim_{\kappa\to\infty} x_\kappa = l$. A sequence (y_κ) is said to be converge to l via weighted A-statistical method if for every given $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\mathcal{P}_n} \sum_{m=1}^{n} \sum_{\kappa:|y_\kappa - l|\geq\epsilon} a_{\kappa m}\rho_m = 0,$$

and it is denoted by $st_A^w - \lim_{n\to\infty} y_n = l$. Observe that if we take $\rho_m = 1,\ m = 1, 2, 3, ..., n$ we get A-statistical convergence[10]. A sequence (x_n) is said to converge to a number l, weighted A-statistically with the rate $o(\alpha_n)$ if for each $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\alpha_n} \left\{ \frac{1}{\mathcal{P}_n} \sum_{m=1}^{n} \sum_{\kappa:|x_\kappa - l|\geq\epsilon} a_{\kappa m}\rho_m \right\} = 0,$$

and is denoted by $x_n - l = st_A^w - o(\alpha_n)$.

The purpose of this segment is to study the convergence of the operators $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ via weighted A-statistical method.

First, we establish the weighted A-statistical convergence of the sequence of positive linear operators $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ using the Korovkin type theorem given by Gadjiev and Orhan[16].

Theorem 4. *For all $f \in \mathcal{C}(I)$, the operators $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ satisfies*

$$st_A^w - \lim_{n\to\infty} ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f) - f|| = 0.$$

Proof. From the Korovkin type theorem proved in [Theorem 1,[16]], it is sufficient to show that

$$st_A^w - \lim_{n\to\infty} ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_i) - e_i|| = 0,\ i = 0, 1, 2.$$

From Lemma 1, it is evident that

$$st_A^w - \lim_{n\to\infty} ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_0) - e_0|| = 0.$$

Lemma 4 gives us

$$|\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_1; x) - e_1(x)| = (1 - \beta_n^{(1)})x,$$

and therefore

$$||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_1) - e_1|| \leq (1 - \beta_n^{(1)}).$$

Let $\epsilon > 0$ be arbitrary, then the set

$$\mathcal{B}^* = \{n \in \mathbb{N} : ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_1) - e_1|| \geq \epsilon\} \subseteq \mathcal{B} = \{n \in \mathbb{N} : 1 - \beta_n^{(1)} \geq \epsilon\}.$$

This shows that

$$\frac{1}{\mathcal{P}_n} \sum_{m=1}^{n} \sum_{\kappa\in\mathcal{B}^*} a_{\kappa m}\rho_m \leq \frac{1}{\mathcal{P}_n} \sum_{m=1}^{n} \sum_{\kappa\in\mathcal{B}} a_{\kappa m}\rho_m.$$

Since the sequence $\beta_n^{(1)} \to 1$, as $n \to \infty$, $\lim_{n\to\infty}(1 - \beta_n^{(1)}) = 0$. In view of the fact that the usual convergence implies weighted A-statistical convergence, it follows that

$$st_A^w - \lim_{n\to\infty} (1 - \beta_n^{(1)}) = 0.$$

Hence,

$$st_A^w - ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_1) - e_1|| = 0.$$

Finally, we show that

$$st_A^w - ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_2) - e_2|| = 0.$$

In equation (1.7) taking $m_1 = 1$, we get

$$\left|\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_2; x) - e_2(x)\right| \leq 2x^2(1 - \beta_n^{(1)}) + \frac{x\beta_n^{(1)}}{[n]_{q_n}}.$$

Taking supremum on both sides on $[0,1]$, we have

$$||\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_2)-e_2|| \leq 2(1-\beta_n^{(1)})+\frac{\beta_n^{(1)}}{[n]_{q_n}}.$$

Consider the sets

$$\begin{aligned}\mathcal{B}_1 &= \{n\in\mathbb{N}: ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_2)-e_2|| \geq \epsilon\}\\ \mathcal{B}_2 &= \{n\in\mathbb{N}: (1-\beta_n^{(1)}) \geq \frac{\epsilon}{4}\}\\ \text{and}\quad \mathcal{B}_3 &= \{n\in\mathbb{N}: \frac{\beta_n^{(1)}}{[n]_{q_n}} \geq \frac{\epsilon}{2}\}.\end{aligned}$$

Therefore, we can write $\mathcal{B}_1 \subseteq \mathcal{B}_2 \cup \mathcal{B}_3$, which implies that

$$\frac{1}{\mathcal{P}_n}\sum_{m=1}^{n}\sum_{\kappa\in\mathcal{B}_1} a_{\kappa m}\rho_m \leq \frac{1}{\mathcal{P}_n}\sum_{m=1}^{n}\sum_{\kappa\in\mathcal{B}_2} a_{\kappa m}\rho_m + \frac{1}{\mathcal{P}_n}\sum_{m=1}^{n}\sum_{\kappa\in\mathcal{B}_3} a_{\kappa m}\rho_m.$$

Observing that

$$st_A^w - \lim_{n\to\infty} 2(1-\beta_n^{(1)}) = 0 \;\; and \;\; st_A^w - \lim_{n\to\infty}\frac{\beta_n^{(1)}}{[n]_{q_n}} = 0,$$

it is evident that

$$st_A^w - ||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(e_2)-e_2|| = 0.$$

Hence the theorem follows. □

Theorem 5. *Let (y_n) be a non-increasing sequence of positive numbers and $f\in\mathcal{C}(I)$. Further, let*

$$\omega(f;\sqrt{\gamma_{n,q_n}}) = st_A^w - o(y_n), \;\; as\; n\to\infty.$$

Then,

$$||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f|| = st_A^w - o(y_n), \;\; as\; n\to\infty.$$

where γ_{n,q_n} is defined as in Theorem 1.

Proof. From Theorem 1, we can write

$$\Big|\Big|\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f\Big|\Big| \leq 2\omega(f;\sqrt{\gamma_{n,q_n}}),$$

where $\gamma_{n,q} = \{1-2(\beta_n^{(1)})+q(\beta_n^{(1)})^2\}+\frac{\beta_n^{(1)}}{[n]_q}$. Hence for a given $\epsilon>0$, we have

$$\frac{1}{y_n}\left\{\frac{1}{\mathcal{P}_n}\sum_{m=1}^{n}\sum_{\kappa:||\mathcal{L}_{\kappa,q_\kappa}^{\beta_\kappa^{(1)},\cdots,\beta_\kappa^{(r)}}(f)-f\big|\big|\geq\epsilon} a_{\kappa m}\rho_m\right\} \leq \frac{1}{y_n}\left\{\frac{1}{\mathcal{P}_n}\sum_{m=1}^{n}\sum_{\kappa:2\omega(f;\sqrt{\gamma_{\kappa,q_\kappa}})\geq\epsilon} a_{\kappa m}\rho_m\right\}. \tag{3.1}$$

From the hypotheses of the theorem, we have

$$\omega(f;\sqrt{\gamma_{n,q_n}}) = st_A^w - o(y_n), \;\; as\; n\to\infty.$$

Hence in view of equation (3.1), we get

$$\lim_{n\to\infty}\frac{1}{y_n}\left\{\frac{1}{\mathcal{P}_n}\sum_{m=1}^{n}\sum_{\kappa:||\mathcal{L}_{\kappa,q_\kappa}^{\beta_\kappa^{(1)},\cdots,\beta_\kappa^{(r)}}(f)-f\big|\big|\geq\epsilon} a_{\kappa m}\rho_m\right\} = 0,$$

which implies that

$$||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f|| = st_A^w - o(y_n), \;\; as\; n\to\infty.$$

□

4. κ-THE ORDER GENERALIZATION OF THE OPERATORS $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$

Voronovskaja[23] proved that the order of approximation by sequence of positive linear operators can not exceed $O(\frac{1}{n^2})$ however smooth the function may be. Using the approach of Kirov and Popova[21], the purpose of this section is to define a generalization of the operators $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ by means of the κ-th order Taylor's polynomial of the function f to achieve a better degree of approximation and to respond to the smoothness of function. Some significant work in this direction can be found in[3],[20],[25] and[35] etc.
Let $\kappa \in \mathbb{N} \cup \{0\}$ and $\mathcal{C}^\kappa(I)$ denote the space of all κ times continuously differentiable functions on I. Then for any $f \in \mathcal{C}^\kappa(I)$ the κ-th order generalization of the operators $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ is given by

$$\begin{aligned} \mathcal{S}_{n,\kappa,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x) &= \left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q\right)_n\right\}\sum_{s=0}^{\infty}\Bigg\{\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=s}\left\{\prod_{k=1}^{r}(q^n,q)_{l_k}\frac{(\beta_n^{(k)})^{l_k}}{(q,q)_{l_k}}\right\} \\ &\quad \sum_{m=0}^{\kappa}\frac{1}{[m]_q!}\left(x-\frac{[l_1]_q}{[n+l_1-1]_q}\right)^m f^{(m)}\left(\frac{[l_1]_q}{[n+l_1-1]_q}\right)\Bigg\}x^s, \end{aligned} \tag{4.1}$$

where $f^{(m)}$ denotes the m-th derivative of f. Note that for $\kappa = 0$, we have $f^{(0)} = f$ and hence the operator $\mathcal{S}_{n,\kappa,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ reduces to $\mathcal{L}_{n,q}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$.
Now, we estimate error in the approximation by the operators $\mathcal{S}_{n,\kappa,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ for the functions $f \in \mathcal{C}^\kappa(I_0^1)$ such that $f^{(\kappa)} \in Lip_M\upsilon$.

Theorem 6. *Let $\kappa \in \mathbb{N} \cup \{0\}$ then for all $f \in \mathcal{C}(I)$ such that $f^{(\kappa)} \in Lip_M\nu$, $0 < \nu \leq 1$, we have*

$$|\mathcal{S}_{n,\kappa,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x) - f(x)| \leq \mathcal{M}\frac{\Gamma(\nu+1)}{\Gamma(\kappa+\nu+1)}\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(h;x),$$

where $h(u) = |x-u|^{\kappa+\nu}$, $\mathcal{M} > 0$ is constant depending on f and $\Gamma(.)$ denotes the Gamma function.

Proof. From equation (1.3) and (4.1), for each $x \in I$, we have

$$\begin{aligned} f(x) - \mathcal{S}_{n,\kappa,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x) &= \left\{\prod_{i=1}^{r}\left(\beta_n^{(i)}x^i;q_n\right)_n\right\}\sum_{s=0}^{\infty}\Bigg\{\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=s}\left\{\prod_{k=1}^{r}(q_n^n,q_n)_{l_k}\frac{(\beta_n^{(k)})^{l_k}}{(q_n,q_n)_{l_k}}\right\} \\ &\quad f(x) - \sum_{m=0}^{\kappa}\frac{1}{m!}\left(x-\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)^m f^{(m)}\left(\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)\Bigg\}x^s. \end{aligned} \tag{4.2}$$

By using Taylor's integral form of remainder, we get

$$f(x) - \sum_{m=0}^{\kappa}\frac{1}{m!}\left(x-\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)^m f^{(m)}\left(\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)$$

$$= \frac{1}{(\kappa-1)!}\left(x-\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)^\kappa \int_0^1 (1-t)^{\kappa-1}\left\{f^{(\kappa)}\left(\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}} + t\left(x-\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)\right) - f^{(\kappa)}\left(\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)\right\}$$

Since $f^{(\kappa)} \in Lip_M\nu$ and using the definition of Beta function, we obtain

$$\begin{aligned} \Bigg|f(x) &- \sum_{m=0}^{\kappa}\frac{1}{m!}\left(x-\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)^m f^{(m)}\left(\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right)\Bigg| \\ &\leq \frac{\mathcal{M}}{(\kappa-1)!}\left|x-\frac{[l_1]_{q_n}}{[n+l_1-1]_{q_n}}\right|^{\kappa+\nu}\int_0^1(1-t)^{\kappa-1}t^\nu dt \\ &= \mathcal{M}\frac{\Gamma(\nu+1)}{\Gamma(\kappa+\nu+1)}\left|x-\frac{[l_1]_q}{[n+l_1-1]_q}\right|^{\kappa+\nu}. \end{aligned} \tag{4.3}$$

From equations (4.2) and (4.3), we achieve the desired result.

$\square$

Note that the function $h(u)=|x-u|^{\kappa+\nu}$ *satisfies* $h(x)=0$ *and hence* $h\in Lip_M 1$ *with Lipschitz constant* $M=\kappa+\nu$. *Consequently,* $h\in\mathcal{C}(I)$ *which in view of equation (1.8), implies that* $||\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(h)||\to 0$, *as* $n\to\infty$. *Thus from Theorem (6), for all* $f\in\mathcal{C}(I_0^1)$ *we obtain* $||\mathcal{S}_{n,\kappa,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f||\to 0$, *as* $n\to\infty$.

Considering the fact that $h\in\mathcal{C}(I)$, from Theorem 1 we are led to the following:

Corollary 2. *Let* $f\in\mathcal{C}^\kappa(I)$ *such that* $f^{(\kappa)}\in Lip_M\nu$, $0<\nu\leq 1$, *then we have*

$$||\mathcal{S}_{n,\kappa,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f||\leq 2\mathcal{M}\frac{\Gamma(\nu+1)}{\Gamma(\kappa+\nu+1)}\omega(h;\sqrt{\gamma_{n,q_n}})$$

where $\mathcal{M}$ *as in Theorem 6 and* γ_{n,q_n} *as in Theorem 1.*

Further since $h\in Lip_{\kappa+\nu}1$, from Corrollary 2, we obtain:

Corollary 3. *Let* $f\in\mathcal{C}^\kappa(I)$ *such that* $f^{(\kappa)}\in Lip_M\nu$, $0<\nu\leq 1$, *then we have*

$$||\mathcal{S}_{n,\kappa,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f)-f||\leq 2\mathcal{M}\frac{(\kappa+\nu)\Gamma(\nu+1)}{\Gamma(\kappa+\nu+1)}\sqrt{\gamma_{n,q_n}}$$

where $\mathcal{M}$ *is as in Theorem 6 and* γ_{n,q_n} *as in Theorem 1.*

5. A-statistical approximation by bivariate extension of $\mathcal{L}_{n,q_n}^{\beta_n^{(1)},\cdots,\beta_n^{(r)}}$ by using four dimensional matrix transformation

Let $\mathcal{C}(I^2)$ denote the space of all continuous functions on $I\times I$, equipped with the supremum norm $||f||=\sup_{(x,y)\in I^2}|f(x,y)|$. For $f\in\mathcal{C}(I^2)$, the bivariate extension of the operators given by (1.3) is defined as follows:

$$\begin{aligned}\mathcal{L}_{m,q,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}^{n,q,\beta_n^{(1)},\cdots,\beta_n^{(r)}}(f;x,y) &= \left\{\prod_{i=1}^{r}\prod_{j=1}^{p}\left(\beta_n^{(i)}x^i;q\right)_n\left(\alpha_m^{(j)}y^j;q\right)_m\right\}\sum_{s=0}^{\infty}\sum_{\eta=0}^{\infty}\Bigg\{\sum_{m_1l_1+m_2l_2+\cdots+m_rl_r=s},\sum_{n_1h_1+n_2h_2+\cdots+n_rh_r=\eta}\\ &\left\{\prod_{k=1}^{r}\prod_{k=1}^{p}(q^n,q)_{l_k}(q^m,q)_{h_k}\frac{(\beta_n^{(k)})^{l_k}}{(q,q)_{l_k}}\frac{(\alpha_m^{(k)})^{h_k}}{(q,q)_{h_k}}\right\}f\left(\frac{[l_1]_q}{[n+l_1-1]_q},\frac{[h_1]_q}{[m+h_1-1]_q}\right)\Bigg\}x^sy^\eta,\end{aligned} \tag{5.1}$$

where $\langle\alpha_m^{(j)}\rangle_{m\in\mathbb{N}}$, $(j=1,2,\cdots p)$ are sequences of real numbers in $(0,1)$ and $p\in\mathbb{N}$. The purpose of this section to verify the Korovkin type approximation theorem studied by Dirik and Demirci[8] via A-statistical convergence for the above double sequence of positive linear operators $\mathcal{L}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}$. The concept of convergence of double sequences was introduced by Pringsheim[31] as follows:

A double sequence $(z_{n,m})$ is said to be convergent to a limit l if for every $\epsilon>0$, there exists a natural number $\mathcal{N}\in\mathbb{N}$ such that

$$|z_{n,m}-l|<\epsilon,\ \text{whenever}\ n,m>\mathcal{N},$$

We denote this convergence as $\mathcal{P}-\lim_{n,m\to\infty}z_{n,m}=l$. Also, a double sequence $(z_{n,m})$ is called bounded if there exists $\mathcal{M}>0$ such that $|z_{n,m}|\leq\mathcal{M}$, $\forall\ n,m\in\mathbb{N}$. Note that unlike the single sequences, the double convergent sequence need not be bounded. For example, let for any $n\in\mathbb{N}$, $z_{n,1}=n$ and $z_{n,m}=\frac{1}{n}+\frac{1}{m}$, for $m\geq 2$. Then clearly $z_{n,m}\to 0$, as $m,n\to\infty$, and $z_{n,m}$ is an unbounded double sequence.

Let $A=(a_{i,j,n,m})$ be a four dimensional summability matrix. Then the A-transform of a double sequence $(z_{n,m})$ is defined as

$$(Az)_{i,j}=\sum_{n,m}a_{i,j,n,m}z_{n,m}$$

provided the above series converges in the sense of Pringsheim for all $i,j\in\mathbb{N}$. Silverman-Toeplitz [18] characterized the regular matrix summability methods for the two dimensional case. Robinson[32] gave the necessary and sufficient conditions for the regularity of the four dimensional matrix transformation by considering an additional assumption on the boundedness of the sequence as every convergent (Pringsheim sense) double sequence is not bounded. These regularity conditions is known as Robinson-Hamilton conditions or RH-regularity[17][32].

A four dimensional matrix transformation is said to be RH-regular if it takes every $\mathcal{P}$ convergent and bounded double sequence to a $\mathcal{P}$-convergent sequence with the same $\mathcal{P}$-limit. The necessary and sufficient conditions for a four dimensional matrix to be RH-regular are

(i) $\mathcal{P}-\lim_{i,j} a_{i,j,n,m}=0$ for all $(n,m)\in\mathbb{N}\times\mathbb{N}$,
(ii) $\mathcal{P}-\lim_{i,j}\sum_{n,m} a_{i,j,n,m}=1$,
(iii) $\mathcal{P}-\lim_{i,j}\sum_{n}|a_{i,j,n,m}|=0$ for all $m\in\mathbb{N}$,
(iv) $\mathcal{P}-\lim_{i,j}\sum_{m}|a_{i,j,n,m}|=0$ for all $n\in\mathbb{N}$,
(v) $\sum_{n,m}|a_{i,j,n,m}|$ is $\mathcal{P}$- convergent,
(vi) there exists two positive integers $\mathcal{N}$ and $\mathcal{M}$ such that $\sum_{n,m>\mathcal{N}}|a_{i,j,n,m}|<\mathcal{M}$ for all $(i,j)\in\mathbb{N}\times\mathbb{N}$.

Let $\mathcal{E}$ be a subset of $\mathbb{N}\times\mathbb{N}$ and $A=(a_{i,j,n,m})$ be a non-negative RH-regular infinite summability matrix. The the A-density of the set $\mathcal{E}$ is defined as

$$\delta_{\mathcal{A}}(\mathcal{E})=\mathcal{P}-\lim_{i,j}\sum_{(n,m)\in\mathcal{E}} a_{i,j,n,m}.$$

whenever the above $\mathcal{P}$-limit exists. A real double sequence $(z_{n,m})$ is said to be converge A-statistically to l, if for every $\epsilon>0$

$$\mathcal{P}-\lim_{i,j}\sum_{(n,m):|z_{n,m}-l|\geq\epsilon} a_{i,j,n,m}=0,$$

and write it as $st_A^2-\lim_{n,m} z_{n,m}=l$. Pringsheim convergence ($\mathcal{P}$-convergence) is obtained by replacing matrix A by the identity matrix of four dimensional matrices. It is worthy to note that every double $\mathcal{P}$-convergent sequence converges A-statistically to the same limit however the converse is not true. For example, the double sequence

$$z_{n,m}=\begin{cases} nm, & \text{if } n \text{ and } m \text{ are perfect cube}\\ 1, & \text{otherwise,}\end{cases}$$

is A-statistically convergent to 1, however $\mathcal{P}-\lim_{n,m\to\infty} z_{n,m}$ does not exist. In order to establish the main result of this section, we need the following basic result:

Lemma 5. *The operators $\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}$ satisfies the following*

(i) $\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(1;x,y)=1$,
(ii) $\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(v;x,y)=y\alpha_m^{(1)}$,
(iii) $\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(v^2;x,y)\leq q_m y^2(\alpha_m^{(1)})^2+\frac{y\alpha_m^{(1)}}{[m]_{q_m}}$,
(iv) $\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(u^2+v^2;x,y)\leq q_n x^2(\beta_n^{(1)})^2+\frac{x\beta_n^{(1)}}{[n]_{q_n}}+q_m y^2(\alpha_m^{(1)})^2+\frac{y\alpha_m^{(1)}}{[m]_{q_m}}$,

The proof of this lemma follows from the Lemmas (1, 2) and (3).

Theorem 7. *Let $A=(a_{i,j,n,m})$ be a non-negative infinite RH-regular summability matrix. Then for any $f\in\mathcal{C}(I^2)$ there holds*

$$st_A^2-\lim_{n,m\to\infty}||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(f)-f||=0$$

Proof. The Korovkin type theorem in A-statistical sense via four dimensional summability matrix is given by Dirik and Demirci[8]. In view of this theorem, it is sufficient to prove that

$$st_A^2-\lim_{n,m\to\infty}||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_i)-g_i||=0,\ i=0,1,2,3,$$

where $g_0(u,v)=1,\ g_1(u,v)=u,\ g_2(u,v)=v,\ g_3(u,v)=u^2+v^2$. For $i=0$, it is not difficult to prove

$$st_A^2-\lim_{n,m\to\infty}||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_0)-g_0||=0.$$

From Lemma(2) taking $m_1=1$, we get

$$\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_1;x,y)-g_1(x,y)=x(\beta_n^{(1)}-1).$$

Therefore

$$|\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_1;x,y)-g_1(x,y)|=x(1-\beta_n^{(1)}).$$

Taking supremum on the right side of the above equation on $[0,1]$, we have

$$||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_1)-g_1||\leq 1-\beta_n^{(1)}.$$

For any given $\epsilon>0$, the set

$$\mathcal{D}=\{(n,m)\in\mathbb{N}\times\mathbb{N}:||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_1)-g_1||\geq\epsilon\}\subseteq\bar{\mathcal{D}}=\{(n,m)\in\mathbb{N}\times\mathbb{N}:(1-\beta_n^{(1)})\geq\epsilon\}.$$

This implies that

$$\sum_{(n,m)\in\mathcal{D}}a_{i,j,n,m}\leq\sum_{(n,m)\in\bar{\mathcal{D}}}a_{i,j,n,m}.$$

Since the sequence $\beta_n^{(1)}\to1$, as $n\to\infty$ in Pringsheim sense, $\mathcal{P}-\lim_{n,m\to\infty}(1-\beta_n^{(1)})=0$, and Pringsheim convergence implies A-statistical convergence, we get

$$st_A^2-\lim_{n,m\to\infty}(1-\beta_n^{(1)})=0.$$

Therefore

$$st_A^2-\lim_{n,m\to\infty}||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_1)-g_1||=0.$$

By giving similar arguments and using the fact that $\mathcal{P}-\lim_{m\to\infty}(1-\alpha_m^{(1)})=0$, we can show that

$$st_A^2-\lim_{n,m\to\infty}||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_2)-g_2||=0.$$

Finally, from Lemma (5), we have

$$|\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_3;x,y)-g_3(x,y)|\leq 2x^2(1-\beta_n^{(1)})+2y^2(1-\alpha_m^{(1)})+\frac{x\beta_n^{(1)}}{[n]_{q_n}}+\frac{y\alpha_m^{(1)}}{[m]_{q_m}},$$

Taking supremum on the right side of the above equation on $[0,1]$, we have

$$||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_3)-g_3||\leq(4-\beta_n^{(1)}-\alpha_m^{(1)})+\frac{\beta_n^{(1)}}{[n]_{q_n}}+\frac{\alpha_m^{(1)}}{[m]_{q_m}}.\tag{5.2}$$

Now, for a given $\epsilon>0$, let us define the following sets:

$$\begin{aligned}
\mathcal{D}^* &= \{(n,m)\in\mathbb{N}\times\mathbb{N}:||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_2)-g_2||\geq\epsilon\},\\
\mathcal{D}_1 &= \{(n,m)\in\mathbb{N}\times\mathbb{N}:(4-\beta_n^{(1)}-\alpha_m^{(1)})\geq\frac{\epsilon}{3}\},\\
\mathcal{D}_2 &= \{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{\beta_n^{(1)}}{[n]_{q_n}}\geq\frac{\epsilon}{3}\},\\
\mathcal{D}_3 &= \{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{\alpha_m^{(1)}}{[m]_{q_m}}\geq\frac{\epsilon}{3}\}.
\end{aligned}$$

Hence in view of equation (5.2), it is easy to see that $\mathcal{D}\subseteq\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3$, consequently we can write

$$\sum_{(n,m)\in\mathcal{D}^*}a_{i,j,n,m}\leq\sum_{(n,m)\in\mathcal{D}_1}a_{i,j,n,m}+\sum_{(n,m)\in\mathcal{D}_2}a_{i,j,n,m}+\sum_{(n,m)\in\mathcal{D}_2}a_{i,j,n,m}.\tag{5.3}$$

Since

$$\lim_{i,j\to\infty}\sum_{(n,m)\in\mathcal{D}_k} a_{i,j,n,m}=0,\ k=1,2,3$$

From equation (5.3), we obtain

$$st_A^2-\lim_{n,m\to\infty}||\mathcal{L}^{n,q_n,\beta_n^{(1)},\cdots,\beta_n^{(r)}}_{m,q_m,\alpha_m^{(1)},\cdots,\alpha_m^{(p)}}(g_3)-g_3||=0.$$

Hence the conclusion of the theorem holds true.

□

6. Conflicts of Interest

The authors declare that there are no any conflicts of interest regarding the publication of this manuscript.

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