The Lie symmetry analysis, exact solutions and conservation laws for variable coefficients Broer-Kaup-Kupershmit equation

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Abstract

In this article, the (2+1) - dimensional variable coefficients Broer-Kaup-Kupershmit equation is studied for the first time by Lie symmetry analysis. The derivation process of generating elements of vcBKK equation is given systematically, and the optimal system of the one-dimensional subalgebras is determined. Furthermore vcBKK equation is reduced based on the optimal system, and then the reduced equations are solved with the help of the (G'/G) - expansion method. The images of various kinds of exact solutions are drawn. Finally, according to the conservation theorem, the conservation laws of vcBKK equation is constructed.

Keywords: Exact solutions; Conservation laws; Lie symmetry analysis; vcBKK equation; (G'/G) - expansion method.

1. Introduction

Nonlinear partial differential equations (NPDEs) arise in many areas of physics. Exhibiting a wealth of nonlinear phenomena, and NPDEs are often used to describe important areas of physics, mechanics, chemistry, dynamic processes biology and other sciences. The study of exact solutions of NPDEs are of great importance and a large number of methods have now been developed, including Bäcklund transformation [1, 2], the Hirota bilinear method [3], Lie symmetry analysis [4, 5], the CK method [6], dynamical system theory [7, 8], and hyperbolic function method [9]. With the help of Lie group, constructing exact solutions of partial differential equations (PDEs) is very effective [10–12] , and we can use Lie group to get different reduced equations, and then get different kinds of solutions, such as periodic solutions and kink solutions, etc [13–15].

The Broer-Kaup-Kupershmidt (BKK) equation is a well-known evolution equation, which can be obtained from symmetry reduction of the KP equation [16, 17]. It has found a wide range of applications in physical fields, such as plasma physics, hydrodynamics and nonlinear fibre optic communication [18]. The BKK equation is used to simulate the nonlinear dispersive long gravity wave propagating along two horizontal directions in uniform water depth. The constant coefficients BKK equation has the following form

$$\begin{aligned} &u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} = 0, \\ &v_t + v_{xx} + 2(uv)_x = 0. \end{aligned} \tag{1.1}$$

Due to the assumption of constant coefficients, the physical situations that give rise to the nonlinear equations are often highly idealised. Therefore, the variable coefficient NPDEs has been widely investigated [19–23]. In this article we discuss the following vcBKK equation

$$\begin{aligned} &u_{ty} - a(t)\,u_{xxy} + b(t)\,(uu_x)_y + c(t)\,v_{xx} = 0, \\ &v_t + d(t)\,v_{xx} + e(t)\,(uv)_x = 0, \end{aligned} \tag{1.2}$$

where $a(t), b(t), c(t), d(t), e(t)$ are arbitrary non-zero functions on t. When $a(t) = d(t) = 1, b(t) = c(t) = e(t) = 2$, the above equation is the BKK equation (1.1).

Through literature review, the vcBKK equation has not been studied by the method of Lie symmetry analysis, the outline of this paper is as follows. In Section 2, using the method of Lie symmetry analysis, the generators of vcBKK equation are obtained, and then optimal system is determined. In Section 3, based on the optimal system, the vcBKK equation is reduced to the (1+1) dimensional equations. In Section 4, the reduced equations are solved by (G'/G) -expansion method [24], and use Maple software to draw figures, and get various types of images. In Section 5, four groups of conservation laws of the vcBKK equation are obtained according to the conservation theorem. In Section 6, we have concluded this paper.

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2. Lie symmetry analysis

Lie symmetry analysis is an effective method for finding invariants and exploring certain properties by reducing the dimensional of the NPDEs [28]. In this section, the generators of vcBKK are obtained with the help of Lie symmetry analysis. The one-parameter Lie group transformation has the form

$$\begin{aligned} x^* &= x + \varepsilon \cdot \xi + o\left(\varepsilon^2\right), \\ y^* &= y + \varepsilon \cdot \eta + o\left(\varepsilon^2\right), \\ t^* &= t + \varepsilon \cdot \tau + o\left(\varepsilon^2\right), \\ u^* &= u + \varepsilon \cdot \phi_1 + o\left(\varepsilon^2\right), \\ v^* &= v + \varepsilon \cdot \phi_2 + o\left(\varepsilon^2\right), \end{aligned} \tag{2.1}$$

where $\xi, \eta, \tau, \phi_1, \phi_2$ are generators of the variables x, y, t, u and v, the vector field $\mathcal{X}$ relevant to the one-parameter transformations of vcBKK can be separately written as

$$\mathcal{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial u} + \phi_2 \frac{\partial}{\partial v}. \tag{2.2}$$

To get the Lie point symmetry of vcBKK, the vector field $\mathcal{X}$ must fulfill the surface condition

$$\begin{aligned} \mathrm{Pr}^{(3)} \mathcal{X} F_1|_{F_1=0,F_2=0} &= 0, \\ \mathrm{Pr}^{(3)} \mathcal{X} F_2|_{F_1=0,F_2=0} &= 0, \end{aligned} \tag{2.3}$$

where $\mathrm{Pr}^{(3)}\mathcal{X}$ denotes the third-order prolongation of the vector field $\mathcal{X}$, which takes the form of

$$\begin{aligned} \mathrm{Pr}^{(3)} \mathcal{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial u} + \phi_2 \frac{\partial}{\partial v} + \psi_x^1 \frac{\partial}{\partial u_x} + \psi_y^1 \frac{\partial}{\partial u_y} + \psi_{xy}^1 \frac{\partial}{\partial u_{xy}} \\ + \psi_{xxy}^1 \frac{\partial}{\partial u_{xxy}} + \psi_{ty}^1 \frac{\partial}{\partial u_{ty}} + \psi_x^2 \frac{\partial}{\partial v_x} + \psi_t^2 \frac{\partial}{\partial v_t} + \psi_{xx}^2 \frac{\partial}{\partial v_{xx}} \end{aligned} \tag{2.4}$$

Applying $\mathrm{Pr}^{(3)}\mathcal{X}$ to vcBKK (1.2) , we obtain the following Lie invariant surface conditions

$$\begin{aligned} &\psi_{yt}^1 + \tau\left[-a'(t)\,u_{xxy} + b'(t)\,u_y u_x + b'(t)\,u u_{xy} + c'(t)\,v_{xx}\right] + a(t)\,\psi_{xxy}^1 + b(t)\left(\psi_x^1 u_y + \psi_y^1 u_x + \phi_1 u_{xy} + \psi_{xy}^1 u\right) + c(t)\,\psi_{xx}^2 = 0, \\ &\psi_t^2 + \tau\left[d'(t)\,v_{xx} + e'(t)\,u_x v + e'(t)\,u v_x\right] + e(t)\,\phi_1 v_x + \phi_2 u_x + \psi_x^1 v + \psi_x^2 u + d(t)_{xx}^2 = 0, \end{aligned} \tag{2.5}$$

where $\phi_1^x, \phi_1^y, \phi_1^{xy}, \phi_1^{xxy}, \phi_1^{ty}, \phi_2^x, \phi_2^t$, and ϕ_2^{xx} can be defined separately

$$\begin{aligned} \psi_x^1 &= D_x\left(\phi_1 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt}, \\ \psi_x^2 &= D_x\left(\phi_2 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt}, \\ \psi_y^1 &= D_y\left(\phi_1 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xy} + \eta u_{yy} + \tau u_{yt}, \\ \psi_t^2 &= D_t\left(\phi_2 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xt} + \eta u_{yt} + \tau u_{tt}, \\ \psi_{xx}^2 &= D_{xx}\left(\phi_2 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xxx} + \eta u_{xxy} + \tau u_{xxt}, \\ \psi_{xy}^1 &= D_{xy}\left(\phi_1 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xxy} + \eta u_{xyy} + \tau u_{xyt}, \\ \psi_{yt}^1 &= D_{yt}\left(\phi_1 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xyt} + \eta u_{yyt} + \tau u_{ytt}, \\ \psi_{xxy}^1 &= D_{xxy}\left(\phi_1 - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xxxy} + \eta u_{xxyy} + \tau u_{xxyt}. \end{aligned} \tag{2.6}$$

where D_x, D_y and D_t denote total differential operator.

Taking Eqs. (1.2) and Eqs. (2.6) into Eqs. (2.5) , setting the derivative coefficients of the same order of u to be 0, we can get determining equations for $\xi, \eta, \tau, \phi_1, \phi_2$, then solve those equations yields

$$\begin{aligned} &\xi = c_1 x + c_3, \quad \eta = c_1 y + c_2, \quad \phi_1 = -c_1 u, \\ &\tau = \frac{c_4}{b(t)} + \frac{2c_1 \int b(t)\, dt}{b(t)}, \qquad \phi_2 = -2c_1 v, \end{aligned} \tag{2.7}$$

where $c_j \, (j = 1, 2, 3, 4)$ are arbitrary constants. On the side, the coefficient functions $a(t), b(t), c(t), d(t), e(t)$ in Eqs. (1.2) must satisfy the following conditions

$$\begin{aligned} a(t)\tau_t + a'(t) - 2a(t)c_1 &= 0, \\ b(t)\tau_t + b'(t) - 2b(t)c_1 &= 0, \\ c(t)\tau_t + c'(t) - 2c(t)c_1 &= 0, \\ d(t)\tau_t + d'(t) - 2d(t)c_1 &= 0, \\ e(t)\tau_t + e'(t) - 2e(t)c_1 &= 0. \end{aligned} \tag{2.8}$$

The vector field $\mathcal{X}$ of the vcBKK equation (1.2) can be reduced to the form of vectors $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{X}_4$

$$\mathcal{X} = a_1\mathcal{X}_1 + a_2\mathcal{X}_2 + a_3\mathcal{X}_3 + a_4\mathcal{X}_4, \tag{2.9}$$

where the vectors $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{X}_4$ are to be defined as

$$\begin{aligned} \mathcal{X}_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2\int b(t)\, dt}{b(t)}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}, \\ \mathcal{X}_2 &= \frac{\partial}{\partial y}, \\ \mathcal{X}_3 &= \frac{\partial}{\partial x}, \\ \mathcal{X}_4 &= \frac{1}{b(t)}\frac{\partial}{\partial t}. \end{aligned} \tag{2.10}$$

The Lie algebra and commutation relations for these vectors are calculated using the Lie bracket $\left[\mathcal{X}_i, \mathcal{X}_j\right] = \mathcal{X}_i\mathcal{X}_j - \mathcal{X}_j\mathcal{X}_i$. The Lie series for calculating accompanying relations can be expressed as

$$Ad\left(\exp(\varepsilon)\mathcal{X}_i\right)\mathcal{X}_j = \mathcal{X}_j - \varepsilon\left[\mathcal{X}_i, \mathcal{X}_j\right] + \frac{\varepsilon^2}{2}[\mathcal{X}_i, [\mathcal{X}_i, \mathcal{X}_j]] - \cdots, \tag{2.11}$$

with ε is an infinitesimal real number.

Table 1: Commutator table.

$\left[\mathcal{X}_i, \mathcal{X}_j\right]$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_1$	0	$-\mathcal{X}_2$	$-\mathcal{X}_3$	$-2\mathcal{X}_4$
$\mathcal{X}_2$	$\mathcal{X}_2$	0	0	0
$\mathcal{X}_3$	$\mathcal{X}_3$	0	0	0
$\mathcal{X}_4$	$2\mathcal{X}_4$	0	0	0

Table 2: Adjoint representation table.

Ad	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_1$	$\mathcal{X}_1$	$\mathcal{X}_2 e^{\varepsilon}$	$\mathcal{X}_3 e^{\varepsilon}$	$\mathcal{X}_4 e^{\varepsilon}$
$\mathcal{X}_2$	$\mathcal{X}_1 - \varepsilon\mathcal{X}_2$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_3$	$\mathcal{X}_1 - \varepsilon\mathcal{X}_3$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_4$	$\mathcal{X}_1 - \varepsilon 2\mathcal{X}_4$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$

According to **Table 1** and **Table 2**, the optimal system for the one-dimensional subalgebra is derived by the method in [29, 30]

$$\begin{aligned} &\text{I:} \quad \mathcal{X}_1, \\ &\text{II:} \quad \mathcal{X}_2, \\ &\text{III:} \quad \mathcal{X}_2 + \alpha_1 \mathcal{X}_3 + \alpha_2 \mathcal{X}_4, \end{aligned} \tag{2.12}$$

where α_1 and α_2 are arbitrary constants.

3. Similarity reductions of the vcBKK equation

In this section, we reduce the vcBKK equation to the (1+1)-dimensional PDEs based on the subalgebra in (2.10) and the optimal system (2.12). Here, we only take **Case III** as an example for symmetric reduction. To simplify calculation, let $\alpha_1 = \alpha_2 = 1$.

CaseIII in optimal system (2.12) can be written as

$$\mathcal{X} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \tau(t) \frac{\partial}{\partial t}, \tag{3.1}$$

according to this Lie vector, the corresponding characteristic equation can be expressed as

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dt}{\tau(t)} = \frac{du}{0} = \frac{dv}{0}. \tag{3.2}$$

The relative similarity variables are obtained by solving (3.2) are

$$X = x - \int \frac{1}{\tau(t)} dt, Y = y - \int \frac{1}{\tau(t)} dt, u = F(X, Y), v = H(X, Y). \tag{3.3}$$

Eqs. (1.2) are simplified by means of Eqs. (3.3) to the following form

$$\begin{aligned} &b(t)\tau(t) F F_{XY} + b(t)\tau(t) F_X F_Y - a(t)\tau(t) F_{XXY} + c(t)\tau(t) H_{XX} - F_{XY} - F_{YY} = 0, \\ &e(t)\tau(t) H F_X + e(t)\tau(t) F H_X + d(t)\tau(t) H_{XX} - H_X - H_Y = 0. \end{aligned} \tag{3.4}$$

To insure that only two independent variables X and Y, we also guarantee that the coefficient functions satisfy Eqs. (2.8), therefore, the coefficient functions are expressed as

$$a(t) = \frac{m_1}{\tau(t)}, b(t) = \frac{m_2}{\tau(t)}, c(t) = \frac{m_3}{\tau(t)}, d(t) = \frac{m_4}{\tau(t)}, e(t) = \frac{m_5}{\tau(t)}, \tag{3.5}$$

where $m_i (i = 1, 2 \cdots 5)$ are arbitrary constants, for ease of calculation, let $m_i = 1$, and substituting Eqs. (3.5) into Eqs. (3.4), yields

$$\begin{aligned} &F F_{XY} + F_X F_Y - F_{XXY} + H_{XX} - F_{XY} - F_{YY} = 0, \\ &H F_X + F H_X + H_{XX} - H_X - H_Y = 0. \end{aligned} \tag{3.6}$$

The reduced equations for different Lie vectors are shown in the following tables. **Table 3** shows similar variables corresponding to different Lie vectors. **Table 4** shows the expressions of coefficient functions corresponding to different Lie vectors, and in **Table 5** shows the corresponding reduced equations.

4. Exact solutions of the vcBKK equation

In this section, we will solve the reduced equations from the **Table 5**. Firstly, introduce the traveling wave transformation, the (1+1)-dimensional PDEs are transformed into ordinary differential equations(ODEs), and then using the (G'/G)-expansion method to solve the ODEs, mathematical expressions and graphical illustrations of the exact solutions are explained.

The next work is to solve the **Case III**, **IV**, and **V** in **Table 5** and draw the relevant images.

Case III $\mathcal{X}_4 + \mathcal{X}_2$

Table 3: Similarity variables.

Case	Similarity variables.
I: $\mathcal{X}_1$	$X = xe^{-\left(\int \frac{1}{\tau(t)}dt\right)}, Y = ye^{-\left(\int \frac{1}{\tau(t)}dt\right)}, u = F(X,Y), v = H(X,Y).$
II: $\mathcal{X}_4$	$X = x, Y = y, u = F(X,Y), v = H(X,Y).$
III: $\mathcal{X}_4 + \mathcal{X}_2$	$X = x, Y = y - \int \frac{1}{\tau(t)}dt, u = F(X,Y), v = H(X,Y).$
IV: $\mathcal{X}_4 + \mathcal{X}_3$	$X = x, Y = y - \int \frac{1}{\tau(t)}dt, u = F(X,Y), v = H(X,Y).$
V: $\mathcal{X}_4 + \mathcal{X}_2 + \mathcal{X}_3$	$X = x - \int \frac{1}{\tau(t)}dt, Y = y - \int \frac{1}{\tau(t)}dt, u = F(X,Y), v = H(X,Y).$

Table 4: The expressions of the coefficient functions.

Case	Coefficient functions expressions.
I: $\mathcal{X}_1$	$a(t) = \frac{e^{\int \frac{2}{\tau(t)}dt}}{\tau(t)}, b(t) = \frac{e^{\int \frac{2}{\tau(t)}dt}}{\tau(t)}, c(t) = \frac{e^{\int \frac{2}{\tau(t)}dt}}{\tau(t)}, d(t) = \frac{e^{\int \frac{2}{\tau(t)}dt}}{\tau(t)}, e(t) = \frac{e^{\int \frac{2}{\tau(t)}dt}}{\tau(t)}.$
II: $\mathcal{X}_4$	$a(t), b(t), c(t), d(t), e(t)$ are any functions that depend on t.
III: $\mathcal{X}_4 + \mathcal{X}_2$	$a(t) = \frac{1}{\tau(t)}, b(t) = \frac{1}{\tau(t)}, c(t) = \frac{1}{\tau(t)}, d(t) = \frac{1}{\tau(t)}, e(t) = \frac{1}{\tau(t)}.$
IV: $\mathcal{X}_4 + \mathcal{X}_3$	$a(t) = \frac{1}{\tau(t)}, b(t) = \frac{1}{\tau(t)}, c(t) = \frac{1}{\tau(t)}, d(t) = \frac{1}{\tau(t)}, e(t) = \frac{1}{\tau(t)}.$
V: $\mathcal{X}_4 + \mathcal{X}_2 + \mathcal{X}_3$	$a(t) = \frac{1}{\tau(t)}, b(t) = \frac{1}{\tau(t)}, c(t) = \frac{1}{\tau(t)}, d(t) = \frac{1}{\tau(t)}, e(t) = \frac{1}{\tau(t)}.$

Table 5: The expressions of the reduced PDEs.

Case	Reduced PDEs
I: $\mathcal{X}_1$	$F_XF_Y + FF_{XY} - F_{XXY} + H_{XX} - F_{XY} - F_{YY}Y - 2F_Y = 0,$ $F_XH + FH_X + H_{XX} - H_YY - H_XX - 2H = 0.$
II: $\mathcal{X}_4$	$b(t)F_YF_X + b(t)FF_{XY} - a(t)F_{XXY} + c(t)H_{XX} = 0,$ $e(t)FH_X + e(t)HF_X + d(t)H_{XX} = 0.$
III: $\mathcal{X}_4 + \mathcal{X}_2$	$F_XF_Y + FF_{XY} - F_{XXY} + H_{XX} - F_{YY} = 0,$ $HF_X + FH_X + H_{XX} - H_Y = 0.$
IV: $\mathcal{X}_4 + \mathcal{X}_3$	$F_XF_Y + FF_{XY} - F_{XXY} + H_{XX} - F_{XY} = 0,$ $HF_X + FH_X + H_{XX} - H_X = 0.$
V: $\mathcal{X}_4 + \mathcal{X}_2 + \mathcal{X}_3$	$F_XF_Y + FF_{XY} - F_{XXY} + H_{XX} - F_{XY} - F_{YY} = 0,$ $HF_X + FH_X + H_{XX} - H_X - H_Y = 0.$

In **Table 5**, we can see that the reduced equations as

$$\begin{aligned} &F_X F_Y + F F_{XY} - F_{XXY} + H_{XX} - F_{YY} = 0, \\ &H F_X + F H_X + H_{XX} - H_Y = 0. \end{aligned} \tag{4.1}$$

Firstly, introduce the traveling wave transformation

$$F(X, Y) = F(\theta), H(X, Y) = H(\theta), \theta = X - VY, \tag{4.2}$$

in which V is denoted as the wave velocity.

The travelling wave transform (4.2) is applied to Eqs.(4.1) can obtain the following equations

$$\begin{aligned} &-(F')^2 V - F F'' V + F''' V + H'' - F'' V^2 = 0, \\ &H F' + F H' + H'' + H' V = 0, \end{aligned} \tag{4.3}$$

we can directly solve the solutions of Eqs. (4.3) as

$$\begin{aligned} &F = \frac{c_2 V + V\theta - 2}{c_2 + \theta}, \\ &H = \left(\frac{4V}{(c_2 + \theta)^3} + c_1\right)(c_2 + \theta), \end{aligned} \tag{4.4}$$

where c_1 and c_2 are arbitrary constants.

Substituting the similar variables in **Table 3** into Eqs. (4.4) to obtain the solutions of Eqs. (1.2)

$$\begin{aligned} &u = \frac{c_2 V + V\left(V\left(\int \frac{1}{\tau(t)} dt - y\right) + x\right) - 2}{V\left(\int \frac{1}{\tau(t)} dt - y\right) + c_2 + x}, \\ &v = \left(\frac{4V}{V\left(\int \frac{1}{\tau(t)} dt - y\right) + c_2 + x} + c_1\right)\left(V\left(\int \frac{1}{\tau(t)} dt - y\right) + c_2 + x\right), \end{aligned} \tag{4.5}$$

we choose parameters $c_1 = 1$, $c_2 = 1$, $V = -1$, $\tau(t) = \sin(t)$, the images of Eqs. (4.5) are presented in **Figure. 1** respectively.

Case IV $\mathcal{X}_4 + \mathcal{X}_3$

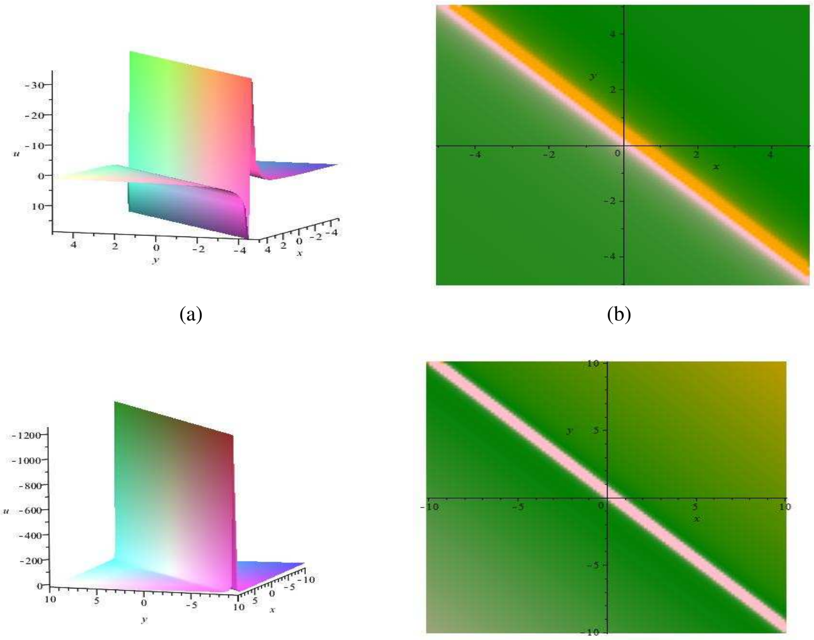
For **Case III**, we adopt the direct solution method after traveling wave transformation. Next we will use the (G'/G) - expansion method to solve the **Case IV** and **Case V**.

In **Table 5**, we can see that the equations for **Case IV** after reduction are

$$\begin{aligned} &F_X F_Y + F F_{XY} - F_{XXY} + H_{XX} - F_{XY} = 0, \\ &H F_X + F H_X + H_{XX} - H_X = 0. \end{aligned} \tag{4.6}$$

Firstly, we perform the traveling wave transformation on Eqs. (4.6) , using (4.2), we can transform the (1+1) dimensional PDEs (4.6) into ODEs

$$\begin{aligned} &-(F')^2 V - F F'' V + F''' V + H'' + F'' V = 0, \\ &H F' + F H' + H'' - H' = 0. \end{aligned} \tag{4.7}$$



(a) (b) (c) (d)

Figure 1: Evolution of solutions (4.5) at (a) $t = 1$, (b) density plot of (a) at $t = 1$, (c) $t = 1$, (d) density plot of (c) at $t = 1$.

With the help of homogeneous balance, we suppose that its solutions take the form

$$\begin{aligned} F(\theta) &= a_1\left(\frac{G'}{G}\right) + a_0, \\ H(\theta) &= b_2\left(\frac{G'}{G}\right)^2 + b_1\left(\frac{G'}{G}\right) + b_0, \end{aligned} \tag{4.8}$$

where a_0, a_1, b_0, b_1, b_2 are the coefficients to be identified, and $G = G(\theta)$ satisfies the second order linear ordinary differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{4.9}$$

in which λ and μ are arbitrary constants.

Substituting Eqs. (4.8) and Eq. (4.9) into Eqs. (4.7), then extracting the coefficients of the same order, and making them to be 0, we can get the following equations

$$\begin{cases} a_1 = 2, \\ b_2 - 4V = 0, \\ 2\lambda b_2 - a_0 b_2 - b_1 + b_2 = 0, \\ \lambda b_1 + 2\mu b_2 - a_0 b_1 - 2b_0 + b_1 = 0, \\ 2V - 10V\lambda - 6V\lambda a_1 - 2Va_0 + 5\lambda b_2 + b_1 = 0, \\ 2V\lambda\mu - 2V\lambda^2\mu - 2V\lambda\mu a_0 - 8V\mu^2 + \lambda\mu b_1 + 2\mu^2 b_2 = 0, \\ 6V\lambda - 22V\lambda^2 - 6V\lambda a_0 - 32V\mu + 4\lambda^2 b_2 + 3\lambda b_1 + 8\mu b_2 = 0, \\ 4\lambda^2 b_2 - 2\lambda a_0 b_2 - \lambda b_1 + 2\lambda b_2 + 2\mu b_2 - a_0 b_1 - 2b_0 + b_1 = 0, \\ \lambda^2 b_1 + 6\lambda\mu b_2 - \lambda a_0 b_1 - 2\lambda b_0 - 2\mu a_0 b_2 - 2\mu b_1 + \lambda b_1 + 2\mu b_2 = 0, \\ 2V\lambda^2 - 2V\lambda^3 - 2V\lambda^2 a_0 - 28V\lambda\mu - 4V\mu a_0 + 4V\mu + \lambda^2 b_1 + 6\lambda\mu b_2 + 2\mu b_1 = 0. \end{cases} \tag{4.10}$$

Solving the system (4.10) , yields

$$V = 1, a_0 = \lambda + 1, a_1 = 2, b_0 = 4\mu, b_1 = 4\lambda, b_2 = 4, \tag{4.11}$$

substituting Eqs. (4.11) into Eqs. (4.8) and reducing it as

$$\begin{aligned} F(\theta) &= 2\left(\frac{G'}{G}\right) + \lambda + 1, \\ H(\theta) &= 4\left(\frac{G'}{G}\right)^2 + 4\lambda\left(\frac{G'}{G}\right) + 4\mu. \end{aligned} \tag{4.12}$$

Substituting the solution of Eq. (4.9) into Eqs. (4.12) [18], we can obtain the exact solutions of the three types of Eqs. (4.6).

When $\lambda^2 > 4\mu$,

$$\begin{aligned} F(\theta) &= \frac{\sqrt{\lambda^2 - 4\mu}\left(k_1 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)\right)}{k_1 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)} + 1, \\ H(\theta) &= \frac{\left(\lambda^2 - 4\mu\right)\left(k_1 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)\right)^2}{\left(k_1 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)\right)^2} - \lambda^2 + 4\mu, \end{aligned} \tag{4.13}$$

where k_1 and k_2 are arbitrary constants, and $\theta = X - VY$.

When $\lambda^2 < 4\mu$,

$$\begin{aligned} F(\theta) &= \frac{\sqrt{-\lambda^2 + 4\mu}\left(-k_3 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right) + k_4 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right)\right)}{k_3 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right) + k_4 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right)} + 1, \\ H(\theta) &= \frac{\left(4\mu - \lambda^2\right)\left(-k_3 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right) + k_4 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right)\right)^2}{\left(k_3 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right) + k_4 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\theta\right)\right)^2} - \lambda^2 + 4\mu, \end{aligned} \tag{4.14}$$

where k_3 and k_4 are arbitrary constants, and $\theta = X - VY$.

When $\lambda^2 = 4\mu$,

$$\begin{aligned} F(\theta) &= \frac{2k_6}{k_6\theta + k_5} + 1, \\ H(\theta) &= \frac{4{k_2}^2}{(k_2\theta + k_1)^2} - \lambda^2 + 4\mu, \end{aligned} \tag{4.15}$$

where k_5 and k_6 are arbitrary constants, and $\theta = X - VY$.

The exact solutions of vcBKK are obtained by substituting the corresponding similar variables into the above solutions, and the procedure is as below.

When $\lambda^2 > 4\mu$,

$$
\begin{aligned}
u(\theta) &= \frac{\sqrt{\lambda^2-4\mu}\left(k_1 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)+k_2\cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)\right)}{k_1\cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)+k_2\sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)}+1,\\
v(\theta) &= \frac{\left(\lambda^2-4\mu\right)\left(k_1 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)+k_2\cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)\right)^2}{\left(k_1\cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)+k_2\sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2-4\mu}\right)\right)^2}-\lambda^2+4\mu,
\end{aligned} \tag{4.16}
$$

where $\theta = x - \int \frac{1}{\tau(t)}\mathrm{d}t - y$, k_1 and k_2 are arbitrary constants. We choose parameters $k_1 = 1, k_2 = 2, \tau(t) = \frac{1}{t}, \lambda = 3, \mu = 2$, the images of solutions (4.16) are presented in **Figure. 2** .

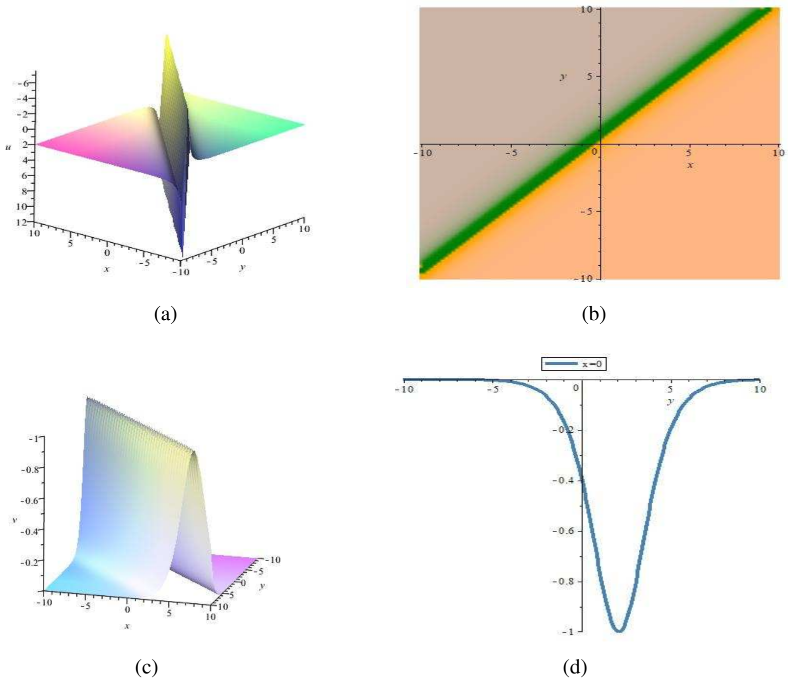


Figure 2: Evolution of the solutions (4.16) at (a) $t = 1$, (b) density plot of (a) at $t = 1$, (c) $t = 1$, (d) 2D plot at $t = 1$.

When $\lambda^2 < 4\mu$,

$$
\begin{aligned}
F(\theta) &= \frac{\sqrt{-\lambda^2+4\mu}\left(-k_3 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)+k_4\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)\right)}{k_3\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)+k_4\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)}+1,\\
H(\theta) &= \frac{\left(4\mu-\lambda^2\right)\left(-k_3 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)+k_4\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)\right)^2}{\left(k_3\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)+k_4\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\theta\right)\right)^2}-\lambda^2+4\mu,
\end{aligned} \tag{4.17}
$$

where $\theta = x - \int \frac{1}{\tau(t)}\mathrm{d}t - y$, k_3 and k_4 are arbitrary constants. We choose parameters $k_3 = 1, k_4 = 1, \tau(t) = t^2, \lambda = 2, \mu = 3$, the images of solutions (4.17) are presented in **Figure. 3** .

When $\lambda^2 = 4\mu$,

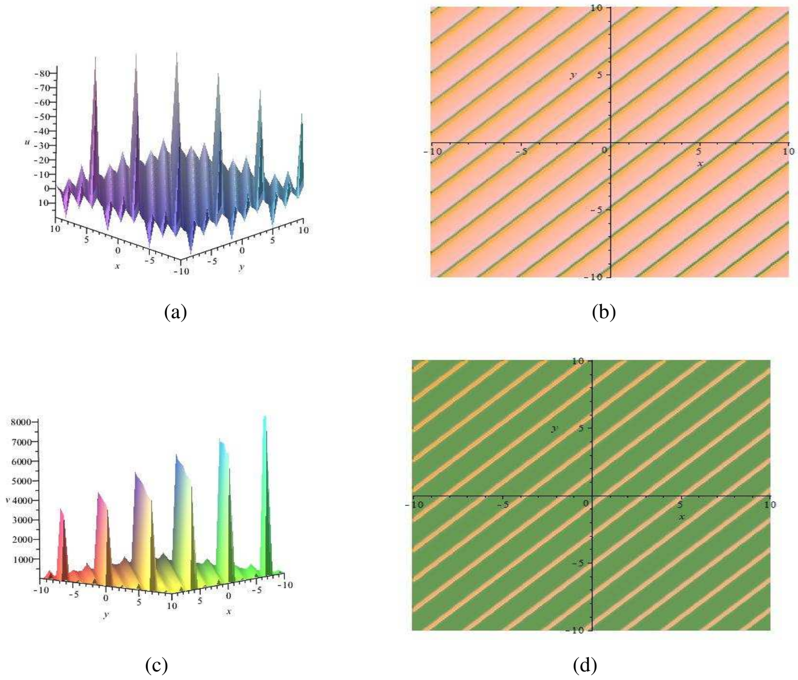


Figure 3: Evolution of the solutions (4.16) at (a) $t = -1$, (b) density plot of (a) at $t = -1$, (c) $t = 1$, (d) density plot of (c) at $t = 1$.

$$\begin{aligned} F(\theta) &= \frac{2k_6}{k_6\theta + k_5} + 1, \\ H(\theta) &= \frac{4{k_2}^2}{(k_2\theta + k_1)^2} - \lambda^2 + 4\mu, \end{aligned} \tag{4.18}$$

where $\theta = x - \int \frac{1}{\tau(t)} \mathrm{d}t - y$, with k_5 and k_6 are arbitrary constants.

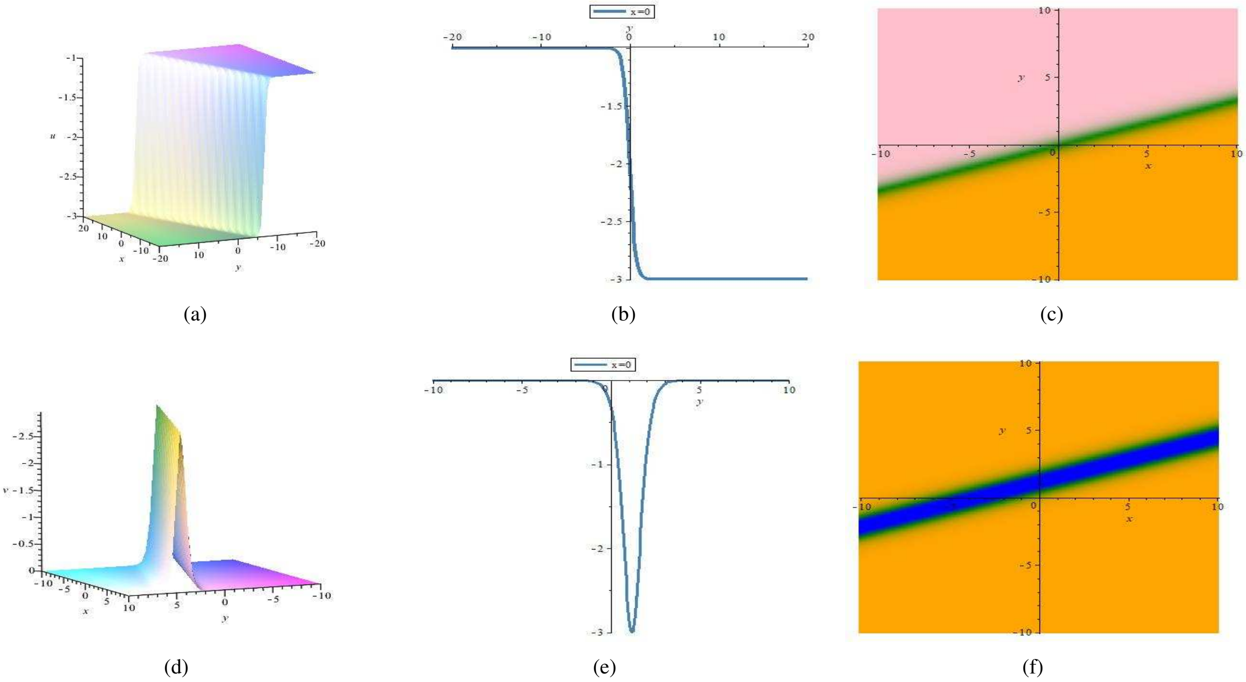
Case V $X_4 + X_2 + X_3$

The following method is used to calculate the exact solutions for **Case V** in the same way as the above cases.

When $\lambda^2 > 4\mu$,

$$\begin{aligned} u(\theta) &= \frac{\sqrt{\lambda^2 - 4\mu}\left(k_1 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)\right)}{k_1 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)} - \lambda + 1, \\ v(\theta) &= \frac{\lambda\left(\lambda^2 - 4\mu\right)\left(k_1 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)\right)^2}{\left(k_1 \cosh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right) + k_2 \sinh\left(\frac{1}{2}\theta\sqrt{\lambda^2 - 4\mu}\right)\right)^2} - \lambda^3 + 4\lambda\mu, \end{aligned} \tag{4.19}$$

where $\theta = -\lambda\left(y - \int \frac{1}{\tau(t)} \mathrm{d}t\right) + x - \int \frac{1}{\tau(t)} \mathrm{d}t$, with k_1 and k_2 are arbitrary constants. We choose parameters $k_1 = 2$, $k_2 = 1$,$\tau(t) = \sin(t)$, $\lambda = 3$, $\mu = 2$, the images of the solutions (4.19) are presented in **Figure. 4** .



(a) (b) (c)

(d) (e) (f)

Figure 4: Evolution of the solutions (4.19) at (a) $t = 1$, (b) 2D plot at $t = 1$, $x = 0$, (c) density plot of (a) at $t = 1$, (d) $t = 1$, (e) 2D plot at $t = 1$, $x = 0$, (f) density plot of (d) at $t = 1$,

When $\lambda^2 < 4\mu$,

$$\begin{aligned} u(\theta) &= \frac{\sqrt{4\mu-\lambda^2}\left(-k_3 \sin\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right) + k_4 \cos\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right)\right)}{k_3 \cos\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right) + k_4 \sin\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right)} - \lambda + 1, \\ v(\theta) &= \frac{\lambda\left(4\mu-\lambda^2\right)\left(-k_3 \sin\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right) + k_4 \cos\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right)\right)^2}{\left(k_3 \cos\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right) + k_4 \sin\left(\frac{1}{2}\theta\sqrt{4\mu-\lambda^2}\right)\right)^2} - \lambda^3 + 4\lambda\mu, \end{aligned} \tag{4.20}$$

where $\theta = -\lambda\left(y - \int \frac{1}{\tau(t)} \mathrm{d}t\right) + x - \int \frac{1}{\tau(t)} \mathrm{d}t$, with k_3 and k_4 are arbitrary constants.

When $\lambda^2 = 4\mu$,

$$\begin{aligned} u(\theta) &= \frac{2k_2}{k_2\theta + k_1} - \lambda + 1, \\ v(\theta) &= \frac{\lambda\left(\left(2k_2\theta\sqrt{\mu} + 2k_1\sqrt{\mu}\right)^2 + 4k_2{}^2\right)}{(k_2\theta + k_1)^2} - \lambda^2(k_2\theta + k_1)^2, \end{aligned} \tag{4.21}$$

where $\theta = -\lambda\left(y - \int \frac{1}{\tau(t)} \mathrm{d}t\right) + x - \int \frac{1}{\tau(t)} \mathrm{d}t$, with k_5 and k_6 are arbitrary constants.

5. Conservation laws of the vcBKK equations

The conservation laws are of great value in exploring the exact solutions of the PDEs. We can use them to explain many of the physical phenomena described by the PDEs, next we derive the conservation laws of the vcBKK. Firstly, we give a standard Lagrangian function as[31–34]

$$L=\Lambda_1\left(u_{ty}-a(t)\,u_{xxy}+b(t)\,(uu_x)_y+c(t)\,v_{xx}\right)+\Lambda_2\left(v_t+d(t)\,v_{xx}+e(t)\,(uv)_x\right), \tag{5.1}$$

where $\Lambda_1=\Lambda_1(x,y,t,u,v)\,,\Lambda_2=\Lambda_2(x,y,t,u,v)$.

For the vector $\mathcal{X}_1=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{2\int b(t)\,dt}{b(t)}\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-2v\frac{\partial}{\partial v}$, we can obtain

$$\begin{aligned} W^1&=-u-xu_x-yu_y-\frac{2\int b(t)\,dt}{b(t)}u_t,\\ W^2&=-2v-xv_x-yv_y-\frac{2\int b(t)\,dt}{b(t)}v_t, \end{aligned} \tag{5.2}$$

thus, the conservation law is obtained

$$\begin{aligned} C^x=&y\left(-\Lambda_1\left(b(t)\left(uu_y\right)_y+v_{xy}c(t)\right)+\Lambda_2\left(e(t)\,(uv)_y+v_{xy}d(t)\right)\right)+x\left(\Lambda_1\left(u_{yt}-a(t)\,u_{xxy}\right)+v_t\Lambda_2\right)\\ &-2\int b(t)\,dt\left(\frac{1}{b(t)}\left(e(t)\,\Lambda_2(uv)_t+v_{xt}\left(c(t)\,\Lambda_1+d(t)\,n\right)\right)+\Lambda_1\left(uu_y\right)_t\right)\\ &-3\left(u\left(b(t)\,u_y\Lambda_1+e(t)\,v\Lambda_2\right)+v_x\left(c(t)\,\Lambda_1+d(t)\,\Lambda_2\right)\right),\\ C^t=&2\int b(t)\,dt\left(\Lambda_1(uu_x)_y+\frac{1}{b(t)}\left(\Lambda_2\left(u_xe(t)\,v+v_{xx}d(t)+ue(t)\,v_x\right)+\Lambda_1\left(c(t)\,v_{xx}-a(t)\,u_{xxy}+u_{yt}\right)\right)\right)\\ &-\Lambda_2\left(2v+v_x+yv_y\right),\\ C^y=&\Lambda_1\left(-3u_t-u_{xt}x+a(t)\left(u_{xxx}x+3u_{xx}\right)\right)+2\Lambda_1\int b(t)\,dt\left(\frac{a(t)\,u_{xxt}}{b(t)}-\left(\frac{u_t}{b(t)}\right)_t\right)\\ &+y\left(b(t)\,\Lambda_1(uu_x)_y+e(t)\,\Lambda_2(uv)_x+c(t)\,v_{xx}\Lambda_1+v_{xx}d(t)\,\Lambda_2+v_t\Lambda_2\right). \end{aligned} \tag{5.3}$$

For the vector $\mathcal{X}_2=\frac{\partial}{\partial y}$, we can obtain

$$\begin{aligned} W^1&=-u_y,\\ W^2&=-v_y, \end{aligned} \tag{5.4}$$

thus, the conservation law is obtained

$$\begin{aligned} C^x&=-u_y\left(b(t)\,u_y\Lambda_1+e(t)\,v\Lambda_2\right)-u_{yy}b(t)\,u\Lambda_1-v_yue(t)\,\Lambda_2-v_{xy}\left(c(t)\,\Lambda_1+d(t)\,\Lambda_2\right),\\ C^t&=-v_y\Lambda_2,\\ C^y&=b(t)\,\Lambda_1\left(u_yu_x+uu_{xy}\right)+e(t)\,\Lambda_2\left(u_xv+v_xu\right)+v_{xx}\left(c(t)\,\Lambda_1+d(t)\,\Lambda_2\right)+v_t\Lambda_2. \end{aligned} \tag{5.5}$$

For the vector $\mathcal{X}_3=\frac{\partial}{\partial x}$, we can obtain

$$\begin{aligned} W^1&=-u_x,\\ W^2&=-v_x, \end{aligned} \tag{5.6}$$

thus, the conservation law is obtained

$$\begin{aligned} C^x&=-a(t)\,u_{xxy}\Lambda_1+u_{yt}\Lambda_1+v_t\Lambda_2,\\ C^t&=-v_x\Lambda_2,\\ C^y&=-u_{xt}\Lambda_1+u_{xxx}a(t)\,\Lambda_1. \end{aligned} \tag{5.7}$$

For the vector $\mathcal{X}_4 = \dfrac{1}{b(t)}\dfrac{\partial}{\partial t}$, we can obtain

$$
\begin{aligned}
W^1 &= -\frac{u_t}{b(t)},\\
W^2 &= -\frac{v_t}{b(t)},
\end{aligned}
\tag{5.8}
$$

thus, the conservation law is obtained

$$
\begin{aligned}
C^x &= -u_t\left(u_y\Lambda_1 + \frac{e(t)\,v\Lambda_2}{b(t)}\right) - v_{xt}\left(\frac{c(t)\,\Lambda_1}{b(t)} + \frac{v_{xt}d(t)\,\Lambda_2}{b(t)}\right) - u_{yt}u\Lambda_1 - \frac{v_t u e(t)\,\Lambda_2}{b(t)},\\
C^t &= \Lambda_1\left(u_y u_x + u u_{xy} + \frac{c(t)\,v_{xx}}{b(t)} - \frac{a(t)\,u_{xxy}}{b(t)} + \frac{u_{yt}}{b(t)}\right) + \Lambda_2\left(\frac{u_x e(t)\,v}{b(t)} + \frac{u e(t)\,v_x}{b(t)} + \frac{v_{xx}d(t)}{b(t)}\right),\\
C^y &= \frac{u_{xxt}a(t)\,\Lambda_1}{b(t)} - \left(\frac{u_t}{b(t)}\right)_t.
\end{aligned}
\tag{5.9}
$$

The above results have been verified using Maple software we can get that $D_x(C^x) + D_t(C^t) + D_y(C^y) = 0$.

6. Conclusions

In this paper, the Lie symmetry method was used to reduce the vcBKK equation on the basis of the optimal system. The (2+1)-dimensional vcBKK equation was reduced to the (1+1)-dimensional PDEs, and then reduced to ODEs by the traveling wave transformation method. Then, the (G'/G)-expansion method was used to solve the corresponding exact solution. We obtain different kinds of solutions, including kink solutions and periodic solutions. Moreover, four conservation laws of the vcBKK equation were obtained at the end of this article.

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