Investigation for Existence, Controllability & Observability of a Fractional order Delay Dynamical System

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Abstract

Recently, several research articles have investigated the existence of solution of dynamical systems with fractional-order and as well as expounded controllability. Nevertheless, very little attention has been given to observability of such dynamical systems. In the present work, we explore the outcomes of controllability and observability regarding a differential system of fractional order with input delay. Laplace and inverse Laplace transforms along with the Mittage-Leffler matrix function are applied to the proposed dynamical system in Caputo's sense and obtain a general solution in the form of an integral equation. Then we set out conditions for the controllability of the underlying model, regarding the linear case. We then expound controllability conditions for the nonlinear case with the aid of fixed point theorem of Schaefer and the Arzola-Ascoli theorem. After converting the problem considered to a fixed point problem, we prove the observability of the linear case using the observability Grammian matrix. The necessary and sufficient conditions, for the nonlinear case, are investigated with the help of the Banach contraction mapping theorem. Finally, we add some examples to elaborate our work.

Keywords: Controllability, Observability, Grammian matrix, Fractional differential equations, Fixed point theorem

1 Introduction

In the recent past, fractional order differential equations have emerged as novel tools for modeling nonlinear phenomena occurring in different branches of science and engineering fields, such as viscoelasticity [1], electronic circuits [2], modified bituminous binders [3], epidemiology mechanism [4], and stochastic models of stock market swing [5]. Models described in this way are more passable and appropriate compared with integer order models for investigation of nonlinear phenomena. The theory of fractional order differential equations has been investigated by Rodino and Delbosco [6] as well as by Lakshmikantam *et al.*[7–10]. Problems regarding stability analysis of fractional order systems have been discussed in [11, 12]. A wide range of partial differential equations and integro-differential equations have been studied in the frame work of fractional calculus, using Banach spaces [13]. The remarkable contributions of D. Baleanu *et al.* [14–23] in different fields of science and engineering further enhanced the role of fractional calculus in modern research. Also, we refer the interested reader to study the work of C. Cattani *et al.* [24–31].

Among other qualitative behaviors of dynamical systems, both controllability and observability are the two key concepts that play a vital role in the analysis of control theory [32–34]. Controllability of linear

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finite dimensional systems and infinite dimensional systems have been discussed in [4, 35]. Controllability of nonlinear systems with input-delay has been studied in [1, 2], while the controllability of fractional order system in finite dimensional space has been investigated in [5, 32]. The time delay systems are the fundamental precipitating factors of the performance degradation and stability of fractional order systems [36–39]. It is therefore vital to investigate such effects on the dynamical behavior of the system. For detail study about such situations see the work of Yan [40], Muthukumar and Rajivganthi [41] and Valliammal *et al.*[42]. To establish connection between our proposed model and the existing literature regarding controllability of systems describing some real world phenomena, we put here a brief history of the recent work of some authors in the following lines.

In [43], the authors have investigated the controllability as well as the observability of two dimensional thermal flow in bulk storage facilities exploiting sensitivity fields. They have considered the convection diffusion reaction (CDR) equation, which describes the dynamics of energy and mass in physical systems like flow systems, heat exchangers, bulk food storage system and almost all kind of chemical reactions, see [44–47] for detail. Physical phenomena, such as transmission of momentum, energy, mass etc occur either inside the system or through the boundaries. The boundary controlled CDR systems investigated by the authors in [43] is described by the PDE given by,

$$\begin{aligned}
\left(\frac{\partial Y}{\partial t} + \nu.\nabla Y\right) &= c\Delta Y + r_Y \in (0,t] \times \Omega_{d1}, \\
Y &= u_{\text{Dirichlet}} \;\text{ on }\; (0,t] \times \partial\Omega_{d1}, \\
\frac{\partial Y}{\partial n} &= u_{\text{Neumann}} \;\text{ on }\; (0,t] \times \partial\Omega_{d2}.
\end{aligned}$$

The last two equations represent Dirichlet and the Neumann boundary conditions at the boundaries $\partial\Omega_{d1}$ and $\partial\Omega_{d2}$, respectively. The symbol, $Y \in \mathbb{R}^n$ shows the state vector, t is the time, ν represents the velocity vector, the diffusion coefficient is denoted by c and the first order reaction vector is symbolized by r_Y. Similarly the symbols $u_{dirichlet}$ and $u_{neumann}$ represent the respective input u and flux through the boundaries $\partial\Omega_{d1}$ and $\partial\Omega_{d2}$. The authors here considered ν constant in their system.

In [48] G. Joseph and C. Murthy presented some novel results regarding the controllability of LDS subject to sparsity constraints on the input. They described that unwinding the sparsity constraint, the classical results can be easily recovered for the unconstraint system. The discrete time LDS have proposed, whose state $y_k \in R$ at any time k is given by

$$y_k = Dy_{k-1} + Hh_k.$$

Here $D \in \mathbb{R}^{n\times n}$ represents the transfer matrix, $H \in \mathbb{R}^{n\times n}$ is the input matrix, $h_k \in \mathbb{R}^L$ isthe input vector being assumed to be sparse i.e $\|h_k\|_0 \leq s$, for all values of k and R_D, R_H represent the respective ranks of D and H. Their definition of s-sparse-controllability states that their underlying LDS is controllable if for any initial and final state x_0 and x_f, respectively there exists an input $\|h_k\|_0 \leq s$, which steers the system from the initial state x_0 to any final state $x_f = x_K$ in a finite duration of time K".

In [49], M. Nawaz *et al.* have recently formulated controllability conditions of a NLFS having time-delay in the state function described by two parameters, delayed Mittage-Leffler matrix functions utilizing Schauder's fixed point theorem. Their proposed fractional differential systems with state delay is defined as

$$\begin{aligned}
{}^cD^{\delta}_{0+}z(t) &= Az(t-\nu) + Bu(t), \qquad t \in I = [0,b], \quad \nu > 0, \quad b \geq 0, \\
z(t) &= \phi(t), \qquad\qquad -\nu < t \leq 0.
\end{aligned}$$

The conforming nonlinear system has the form

$$\begin{cases} {}^cD^{\delta}_{0+}z(t) = Az(t-\nu) + Bu(t) + f(t, z(t-\nu), u(t)), z(t) \in R^n, t \in I, \\ \nu > 0, b \geq 0, z(t) = \phi(t), -\nu < t \leq 0, \end{cases}$$

where $z : [-\nu, b] \to \mathbb{R}^n$ is continuously differentiable on the interval $[0, b]$ such that $b > (n-1)\nu, 0 < \delta \leq 1$. Matrices A and B have respectively orders $n \times n$ and $n \times m$ while $\nu > 0$ denotes the time-delay. The state vector is represented by the symbol $z(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ is the control function. Similarly the initial state function is symbolized as $\phi(t)$ and $f : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and non-linear.

In [50] Y. Yi *et al.* investigated the controllability concerning non-linear fractional integro-differential systems with input-delay exploiting the so-called Schauder's fixed point theorem. Their proposed fractional order integro-differential inclusion is given by

$$\begin{cases} {}^cD^j z(t) = Lz(t) + Mu(t) + Nu(t-\mu) + f(t, z(t)) \\ \quad +h(t, z(t), \int_0^t g(t, s, z(s))ds), t \in I = [0, c], \\ \quad z(0) = z_0, \quad u(t) = \phi(t), -\mu \le t \le 0. \end{cases}$$

In the above system $0 < j \le 1$, $L \in \mathbb{R}^{n\times n}$, M and $N \in \mathbb{R}^{n\times m}$ and $f : I \times R^n \to \mathbb{R}^n$, $h : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : I \times I \times \mathbb{R}^n \to \mathbb{R}^n$ are all continuous functions.

In [51], Balachandran *et al.* reported the observability of a linear and nonlinear system of fractional order $0 < \nu < 1$, using the Laplace transform, the Mittag-Leffler matrix function and the Banach contraction mapping theorem. Their proposed dynamical fractional order system is given by

$${}^cD^\nu y(t) = My(t) + g(t.y(t)), t \in J = [0, T],$$

where $M \in \mathbb{R}^{n\times n}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function with linear observation

$$z(t) = Hy(t).$$

Where H is an appropriate order matrix.

In [52], D. Xu, Y. Li and W. Zhou established sufficient and necessary conditions for controllability and observability of a linear systems with non-integer distinct orders. Their proposed dynamical system is given by

$$\begin{pmatrix} {}^cD^\nu y_1(t) \\ {}^cD^\mu y_1(t) \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t),$$

where ${}^cD^\nu y_1(t)$ and ${}^cD^\mu y_1(t)$ are the Caputo derivatives of orders $0 < \nu < 1$ and $0 < \mu < 1$, respectively. Here, $y_1 \in \mathbb{R}^{n_1}$ and $y_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$ are the state vectors, $M_{ij}, B_{ij}, i, j = 1, 2$ are constant matrices and $u \in \mathbb{R}$ is the input vectors. For some recent results on controllability and observability we also refer the interested readers to see [53–57].

Inspired by the above work, in this paper we investigate the controllability and observability of fractional order systems having input-delay using Schaefer's Fixed point theorem, the Arzela-Ascoli theorem and the Banach contracting mapping theorem. We add some examples to support our work at the end of the manuscript.

2 Preliminaries

In this part of our manuscript we include some important definitions, lemmas, notations and preliminary facts regarding fractional order derivatives, fractional order integrals, the Mittag-Leffler matrix function and its derivative and a class of linear fractional order system having input-delay.

Definition 1. *[8] "The Riemann-Liouville fractional derivative of a suitable function $f(t)$ of order $\alpha > 0$ with $j - 1 < \alpha \le j, j \in \mathcal{N}$ is define as*

$$D^\alpha f(t) = \frac{1}{\Gamma(j-\alpha)} \frac{d^j}{dt^j} \int_0^t (t-s)^{j-1-\alpha} f(s)\, ds.$$

Here, $j = 1 + [\alpha]$, $[\alpha]$ is the integer part of α".

Definition 2. *[8] "The Caputo fractional derivative of a suitable function $f(t)$ of order $\alpha > 0$, $j-1 < \alpha \leq j, j \in \mathcal{N}$ is define as*

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(j-\alpha)}\int_0^t (t-s)^{j-\alpha-1}f^{(j)}(s)ds, j-1 < \alpha \leq j.$$

Here $j = 1 + [\alpha]$, $[\alpha]$ is the integer part of α. If $0 < \alpha \leq 1$, then

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)\,ds".$$

Definition 3. *[8] "The Riemann-Liouville fractional order integral of a function $f(t)$ of order $\alpha > 0$ is defined as*

$$I_0^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds".$$

Definition 4. *[8] "The Mittag-Leffler matrix function for two parameters is expressed as*

$$E_{\alpha,\beta}(At^{\alpha}) = \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+\beta)}, \ \alpha, \beta > 0,$$

Here, A is an arbitrary square matrix. The monoparameteric Mittag-Leffler function can be achieved by putting $\beta = 1$ in the last equation, i.e

$$E_{\alpha,1}(At^{\alpha}) = E_{\alpha}(At^{\alpha}) = \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

The Mittag-Leffler function satisfies the property: $D_t^{\alpha}E_{\alpha}(At^{\alpha}) = AE_{\alpha}(At^{\alpha})$".

Definition 5. *[8] "The jth order derivative of the two parameter Mittag-Leffler function can be defined by following expression*

$$\frac{d^j}{dt^j}(t^{\alpha-1}E_{\alpha,\beta}(At^{\alpha})) = t^{\alpha-j-1}E_{\alpha,\beta-j}(At^{\alpha}), j \in N".$$

Definition 6. *[58] "A mapping $T : X \to Y$ from one Banach space to another is said to be continuous if for $\epsilon > 0$ and each $x \in X$ one can find a small positive δ in such a way that for each $y \in X$*

$$\|T(y) - T(x)\|_Y < \epsilon, \ \text{whenever } \|y - x\|_X < \delta \Rightarrow .$$

The mapping T is said to be uniformly continuous on $A \subset X$ provided for every $\epsilon > 0$ there corresponds a small positive δ such that for all $x, y \in A$

$$\|T(y) - T(x)\|_Y < \epsilon \text{ whenever } \|y - x\|_X < \delta.$$

Let us suppose that $T_{\lambda} : X \to Y, \lambda \in \Lambda$ is a (finite or infinite) class of mappings from one Banach space to another. These mappings are said to be equicontinuous on the set A, where $A$is a subset of X, if for every $\lambda > 0$ one can associate a positive δ however small, such that for any $\lambda \in A$ and every two elements $x, y \in A$ the following holds

$$\|y - x\|_X < \delta \text{ implies } \|T(y) - T(x)\|_Y < \epsilon".$$

Definition 7. *[59] "Assume X and Y are two Banach spaces and $D \subset X$. Further let $T : D \to Y$. We say that the operator T is completely continuous if T is continuous and associates every bounded subset of the set D to a relatively compact subset of the space Y".*

Theorem 1. *(Schaefer's fixed point theorem)[58] "Condider X be a Banach space and $T : X \to X$ be a continuous and compact mapping. Further assume the set*

$$\bigcup_{0\le\xi\le1}\{x\in X : x=\xi T(x)\}.$$

is bounded for some $\lambda \in [0,1]\}$. Then the operator T has a fixed point".

Theorem 2. *[58](Arzela-Ascoli theorem) "Consider $K=[a,b]$ be a compact set in $\mathbb{R}^n$, $n \ge 1$. A set $S \subset C(K)$ is said to be relatively compact in $C(K)$ if and only if S contains uniformly bounded and equicontinuous functions on K.*
Uniformly bounded means, one can find $M > 0$ such that

$$\|f\| = \sup_{x\in K}|f(x)| \le M, \forall f \in S.$$

Equicontinuous means, for every $\epsilon > 0$, one can find $\delta > 0$ such that $\forall\ x, y \in K$ and $\forall\ f \in S$

$$|f(x) - f(y) < \epsilon \text{ whenever } |x-y| < \delta.$$

The Arzela-Ascoli theorem can be extended to the whole of $\mathbb{R}^n$, if it is assumed that the functions uniformly approaches zero at infinity i.e. as $|x| \to \infty$".

Definition 8. *[60] "A system is deemed to be controllable on $[t_0, t_f]$, if for an initial state $x_0 \in \mathbb{R}^n$ at $t = t_0$, and final state $x_f \in \mathbb{R}^n$ at $t = t_f$, there corresponds an input control signal $u(t) : [0,T] \to \mathbb{R}^m$, such that the corresponding solution of the system satisfies $x(t_0) = x_0$ and $x(t_f) = x_f, t \in [t_0, t_f]$".*

3 Main results

Consider the fractional order system given below on a bounded domain,

$$\begin{cases} {}^cD^\nu y(t) = Ky(t) + Lu(t) + Mu(t-q) + f(t, y(t), {}^cD^{\nu-1}y(t)), t \in I = [0,d], \\ y(0) = y_0,\ y'(0) = 0, \\ u(t) = \phi(t),\ \ -q \le t \le 0, \end{cases} \tag{1}$$

where $1 < \nu \le 2$; K is $n \times n$ matrix; L and M are $n \times m$ matrices and f is a nonlinear continuous function. Utilizing the Laplace transform and its inverse along with the Mittag-Leffler function, the general solution of the fractional order system (1) can be expressed as

$$y(t) = E_\nu(Kt^\nu)y_0 + \int_0^t (t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^{\nu-1})(Lu(t) + Mu(t-q) + f(t, y(t), {}^cD^{\nu-1}y(t)))ds. \tag{2}$$

Lemma 1. *[61] For the case $0 \le t \le q$, the solution (2) can be expressed as*

$$\begin{aligned} y(t) &= E_\nu(Kt^\nu)y_0 + \int_0^t (t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)Lu(s)ds \\ &+ \int_{-q}^{t-q} (t-q-s)^{\nu-1}E_{\nu,\nu}(K(t-q-s)^\nu)M\phi(s)ds \\ &+ \int_0^t E_{\nu,\nu}(K(t-q-s)^\nu) \times f(s, y(s), {}^cD^{\nu-1}y(s))ds. \end{aligned} \tag{3}$$

While for the case $t > q$, this solution can be expressed as

$$
\begin{aligned}
y(t) &= E_\nu(Kt^\nu)y_0 + \int_{-q}^{0} (t-q-s)^{\nu-1} E_{\nu,\nu}(K(t-q-s)^\nu) M\phi(s)ds \\
&+ \int_0^{t-q} [(t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu)L + (t-q-s)^{\nu-1} E_{\nu,\nu}(K(t-q-s)^\nu)M]u(s)ds \\
&+ \int_{t-q}^{t} (t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu)Lu(s)ds \\
&+ \int_0^t [(t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu) \times f(s, y(s), {}^cD^{\nu-1}y(s))]ds. \qquad (4)
\end{aligned}
$$

Lemma 2. *[61] The fractional linear system given by*

$$
\begin{cases}
{}_0^cD_t^\nu y(t) = Ky(t) + Lu(t) + Mu(t), \quad t \in I = [0, d], \\
y(0) = y_0, \qquad y'(0) = 0, \\
u(t) = \psi(t), \qquad -q \le t \le 0,
\end{cases} \qquad (5)
$$

is controllable on I, if and only if the controllability Grammian matrices $W(t)$ in each of the following cases are invertible.

Case (1): When $0 \le t \le q$

$$
\begin{aligned}
W(t) &= \int_0^d (d-s)^{\nu-1} E_{\nu,\nu}(K(d-s)^\nu) LL^T ((d-s)^{\nu-1} E_{\nu,\nu}(K(d-s)^\nu))^T ds \\
&+ \int_{-q}^{d-q} (d-q-s)^{\nu-1} E_{\nu,\nu}(K(d-q-s)^\nu) MM^T \\
&\times ((d-q-s)^{\nu-1} E_{\nu,\nu}(K(d-q-s)^\nu))^T ds. \qquad (6)
\end{aligned}
$$

$$(7)$$

Case (2): When $t > q$

$$
\begin{aligned}
W(t) &= \int_0^{d-q} [(d-s)^{\nu-1} E_{\nu,\nu}(K(d-s)^\nu)L + (d-q-s)^{\nu-1} E_{\nu,\nu}(K(d-q-s)^\nu)M] \\
&\times [(d-s)^{\nu-1} E_{\nu,\nu}(K(d-s)^\nu)L + (d-q-s)^{\nu-1} E_{\nu,\nu}(K(d-q-s)^\nu)M]^T ds \\
&+ \int_{d-q}^{d} (d-s)^{\nu-1} E_{\nu,\nu}(K(d-s)^\nu) LL^T ((d-s)^{\nu-1} E_{\nu,\nu}(K(d-s)^\nu))^T ds. \qquad (8)
\end{aligned}
$$

$$(9)$$

Lemma 3. *[61] "The linear fractional system (5) is said to be controllable on I, iff*

$$rank \begin{bmatrix} L & KL & K^2L & \ldots & K^{n-1}L & M & KM & K^2M & \ldots & K^{n-1}M \end{bmatrix} = n."$$

Lemma*. *(Schaefer's theorem): Let X be a Banach space, $f : X \to X$ be continuous and compact. Moreover assume the set $S = \{x \in X : x = \lambda f(x)\}, \lambda \in [0, 1]\}$, has a solution for $\lambda = 1$ and all other solutions for $0 < \lambda < 1$ are unbounded.*

To investigating the controllability of the system (1), we make the underlying hypothesis.
H_1. The non-linear function $f : I \times R^n \times R^n \to R^n$ is measurable and continuous, a positive constant p exists such that

$$\|f(s, y(s), {}^cD^{\nu-1}y(s))\| \le p, \forall\ t \in I. \qquad (10)$$

H_2. For brevity we assume the following:

$$\begin{cases} \psi_0(t) = E_\nu(Kt^\nu), t \in I, & sup\|\psi_0\| = k_0; \\ \psi_1(t,s) = (t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu), t \in I, & sup\|\psi_1\| = k_1; \\ \psi_2(t,q,s) = (t-q-s)^{\nu-1}E_{\nu,\nu}(K(t-q-s)^\nu), t \in I, & sup\|\psi_2\| = k_2; \\ \psi_3(t) = KE_{\nu,1-j}(Kt^\nu), t \in I, & sup\|\psi_3\| = k_3; \\ \psi_4(t,s) = (t-s)^{\nu-j-1}E_{\nu,\nu-j}(K(t-s)^{\nu-j}), t \in I & sup\|\psi_4\| = k_4; \\ \psi_5(t,q,s) = (t-q-s)^{\nu-j-1}E_{\nu,\nu-j}(K(t-q-s)^{\nu-j}), t \in I, & sup\|\psi_5\| = k_5; \\ \psi_6(t,s) = (t-s)^{j-\nu-1}, t \in I, & sup\|\psi_6\| = k_6; \\ \psi_7(s,y) = f(s,y(s),{}^cD^{\nu-1}y(s)). \end{cases}$$

Theorem 3. *If the linear system in (1) is assumed controllable on the interval I and the hypothesis H_1, H_2 hold, then the nonlinear system of fractional order (1) is also controllable on I.*

Proof. Case I. When $t > q$

To prove the theorem, we define the Banach space $Y = \{y : y^{(q)}, {}^cD^\nu(y) \in (I, R^n)\}$, with norm $\|y\| = \max\{\|y(t)\|, \|{}^cD^\nu y(t)\|, \|u\|\}$. Further utilizing the hypothesis H_1, H_2, the input $u(t)$ of the system (1) for an arbitrary solution $y(.)$ can be defined as

$$u(t) = \begin{cases} 0, & -q \le t \le 0; \\ (\psi_1(d,t)L + \psi_2(d,q,t)M)^T W^{-1}\Phi, & 0 \le t \le d-q; \\ (\psi_1(d,t)L)^T W^{-1}\Phi, & d-q \le t \le d; \end{cases} \tag{11}$$

where

$$\Phi = y_1 - \psi_0(t)y_0 - \int_0^d \psi_1(t,s)\psi_7(s,y)ds.$$

and

$$\begin{aligned} W(t) &= \int_0^{d-q} (\psi_1(d,s)L + \psi_2(d,q,s)M) \times (\psi_1(d,s)L + \psi_2(d,q,s)M)^T ds \\ &+ \int_{d-q}^d \psi_1(d,s)L(L\psi_1(d,s))^T ds + \int_0^d \psi_1(d,s)\psi_7(s,y)ds. \end{aligned}$$

We define the nonlinear operator $T : Y \to Y$, given by

$$\begin{aligned} Ty(t) &= \psi_0(t)y_0 + \int_{-q}^0 \psi_2(t,q,s)M\phi(s)ds + \int_0^{t-q} [\psi_1(t,s)L + \psi_2(t,q,s)M]u(s)ds \\ &+ \int_{t-q}^t \psi_1(t,s)Lu(s)ds + \int_0^t [\psi_1(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{12}$$

The operator defined above possesses a fixed point and this fixed point comprises a particular solution of (1). Inserting (11) in (12) we obtain

$$\begin{aligned} (Ty)(t) &= \psi_0(t)y_0 + \int_0^{t-q} [\psi_1(t,s)L + \psi_2(t,q,s)M] \\ &\times [\psi_1(d,s)L + \psi_2(d,q,s)M]^T W^{-1}\Phi ds \\ &+ \int_{t-q}^t \psi_1(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds \\ &+ \int_0^t [\psi_1(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{13}$$

Clearly $Ty(d) = y_1$. Further it means that if the nonlinear operator has a fixed point, then there exists an input $u(t)$ that steers the system from the initial state y_0 to the final state y_1 in time d.

Next we show that the operator T satisfies the Schaefer's fixed point theorem. Our proof consists of three steps:

Step I. In first step we show boundedness of the set $\xi(T) = \{y \in Y : y = \eta Ty, \eta \in [0,1]\}$, in I. For an arbitrary $y \in \xi(T)$ and $0 < \eta < 1$, we have

$$\begin{aligned} y(t) &= \eta\psi_0(t)y_0 + \eta\int_0^{t-q}[\psi_1(t,s)L + \psi_2(t,q,s)M] \\ &\times [\psi_1(d,s)L + \psi_2(d,q,s)M]^T W^{-1}\Phi ds \\ &+ \eta\int_{t-q}^{t}\psi_1(t,s)K(\psi_1(d,s)L)^T W^{-1}\Phi ds \\ &+ \eta\int_0^t[\psi_1(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{14}$$

Then utilizing hypothesis H_1 and H_2 we have

$$\begin{aligned} \|\Phi\| &\leq \|y_1\| + \|\psi_0(t)\|\|y_0\| + \int_0^d(\|\psi_1(t,s)\|\|\psi_7\|)ds, \\ &\leq \|y_1\| + k_0\|x_0\| + dk_1p. \end{aligned} \tag{15}$$

and

$$\|u(t)\| = \begin{cases} 0, & -q \leq t \leq 0; \\ [k_1\|L\| + k_2\|M\|]^T\|W^{-1}\|\|\Phi\|, & 0 \leq t \leq d-q; \\ (k_1\|L\|)^T\|W^{-1}\|\|\Phi\|, & d-q \leq t \leq d. \end{cases} \tag{16}$$

In view of (15) and (16), (14) will give

$$\begin{aligned} \|y(t)\| &\leq k_0\|y_0\| + \int_0^{t-q}[k_1\|L\| + k_2\|M\|][k_1\|L\| + k_2\|M\|]^T\|W^{-1}\|\|\Phi\|ds \\ &+ \int_{t-q}^{t}k_1\|L\|(k_1\|L\|)^T\|W^{-1}\|\|\Phi\|ds + \int_0^t[k_1p]ds, \\ &\leq k_0\|y_0\| + d\|W^{-1}\|[(k_1\|L\| + k_2\|M\|)(k_1\|L\| + k_2\|M\|)^T + K_1^2\|L\|\|L^T\|] \\ &\times (\|y_1\| + k_1\|y_0\| + dk_1p) + dk_1p = \gamma_1. \end{aligned} \tag{17}$$

Further, by Definition (5), we can obtain

$$\begin{aligned} y^{(j)}(t) &= \eta\psi_3(t)y_0 + \eta\int_0^{t-q}[\psi_4(t,s)L + \psi_5(t,q,s)M] \\ &\times [\psi_4(d,s)L + \psi_5(d,q,s)M]^T W^{-1}\Phi ds + \eta\int_{t-q}^{t}\psi_4(t,s)L(\psi_4(d,s)L)^T W^{-1}\Phi ds \\ &+ \eta\int_0^t[\psi_4(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{18}$$

Which further gives

$$\begin{aligned} \|y^{(j)}(t)\| &\leq k_3\|y_0\| + d[(k_4\|L\| + k_5\|M\|)(k_4\|L\| + k_5\|M\|)^T + k_4^2\|L\|\|L\|] \\ &\times \|W^{-1}\|(\|y_1\| + k_1\|y_0\| + dk_1p) + dk_4p = \gamma_2. \end{aligned} \tag{19}$$

Utilizing definition (2) we have

$$\|{}^cD^\nu y(t)\| \leq \|\frac{1}{\Gamma(j-\nu)}\|\|\|\int_0^t (k_6\gamma_2)ds\|. \tag{20}$$

Hence ${}^cD^\nu y(t)$ is bounded. It means that ξT is bounded as well because $\|y\| = max[\|y\|, \|{}^cD^\nu y\|, \|u\|]$.
Step II. In this step we prove that the operator T is completely continuous. To do this we assume that $B_s = \{y \in Y; \|y\| \leq s\}$, which is mapped into equicontinuous family by T. Then for any $y \in B_s$ and $t_1, t_2 \in I$ with $0 < t_1 < t_2 < d$ one gets

$$\begin{aligned}
\|Ty(t_2) - Ty(t_1)\| &\leq \|\psi_0(t_2) - \psi_0(t_1)\|\|y_0\| \\
&+ \|\int_{t_{1-q}}^{t_2-q} [(\psi_1(t_2,s)L + \psi_2(t_2,q,s)M)] \\
&\times (\psi_1(d,s)L + \psi_2(d,q,s)M)^T W^{-1}\Phi ds\| \\
&+ \|\int_0^{t_1-q} [(\psi_1(t_2,s)L + \psi_2(t_2,q,s)M) - (\psi_1(t_1,s)L + \psi_2(t_1,q,s)M)] \\
&\times (\psi_1(d,s)L + \psi_2(d,q,s)M)^T W^{-1}\Phi ds\| \\
&+ \|\int_{t_1-q}^{t_1} [\psi_1(t_2,s)L - \psi_1(t_1,s)L](\psi_1(d,s)L)^T W^{-1}\Phi ds\| \\
&+ \|\int_0^{t_1} [\psi_1(t_2,s) - \psi_1(t_1,s)]\psi_7(s,y)ds\| \\
&+ \|\int_{t_1}^{t_2} \psi_1(t_2,s)\psi_7(s,y)ds\| \\
&+ \|\int_{t_1}^{t_2} \psi_1(t_2,s)L(\psi_1(d,s)L)^T W^{-1}\Phi\|.
\end{aligned} \tag{21}$$

In view of (21), (11) can be written as

$$\begin{gathered}
\|Tu(t_2) - Tu(t_1)\| \leq \\
\begin{cases}
0, & -q \leq t \leq 0; \\
\|[(\psi_1(d,t_2)L + \psi_2(d,q,t_2)M) - & \\
(\psi_1(d,t_1)L + \psi_2(d,q,t_1)M)]^T\|\|W^{-1}\|\|\Phi\|, & 0 \leq t \leq d-q; \\
\|[(\psi_1(d,t_2)L) - (\psi_1(d,t_1)L)]^T\|\|W^{-1}\|\|\Phi\|, & d-q \leq t \leq d.
\end{cases}
\end{gathered} \tag{22}$$

This further implies that

$$\begin{aligned}
&\|{}^cD^\nu Ty(t_2) - {}^cD^\nu Ty(t_1)\| \leq \\
&\|\frac{1}{\Gamma(j-\nu)}\|\|\|\int_{t_1}^{t_2} (\psi_6(t_2,s))(Ty)^{(j)}ds\| + \\
&\|\frac{1}{\Gamma(j-\nu)}\|\|\|\int_0^{t_1} (\psi_6(t_2,s)(Ty)^{(j)} - \psi_6(t_1,s)(Ty)^{(j)})ds\|.
\end{aligned} \tag{23}$$

Evidently,

$$\begin{aligned}
&\lim_{t_2 \to t_1} \|(Ty)(t_2) - (Ty)(t_1)\| \to 0, \\
&\lim_{t_2 \to t_1} \|(Tx)^{(j)}(t_2) - (Tx)^{(j)}(t_1))\| \to 0 \\
&\lim_{t_2 \to t_1} \|{}^cD^\nu(Ty)(t_2) - ({}^cD^\nu Ty)(t_1)\| \to 0.
\end{aligned}$$

Hence, the equicontinuous family of functions, $\{(Ty) : y \in B_s\}$ is uniformly bounded. Next we show that the operator T is compact. For any $y \in B_s$ and a real number ϵ such that $0 < \epsilon < t$ where $t \in [0, d]$ we define

$$\begin{aligned}
(T_\epsilon y)(t) &= \psi_0(t)y_0 + \int_0^{t-\epsilon-q} [\psi_1(t,s)L + \psi_2(t,q,s)M] \\
&\times (\psi_1(d,s)L + \psi_2(d,q,s)M)^T W^{-1}\Phi ds \\
&+ \int_{t-q}^{t-\epsilon} \psi_1(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds \\
&+ \int_0^{t-\epsilon} [\psi_1(t,s)\psi_7(s,y)]ds. \qquad (24)
\end{aligned}$$

as above we obtain that $\{(T_\epsilon y) : y \in B_s\}$ is an equicontinuous family of functions that fulfill the uniform bounded condition. Therefore, one has

$$\begin{aligned}
\|(Ty)(t) - (T_\epsilon y)(t)\| &\leq \|\int_{t-\epsilon-q}^{t-q} [\psi_1(t,s)L + \psi_2(t,q,s)M] \\
&\times (\psi_1(d,s)L + \psi_2(d,q,s)M)^T W^{-1}\Phi ds\| \\
&+ \|\int_{t-\epsilon}^{t} \psi_1(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds\| \\
&+ \|\int_{t-\epsilon}^{t} [\psi_1(t,s)\psi_7(s,y)]ds\|, \\
&\leq \epsilon\|W^{-1}\|\|\Phi\|[(k_1\|L\| + k_2\|M\|)(k_1\|L\| + k_2\|M\|)^T + k_1^2\|L\|\|L^T\|] \\
&+ \epsilon k_1 p. \qquad (25)
\end{aligned}$$

Utilizing the above we obtain

$$\begin{aligned}
\|(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)\| &\leq \|\int_{t-\epsilon-q}^{t-q} [\psi_4(t,s)L + \psi_5(t,q,s)M] \\
&\times (\psi_1(d,s)L + \psi_2(d,q,s)M)^T W^{-1}\Phi ds\| \\
&+ \|\int_{t-\epsilon}^{t} \psi_4(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds\| \\
&+ \|\int_{t-\epsilon}^{t} [\psi_4(t,s)\psi_7(s,y)]ds\|, \\
&\leq \epsilon\|W^{-1}\|\|\Phi\|[(k_4\|L\| + k_5\|M\|)(k_1\|L\| + k_2\|M\|)^T \\
&+ k_1k_4\|L\|\|L^T\|] + \epsilon k_4 p. \qquad (26)
\end{aligned}$$

Applying definition of Caputo derivative we have

$$\begin{aligned}
&\|{}^cD^\nu((Ty)(t_2) - {}^cD^\nu(T_\epsilon y)(t_1))\| \leq \\
&\|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^t \psi_6(t,s)[(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}](t)ds\|. \qquad (27)
\end{aligned}$$

Distinctly,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \|(Ty)(t) - (T_\epsilon y)(t)\| \to 0, \\
&\lim_{\epsilon\to 0} \|(Ty)^{(j)}(t) - (T_\epsilon y)^{(j)}(t)\| \to 0, \\
&\lim_{\epsilon\to 0} \|{}^cD^\nu(Ty)(t) - {}^cD^\nu(T_\epsilon y)(t)\| \to 0.
\end{aligned}$$

Hence, by the Arzela-Ascoli theorem $\{(Ty)(t) : y \in B_s\}$ is compact in Y.
Step III. The last step to show that T is continuous. We make two more hypothesis:
(H_3) Let $Y = \{y_1, y_2, ..., y_n\}, \lim_{n\to\infty} \|y_n - y(t)\| = 0$.
(H_4) Let $z = max\{\|y_n\|, \|u_n\|, \|^c D^\nu y_n\|\}$, z is a positive constant. Utilizing the above hypothesis we obtain

$$\begin{aligned} f(t, y_n(t), {}^c D^{(\nu-1)} y_n(t)) &\leq f(t, y(t), {}^c D^{(\nu-1)} y(t)), i.e \\ \psi_7(s, y_n) &\leq \psi_7(s, y). \end{aligned}$$

Now by Fatou-Lebesgue theoerm

$$\begin{aligned} \|(Ty_n)(t) - (Ty)(t)\| &\leq \| \int_0^t k_1[(k_1\|L\| + k_2\|M\|)(k_1\|L\| + k_2\|M\|)^T + k_1^2\|L\|\|L^T\|] \\ &\times \|W^{-1}\|\| \int_0^s ((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta))))d\vartheta\|ds\| \\ &+ k_1\| \int_0^t ((\psi_7(s, y_n)) - (\psi_7(s, y)))ds\|. \end{aligned} \tag{28}$$

Utilizing similar approach as above we also have

$$\begin{aligned} \|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| &\leq \| \int_0^t k_4[(k_4\|L\| + k_5\|M\|)(k_4\|L\| + k_5\|M\|)^T + k_1k_4\|L\|\|L^T\|] \\ &\times \|W^{-1}\|\| \int_0^s ((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta))))d\vartheta\|ds\| \\ &+ k_4\| \int_0^t ((\psi_7(s, y_n)) - (\psi_7(s, y)))ds\|. \end{aligned} \tag{29}$$

Making use of definition (2), one obtain

$$\begin{aligned} \|^c D^\nu (Ty_n)(t) - {}^c D^\nu (Ty(t))\| &\leq \\ \|\frac{1}{\Gamma(j-\nu)}\|\| \int_0^t \psi_6(t, s)[(Ty_n)^{(j)}(t) - (Ty)^{(j)}](t)ds\|. \end{aligned} \tag{30}$$

Clearly

$$\begin{aligned} &\lim_{n\to\infty} \|(Ty_n)(t) - (Ty)(t)\| = 0, \\ &\lim_{n\to 0} \|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| = 0, \\ &\lim_{n\to\infty} \|^c D^\nu (Ty_n)(t) - {}^c D^\nu (Ty)(t)\| = 0. \end{aligned}$$

This clearly indicates the continuity of T. Hence following the Arzela-Ascoli and Schaefer's fixed point theorems, one may easily deduce that the operator T is continuous and possesses a fixed point Y in B_s. Further this fixed point Y is the solution of the system (1). It is therefore concluded that (1) is controllable on I for the case $t > q$.

Case II. When $0 \leq t \leq q$:
We define the Banach space $Y = \{y : y^{(j)}, {}^c D^{(\nu-1)} y, t \in (I, R^n)\}$, with norm $\|y\| = max\{\|y\|, \|^c D^\nu (y)\|, \|u\|\}$. Then utilizing an arbitrary solution $y(.)$ of (1) and the hypothesis H_1, H_2 the input signal $u(t)$ can be obtained as

$$u(t) = \begin{cases} (\psi_2(d, q, t)M)^T W^{-1}\Phi, & -q \leq t \leq d - q; \\ 0, & d - q \leq t \leq 0; \\ (\psi_1(d, t)L)^T W^{-1}\Phi, & 0 < t \leq d; \end{cases} \tag{31}$$

where

$$\Phi = y_1 - \psi_0(t)y_0 - \int_0^d \psi_1(d,s)\psi_7(s)ds.$$

Define the nonlinear operator $T : Y \to Y$ by

$$\begin{aligned} Ty(t) &= \psi_0(t)y_0 + \int_0^t \psi_1(t,s)Lu(s)ds + \int_{-q}^{t-q} \psi_2(t,q,s)M\phi(s)ds \\ &+ \int_0^t [\psi_1(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{32}$$

The above defined operator has a fixed point that is a particular solution of (1). Plugging (31) in (32) we obtain

$$\begin{aligned} Ty(t) &= \psi_0(t)y_0 + \int_0^t \psi_1(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds \\ &+ \int_{-q}^{t-q} \psi_2(t,q,s)M(\psi_2(d,q,s)M)^T W^{-1}\Phi ds + \int_0^t [\psi_1(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{33}$$

Clearly $Ty(d) = y_1$. Further it means that if the nonlinear operator has a fixed point, then there exists an input $u(t)$ that steers the system from the initial state y_0 to the final state y_1 in time d.

Next we show that the operator T satisfies the Schaefer's fixed point theorem. Our proof consists of three steps

Step I. In first step we show boundedness of the set $\xi(T) = \{y \in Y : y = \eta Ty, \eta \in [0,1]\}$, on I.

For an arbitrary $y \in \xi(T)$ and $0 < \eta < 1$ we have

$$\begin{aligned} y(t) &= \eta\psi_0(t)y_0 + \eta\int_0^t \psi_1(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds \\ &+ \eta\int_{-q}^{t-q} \psi_2(t,q,s)M(d,q,s)M)^T W^{-1}\Phi ds + \eta\int_0^t [\psi_1(t,s)\psi_7(s,y)]ds. \end{aligned} \tag{34}$$

Then utilizing hypothesis H_1 and H_2 we have

$$\begin{aligned} \|\Phi\| &\leq \|y_1\| + \|\psi_0(t)\|\|y_0\| + \int_0^d (\|\psi_1(t,s)\|\|\psi_7(s)\|)ds \\ &\leq \|y_1\| + k_1\|y_0\| + dk_1 p, \end{aligned} \tag{35}$$

and

$$\|u(t)\| = \begin{cases} k_2 M^T\|W^{-1}\|\|\Phi\|, & -q \leq t \leq d-q; \\ 0, & d-q \leq t \leq 0; \\ k_1 L^T\|W^{-1}\|\|\Phi\|, & 0 < t \leq d. \end{cases} \tag{36}$$

In view of (35) and (36), (34) will give

$$\begin{aligned} \|y(t)\| &\leq k_0\|y_0\| + \int_0^t k_1\|L\|\|(k_1L)^T\|\|W^{-1}\|\|\Phi\|ds \\ &+ \int_{-q}^{t-q} k_2\|M\|\|(k_2M)^T\|\|W^{-1}\|\Phi\|ds + \int_0^t [k_1 p]ds, \\ &\leq k_0\|y_0\| + [k_1^2\|L\|\|L^T\| + k_2^2\|M\|\|M^T] \\ &\times d\|W^{-1}\|[\|y_1\| + k_1\|y_0\| + dk_1 p] + dk_1 p = \gamma_3. \end{aligned} \tag{37}$$

Further, by Definition (5)

$$\begin{aligned} y^{(j)}(t) &= \eta\psi_3(t)y_0 + \eta\int_0^t \psi_4(t,s)L(\psi_1(d,s)L)^T W^{-1}\Phi ds \\ &+ \eta\int_{-q}^{t-q} \psi_5(t,q,s)M(d,q,s)M)^T W^{-1}\Phi ds \\ &+ \eta\int_0^t [\psi_4(t,s)\psi_7(s,y)]ds, \end{aligned} \tag{38}$$

which gives

$$\begin{aligned} \|y^{(j)}(t)\| &\leq k_3\|y_0\| + [dk_1k_4\|L\|\|L^T\| + dk_2k_5\|M\|\|M^T\|] \\ &\times \|W^{-1}\|(\|y_1\| + k_1\|y_0\| + dk_1p) + dk_4p = \gamma_4. \end{aligned} \tag{39}$$

Utilizing definition (2) we have

$$\|^cD^\nu y(t)\| \leq \|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^t (k_6\gamma_4)ds\|. \tag{40}$$

Hence, ${}^cD^\nu y(t)$ is bounded. It means that ξT is bounded as well because $\|y\| = max[\|y\|, \|^cD^\nu y\|, \|u\|]$.

Step II. Here we show that T is completely continuous operator. Suppose $B_s = \{y \in Y; \|y\| \leq s\}$, which is mapped into equicontinuous family by T. Then for any $y \in B_s$ and $t_1, t_2 \in I$ with $0 < t_1 < t_2 < d$ we show that TB_s is uniformly bounded

$$\begin{aligned} \|Ty(t_2) - Ty(t_1)\| &\leq \|\psi_0(t_2) - \psi_0(t_1)\|\|y_0\| \\ &+ \int_0^{t_1} [\psi_1(t_2,s)L - \psi_1(t_1,s)L] \times (\psi_1(d,s)L)^T W^{-1}\Phi \\ &+ \int_{-q}^{t_1-q} [\psi_2(t_2,q,s)M - \psi_2(t_1,q,s)M](\psi_2(d,q,s)M)^T W^{-1}\Phi \\ &+ \int_0^{t_1} [\psi_1(t_2,s) - \psi_1(t_1,s)]\psi_7(s,y)ds \\ &+ \int_{t_1}^{t_2} [\psi_2(t_2,s)L(\psi_2(d,s)L)^T]W^{-1}\Phi \\ &+ \int_{t_1-q}^{t_2-q} [\psi_2(t_2,q,s)M(\psi_2(d,q,s)M)^T]W^{-1}\Phi \\ &+ \int_{t_1}^{t_2} \psi_1(t_2,s)\psi_7(s,y)ds. \end{aligned} \tag{41}$$

In view of (41), (31) can be written as

$$\|(Tu)(t_2) - (Tu)(t_1)\| \leq \begin{cases} [(\psi_2(d,q,t_2)M) - (\psi_2(d,q,t_1)M)]^T W^{-1}\Phi, \ -q \leq t \leq d-q, \\ 0, \ d-q \leq t \leq 0, \\ [(\psi_1(d,t_2)L) - (\psi_1(d,t_1)L)]^T W^{-1}\Phi, 0 < t \leq d. \end{cases} \tag{42}$$

This further implies that

$$\begin{aligned} &\|^cD^\nu Ty(t_2) - {}^cD^\nu Ty(t_1)\| \leq \\ &\|\frac{1}{\Gamma(j-\nu)}\|\|\int_{t_1}^{t_2} (\psi_6(t_2,s))(Ty)^{(j)}ds\| + \\ &\|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^{t_1} (\psi_6(t_2,s)(Ty)^{(j)} - \psi_6(t_1,s)(Ty)^{(j)})ds\|. \end{aligned} \tag{43}$$

Consequently,

$$\begin{aligned}
&\lim_{t_2\to t_1}\|(Ty)(t_2)-(Ty)(t_1)\|\to 0,\\
&\lim_{t_2\to t_1}\|(Tx)^{(q)}(t_2)-(Tx)^{(q)}(t_1))\|\to 0\\
&\lim_{t_2\to t_1}\|{}^cD^{\nu}(Ty)(t_2)-({}^cD^{\nu}Ty)(t_1)\|\to 0.
\end{aligned}$$

Hence, the equicontinuous family of functions,$\{(Ty):y\in B_s\}$ is uniformly bounded. Next we show that the operator T is compact. For any $y\in B_s$ and a real number ϵ such that $0<\epsilon<t$ where $t\in[0,d]$ we define

$$\begin{aligned}
(Ty_\epsilon)(t) &= \psi_0(t)y_0+\int_0^{t-\epsilon}\psi_1(t,s)L(\psi_1(d,s)L)^TW^{-1}\Phi ds\\
&+ \int_{-q}^{t-\epsilon-q}\psi_2(t,q,s)M(\psi_2(d,q,s)M)^TW^{-1}\Phi ds\\
&+ \int_0^{t-\epsilon}[\psi_1(t,s)\psi_7(s,y)]ds. \qquad (44)
\end{aligned}$$

as above we obtain that $\{(T_\epsilon y):y\in B_s\}$ is an equicontinuous family of functions that fulfill the uniform bounded condition. Therefore, we can infer

$$\begin{aligned}
\|(Ty)(t)-(T_\epsilon y)(t)\| &\le \|\int_{t-\epsilon}^{t}\psi_1(t,s)L(\psi_1(d,s)L)^TW^{-1}\Phi ds\|\\
&+ \|\int_{t-\epsilon-q}^{t-q}\psi_2(t,q,s)M(\psi_2(d,q,s)M)^TW^{-1}\Phi ds\|\\
&+ \|\int_{t-\epsilon}^{t}[\psi_1(t,s)\psi_7(s,y)]ds\|,\\
&\le \epsilon[k_1^2\|L\|\|L^T\|+k_2^2\|M\|\|M^T\|]\|W^{-1}\|\|\Phi\|+\epsilon k_1p. \qquad (45)
\end{aligned}$$

Utilizing the above we obtain

$$\begin{aligned}
\|(Ty)^{(j)}(t)-(T_\epsilon y)^{(j)}(t)\| &\le \|\int_{t-\epsilon}^{t}\psi_4(t,s)L(\psi_1(d,s)L)^TW^{-1}\Phi ds\|\\
&+ \|\int_{t-\epsilon-q}^{t-q}\psi_5(t,q,s)M(\psi_2(d,q,s)M)^TW^{-1}\Phi ds\|\\
&+ \|\int_{t-\epsilon}^{t}[\psi_4(t,s)\psi_7(s,y)]ds\|,\\
&\le \epsilon[k_1k_4\|L\|\|L^T\|+k_2k_5\|M\|\|M^T\|]\|W^{-1}\|\|\Phi\|+\epsilon k_4p. \qquad (46)
\end{aligned}$$

Applying definition of Caputo derivative we have

$$\begin{aligned}
&\|{}^cD^{\nu}((Ty)^{(j)}(t_2)-(T_\epsilon y)^{(j)}(t_1))\|\le\\
&\|\frac{1}{\Gamma(j-\nu)}\|\|\int_0^t\psi_6[(Ty)^{(j)}(t)-(T_\epsilon y)^{(j)}(t)]ds\|.
\end{aligned}\qquad (47)$$

Distinctly,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\|(Ty)(t)-(T_\epsilon y)(t)\|\to 0,\\
&\lim_{\epsilon\to 0}\|(Ty)^{(j)}(t)-(T_\epsilon y)^{(j)}(t)\|\to 0,\\
&\lim_{\epsilon\to 0}\|{}^cD^{\nu}(Ty)(t)-{}^cD^{\nu}(T_\epsilon y)(t)\|\to 0.
\end{aligned}$$

It follows form Arzela-Ascoli theorem $\{(Ty)(t) : y \in B_s\}$ is compact in Y.

Step III. To show that T is continuous, we assume the following:
(H_3) Let $Y = \{y_1, y_2, ..., y_n\}, \lim_{n\to\infty} \|y_n - y(t)\| = 0$.
(H_4) Let $z = max\{\|y_n\|, \|u_n\|, \|^c D^\nu y_n\|\}$, z is a positive constant. Utilizing $H_3 - H_4$, we have

$$\begin{aligned} f(t, y_n(t), {}^cD^{(\nu-1)}y_n(t)) &\leq f(t, y(t), {}^cD^{(\nu-1)}y(t)), i.e \\ \psi_7(s, y_n) &\leq \psi_7(s, y). \end{aligned}$$

Now by Fatou-Lebesgue theorem

$$\begin{aligned} \|(Ty_n)(t) - (Ty)(t)\| &\leq \int_0^t [k_1^2\|L\|\|L^T\| + k_2^2\|M\|\|M^T\|] \\ &\times \|W^{-1}\|k_1\| \int_0^s ((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta))))d\vartheta\|ds \\ &+ k_1\| \int_0^t ((\psi_7(s, y_n)) - (\psi_7(s, y)))ds\|. \end{aligned} \tag{48}$$

Utilizing similar approach as above we also have

$$\begin{aligned} \|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| &\leq \int_0^t [k_1k_4\|L\|\|L^T\| + k_2k_5\|M\|\|M^T\|] \\ &\times \|W^{-1}\|k_4\| \int_0^s ((\psi_7(\vartheta, y_n(\vartheta))) - (\psi_7(\vartheta, y(\vartheta))))d\vartheta\|ds \\ &+ k_4\| \int_0^t ((\psi_7(s, y_n)) - (\psi_7(s, y)))ds\|. \end{aligned} \tag{49}$$

Making use of definition (2), one obtain

$$\begin{aligned} &\|^cD^\nu(Ty_n)(t) - {}^cD^\nu(Ty)(t)\| \leq \\ &\|\frac{1}{\Gamma(j-\nu)}\|\| \int_0^t \psi_6(t, s)((Ty_n)^{(j)}(t) - (Ty)^{(j)}(t))ds\|. \end{aligned} \tag{50}$$

Clearly

$$\begin{aligned} &\lim_{n\to\infty} \|(Ty_n)(t) - (Ty)(t)\| = 0, \\ &\lim_{n\to 0} \|(Ty_n)^{(j)}(t) - (Ty)^{(j)}(t)\| = 0, \\ &\lim_{n\to\infty} \|^cD^\nu(Ty_n)(t) - {}^cD^\nu(Ty)(t)\| = 0. \end{aligned}$$

Hence, T is continuous. So, by Arzela-Ascoli and Schaefer's fixed point theorem, it can be concluded the operator T is continuous and has a fixed point Y in B_s. Further, this fixed point Y is the solution of the system (1). We conclud that (1) is controllable for $0 \leq t \leq q$ on I . □

4 Observability

A property of dynamical system that measure how well internal states of a system can be obtained from the information of its external outputs. For observability of our proposed model we assume that $u(t) = 0$, because it has been shown in [62] that observability of a system is independent of the input signal $u(t)$. After this change is made and adding a linear observer, the system (1) obtains the form,

$$\begin{cases} {}^cD^\nu y(t) = Ky(t) + f(t, y(t), {}^cD^{\nu-1}y(t)), t \in I = [0, d], \\ y(0) = y_0, \qquad y'(0) = 0, \\ z(t) = Hy(t). \end{cases} \tag{51}$$

Where $1 < \nu \leq 2$; K is $n \times n$ matrix and f is a nonlinear continuous function.

4.1. Linear Case:

Definition 9. *The time invariant linear system in (51) is said to be observable at time $t \in I$, if $z(t) = Hy(t) = 0$ implies that $y(t) = 0$.*

Theorem 4. *The linear system in (51) is observable in I, if and only if the observability Grammian matrix*

$$Q(0,d) = \int_0^d E_\nu(K^* t^\nu) H^* H E_\nu(K t^\nu) dt \tag{52}$$

is possitive definite.

Proof. By applying the Laplace transform, the Mittag-Leffler function and the initial conditions, the solution of the linear system in (51), is given by

$$y(t) = E_\nu(K t^\nu) y_0, \tag{53}$$

using this equation, we have $z(t) = H E_\nu(K t^\nu) y_0$, and

$$\begin{aligned} \|z(t)\|^2 &= \int_0^d z^*(t) z(t) dt \\ &= y_0^* \int_0^t E_\nu(K t^\nu) H^* H E_\nu(K t^\nu) y_0 dt \\ &= y_0^* Q(0,d) y_0, \end{aligned}$$

clearly $Q(0,d)$ is symmetric and the equation is quadratic in y_0. If $Q(0,d)$ is positive definite and $z(t) = y_0^* Q(0,d) y_0 = 0$, then $y_0 = 0$. Hence, the linear system in (51) is observable. If $Q(0,d)$ is not positive definite, then there exist some non-zero y_0 such that $y_0^* Q(0,d) y_0 = 0$. Which implies that $y(t) = E_\nu(K t^\nu) y_0 \neq 0$, but $\|z\| = 0 \Rightarrow y = 0$, which in turn implies that the system is not observable. Hence the required proof. □

4.2. Nonlinear Case:

For the observability of the nonlinear system (51), one needs to estimate the unidentified state $y(t)$ at the current time t from the information of the system output $z(t)$ in $[\bar{t}, \text{t}]$, where $\bar{t}$ denotes past time.

Definition 10. *The nonlinear system (51) is callee observable at time t if one can determine $\bar{t} < t$, in such a way that state of the system at time t can be estimated from the information of the system's output through the interval $[\bar{t}, t]$. If a given system is observable for all $t \in I$, we call it completely observable.*

Let the nonlinear system (51) possesses a distinctive solution for some initial condition $y = y(t_0)$, $t_0 \in (\bar{t}, t)$, and is given by

$$y(t) = E_\nu(K(t-t_0)^\nu) y(t_0) + \int_{t_0}^t (t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu) f(t, y(t), {}^c D^{\nu-1} y(s)) ds, \tag{54}$$

it is solved for $y(t_0)$ by assuming $[E_\nu(K(t-t_0)^\nu)]$ is invertible, we obtain

$$y(t_0) = [E_\nu(K(t-t_0)^\nu)]^{-1} [y(t) - \int_{t_0}^t (t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu) f(t, y(t), {}^c D^{\nu-1} y(s)) ds], \tag{55}$$

it in turn will give

$$\begin{aligned} z(t_0) &= [E_\nu(K(t-t_0)^\nu)]^{-1} [H y(t) - H \int_{t_0}^t (t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu) f(t, y(t), {}^c D^{\nu-1} y(s)) ds], \\ &= \frac{1}{[E_\nu(K(t-t_0)^\nu)]^2} [H y(t) - H \int_{t_0}^t (t-s)^{\nu-1} E_{\nu,\nu}(K(t-s)^\nu) f(t, y(t), {}^c D^{\nu-1} y(s)) ds] \times E_\nu(K(t-t_0)^\nu). \end{aligned}$$

Integrating the above equation form $\bar{t}$ to t, after multiplying it by $E_\nu(K^*(t-t_0)^\nu)H^*$, we obtain

$$
\begin{aligned}
&\int_{\bar{t}}^{t} [E_\nu(K(t-t_0)^\nu)]^2 E_\nu(K^*(t-t_0)^\nu)H^* z(t_0)dt_0 \\
&= \int_{\bar{t}}^{t} E_\nu(K^*(t-t_0)^\nu)H^*Hy(t)E_\nu(K(t-t_0)^\nu)dt_0 \\
&- \int_{\bar{t}}^{t} E_\nu(K^*(t-t_0)^\nu)H^*H \times \left(\int_{t_0}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu) \times f(t,y(t),{}^cD^{\nu-1}y(s))ds\right) \\
&\times E_\nu(K(t-t_0)^\nu)dt_0 \\
&= \int_{\bar{t}}^{t} E_\nu(K^*(t-t_0)^\nu)H^*HE_\nu(K(t-t_0)^\nu)dt_0 y(t) \\
&- \int_{\bar{t}}^{t} (t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu) \times f(t,y(t),{}^cD^{\nu-1}y(s)) \\
&\times \left(\int_{\bar{t}}^{s} E_\nu(K^*(t-t_0)^\nu)H^*HE_\nu(K(t-t_0)^\nu)dt_0\right)ds.
\end{aligned}
$$

Which implies that,

$$
\begin{aligned}
&\int_{\bar{t}}^{t} [E_\nu(K(t-t_0)^\nu)]^2 E_\nu(K^*(t-t_0)^\nu)H^* z(t_0)dt_0 \\
&= Q(\bar{t},t)y(t) - \int_{\bar{t}}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu) \times f(t,y(t),{}^cD^{\nu-1}y(s)) \times Q(\bar{t},s)ds.
\end{aligned} \tag{56}
$$

Now, in case the matrix $Q(\bar{t},t)$ is invertible, i.e., the linear system in (51) is observable, then from the last equation we obtain

$$
\begin{aligned}
&y(t) = Q^{-1}(\bar{t},t)\int_{\bar{t}}^{t} [E_\nu(K(t-s)^\nu)]^2 E_\nu(K^*(t-s)^\nu)H^* z(s)ds \\
&+ Q^{-1}(\bar{t},t)\int_{\bar{t}}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu) \times f(t,y(t),{}^cD^{\nu-1}y(s)) \times Q(\bar{t},s)ds.
\end{aligned} \tag{57}
$$

Let

$$
\begin{aligned}
G_1(t,\bar{t},s) &= Q^{-1}(\bar{t},t)[E_\nu(K(t-s)^\nu)]^2 E_\nu(K^*(t-s)^\nu)H^*, \\
G_2(t,\bar{t},s) &= Q^{-1}(\bar{t},t)E_{\nu,\nu}(K(t-s)^\nu)Q(\bar{t},s),
\end{aligned}
$$

we obtain

$$
y(t) = \int_{\bar{t}}^{t} G_1(t,\bar{t},s)z(s)ds + \int_{\bar{t}}^{t}(t-s)^{\nu-1}G_2(t,\bar{t},s) \times f(t,y(t),{}^cD^{\nu-1}y(s))ds. \tag{58}
$$

The above equation represents relation between the state variable $y(t)$ and the system output $z(t)$ over the interval $[\bar{t},t]$, hence the following is deduced.

Theorem 5. *The nonlinear system (51) is* (*a*) *observable globally at time t and* (*b*) *observable completely if the conditions given below fulfil.*

- *$det(Q(\bar{t},t)) \geq c$, for some positive c.*
- *One can associate a unique and continuous solution for any z of (57) in $[\bar{t},t]$, for some $\bar{t} < t$,*

    1. *The situation of an observable system at time t, and*

2. The situation of completely observable system $\forall t$.

The time $\bar{t}$ in (58), is not necessarily fixed, so it can be replaced by t_0. After this change is incorporated and the resultant equation is substituted in (55), one obtain

$$y(t_0) = [E_\nu(K(t-t_0)^\nu)]^{-1}\Big[\int_{t_0}^{t} G_1(t,t_0,s)z(s)ds + \int_{t_0}^{t}(t-s)^{\nu-1}G_2(t,t_0,s)\times f(t,y(t),{}^cD^{\nu-1}y(s))ds \\ - \int_{t_0}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)f(t,y(t),{}^cD^{\nu-1}y(s))ds\Big]. \tag{59}$$

Let

$$G_3(t,t_0,s) = [E_\nu(K(t-t_0)^\nu)]^{-1}G_1(t,t_0,s),$$
$$G_4(t,t_0,s) = [E_\nu(K(t-t_0)^\nu)]^{-1}[G_2(t,t_0,s) - E_{\nu,\nu}(K(t-s)^\nu)].$$

After these assumptions are made, (58) reduce to

$$y(t_0) = \int_{t_0}^{t} G_3(t,t_0,s)z(s)ds + \int_{t_0}^{t}(t-s)^{\nu-1}G_4(t,t_0,s)f(t,y(t),{}^cD^{\nu-1}y(s))ds. \tag{60}$$

This equation demonstrate that the same results are also valid if we replace (58) by (60) in theorem (5) with a simple change of variable. Next, we apply the Banach's contraction theorem to the nonlinear system given by

$$\begin{cases} {}^cD^\nu y(t) = Ky(t) + \epsilon f(t,y(t),{}^cD^{\nu-1}y(t)), \\ z(t) = Hy(t). \end{cases} \tag{61}$$

where ϵ is a positive constant. Assume that there exist constants $\mathcal{K} > 0$ and $0 < \mathcal{L} < 1$, such that

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \leq \mathcal{K}\|u-\bar{u}\| + \mathcal{L}\|v-\bar{v}\|. \tag{62}$$

Theorem 6. *The nonlinear system (61) is* (a) *observable globally at time t and* (b) *observable completely if the conditions given below fulfil.*

- *$det(Q(\bar{t},t)) \geq c$, for some $c > 0$.*
- *A positive constant $\epsilon < \frac{\nu(1-\mathcal{L}I)}{(t-\bar{t})^\nu \ell(\bar{t},t)\mathcal{K}}$ in $[\bar{t},t]$, for some $\bar{t} < t$,*

  1. The situation of an observable system at time t, and

  2. The situation of completely observable system for all t.

Proof. A general solution of the nonlinear system (61) with $y = y(t_0)$ as an initial condition, using Laplace transform, inverse Laplace transform and the Mittag-Leffler matrix function is given by

$$y(t) = E_\nu(K(t-t_0)^\nu)y(t_0) + \epsilon\int_{t_0}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)f(t,y(t),{}^cD^{\nu-1}y(s))ds, \tag{63}$$

it is solved for $y(t_0)$, obtaining

$$y(t_0) = [E_\nu(K(t-t_0)^\nu)]^{-1}[y(t) - \epsilon\int_{t_0}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)f(t,y(t).{}^cD^{\nu-1}y(s))ds], \tag{64}$$

After some calculation just like we obtain (57) from (55), the next equation is derived from (64) is given by

$$y(t) = Q^{-1}(\bar{t},t)\int_{\bar{t}}^{t}[E_\nu(K(t-s)^\nu)]^2E_\nu(K^*(t-s)^\nu)H^*z(s)ds \\ + \epsilon Q^{-1}(\bar{t},t)\int_{\bar{t}}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)\times f(t,y(t),{}^cD^{\nu-1}y(s))\times Q(\bar{t},s)ds. \tag{65}$$

Using (65) in equation (64), we obtain

$$
\begin{aligned}
& y(t_0) = [E_\nu(K(t-t_0)^\nu)]^{-1}[Q^{-1}(\bar{t},t)\int_{\bar{t}}^{t}[E_\nu(K(t-s)^\nu)]^2 E_\nu(K^*(t-s)^\nu)H^* z(s)ds \\
& + \epsilon Q^{-1}(\bar{t},t)\int_{\bar{t}}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)\times f(s,y(s),{}^cD^{\nu-1}y(s))\times Q(\bar{t},s)ds \\
& - \epsilon\int_{t_0}^{t}(t-s)^{\nu-1}E_{\nu,\nu}(K(t-s)^\nu)f(s,y(s){}^cD^{\nu-1}y(s))ds].
\end{aligned} \tag{66}
$$

It can be concluded from the last equation, that the system (61) is observable. For this it is sufficient to prove that $Q(\bar{\cdot},\ \cdot)$ is invertible and there exists a unique solution of (66). If it is assume that there exists two such solutions say $y,\bar{y},y\neq\bar{y}$ of (66) for a given z then utilizing (62) we have

$$
\begin{aligned}
& |y(t_0)-\bar{y}(t_0)| \leq \epsilon\left|[E_\nu(K(t-t_0)^\nu)]^{-1}\right|\left|Q^{-1}(\bar{t},t)\right|\int_{\bar{t}}^{t}(t-s)^{\nu-1}\left|E_{\nu,\nu}(K(t-s)^\nu)\right|\left|Q(\bar{t},s)\right| \\
& \times[\left|f(s,y(s),{}^cD^{\nu-1}y(s))-f(s,\bar{y}(s),{}^cD^{\nu-1}\bar{y}(s))\right|]ds \\
& + \epsilon\left|[E_\nu(K(t-t_0)^\nu)]^{-1}\right|\int_{t_0}^{t}(t-s)^{\nu-1}\left|E_{\nu,\nu}(K(t-s)^\nu)\right| \\
& \times[\left|f(s,y(s){}^cD^{\nu-1}y(s))-f(s,\bar{y}(s){}^cD^{\nu-1}\bar{y}(s))\right|]ds \\
& \leq \frac{\epsilon}{\nu}\ell_1(\bar{t},t)\times(t-\bar{t})^\nu\times[\frac{\mathcal{K}}{1-\mathcal{L}I}]\left|y-\bar{y}\right| \\
& + \frac{\epsilon}{\nu}\ell_2(\bar{t},t)\times(t-\bar{t})^\nu\times[\frac{\mathcal{K}}{1-\mathcal{L}I}]\left|y-\bar{y}\right|
\end{aligned} \tag{67}
$$

where

$$
\begin{aligned}
& \ell_1(\bar{t},t) = \max_{\bar{t}<t_0<s<t}\left|[E_\nu(K(t-t_0)^\nu)]^{-1}Q^{-1}(\bar{t},t)\right|\times\left|E_{\nu,\nu}(K(t-s)^\nu)Q(\bar{t},s)\right|, \\
& \ell_2(\bar{t},t) = \max_{\bar{t}<t_0<s<t}\left|[E_\nu(K(t-t_0)^\nu)]^{-1}E_{\nu,\nu}(K(t-s)^\nu)\right|, \\
& |I| = \left|\int_{\bar{t}}^{t}dt\right|.
\end{aligned} \tag{68}
$$

A little bit simplification will give

$$
\|y(t_0)-\bar{y}(t_0)\| \leq \frac{\epsilon(t-\bar{t})^\nu\ell(\bar{t},t)\mathcal{K}}{\nu(1-\mathcal{L}I)}\left|y-\bar{y}\right|. \tag{69}
$$

Where $\ell(\bar{t},t)=\ell_1(\bar{t},t)+\ell_1(\bar{t},t)$. If

$$
\frac{\epsilon(t-\bar{t})^\nu\ell(\bar{t},t)\mathcal{K}}{\nu(1-\mathcal{L}I)} < 1, \tag{70}
$$

then the equation (66) has a unique solution and the system (61) is observable. □

5 Example:

Given a non-linear fractional order system

$$
\begin{cases}
{}^cD^\nu y(t) = Ky(t)+Lu(t)+Mu(t-q)+f(t,y(t),{}^cD^{\nu-1}y(t)), t\geq 0, \\
y(0)=y_0, y'(0)=0, \\
u(t)=\phi(t), -j\leq t\leq 0,
\end{cases} \tag{71}
$$

where $j-1 \leq \nu \leq j$, $t \in I$ and

$$K = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & 2 \\ 5 & 1 & 7 \end{bmatrix}, L = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, M = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, y(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

And the nonlinear function f

$$f(t, y(t), {}^cD^{\nu-1}y(t)) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2e^{t+1}(1+|y(t)|+|{}^cD^{\nu-1}y(t))} \end{bmatrix}.$$

Also by the Mittage-Leffler matrix function

$$E_{\nu.\nu}(At^\nu) = \sum_{k=0}^{\infty} \frac{A^k t^{k\nu}}{\Gamma\left((k+1)\,\nu\right)}.$$

We obtain

$$E_{\nu.\nu}(K(d-s)^\nu)L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}, E_{\nu.\nu}(K(d-q-s)^\nu)M = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix},$$

where

$$\begin{aligned}
l_1 &= \frac{1}{\Gamma(\nu)} + \frac{4\,(d-s)^\nu}{\Gamma\,(2\,\nu)} + \frac{6\,(d-s)^{2\,\nu}}{\Gamma\,(3\,\nu)} + ...,\\
l_2 &= \frac{7\,(d-s)^\nu}{\Gamma\,(2\,\nu)} + \frac{85\,(d-s)^{2\,\nu}}{\Gamma\,(3\,\nu)} + ...,\\
l_3 &= \frac{2}{\Gamma(\nu)} + 19\frac{(d-s)^\nu}{\Gamma\,(2\,\nu)} + \frac{160\,(d-s)^{2\,\nu}}{\Gamma\,(3\,\nu)} + ...,\\
s_1 &= \frac{2}{\Gamma(\nu)} + \frac{7\,(d-q-s)^\nu}{\Gamma\,(2\,\nu)} + \frac{9\,(d-q-s)^{2\,\nu}}{\Gamma\,(3\,\nu)} + ...,\\
s_2 &= \frac{12\,(d-q-s)^\nu}{\Gamma\,(2\,\nu)} + \frac{143\,(d-q-s)^{2\,\nu}}{\Gamma\,(3\,\nu)} + ...,\\
s_3 &= \frac{3}{\Gamma(\nu)} + \frac{31\,(d-q-s)^\nu}{\Gamma\,(2\,\nu)} + \frac{264\,(d-q-s)^{2\,\nu}}{\Gamma\,(3\,\nu)} + ...
\end{aligned}$$

Now the Grammian $W(t)$ in both the following cases for arbitrary $d > 0$ is nonsingular.

Case I. $0 \leq t \leq q$

$$\begin{aligned}
W(t) &= \int_0^d (d-s)^{2(\nu-1)} \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} ds \\
&+ \int_{-q}^{d-q} (d-q-s)^{2(\nu-1)} \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}^T \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} ds, \\
&= \int_0^d (d-s)^{2(\nu-1)} \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & l_3^2 \end{bmatrix} ds \\
&+ \int_{-q}^{d-q} (d-q-s)^{2(\nu-1)} \begin{bmatrix} s_1^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & s_2^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & s_3^2 \end{bmatrix} ds.
\end{aligned}$$

Case II. $t > q$

$$\begin{aligned} W(t) &= \int_0^{d-q} [(d-s)^{\nu-1}(\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T + (d-q-s)^{\nu-1}\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}^T) \\ &\times ((d-s)^{\nu-1}\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} + (d-q-s)^{\nu-1}\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix})]ds \\ &+ \int_{d-q}^{d} [(d-s)^{\nu-1}\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} ds, \\ &= \int_0^{d-q} [(d-s)^{2(\nu-1)} \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & l_3^2 \end{bmatrix} \\ &+ ((d-s)(d-q-s))^{\nu-1} \begin{bmatrix} l_1 s_1 & l_1 s_2 & l_1 s_3 \\ l_2 s_1 & l_2 s_2 & l_2 s_3 \\ l_3 s_1 & l_3 s_2 & l_3 s_3 \end{bmatrix} \\ &+ ((d-s)(d-q-s))^{\nu-1} \begin{bmatrix} l_1 s_1 & l_2 s_1 & l_3 s_1 \\ l_1 s_2 & l_2 s_2 & l_3 s_2 \\ l_1 s_3 & l_2 s_3 & l_3 s_3 \end{bmatrix} \\ &+ (d-q-s))^{2(\nu-1)} \begin{bmatrix} s_1^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & s_2^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & s_3^2 \end{bmatrix}]ds \\ &+ \int_{d-q}^{d} (d-s)^{2(\nu-1)} \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & l_3^2 \end{bmatrix} ds. \end{aligned}$$

Since the nonlinear fractional differential function f satisfy the aforementioned hypothesis, and the Grammian matrices $W(t)$ in both the cases are nonsingular. Hence by theorem (3) the system (71) is controllable on I.

6 Example

We here construct a fractional order system (51) as follow,

$$K = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & 2 \\ 5 & 1 & 7 \end{bmatrix}, f(t, y(t), {}^cD^{\nu-1}y(t)) = \begin{bmatrix} 0 \\ \frac{1}{2e^{t+1}(1+|y(t)|+|{}^cD^{\nu-1}y(t))} \end{bmatrix}, t \in [0,1].$$

Clearly f is continuous and for any $y_1, y_2, \bar{y_1}, \bar{y_2} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t, y_1, y_2) - f(t, \bar{y_1}, \bar{y_2})| \leq \frac{1}{2e}(|y_1 - y_2| + |\bar{y_1} - \bar{y_2}|), \mathcal{K} = \mathcal{L} = \frac{1}{2e}.$$

Also the Mittag-Leffler function for the given K and $\nu = 3/2$ is given by

$$\begin{aligned} E_{3/2}(Kt^{3/2}) &= \sum_{i=0}^{\infty} \frac{K^i t^{\frac{3}{2}i}}{\Gamma(i\alpha + 1)} \\ &= \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} \end{aligned}$$

and

$$
\begin{aligned}
E_{3/2}(K^* t^{3/2}) &= \sum_{i=0}^{\infty} \frac{{K^*}^i t^{\frac{3}{2}i}}{\Gamma(i\alpha+1)} \\
&= \begin{bmatrix} a_{11}(t) & a_{21}(t) & a_{31}(t) \\ a_{12}(t) & a_{22}(t) & a_{32}(t) \\ a_{13}(t) & a_{23}(t) & a_{33}(t) \end{bmatrix}
\end{aligned}
$$

Where

$$
\begin{aligned}
a_{11}(t) &= \frac{8}{3}\frac{t^{3/2}}{\sqrt{\pi}} - \frac{32}{21}\frac{t^{9/2}}{\sqrt{\pi}} + 1 + ..., \\
a_{12}(t) &= \frac{-4t^{3/2}}{\sqrt{\pi}} \frac{-10}{3} t^3 - \frac{3104}{945}\frac{t^{9/2}}{\sqrt{\pi}} + ..., \\
a_{13}(t) &= \frac{4}{3}\frac{t^{3/2}}{\sqrt{\pi}} + \frac{1}{2}t^3 - \frac{608}{945}\frac{t^{9/2}}{\sqrt{\pi}} + ..., \\
a_{21}(t) &= \frac{4t^{3/2}}{\sqrt{\pi}} + \frac{31}{6}t^3 + \frac{8032}{945}\frac{t^{9/2}}{\sqrt{\pi}} + ..., \\
a_{22}(t) &= \frac{20}{3}\frac{t^{3/2}}{\sqrt{\pi}} + 3\,t^3 + \frac{256}{315}\frac{t^{9/2}}{\sqrt{\pi}} + 1 + ..., \\
a_{23}(t) &= \frac{8}{3}\frac{t^{3/2}}{\sqrt{\pi}} + \frac{9}{2}t^3 + \frac{8192}{945}\frac{t^{9/2}}{\sqrt{\pi}} + ..., \\
a_{31}(t) &= \frac{20}{3}\frac{t^{3/2}}{\sqrt{\pi}} + 8\,t^3 + \frac{11744}{945}\frac{t^{9/2}}{\sqrt{\pi}} + ..., \\
a_{32}(t) &= \frac{4}{3}\frac{t^{3/2}}{\sqrt{\pi}} - \frac{1}{2}t^3 - \frac{3296}{945}\frac{t^{9/2}}{\sqrt{\pi}} + ..., \\
a_{33}(t) &= \frac{28}{3}\frac{t^{3/2}}{\sqrt{\pi}} + \frac{28}{3}t^3 + \frac{1984}{135}\frac{t^{9/2}}{\sqrt{\pi}} + 1 + ...
\end{aligned}
$$

We have

$$
\begin{aligned}
Q(0,1) &= \int_0^1 E_\nu(K^* t^\nu) H^* H E_\nu(K t^\nu) dt \\
&= \int_0^1 \begin{bmatrix} a_{11}(t) & a_{21}(t) & a_{31}(t) \\ a_{12}(t) & a_{22}(t) & a_{32}(t) \\ a_{13}(t) & a_{23}(t) & a_{33}(t) \end{bmatrix} H^* H \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} dt.
\end{aligned}
$$

Which is positive definite for suitable H, i.e $Q^{-1}(0,1)$ exists. Hence by theorem (6), the system (51), is observable.

7 Concluding Comments.

The present article explored the dynamical aspects & qualitative study of nonlinear fractional-order system with input delay. We found general solution of the proposed dynamical system in the form of an integral equation and proved controllability as well as observability for the linear case. The non-linear problem has been transformed into a fixed-point problem and a set of necessary and sufficient conditions for the controllability within two different domains $0 \leq t \leq q$ and $0 > t \geq q$, utilizing Schaefer's fixed-point theorem together with theorem of Arzela-Ascoli, have been established. We also explored the observability of the nonlinear case of our proposed dynamical system in the absence of control input $u(t)$ using the Banach contraction mapping theorem. For authentication of the method, we put an example at the end of the paper.

In the study of dynamical systems, the observability property plays and important role. In sensor networking, it is used in the controller configuration of closed-loop feedback system as well as to reduce the number of output sensors. Both the dynamical properties, controllability, and observability assist in "Actuator & "Sensor selection. This further suggests that in minimum components we can get maximum stability in the system and can observe a less noisy system. From a mathematical point of view, the Gramain criterion is used to check the observability of a system. Observability Gramain informs us about the order. It means that from Gramain criteria we get information from the most observable to the least observable state. In a dynamical system, some states "p" can be easily observed given a state "q" or in certain situations, some states possess less noise measurement as compared to other states [63, 64].

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