GLOBAL REGULARITY FOR THE 2D MAGNETIC BÉNARD SYSTEM WITH FRACTIONAL PARTIAL DISSIPATION

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Abstract. In this paper, we considered the global regularity for the 2D incompressible anisotropic magnetic Bénard system with fractional partial dissipation. More precisely, we established the global existence and regularity for the 2D incompressible anisotropic magnetic Bénard system with only vertical hyperdiffusion $\Lambda_2^{2\beta} b_1$ and horizontal hyperdiffusion $\Lambda_1^{2\beta} b_2$ and $(-\Delta)^{\alpha}\theta$, where Λ_1 and Λ_2 are directional Fourier multiplier operators with the symbols being $|\xi_1|$ and $|\xi_2|$, respectively. We prove that, for $\beta > 1$ and $0 < \alpha < 1$, this system always possesses a unique global-in-time classical solution when the initial data is sufficiently smooth.

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1. Introduction and Main Results

The 2D magnetic Bénard problem with full viscosity can be stated as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \mu\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b} + \theta\mathbf{e}_2, \\ \partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \nu\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u}, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta + \mathbf{u}\cdot\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0. \end{cases} \tag{1.1}$$

Where $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^2$ denotes the fluid velocity, $\mathbf{b} = \mathbf{b}(x,t) \in \mathbb{R}^2$ the magnetic field, $\pi(x,t)$ the scalar pressure and $\theta(x,t)$ is the scalar temperature. The term $\mathbf{e}_2 = (0,1)^T$ is

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the vertical unit vector. The positive parameters μ, ν, κ are the coefficients of dissipation, magnetic diffusion and thermal diffusivity. The forcing term $\theta\mathbf{e}_2$ in the momentum equation describes the acting of the buoyancy force on fluid motion and $\mathbf{u}\cdot\mathbf{e}_2$ models the Rayleigh-Bénard convection in a heated inviscid fluid.

If we ignore the thermal effects in the fluid motion, i.e. $\theta = 0$, the 2D magnetic Bénard problem (1.1) reduces to the well-known 2D magnetohydrodynamics (MHD) equations,

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \mu\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b}, \\ \partial_t\mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \nu\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0. \end{cases} \tag{1.2}$$

which describes the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. The global regularity issue for the 2D MHD system has attracted much attention (see, e.g., [10, 11, 28, 45, 54, 55, 57]). Very recently, Dong-Li-Wu in [14] obtained the global regularity for the 2D MHD equations with partial hyperresistivity. The global regularity for the 2D MHD equations with fractional dissipation and partial magnetic diffusion was established by Dong-Jia- Li-Wu in [13]. There have been significant recent developments on the MHD equations with partial or fractional dissipation. One can refer to (see, e.g, [2–4, 6, 7, 9, 15–19, 25–27, 31–33, 43, 48–53]) for details and the references therein.

If we ignore the magnetic field, that is $\mathbf{b} = 0$, the system (1.1) becomes the standard Bénard system, which has widely used to deal with convective motions in a heated fluid (see,e.g., [1, 8, 12, 22, 30, 37, 47]) and references therein.

If we ignore the Rayleigh-Bénard convection term $\mathbf{u}\cdot\mathbf{e}_2$ and let $\mathbf{b} = 0$, then the system (1.1) reduces to the 2D anisotropic Boussinesq equations.

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\pi + \mu\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.3}$$

The Boussinesq equations, which model geophysical flows such as atmospheric fronts and oceanic circulation, play an important role in the study of Raleigh-Bernard convection (see [34, 46] etc. for more details). Similar to the 2D incompressible Navier-Stokes equations[34], one can obtain the global well-posedness for the 2D standard Boussinesq equations . But the global regularity result is open to the 2D inviscid Boussinesq equations expect the thermal is zero. Recently, mathematicians began to study the 2D Boussinesq system with fractional dissipations, please see [20, 23, 24, 38] and the references therein. When both $\mathbf{b} = 0$ and $\theta = 0$, the system (1.1) is reduced to the 2D incompressible Navier-Stokes equations which have been studied intensively.

The magnetic Bénard problem as a toy model comes from the convective motions in a heated and incompressible fluid. As we know, in a homogeneous, viscous, and electrically conducting fluid, the convection will occur if the temperature gradient passes a certain critical threshold in two horizontal layers and the convection is permeated by an imposed uniform magnetic field, normal to the layers, and heated from below. The magnetic Bénard problem illuminates the heat convection phenomenon under the presence of the magnetic field (see [21, 36, 41] for details). The magnetic Bénard problem couples the Boussinesq, magnetic induction and thermal convection equations, and the system includes as particular cases the Navier-Stokes and magnetohydrodynamics equations. If the gradients of the velocity, magnetic field and temperature remain bounded in all space, and the pressure decreases at infinity at most like $|x|^{-\frac{1}{2}}$, Miao[35] established the uniqueness theorems for the unbounded classical solution of the magnetic Bénard system. By using the Galerkin method, the regularity and analyticity of the solutions of the magnetic Bénard problem in $\mathbb{R}^n (n = 2, 3)$ were obtained by Nakamura[42]. There are more works on magnetic Bénard problem in two dimension such as [58–60] and the reference therein.

Very recently, the global well-posedness of 2D magnetic Bénard problem without thermal diffusivity and with vertical or horizontal magnetic diffusion and the global regularity and some conditional regularity of strong solutions are obtained for 2D magnetic Bénard problem with mixed partial viscosity were considered by Cheng-Du in[5]. Ye[58] studied the global regularity of the 2D anisotropic magnetic Bénard system with vertical dissiption. The global regularity of the 2D magnetic Bénard system with zero thermal conductivity obtained by Ye[59]. Yamazaki-Kazuo[56] established global regularity of generalized magnetic Bénard problem. Ma[40] investigated the global regularity for the 2D magnetic Bénard fluid system with mixed partial viscosity.

In this paper, we focus on the 2D incompressible magnetic Bénard problem with only partial fractional magnetic diffusion and thermal diffusivity as follows

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\pi + \mathbf{b}\cdot\nabla\mathbf{b} + \theta\mathbf{e}_2, \\ \partial_t b_1 + \mathbf{u}\cdot\nabla b_1 = -\nu\Lambda_2^{2\beta} b_1 + \mathbf{b}\cdot\nabla u_1, \\ \partial_t b_2 + \mathbf{u}\cdot\nabla b_2 = -\nu\Lambda_1^{2\beta} b_2 + \mathbf{b}\cdot\nabla u_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = -\kappa\Lambda^{2\alpha}\theta + \mathbf{u}\cdot\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x),\ \mathbf{b}(x,0) = \mathbf{b}_0(x),\ \theta(x,0) = \theta_0(x). \end{cases} \tag{1.4}$$

Where $\mathbf{u} = (u_1,\ u_2)$ denotes the velocity field and $\mathbf{b} = (b_1,\ b_2)$ the magnetic feild. Here, $\Lambda_i^\gamma (i = 1, 2)$ denote the directional fractional operators defined via the Fourier transform

$$\widehat{\Lambda_i^\gamma f}(\xi) = |\xi_i|^\gamma \hat{f}(\xi),\ i = 1, 2.$$

Where $\xi = (\xi_1,\ \xi_2)$ and $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator. In addition, we use $\Lambda^s (s > 0)$ to denote the 2D fractional Laplace operator,

$$\widehat{\Lambda^s f(\xi)} = |\xi|^s \hat{f}(\xi).$$

For simplicity, we take $\nu = \kappa = 1$.

The purpose of this paper is to establish the global well-posedness for the system (1.4) with any sufficiently smooth initial data $(\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0)$, when $\beta > 1$ and $0 < \alpha < 1$. More precisely, the main results of this paper states as follows

Theorem 1.1. *Consider the system* (1.4) *with* $\beta > 1$ *and* $0 < \alpha < 1$. *Assume the initial data* $(\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0) \in H^s(\mathbb{R}^2)$ *with* $s \geq 3$ *and* $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$. *Then system* (1.4) *has a unique global solution* $(\mathbf{u},\ \mathbf{b},\ \theta)$ *satisfying, for any* $T > 0$,

$$(\mathbf{u},\ \mathbf{b},\ \theta) \in L^\infty(0,\ T;\ H^s(\mathbb{R}^2)),\ \mathbf{b} \in L^2(0,\ T;\ \dot{H}^{s+\beta}(\mathbb{R}^2)),\ \theta \in L^2(0,\ T;\ \dot{H}^{s+\alpha}(\mathbb{R}^2)). \tag{1.5}$$

The rest of this paper is constructed as follows. In section 2, we will give some notation and preliminaries. In section 3, we will prove our main result. The proof of Theorem 1.1 will be divided into three subsections.

2. Notation and preliminaries

For convenience, we will give some notations before we prove our main result, which are used throughout this paper. We denote

$$\|f\|_{L^p(\mathbb{R}^2)} = \|f\|_p, \ \frac{\partial f}{\partial x_i} = \partial_i f,$$
$$\int f dxdy = \iint_{\mathbb{R}^2} f dxdy,$$

and

$$\|f_1, f_2, \cdots, f_n\|^2_{L^2(\mathbb{R}^2)} = \|f_1\|_2^2 + \|f_2\|_2^2 + \cdots + \|f_n\|_2^2.$$

Next we will give some auxiliary lemmas. First we recall the classical commutator estimate(See,e.g., [29]).

Lemma 2.1. *Assume that $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,*

$$\|[\Lambda^s, \ f]g\|_{L^r} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}). \tag{2.1}$$

Where C is a constant depending on the indices $s, \ r, \ p_1, \ q_1, \ p_2$ and q_2.

The next lemma is very useful to establish the global bound of $\|\nabla \mathbf{b}\|_{L^p}$.

Lemma 2.2. *Assume that $\beta > 0, \ t > 0$. Considering the following equations,*

$$\begin{cases} \partial_t \mathbf{u} + (-\Delta)^\beta \mathbf{u} = f, \\ \mathbf{u}(x, \ 0) = \mathbf{u}_0(x). \end{cases} \tag{2.2}$$

Then the solution of it can be expressed as

$$\mathbf{u}(x, \ t) = K_\beta(\cdot, \ t) * \mathbf{u}_0 + \int_0^t K_\beta(\cdot, \ t-\tau) * f(\cdot, \ \tau) d\tau.$$

Where the kernel function is defined via the Fourier transform

$$K_\beta(x, \ t) = \int_{\mathbb{R}^n} e^{-t|\xi|^{2\beta}} e^{ix\cdot\xi} d\xi,$$

and $K_\beta(x, \ t)$ satisfies the following properties:

(i)For any $t > 0$,

$$K_\beta(x, \ t) = t^{-\frac{n}{2\beta}} K_\beta(xt^{-\frac{1}{2\beta}}, \ 1).$$

(ii)For any integer $m > 0$, $1 \leq r \leq \infty$ and any $t > 0$,

$$\|\nabla^m K_\beta(x, \ t)\|_{L^r(\mathbb{R}^n)} \leq C t^{-\frac{m}{2\beta} - \frac{n}{2\beta}(1-\frac{1}{r})}.$$

In particular, when $\beta = 1$, $K_1(x,\ t)$ is the classical heat equation kernel. One can refer to [39] for the proof. We omit it here.

The following Hörmander-Mikhlin multiplier lemma plays an important role in estimating the global bound of the current density(see,e.g.,[44]).

Lemma 2.3. *Let h be a bounded function on $\mathbb{R}^n$ which is smooth except at the origin. Let k be a multi-index. Such that*

$$|\nabla^k h(\xi)| \leq C|\xi|^{-|k|},\ 0 \leq |k| \leq \frac{n}{2} + 1. \tag{2.3}$$

Then h is a L^p multiplier for all $1 < p < \infty$, or the operator T_h defined by

$$\widehat{T_h f} = h\hat{f}, \quad f \in L^2 \cap L^p,$$

is bounded from $L^2 \cap L^p$ to $L^2 \cap L^p$.

3. Global regularity for the 2D incompressible magnetic Bénard problem

In this section, we will prove Theorem 1.1. The Theorem 1.1 is proved through three stages. The first step is to establish the H^1-estimate, which relies on the equations of the vorticity $\omega = \nabla \times \mathbf{u}$ and the current density $j = \nabla \times \mathbf{b}$. Second we will prove the global bound of $\|\nabla \mathbf{b}\|_{L_t^\infty L^p}$, $\|\Delta \mathbf{b}\|_{L_t^1 L^p}$, $\|\omega\|_{L_t^\infty L^p}$ and $\|\nabla\theta\|_{L_t^\infty L^p}$ with any $2 < p < \infty$. Finally, we will achieve the global bounds of $\|\Delta j\|_{L_t^1 L^\infty}$, $\|\omega\|_{L_t^\infty L^\infty}$ and $\|\nabla\theta\|_{L_t^\infty L^\infty}$, and then complete the proof of Theorem1.1.

Step1. Global H^1 bound for $(\mathbf{u},\ \mathbf{b},\ \theta)$.

Proposition 3.1. *Assume that $(\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0)$ satisfies the conditions stated in Theorem1.1. Then system (1.4) has a global solution $(\mathbf{u},\ \mathbf{b},\ \theta)$ obeys the following bounds uniformly, for any $t > 0$,*

$$\begin{aligned} &\|\mathbf{u},\ \mathbf{b},\ \theta\|_2^2 + \int_0^t \|\Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2,\ \Lambda^\alpha\theta\|_2^2\, dt \leq C(\|\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0\|_2^2) \\ &\|\omega,\ j,\ \nabla\theta\|_2^2 + \int_0^t \|\Lambda_2^\beta \nabla b_1,\ \Lambda_1^\beta \nabla b_2,\ \Lambda^{\alpha+1}\theta\|_2^2\, dt \leq C(\|\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0\|_{H^1}^2), \end{aligned} \tag{3.1}$$

where $C > 0$ is a constant, depending on t and the initial data.

Proof. Multiplying the equations $(1.4)_{1\sim 4}$ by $\mathbf{u},\ b_1,\ b_2$ and θ, respectively and taking the L^2 inner product, integrating by parts, using the divergence-free conditions $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{b} = 0$, adding the resulting equations together, yield that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u},\ \mathbf{b},\ \theta\|_2^2 + \|\Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2,\ \Lambda^\alpha\theta\|_2^2 = 2\int u_2\theta\, dxdy \leq C\|\mathbf{u},\ \theta\|_2^2. \tag{3.2}$$

Where we have used the fact that

$$\int \mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{u}\, dxdy + \int \mathbf{b} \cdot \nabla \mathbf{u} \cdot \mathbf{b}\, dxdy = 0.$$

Applying the Gronwall's inequality, we obtain the L^2 bound for $\mathbf{u},\ \mathbf{b},\ \theta$ as follow

$$\|\mathbf{u},\ \mathbf{b},\ \theta\|_2^2 + \int_0^t \|\Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2,\ \Lambda^\alpha \theta\|_2^2\, dt \le C(\|\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0\|_2^2).$$

To establish the global H^1 bound, we consider the equation of the vorticity $\omega = \nabla \times \mathbf{u}$ and the current density $j = \nabla \times \mathbf{b}$, combining the equation $(1.4)_4$, which satisfy

$$\begin{cases} \partial_t \omega + \mathbf{u}\cdot\nabla\omega = \mathbf{b}\cdot\nabla j + \partial_1\theta, \\ \partial_t j + \mathbf{u}\cdot\nabla j + \Lambda_1^{2\beta}\partial_1 b_2 - \Lambda_2^{2\beta}\partial_2 b_1 = \mathbf{b}\cdot\nabla\omega + Q(\mathbf{u},\ \mathbf{b}), \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = -\Lambda^{2\alpha}\theta + u_2. \end{cases} \tag{3.3}$$

Where

$$Q(\mathbf{u},\ \mathbf{b}) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - 2\partial_1 u_1(\partial_1 b_2 + \partial_2 b_1).$$

Dotting the equations $(3.3)_{1\sim 3}$ with $\omega,\ j$ and $\Lambda^2\theta$, respectively and integrating by parts yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\omega,\ j,\ \Lambda\theta\|_2^2 + \|\Lambda_2^\beta\nabla b_1,\ \Lambda_1^\beta\nabla b_2,\ \Lambda^{\alpha+1}\theta\|_2^2 = \int \partial_1\theta\omega\, dxdy + \int u_2\Lambda^2\theta\, dxdy \\ &- \int [\Lambda,\ \mathbf{u}\cdot\nabla]\theta\Lambda\theta\, dxdy + \int Q(\mathbf{u},\ \mathbf{b}) j\, dxdy \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.4}$$

Where we have used the fact

$$\int \mathbf{b}\cdot\nabla j\cdot\omega\, dxdy + \int \mathbf{b}\cdot\nabla\omega\cdot j\, dxdy = 0.$$

Using Hölder's and Young's inequality, the term I_1 can be bounded as

$$I_1 = \int \partial_1\theta\omega\, dxdy \le \|\omega,\ \Lambda\theta\|_2^2,$$

similarly,

$$I_2 = \int u_2\Lambda^2\theta\, dxdy \le \|\omega,\ \Lambda\theta\|_2^2.$$

Applying Hölder's, Young's and Gagliardo-Nirenberg inequalities, we can estimate the term I_3 as follow

$$\begin{aligned} I_3 = -\int [\Lambda,\ \mathbf{u}\cdot\nabla]\theta\Lambda\theta\, dxdy &\le \int |\nabla\mathbf{u}||\Lambda\theta|^2\, dxdy \le \|\nabla\mathbf{u}\|_2\|\Lambda\theta\|_4^2 \\ &\le \varepsilon\|\Lambda^{\alpha+1}\theta\|_2^2 + C_\varepsilon\|\omega,\ \Lambda\theta\|_2^2 + C_\varepsilon\|\theta\|_2^2. \end{aligned}$$

Now we will estimate the term I_4, we rewrite it as

$$I_4 = 2\int \partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) j\, dxdy - 2\int \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1) j\, dxdy = I_{41} + I_{42}.$$

Applying Hölder's, Young's and Gagliardo-Nirenberg inequalities, we can find that

$$I_{41} = 2\int \partial_1 b_1(\partial_1 u_2+\partial_2 u_1)j\,dxdy \le C\|\omega\|_2\|j\|_4^2 \le \varepsilon\|\nabla j\|_2^2 + C_\varepsilon\|j\|_2^2\|\omega\|_2^2,$$

furthermore,

$$\|j\|_2^2 \le C\|\partial_1 b_2,\ \partial_2 b_1\|_2^2 \le C(\beta)\|\mathbf{b},\ \Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2\|_2^2.$$

And

$$\|\nabla j\|_2^2 \le C\|\partial_1 j,\ \partial_2 j\|_2^2 \le C_\varepsilon\|j\|_2^2 + \varepsilon\|\Lambda_2^\beta\nabla b_1,\ \Lambda_1^\beta\nabla b_2\|_2^2.$$

Combining the above two estimates with I_{41}, we infer that

$$I_{41} \le \varepsilon\|\Lambda_2^\beta\nabla b_1,\ \Lambda_1^\beta\nabla b_2\|_2^2 + C_\varepsilon\|\mathbf{b},\ \Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2\|_2^2(1+\|\omega\|_2^2). \tag{3.5}$$

Similarly,

$$I_{42} \le \varepsilon\|\Lambda_2^\beta\nabla b_1,\ \Lambda_1^\beta\nabla b_2\|_2^2 + C_\varepsilon\|\mathbf{b},\ \Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2\|_2^2(1+\|\omega\|_2^2). \tag{3.6}$$

Inserting the estimates of $I_1 \sim I_3$ and (3.5) $\sim$ (3.6), choosing ε small enough, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega,\ j,\ \Lambda\theta\|_2^2 + \|\Lambda_2^\beta\nabla b_1,\ \Lambda_1^\beta\nabla b_2,\ \Lambda^{\alpha+1}\theta\|_2^2 \le C_\varepsilon\|\mathbf{b},\ \Lambda_2^\beta b_1,\ \Lambda_1^\beta b_2,\ \theta\|_2^2(1+\|\omega,\ j,\ \Lambda\theta\|_2^2).$$

Applying the Gronwall's inequality leads to

$$\|\omega,\ j,\ \nabla\theta\|_2^2 + \int_0^t \|\Lambda_2^\beta\nabla b_1,\ \Lambda_1^\beta\nabla b_2,\ \Lambda^{\alpha+1}\theta\|_2^2\,dt \le C(\|\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0\|_{H^1}^2).$$

In addition, according the above inequality, one can easily check that

$$\int_0^t \|\nabla\theta\|_\infty\,dt \le \int_0^t \|\theta,\ \Lambda^{\alpha+1}\theta\|_2^2\,dt \le C. \tag{3.7}$$

Where C is a constant depending on t and the initial data.

Step2. Global bound for $\|\nabla\mathbf{b}\|_{L_t^\infty L^p}$, $\|\Delta\mathbf{b}\|_{L_t^1 L^p}$, $\|\omega\|_{L_t^\infty L^p}$ and $\|\nabla\theta\|_{L_t^\infty L^p}$ with any $2<p<\infty$.

In this section, we will establish global bound of $\|\nabla\mathbf{b}\|_{L_t^\infty L^p}$, $\|\Delta\mathbf{b}\|_{L_t^1 L^p}$, $\|\omega\|_{L_t^\infty L^p}$ and $\|\nabla\theta\|_{L_t^\infty L^p}$ with any $2<p<\infty$. The integral form of b_1 and b_1 will be used to establish some a priori estimates. The process of this section is more complex.

Proposition 3.2. *Assume that* $(\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0)$ *satisfies the conditions stated in Theorem1.1. Then system* (1.4) *has a global solution* $(\mathbf{u},\ \mathbf{b},\ \theta)$ *obeys the following bounds, for any* $T>0$,

$$\begin{aligned}
&\nabla\mathbf{b}\in L^\infty(0,\ T;\ L^p(\mathbb{R}^2)),\ \mathbf{b}\in L^\infty(0,\ T;\ L^\infty(\mathbb{R}^2)),\\
&\theta\in L^\infty(0,\ T;\ L^\infty(\mathbb{R}^2)),\ \nabla\theta\in L^\infty(0,\ T;\ L^p(\mathbb{R}^2)),\\
&\omega\in L^\infty(0,\ T;\ L^p(\mathbb{R}^2)), \nabla j\in L^1(0,\ T;\ L^p(\mathbb{R}^2)).
\end{aligned} \tag{3.8}$$

Where $2 < p < \infty$.

Proof. To show this proposition, we will make full use of the special structure of the nonlinear terms in the equation of $\mathbf{b}$, which was previously considered in [13], we give details for the completeness. We write the equations of b_1 and b_2 in the integral form

$$b_1 = K_\beta^2(t) *_2 b_{01} + \int_0^t K_\beta^2(t-\tau) *_2 (\mathbf{b}\cdot\nabla u_1 - \mathbf{u}\cdot\nabla b_1)\, d\tau, \tag{3.9}$$

$$b_2 = K_\beta^1(t) *_1 b_{02} + \int_0^t K_\beta^1(t-\tau) *_1 (\mathbf{b}\cdot\nabla u_2 - \mathbf{u}\cdot\nabla b_2)\, d\tau, \tag{3.10}$$

where K_β^2 and K_β^1 denote the 1D inverse Fourier transform of $e^{-|\xi_2|^{2\beta}t}$ and $e^{-|\xi_1|^{2\beta}t}$, respectively, namely

$$K_\beta^2(x_2,\ t) = \int_{\mathbb{R}} e^{ix_2\xi_2} e^{-|\xi_2|^{2\beta}t}\, d\xi_2, \quad K_\beta^1(x_1,\ t) = \int_{\mathbb{R}} e^{ix_1\xi_1} e^{-|\xi_1|^{2\beta}t}\, d\xi_1,$$

and the convolution notations are defined as follow

$$\begin{aligned} K_\beta^2(t) *_2 b_{01} &= \int_{\mathbb{R}} K_\beta^2(x_2 - y_2,\ t) b_{01}(x_1,\ y_2)\, dy_2, \\ K_\beta^1(t) *_1 b_{02} &= \int_{\mathbb{R}} K_\beta^1(x_1 - y_1,\ t) b_{01}(y_1,\ x_2)\, dy_1. \end{aligned}$$

If we know the bound of $\|\partial_2 b_1\|_p$ and $\|\partial_1 b_2\|_p$, by divergence free condition $\nabla\cdot\mathbf{b} = 0$, one can be easily to obtain the estimate for $\|\nabla\mathbf{b}\|_p$. Due to the divergence free conditions $\nabla\cdot\mathbf{u} = 0$ and $\nabla\cdot\mathbf{b} = 0$, we find that

$$\mathbf{b}\cdot\nabla u_1 - \mathbf{u}\cdot\nabla b_1 = \partial_1(b_1u_1) + \partial_2(b_2u_1) - \partial_1(b_1u_1) - \partial_2(u_2b_1) = \partial_2(b_2u_1 - u_2b_1), \tag{3.11}$$

similarly,

$$\mathbf{b}\cdot\nabla u_2 - \mathbf{u}\cdot\nabla b_2 = \partial_1(b_1u_2 - b_2u_1). \tag{3.12}$$

By virtues of (3.9) and (3.11), one has

$$\partial_2 b_1 = \partial_2(K_\beta^2(t) *_2 b_{01}) + \int_0^t \partial_{22}K_\beta^2(t-\tau) *_2 (b_2u_1 - u_2b_1)(\tau)\, d\tau,$$

and then taking the L^p norm with respect to x each side, we obtain

$$\|\partial_2 b_1\|_p \le \|K_\beta^2(t)\|_1 \|\partial_2 b_{01}\|_p + \int_0^t \|\partial_{22}K_\beta^2(t-\tau)\|_{L^1_{x_2}} \|(b_2u_1 - u_2b_1)(\tau)\|_p\, d\tau. \tag{3.13}$$

According to lemma2.2, we have

$$\|\partial_{22}K_\beta^2(t-\tau)\|_{L^1_{x_2}} \le C(t-\tau)^{-\frac{1}{\beta}}. \tag{3.14}$$

Where C is a constant depending only on β. Thanks to the Hölder's inequality and Sobolev's inequality, we deduce

$$\|(b_2u_1 - u_2b_1)(\tau)\|_p \leq C\|\mathbf{u}\|_{2p}\|\mathbf{b}\|_{2p} \leq C(\|\mathbf{u}\|_2 + \|\omega\|_2)(\|\mathbf{b}\|_2 + \|j\|_2). \tag{3.15}$$

Furthermore, inserting the estimates (3.14) and (3.15) into (3.13) yields

$$\begin{aligned}\|\partial_2 b_1\|_p &\leq C\|\partial_2 b_{01}\|_p + Ct^{1-\frac{1}{\beta}}(\|\mathbf{u}\|_{L_t^\infty L^2} + \|\omega\|_{L_t^\infty L^2})(\|\mathbf{b}\|_{L_t^\infty L^2} + \|j\|_{L_t^\infty L^2}) \\ &\leq Ct^{1-\frac{1}{\beta}} + C.\end{aligned} \tag{3.16}$$

Similarly,

$$\|\partial_1 b_2\|_p \leq Ct^{1-\frac{1}{\beta}} + C. \tag{3.17}$$

Consequently, one has

$$\|\nabla \mathbf{b}\|_p \leq C(\|\partial_2 b_1\|_p + \|\partial_1 b_2\|_p) \leq Ct^{1-\frac{1}{\beta}} + C. \tag{3.18}$$

Furthermore, thanks to the Sobolev's inequality, we infer that,

$$\|\mathbf{b}\|_\infty \leq C(\|\mathbf{b}\|_2 + \|\nabla \mathbf{b}\|_p) \leq Ct^{1-\frac{1}{\beta}} + C, \tag{3.19}$$

for any $p > 2$. In addition, one can easily check that

$$\|j\|_p \leq \|\nabla \mathbf{b}\|_p \leq C(\|\partial_2 b_1\|_p + \|\partial_1 b_2\|_p) \leq Ct^{1-\frac{1}{\beta}} + C.$$

Next we will establish the bound for $\|\theta\|_{L_t^\infty L^\infty}$. Multiplying the equation $(1.4)_4$ by $|\theta|^{p-2}\theta$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta\|_p^p + \int \Lambda^{2\alpha}\theta|\theta|^{p-2}\theta\, dxdy = \int u_2|\theta|^{p-2}\theta\, dxdy \leq C\|u_2\|_p\|\theta\|_p^{p-1} \leq C(\|\mathbf{u}\|_2 + \|\omega\|_2)\|\theta\|_p^{p-1}. \tag{3.20}$$

One can refer to [dong-wu-xu-ye8] to find

$$\int \Lambda^{2\alpha}\theta|\theta|^{p-2}\theta\, dxdy > 0.$$

Applying the Gronwall's inequality, we obtain

$$\|\theta\|_p \leq C(t, \mathbf{u}_0, \mathbf{b}_0, \theta_0).$$

Taking the limit as $p \to \infty$ yields

$$\|\theta\|_\infty \leq C(t, \mathbf{u}_0, \mathbf{b}_0, \theta_0). \tag{3.21}$$

Applying the operator ∇ to both sides of the equation $(3.3)_3$, taking the inner product with the resulting equating by $|\nabla\theta|^{p-2}\nabla\theta$ and multiplying the equation $(3.3)_1$ by $|\omega|^{p-2}\omega$, integrating over $\mathbb{R}^2$ and adding them together, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\omega,\ \nabla\theta\|_p^p+\int\Lambda^{2\alpha}\nabla\theta|\nabla\theta|^{p-2}\nabla\theta\,dxdy\\
=\ &\int\nabla u_2|\nabla\theta|^{p-2}\nabla\theta\,dxdy+\int\nabla(\mathbf{u}\cdot\nabla\theta)|\nabla\theta|^{p-2}\nabla\theta\,dxdy\\
&+\int\mathbf{b}\cdot\nabla j|\omega|^{p-2}\omega\,dxdy+\int\partial_1\theta|\omega|^{p-2}\omega\,dxdy\\
=\ &J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.22}$$

Employing the Hölder's inequality, the term J_1 can be bounded as

$$J_1=\int\nabla u_2|\nabla\theta|^{p-2}\nabla\theta\,dxdy\le C\|\nabla u_2\|_p\|\nabla\theta\|_p^{p-1}.$$

Integrating by parts, the term J_2 can be bounded as

$$\begin{aligned}
J_2&=\int\nabla(\mathbf{u}\cdot\nabla\theta)|\nabla\theta|^{p-2}\nabla\theta\,dxdy=\int\partial_k(u_i\partial_i\theta)|\nabla\theta|^{p-2}\partial_k\theta\,dxdy\\
&=\int\partial_k u_i\partial_i\theta|\nabla\theta|^{p-2}\partial_k\theta\,dxdy+\int u_i\partial_{ik}\theta|\nabla\theta|^{p-2}\partial_k\theta\,dxdy\\
&=\int\partial_k u_i\partial_i\theta|\nabla\theta|^{p-2}\partial_k\theta\,dxdy\\
&\le\|\nabla\theta\|_\infty\|\nabla\mathbf{u}\|_p\|\nabla\theta\|_p^{p-1}.
\end{aligned}$$

According the estimate (3.19), we find that

$$J_3=\int\mathbf{b}\cdot\nabla j|\omega|^{p-2}\omega\,dxdy\le C\|\mathbf{b}\|_{L^\infty_{x,t}}\|\nabla j\|_p\|\omega\|_p^{p-1}.$$

Similar as J_1, one can easily check that

$$J_4=\int\partial_1\theta|\omega|^{p-2}\omega\,dxdy\le C\|\partial_1\theta\|_p\|\omega\|_p^{p-1}.$$

Combining the estimate for $J_1,\ J_2,\ J_3$ and J_4 with (3.7) and (3.22) , we have

$$\frac{d}{dt}\|\omega,\ \nabla\theta\|_p\le C\|\nabla j\|_p+C\|\omega,\ \nabla\theta\|_p,\tag{3.23}$$

where we have used the fact

$$\int\Lambda^{2\alpha}\nabla\theta|\nabla\theta|^{p-2}\nabla\theta\,dxdy>0.$$

If $\int_0^t\|\nabla j(\tau)\|_p\,d\tau>0$, the Gronwall's inequality implies the desired estimate. Next we will establish the global bound for $\|\nabla j(\tau)\|_{L^1_tL^p}$. We are first to estimate $\|\partial_{22}b_1\|_p$. Applying the operator ∂_{22} to both sides of the equation (3.9) leads to

$$\partial_{22}b_1 = \partial_{22}K^2_\beta(t) *_2 b_{01} + \int_0^t \partial_{22}K^2_\beta(t-\tau) *_2 (\mathbf{b}\cdot\nabla u_1 - \mathbf{u}\cdot b_1)\,d\tau.$$

Similar as we handled for $\|\partial_2 b_1\|_p$, taking the L^p-norm each side, we find that

$$\|\partial_{22}b_1\|_p = \|\partial_{22}K^2_\beta(t) *_2 b_{01}\|_p + \int_0^t \|\partial_{22}K^2_\beta(t-\tau)\|_{L^1_{x_2}}\|(\mathbf{b}\cdot\nabla u_1 - \mathbf{u}\cdot b_1)\|_p\,d\tau.$$

Thanks to the Hölder's and Sobolev's inequalities, we have

$$\|(\mathbf{b}\cdot\nabla u_1 - \mathbf{u}\cdot\nabla b_1)\|_p \le \|\mathbf{b}\|_\infty\|\omega\|_p + \|\mathbf{u}\|_{2p}\|\nabla b_1\|_{2p} \le C\|\omega\|_p + C\|\mathbf{u},\ \omega\|_2\|j,\ \nabla j\|_2.$$

By lemma2.2 and invoking (3.14), we deduce that

$$\|\partial_{22}b_1\|_{L^1_tL^p} = \|\partial_{22}K^2_\beta(t) *_2 b_{01}\|_{L^1_tL^p} + Ct^{1-\frac{1}{\beta}}(\|\omega\|_{L^1_tL^p}+1).$$

Thanks to the Young's inequality for convolution, on has

$$\|\partial_{22}K^2_\beta(t) *_2 b_{01}\|_{L^1_tL^p} \le \|\partial_{22}K^2_\beta(t)\|_{L^1_tL^1_{x_2}}\|b_{01}\|_p \le Ct^{1-\frac{1}{\beta}}\|b_{01}\|_p.$$

Furthermore,

$$\|\partial_{22}b_1\|_{L^1_tL^p} \le Ct^{1-\frac{1}{\beta}}(\|b_{01}\|_p + \|\omega\|_{L^1_tL^p}+1). \tag{3.24}$$

Similarly, applying the operator ∂_{11} to both sides of the equation (3.10), one can easily check that

$$\|\partial_{11}b_2\|_{L^1_tL^p} \le Ct^{1-\frac{1}{\beta}}(\|b_{02}\|_p + \|\omega\|_{L^1_tL^p}+1). \tag{3.25}$$

Using the similar methods to $\partial_{22}b_1$, we can infer that

$$\begin{aligned}\|\partial_{12}b_1\|_{L^1_tL^p} &= \|\partial_2K^2_\beta(t)\|_{L^1_tL^1_x}\|\partial_1 b_{01}\|_p + \|\partial_{22}K^2_\beta(t-\tau)\|_{L^1_tL^1_{x_2}}\|\partial_1(b_2u_1 - u_2b_1)\|_{L^1_tL^p}\\ &\le Ct^{1-\frac{1}{2\beta}}\|\partial_1 b_{01}\|_p + Ct^{1-\frac{1}{\beta}}(\|\omega\|_{L^1_tL^p}+1).\end{aligned} \tag{3.26}$$

Similarly,

$$\|\partial_{12}b_2\|_{L^1_tL^p} \le Ct^{1-\frac{1}{2\beta}}\|\partial_2 b_{02}\|_p + Ct^{1-\frac{1}{\beta}}(\|\omega\|_{L^1_tL^p}+1). \tag{3.27}$$

According to (3.24) $\sim$ (3.27) and the divergence free condition $\nabla\cdot\mathbf{b}=0$, we obtain

$$\begin{aligned}\|\nabla j\|_{L^1_tL^p} &\le \|\partial_{22}b_1\|_{L^1_tL^p} + \|\partial_{11}b_2\|_{L^1_tL^p} + \|\partial_{12}b_1\|_{L^1_tL^p} + \|\partial_{12}b_2\|_{L^1_tL^p}\\ &\le Ct^{1-\frac{1}{2\beta}} + Ct^{1-\frac{1}{\beta}}(\|\omega\|_{L^1_tL^p}+1).\end{aligned} \tag{3.28}$$

Which together with (3.23) and the Gronwall's inequality implies that

$$\|\omega,\ \nabla\theta\|_p \le C\int_0^t \|\nabla j\|_p\,d\tau + C. \tag{3.29}$$

Step3. Global bound for $\|\nabla j\|_{L^1_tL^\infty}$, $\|\omega\|_{L^\infty_{x,t}}$ and $\|\nabla\theta\|_{L^\infty_{x,t}}$ and the proof of Theorem1.1.

In this section, we will prove the crucial global bound for $\|\omega\|_{L^\infty_{x,t}}$, which make sure the global bound of $\|\mathbf{u},\ \mathbf{b},\ \theta\|_{H^s}$.

Proposition 3.3. *Assume that* $(\mathbf{u}_0,\ \mathbf{b}_0,\ \theta_0)$ *satisfies the conditions stated in Theorem1.1. Then system* (1.4) *has a global solution* $(\mathbf{u},\ \mathbf{b},\ \theta)$ *obeys the following bounds, for any* $T>0$,

$$\nabla j \in L^1(0,\ T;\ L^\infty(\mathbb{R}^2)),\quad \omega\in L^\infty(0,\ T;\ L^\infty(\mathbb{R}^2)), \nabla\theta\in L^\infty(0,\ T;\ L^\infty(\mathbb{R}^2)). \tag{3.30}$$

Proof. We will use the special structure of the nonlinear terms in the equation of $\mathbf{b}$ and the Hörmander-Mikhlin multiplier theorem to prove this proposition, which was previously used in [13], for the completeness, we give the proof as follows.

We first to estimate $\|\nabla j\|_{L^1_tL^\infty}$. Thanks to the embedding inequality, for any $\sigma > 0,\ p>\dfrac{2}{\sigma}$, we have

$$\|\nabla j\|_\infty \le C(\|\nabla j\|_2 + \|\Lambda^\sigma \nabla j\|_p). \tag{3.31}$$

Therefore, it suffices to prove that for $\sigma>0$,

$$\|\Lambda^\sigma\nabla j\|_{L^1_tL^p}<\infty.$$

We are first to show $\|\Lambda_2^\sigma\partial_{22}b_1\|_{L^1_tL^p}<\infty$. Applying the operator $\Lambda_2^\sigma\partial_{22}$ to the integral form of b_1, taking $L^1_tL^p_x$-norm, and using Young's inequality for convolution, we find that

$$\|\Lambda_2^\sigma\partial_{22}b_1\|_{L^1_tL^p_x}\le \|\Lambda_2^\sigma\partial_{22}K_\beta^2(t)\|_{L^1_tL^1_x}\|b_{01}\|_p + \|\Lambda_2^\sigma\partial_{22}K_\beta^2(t)\|_{L^1_tL^1_x}\|\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1\|_{L^1_tL^p_x}.$$

By lemma2.2, one has

$$\|\Lambda_2^\sigma\partial_{22}K_\beta^2(t)\|_{L^1_x}\le Ct^{-\frac{\sigma+2}{2\beta}},$$

furthermore, for $0<\sigma<2(\beta-1)$, we have

$$\|\Lambda_2^\sigma\partial_{22}K_\beta^2(t)\|_{L^1_tL^1_x}\le Ct^{1-\frac{\sigma+2}{2\beta}}.$$

Employing Hölder's and Sobolev's inequalities, we infer that

$$\begin{aligned}\|\mathbf{b}\cdot\nabla u_1-\mathbf{u}\cdot\nabla b_1\|_{L^1_tL^p_x} &\le \|\mathbf{b}\|_{L^\infty_{x,t}}\|\omega\|_{L^1_tL^p_x}+\|\mathbf{u}\|_{L^\infty_tL^{2p}_x}\|j\|_{L^1_tL^{2p}_x}\\ &\le C\|\omega\|_{L^1_tL^p_x}+C(\|\mathbf{u}\|_{L^\infty_tL^2}+\|\omega\|_{L^\infty_tL^2})(\|j\|_{L^1_tL^2}+\|\nabla j\|_{L^1_tL^2})\\ &\le C.\end{aligned}$$

Combining the above inequalities, we obtain

$$\|\Lambda_2^\sigma\partial_{22}b_1\|_{L^1_tL^p_x}\le Ct^{1-\frac{\sigma+2}{2\beta}}. \tag{3.32}$$

Similarly,

$$\begin{aligned}\|\Lambda_1^\sigma\partial_{11}b_2\|_{L_t^1L_x^p} &\le \|\Lambda_1^\sigma\partial_{11}K_\beta^1(t)\|_{L_t^1L_x^1}\|b_{02}\|_p + \|\Lambda_1^\sigma\partial_{11}K_\beta^1(t)\|_{L_t^1L_x^1}\|\mathbf{b}\cdot\nabla u_2 - \mathbf{u}\cdot\nabla b_2\|_{L_t^1L_x^p}\\ &\le Ct^{1-\frac{\sigma+2}{2\beta}}.\end{aligned} \tag{3.33}$$

Next we will estimate the term $\|\Lambda_2^\sigma\partial_{12}b_1\|_{L_t^1L_x^p}$. Using the similar methods to $\|\Lambda_2^\sigma\partial_{22}b_1\|_{L_t^1L_x^p}$, we find that

$$\begin{aligned}\|\Lambda_2^\sigma\partial_{12}b_1\|_{L_t^1L_x^p} &\le \|\Lambda_2^\sigma\partial_2K_\beta^2(t)\|_{L_t^1L_x^1}\|\partial_1b_{01}\|_p + \|\Lambda_2^\sigma\partial_{22}K_\beta^2(t)\|_{L_t^1L_x^1}\|\partial_1(b_2u_1-u_2b_1)\|_{L_t^1L_x^p}\\ &\le Ct^{1-\frac{\sigma+1}{2\beta}} + Ct^{1-\frac{\sigma+2}{2\beta}}.\end{aligned} \tag{3.34}$$

According to the special structure of the nonlinear terms in the equation of $\mathbf{b}$, one has

$$\begin{aligned}\|\Lambda_1^\sigma\partial_{12}b_2\|_{L_t^1L_x^p} &\le \|\Lambda_1^\sigma\partial_1K_\beta^1(t)\|_{L_t^1L_x^1}\|\partial_2b_{02}\|_p + \|\Lambda_1^\sigma\partial_{11}K_\beta^2(t)\|_{L_t^1L_x^1}\|\partial_2(b_1u_2-u_1b_2)\|_{L_t^1L_x^p}\\ &\le Ct^{1-\frac{\sigma+1}{2\beta}} + Ct^{1-\frac{\sigma+2}{2\beta}}.\end{aligned} \tag{3.35}$$

Next we will bound the difficult terms $\|\Lambda_1^\sigma\partial_{12}b_1\|_{L_t^1L_x^p}$, $\|\Lambda_1^\sigma\partial_{22}b_1\|_{L_t^1L_x^p}$, $\|\Lambda_2^\sigma\partial_{12}b_2\|_{L_t^1L_x^p}$ and $\|\Lambda_2^\sigma\partial_{11}b_2\|_{L_t^1L_x^p}$. These terms can not be handled as the above estimates. Taking $\|\Lambda_1^\sigma\partial_{22}b_1\|_{L_t^1L_x^p}$ for example, when we apply the operator $\Lambda_1^\sigma\partial_{22}$ to the integral form of b_1, Λ_1^σ has to be applied to $\mathbf{b}\cdot\nabla u_1 - \mathbf{u}\cdot\nabla b_1$. However, we have no bound for $\Lambda_1^\sigma\nabla u_1$. Fortunately, we can resort to the Hörmander- Mikhlin multiplier theorem stated in Lemma2.3. We can define it as

$$\widehat{T_hf}(\xi) = h(\xi)\hat{f}(\xi), \quad h(\xi) = \frac{|\xi_1|^\sigma|\xi_2|^2}{|\xi_1^2|^{1+\frac{\sigma}{2}} + |\xi_2^2|^{1+\frac{\sigma}{2}}}.$$

One can easily check that $h(\xi)$ satisfies the conditions stated in lemma2.3. Then we can handle the term $\|\Lambda_1^\sigma\partial_{22}b_1\|_p$ as

$$\|\Lambda_1^\sigma\partial_{22}b_1\|_p = \|T_h\Lambda_2^\sigma\partial_{22}b_1 + T_h\Lambda_1^\sigma\partial_{12}b_2\|_p \le C(\|\Lambda_2^\sigma\partial_{22}b_1\|_p + \|\Lambda_1^\sigma\partial_{12}b_2\|_p).$$

Combining the global bounds (3.32) and (3.35), we obtain

$$\|\Lambda_1^\sigma\partial_{22}b_1\|_{L_t^1L_x^p} \le C(\|\Lambda_2^\sigma\partial_{22}b_1\|_{L_t^1L_x^p} + \|\Lambda_1^\sigma\partial_{12}b_2\|_{L_t^1L_x^p}) \le Ct^{1-\frac{\sigma+1}{2\beta}} + Ct^{1-\frac{\sigma+2}{2\beta}}. \tag{3.36}$$

Using the similar method to $\|\Lambda_1^\sigma\partial_{22}b_1\|_p$, and according to (3.33) and (3.34), one can easily check that

$$\|\Lambda_2^\sigma\partial_{11}b_2\|_{L_t^1L_x^p} \le C(\|\Lambda_1^\sigma\partial_{11}b_2\|_{L_t^1L_x^p} + \|\Lambda_2^\sigma\partial_{12}b_1\|_{L_t^1L_x^p}) \le Ct^{1-\frac{\sigma+1}{2\beta}} + Ct^{1-\frac{\sigma+2}{2\beta}}. \tag{3.37}$$

Due to the divergence free condition $\nabla\cdot\mathbf{b}=0$, in a same manner as $\|\Lambda_1^\sigma\partial_{22}b_1\|_p$, we have

$$\|\Lambda_2^\sigma\partial_{12}b_2\|_{L_t^1L_x^p}\le C(\|\Lambda_1^\sigma\partial_{12}b_2\|_{L_t^1L_x^p}+\|\Lambda_2^\sigma\partial_{22}b_1\|_{L_t^1L_x^p})\le Ct^{1-\frac{\sigma+1}{2\beta}}+Ct^{1-\frac{\sigma+2}{2\beta}}. \tag{3.38}$$

Similarly,

$$\|\Lambda_1^\sigma\partial_{12}b_1\|_{L_t^1L_x^p}\le Ct^{1-\frac{\sigma+1}{2\beta}}+Ct^{1-\frac{\sigma+2}{2\beta}}. \tag{3.39}$$

Combining the estimates of (3.32) $\sim$ (3.39), we obtain

$$\|\nabla j\|_{L_t^1L_x^\infty}\le C(\|\nabla j\|_{L_t^1L_x^2}+\|\Lambda^\sigma\nabla j\|_{L_t^1L_x^p})<\infty.$$

Furthermore, taking the limit as $p\to\infty$ in (3.29), one has

$$\|\omega,\ \nabla\theta\|_\infty\le C\int_0^t\|\nabla j\|_\infty\,d\tau+C<C. \tag{3.40}$$

Where C is a constant depending on t and the initial data. Finally, we will prove the Theorem1.1. Applying the operator Λ^s to the equations of (1.4), dotting the resulting equations with $\Lambda^s\mathbf{u},\ \Lambda^s b_1,\ \Lambda^s b_2$ and $\Lambda^s\theta$ respectively and taking the L^2 inner product, we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\Lambda^s\mathbf{u},\ \Lambda^s\mathbf{b},\ \Lambda^s\theta\|_2^2+\|\Lambda^s\Lambda_2^\beta\nabla b_1,\ \Lambda^s\Lambda_1^\beta\nabla b_2,\ \Lambda^{s+\alpha}\theta\|_2^2\\
=\ &-\int[\Lambda^s,\mathbf{u}\cdot\nabla]\mathbf{u}\Lambda^s\mathbf{u}\,dxdy+\int\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b})\Lambda^s\mathbf{u}\,dxdy+\int\Lambda^s\theta\Lambda_2^s\mathbf{u}\,dxdy\\
&-\int[\Lambda^s,\ \mathbf{u}\cdot\nabla]b_1\Lambda^s b_1\,dxdy+\int\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{u})\Lambda^s\mathbf{b}\,dxdy-\int[\Lambda^s,\ \mathbf{u}\cdot\nabla]b_2\Lambda^s b_2\,dxdy\\
&-\int[\Lambda^s,\ \mathbf{u}\cdot\nabla]\theta\Lambda^s\theta\,dxdy+\int\Lambda^s u_2\Lambda_2^s\theta\,dxdy\\
=\ &\sum_{i=1}^8 L_i.
\end{aligned} \tag{3.41}$$

To start with L_1, thanks to the lemma2.1, we find that

$$L_1=-\int[\Lambda^s,\mathbf{u}\cdot\nabla]\mathbf{u}\Lambda^s\mathbf{u}\,dxdy\le C\|\nabla\mathbf{u}\|_\infty\|\Lambda^s\mathbf{u}\|_2^2\le C\|\omega\|_\infty\|\Lambda^s\mathbf{u}\|_2^2.$$

Due to the divergence free conditions $\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{b}=0$, employing lemma2.1, we can estimate the terms L_2 and L_5 together as

$$\begin{aligned}
L_2+L_5&=\int\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b})\Lambda^s\mathbf{u}\,dxdy+\int\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{u})\Lambda^s\mathbf{b}\,dxdy\\
&=\int[\Lambda^s,\ \mathbf{b}\cdot\nabla]\mathbf{b}\Lambda^s\mathbf{u}\,dxdy+\int[\Lambda^s,\ \mathbf{b}\cdot\nabla]\mathbf{u}\Lambda^s\mathbf{b}\,dxdy\\
&\le C\|\nabla\mathbf{b}\|_\infty\|\Lambda^s\mathbf{b}\|_2\|\Lambda^s\mathbf{u}\|_2+C(\|\nabla\mathbf{b}\|_\infty\|\Lambda^s\mathbf{u}\|_2+\|\nabla\mathbf{u}\|_\infty\|\Lambda^s\mathbf{b}\|_2)\|\Lambda^s\mathbf{b}\|_2\\
&\le C(1+\|\omega\|_\infty)\|\Lambda^s\mathbf{u},\ \Lambda^s\mathbf{b}\|_2^2.
\end{aligned}$$

Applying lemma2.1, we infer that

$$\begin{aligned} L_4 &= -\int [\Lambda^s,\ \mathbf{u}\cdot\nabla]b_1\Lambda^s b_1\, dxdy \\ &\le C(\|\nabla\mathbf{u}\|_\infty\|\Lambda^s b_1\|_2+\|\nabla b_1\|_\infty\|\Lambda^s\mathbf{u}\|_2)\|\Lambda^s b_1\|_2 \\ &\le C(1+\|\omega\|_\infty)\|\Lambda^s\mathbf{u},\ \Lambda^s\mathbf{b}\|_2^2. \end{aligned}$$

Similarly,

$$\begin{aligned} L_6 &= -\int [\Lambda^s,\ \mathbf{u}\cdot\nabla]b_2\Lambda^s b_2\, dxdy \\ &\le C(\|\nabla\mathbf{u}\|_\infty\|\Lambda^s b_2\|_2+\|\nabla b_2\|_\infty\|\Lambda^s\mathbf{u}\|_2)\|\Lambda^s b_2\|_2 \\ &\le C(1+\|\omega\|_\infty)\|\Lambda^s\mathbf{u},\ \Lambda^s\mathbf{b}\|_2^2. \end{aligned}$$

And

$$\begin{aligned} L_7 &= -\int [\Lambda^s,\ \mathbf{u}\cdot\nabla]\theta\Lambda^s \theta\, dxdy \\ &\le C(\|\nabla\mathbf{u}\|_\infty\|\Lambda^s \theta\|_2+\|\nabla \theta\|_\infty\|\Lambda^s\mathbf{u}\|_2)\|\Lambda^s \theta\|_2 \\ &\le C(1+\|\omega\|_\infty)\|\Lambda^s\mathbf{u},\ \Lambda^s\theta\|_2^2. \end{aligned}$$

Using Hölder's inequality, the terms L_3 and L_8 can be bounded as

$$L_3 = \int \Lambda^s\theta\Lambda_2^s\mathbf{u}\, dxdy \le C\|\Lambda^s\mathbf{u},\ \Lambda^s\theta\|_2^2,$$

and

$$L_3 = \int \Lambda^s u_2\Lambda_2^s\theta\, dxdy \le C\|\Lambda^s\mathbf{u},\ \Lambda^s\theta\|_2^2.$$

Inserting the estimates of $L_1 \sim L_8$ into (3.41), combining the proposition3.1, proposition3.2 and proposition3.3 with the Gronwall's inequality, we obtain, for any $T > 0$,

$$\|\Lambda^s\mathbf{u},\ \Lambda^s\mathbf{b},\ \Lambda^s\theta\|_2^2+\int_0^T \|\Lambda^s\Lambda_2^\beta\nabla b_1,\ \Lambda^s\Lambda_1^\beta\nabla b_2,\ \Lambda^{s+\alpha}\theta\|_2^2\, dt \le C. \tag{3.42}$$

Where C is a constant depending on t and the initial data.

In addition, we infer that

$$\begin{aligned} \|\Lambda^{s+\beta}\mathbf{b}\|_2 &\le \|\Lambda_1^{s+\beta}\mathbf{b}\|_2+\|\Lambda_2^{s+\beta}\mathbf{b}\|_2 \\ &\le \|\Lambda_1^{s+\beta}b_1\|_2+\|\Lambda_1^{s+\beta}b_2\|_2+\|\Lambda_2^{s+\beta}b_1\|_2+\|\Lambda_2^{s+\beta}b_2\|_2 \end{aligned} \tag{3.43}$$

Due to the estimates (3.1) and (3.42) and the divergence free condition $\nabla\cdot\mathbf{b}=0$, we find that

$$\begin{aligned}\|\Lambda_1^{s+\beta}b_1\|_2^2 &= \|\Lambda_1^{s-1}\partial_2\Lambda_1^{\beta}b_2\|_2^2 \le \|\Lambda_1^{s}\Lambda_1^{\beta}b_2\|_2^2 + \|\Lambda_2^{s}\Lambda_1^{\beta}b_2\|_2^2\\ &\le C\|\Lambda^{s}\Lambda_1^{\beta}b_2\|_2^2 \le C\|\Lambda_1^{\beta}b_2,\ \Lambda^{s+1}\Lambda_1^{\beta}b_2\|_2^2\\ &\le C\|\Lambda_1^{\beta}b_2,\ \Lambda^{s}\Lambda_1^{\beta}\nabla b_2\|_2^2.\end{aligned}$$

Similarly,

$$\|\Lambda_1^{s+\beta}b_2\|_2^2 \le \|\Lambda^{s}\Lambda_1^{\beta}b_2\|_2^2 \le C\|\Lambda_1^{\beta}b_2,\ \Lambda^{s+1}\Lambda_1^{\beta}b_2\|_2^2 \le C\|\Lambda_1^{\beta}b_2,\ \Lambda^{s}\Lambda_1^{\beta}\nabla b_2\|_2^2,$$

and

$$\|\Lambda_2^{s+\beta}b_1\|_2^2 \le \|\Lambda^{s}\Lambda_2^{\beta}b_1\|_2^2 \le C\|\Lambda_2^{\beta}b_1,\ \Lambda^{s}\Lambda_2^{\beta}\nabla b_1\|_2^2.$$

Applying the similar method to handle $\|\Lambda_1^{s+\beta}b_1\|_2^2$, we have

$$\begin{aligned}\|\Lambda_2^{s+\beta}b_2\|_2^2 &= \|\Lambda_2^{s-1}\partial_1\Lambda_2^{\beta}b_1\|_2^2 \le \|\Lambda_2^{s}\Lambda_2^{\beta}b_1\|_2^2 + \|\Lambda_1^{s}\Lambda_2^{\beta}b_1\|_2^2\\ &\le C\|\Lambda^{s}\Lambda_2^{\beta}b_1\|_2^2 \le C\|\Lambda_2^{\beta}b_1,\ \Lambda^{s+1}\Lambda_2^{\beta}b_1\|_2^2\\ &\le C\|\Lambda_2^{\beta}b_1,\ \Lambda^{s}\Lambda_2^{\beta}\nabla b_1\|_2^2.\end{aligned}$$

Inserting the above four estimates into (3.43), we obtain

$$\int_0^T \|\Lambda^{s+\beta}\mathbf{b}\|_2\,dt \le C\int_0^T \|\Lambda^{s}\Lambda_2^{\beta}\nabla b_1,\ \Lambda^{s}\Lambda_1^{\beta}\nabla b_2\|_2^2\,dt + C \le C.$$

Where C is a constant depending on t and the initial data. Which together with (3.42), we complete the proof of Theorem1.1.

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