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The characterization of B-Riesz potential and its commutators on generalized weighted B-Morrey spaces

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Summary

In this study, we investigate the boundedness of B-Riesz potential and its commutators on generalized weighted B-Morrey spaces. The primary aim of this study is to investigate the boundedness of Riesz potential and its commutators generated by generalized translate operator on generalized weighted B-Morrey spaces. In particular, B-Riesz potential, its commutators and B-Morrey spaces are related to generalized translate operators that is associated with Laplace Bessel differential operator. Then, two-weighted inequality for these operators and its commutators on generalized weighted B-Morrey spaces is also provided. As a corollary, the boundedness of B-Riesz potential and its commutators on generalized weighted B-Morrey spaces is proved.

KEYWORDS:
Generalized translate operator, B-maximal operator, B-Riesz potential operator, Commutators, Generalized weighted B-Morrey space.

1 | INTRODUCTION

The theory of boundedness of classical operators, such as maximal operator, fractional maximal operator, Riesz potential, singular integral operator etc, studied by many mathematicians. These results are important in Harmonic analysis and can be applied fruitful in many applications.

In this paper, we consider the generalized translate operator associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \dots, \gamma_k > 0.$$

The Riesz potential $I_{\alpha,\gamma}$ (B-Riesz potential), associated with the generalized translate operator are investigated. At first, we prove that the B-Riesz potential $I_{\alpha,\gamma}$ and their commutators for $0 < \alpha < n + |\gamma|$ is bounded from the generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$, where $\alpha/(n+|\gamma|) = 1/p - 1/q$, $1 < p < (n+|\gamma|)/\alpha$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$, $\frac{1}{p} + \frac{1}{p'} = 1$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r.

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The maximal operator M and the Riesz potential I^α are defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$. The operators M and I^α play an important role in real and harmonic analysis (see, for example[42], [43] and[38]).

In the theory of partial differential equations, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role. In[33], they were introduced by C. Morrey and defined as follows: For $0 \le \lambda \le n$ and $1 \le p < \infty$, $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak $L_p(\mathbb{R}^n)$ spaces.

F. Chiarenza and M. Frasca[9] studied the boundedness of the maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ (see, also[6,7]). Their results can be summarized as follows:

Theorem 1. Let $0 < \alpha < n$, $0 \le \lambda < n$ and $1 \le p < \infty$.

i) If $1 < p < \infty$, then M is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$.

ii) If $p = 1$, then M is bounded from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$.

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I^α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$ and for $p = 1 < q < \infty$, I^α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n - \frac{n}{q}$. In[1], D. R. Adams studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement (see, also[7])

Theorem 2. Let $0 < \alpha < n$, $0 \le \lambda < n$ and $1 \le p < \frac{n-\lambda}{\alpha}$.

i) If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness I^α from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

ii) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness I^α from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of Theorem 2 reduces to the abovementioned result by Hardy-Littlewood-Sobolev.

If in place of the power function r^λ in the definition of $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we consider any positive measurable weight function $\omega(r)$, then it becomes generalized Morrey space $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

Definition 1. Let $\omega(r)$ positive measurable weight function on $(0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ the generalized Morrey spaces, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(r)} \|f\|_{L_p(B(x,r))}.$$

T. Mizuhara[32], E. Nakai[35]([36]) and V. S. Guliyev[14] obtained sufficient conditions on weights ω_1 and ω_2 ensuring the boundedness of integral operators T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In[35], the following statement was proved, containing the result in[32] and in the general setting of metric measure spaces obtained in[39]([40]).

In[14], [32] and[35], the authors obtained sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular integral operator T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In[35], the following doubling conditions were imposed on $\omega(r)$:

$$c^{-1}\omega(r) \le \omega(t) \le c\,\omega(r), \tag{1}$$

whenever $r \le t \le 2r$, where $c \ge 1$ does not depend on t and r, jointly with the condition:

$$\int_r^\infty \omega^p(t)\frac{dt}{t} \le C\,\omega^p(r) \tag{2}$$

for the maximal or singular integral operator and the condition

$$\int_r^\infty t^{\alpha p}\omega^p(t)\frac{dt}{t} \le C\, r^{\alpha p}\omega^p(r) \tag{3}$$

for potential and fractional maximal operators, where $C > 0$ does not depend on r.

In [35], the following statements were proved.

Theorem 3. [35] Let $1 < p < \infty$ and $\omega(r)$ satisfy conditions (1)-(2). Then the operators M and singular integral operator T are bounded in $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

Theorem 4. [35] Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $\omega(t)$ satisfy conditions (1) and (3). Then the operators M^α and I^α are bounded from $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega}(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The following statement, containing the results in [32], [35], was proved in [14]. Note that Theorems 5 and 6 do not require condition (1)

Theorem 5. [14] Let $1 < p < \infty$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition

$$\int_r^\infty \omega_1(t)\frac{dt}{t} \le c_1\, \omega_2(r) \tag{4}$$

with $c_1 > 0$ not depending on $t > 0$. Then the operators M and singular integral operator T are bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$.

Theorem 6. [14] Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition

$$\int_r^\infty t^\alpha\omega_1(t)\frac{dt}{t} \le c_1\, \omega_2(r). \tag{5}$$

Then the operators M^α and I^α are bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$.

The maximal operator and potential operator related topics associated with the Laplace-Bessel differential operator have been investigated by many researchers, see B. Muckenhoupt and E.Stein [34], I. Kipriyanov [26], K. Trimeche [45], L. Lyakhov [31], K. Stempak [44], A.D. Gadjiev and I.A. Aliev [12], V.S. Guliyev [18,19], V.S. Guliyev and J.J. Hasanov [20], J.J. Hasanov [23], A. Serbetci, I. Ekincioglu [15,16,41] and others.

In this paper we consider the generalized translate operator generated by the Laplace-Bessel differential operator Δ_B in terms of which the B-maximal operator and the B-Riesz potential are investigated in the generalized weighted B-Morrey spaces.

We obtain for the operator $I_{\alpha,\gamma}$ to be bounded from generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$ and from generalized weighted B-Morrey space $\mathcal{M}_{1,\omega_1,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ to weak generalized weighted B-Morrey space $W\mathcal{M}_{q,\omega_2,\varphi,\gamma}(\mathbb{R}^n_{k,+})$.

The structure of the paper is as follows. In first section, we present some definitions and auxiliary results. In second section, we introduced generalized B-Morrey spaces. In Section 3, the boundedness of the fractional B-maximal operator on generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ is proved. In Section 4, the main result of the paper which is the Hardy-Littlewood-Sobolev theorem for B-Riesz potential in the generalized weighted B-Morrey space established. In finaly section, the boundedness of the B-singular integral operator from generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{p,\omega_2,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ is proved.

2 | DEFINITIONS, NOTATION AND PRELIMINARIES

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \ge 2$, $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n;\ x_1 > 0,\ \dots,\ x_k > 0\}$, $1 \le k \le n$, $E(x,r) = \{y \in \mathbb{R}^n_{k,+};\ |x-y| < r\}$, $E_r = E(0,r)$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$ and $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$, suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int\limits_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Denote by T^x the generalized translate operator (B-translate operator) acting according to the law

$$T^x f(y) = C_{\gamma,k} \int\limits_0^\pi \ldots \int\limits_0^\pi f\left((x',y')_\beta, x'' - y''\right) \, d\nu(\beta),$$

where $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x',y')_\beta = \left((x_1,y_1)_{\beta_1}, \ldots, (x_k,y_k)_{\beta_k}\right)$, $d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i - 1} \beta_i \, d\beta_1 \ldots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k} \omega(2k,k,\gamma).$$

We remark that the generalized translate operator T^x is closely connected with the Bessel differential operator B (for example, $n = k = 1$ see[30], $n > 1$, $k = 1$ see[26] and $n, k > 1$ see[31] for details).

Let $L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\varphi,\gamma}} = \|f\|_{L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \varphi^p(x) (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_{\infty,\varphi}} = \underset{x \in \mathbb{R}^n_{k,+}}{ess\ sup}\, \varphi(x)|f(x)|.$$

Definition 2. The weight function φ belongs to the class $A_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le p < \infty$, if

$$\sup_{x \in \mathbb{R}^n_{k,+}, r>0} \left(|E(x,r)|_\gamma^{-1} \int\limits_{E(x,r)} \varphi^p(y) (y')^\gamma dy \right)^{\frac{1}{p}}$$

$$\times \left(|E(x,r)|_\gamma^{-1} \int\limits_{E(x,r)} \varphi^{-p'}(y) (y')^\gamma dy \right)^{\frac{1}{p'}} < \infty$$

and φ belongs to $A_{1,\gamma}(\mathbb{R}^n_{k,+})$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n_{k,+}$ and $r > 0$

$$|E(x,r)|_\gamma^{-1} \int\limits_{E(x,r)} \varphi(y) (y')^\gamma dy \le C \operatorname*{ess\,sup}_{y \in E(x,r)} \frac{1}{\varphi(y)}.$$

Definition 3. The weight function (φ_1, φ_2) belongs to the class $\widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$, if

$$\sup_{x \in \mathbb{R}^n_{k,+}, r>0} \left(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi_2^p(y) (y')^\gamma dy \right)^{\frac{1}{p}}$$

$$\times \left(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi_1^{-p'}(y) (y')^\gamma dy \right)^{\frac{1}{p'}} < \infty.$$

The generalized translate operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int\limits_{\mathbb{R}^n_{k,+}} f(y) [T^x g(y)] (y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \leq \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \leq p, q \leq r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1. For all $x \in \mathbb{R}^n_{k,+}$ the following equality is valid

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0),t)} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \dots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) d\mu(z, \overline{z'}),$$

where $E((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ : \ \left|\left(x - z, \overline{z'}\right)\right| < t\}$.

Lemma 2. For all $x \in \mathbb{R}^n_{k,+}$ the following equality is valid

$$\int_{\mathbb{R}^n_{k,+}} T^y g(x)\varphi(y) M_\gamma \chi_{E_r}(y)(y')^\gamma dy$$

$$= \int_{\mathbb{R}^n \times (0,\infty)^k} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \dots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) \varphi(z, \overline{z'}) M_\nu \chi_{E((x,0),r)}(z, \overline{z'}) d\nu(z, \overline{z'}),$$

where $E((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ : \ \left|\left(x - z, \overline{z'}\right)\right| < t\}$.

Lemmas 1 and 2 are straightforward via the following substitutions

$$z'' = x'', z_i = y_i \cos\alpha_i, \ \ \overline{z_i} = y_i \sin\alpha_i, \ \ 0 \leq \alpha_i < \pi, \ i = 1, \dots, k,$$
$$y \in \mathbb{R}^n_{k,+}, \ \overline{z'} = (\overline{z}_1, \dots, \overline{z}_k), \ (z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k, \ 1 \leq k \leq n.$$

Definition 4. [18] Let $1 \leq p < \infty$ and $0 \leq \lambda \leq Q$. We denote by $\mathcal{M}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ Morrey space ($\equiv$ B-Morrey space), associated with the Laplace-Bessel differential operator the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm

$$\|f\|_{\mathcal{M}_{p,\lambda,\gamma}} = \sup_{t>0,\, x \in \mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{E_t} T^y[|f|\,]^p(x)(y')^\gamma dy\right)^{1/p}.$$

Define the B-maximal operator of f by

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y[|f|\,](x)(y')^\gamma dy,$$

and the fractional B-maximal operator by

$$M_{\alpha,\gamma} f(x) = \sup_{r>0} |E_r|_\gamma^{\frac{\alpha}{Q}-1} \int_{E_r} T^y[|f|\,](x)(y')^\gamma dy, \quad 0 \leq \alpha < Q,$$

and the B-Riesz potential by

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y[|f|\,](x)|y|^{\alpha-Q}(y')^\gamma dy, \quad 0 < \alpha < Q.$$

We write $M_{0,\gamma} f(x) = M_\gamma f(x)$ in the case where $\alpha = 0$.

Let ω and φ positive measurable weight functions. The norm in the spaces $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and $\mathcal{M}_{p,\omega,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ defined in two forms,

$$\|f\|_{\mathcal{M}_{p,\omega,\gamma}} = \sup_{x \in \mathbb{R}^n_{k,+},\, t>0} \frac{t^{-\frac{Q}{p}}}{\omega(t)} \left(\int_{E_t} T^y[|f|\,]^p(x)(y')^\gamma dy\right)^{1/p},$$

and

$$\|f\|_{\mathcal{M}_{p,\omega,\varphi,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\ t>0} \frac{1}{\omega(t)\|\varphi\|_{L_{p,\gamma}(E(0,t))}} \left(\int_{E_t} T^y[|f|\,]^p(x)\varphi(y)(y')^\gamma dy\right)^{1/p}.$$

If $\omega(t)\equiv r^{-\frac{Q}{p}}$ then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})\equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$, if $\omega(t)\equiv t^{\frac{\lambda-Q}{p}}$, $0\le\lambda<Q$, then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})\equiv\mathcal{M}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Denote by $M^\sharp_\gamma$, the sharp maximal function defined by

$$M^\sharp_\gamma f(x) = \sup_{t>0} |E(0,t)|^{-1}_\gamma \int_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy,$$

where $f_{E(0,t)}(x) = |E(0,t)|^{-1}_\gamma \int_{E(0,t)} T^y f(x)(y')^\gamma dy$.

$B-BMO$ space, $BMO_\gamma(\mathbb{R}^n_{k,+})$, defined as the space of locally integrable functions f with finite norm

$$\|f\|_{BMO_\gamma} = \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|^{-1}_\gamma \int_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy < \infty,$$

or

$$\|f\|_{BMO_\gamma} = \inf_C \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|^{-1}_\gamma \int_{E(0,t)} |T^y f(x) - C|(y')^\gamma dy < \infty.$$

The following theorem was proved in[2].

Theorem 7. **i)** Let $f\in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$. If

$$\sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} \left(|E(0,t)|^{-1}_\gamma \int_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|^p (y')^\gamma dy\right)^{1/p}$$
$$= \|f\|_{BMO_{p,\gamma}} < \infty,$$

then for any $1<p<\infty$

$$\|f\|_{BMO_\gamma} \le \|f\|_{BMO_{p,\gamma}} \le A_p \|f\|_{BMO_\gamma}$$

where the constant A_p depends only on p.

ii) Let $f\in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then, there is a constant $C>0$ such that

$$|f_{E(0,r)} - f_{E(0,t)}| \le C\|f\|_{BMO_\gamma} \ln\frac{t}{r}, \quad 0<2r<t$$

where C is independent of f, x, r and t.

Lemma 3. [25] Let $1<p<\infty$, $\varphi\in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $b\in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then

$$\|b\|_{BMO_\gamma} \approx \sup_{x\in\mathbb{R}^n_{k,+}, r>0} \frac{\left\|T^\cdot b(x) - b_{E(0,r)}\right\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.$$

3 | *B*-RIESZ POTENTIALS IN GENERALIZED *B*-MORREY SPACES WITH TWO-WEIGHTS

Theorem 8. Let $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$, $(\varphi_1,\varphi_2)\in\widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition

$$\int_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}}\frac{dr}{r} \le C\omega_2(t). \tag{6}$$

Then $I_{\alpha,\gamma}$ is bounded from $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}$.

Proof. Let $f \in \mathcal{M}_{p,\omega_1,\varphi_1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then

$$I_{\alpha,\gamma} f(x) = I_{\alpha,\gamma} f_1(x) + I_{\alpha,\gamma} f_2(x). \tag{7}$$

Firstly, we estimate $I_{\alpha,\gamma} f_1(x)$. By using the Hölder's inequality we have

$$\begin{aligned}
|I_{\alpha,\gamma} f_1(x)| &\le \int\limits_{E(0,t)} T^y |f(x)|\, |y|^{\alpha-Q} (y')^\gamma dy \\
&\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y |f(x)| (y')^\gamma dy \\
&\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \left(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y\, |f(x)|^p\, \varphi_1^p(y) (y')^\gamma dy \right)^{1/p} \\
&\times \left(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} \varphi_1^{-p'}(y) (y')^\gamma dy \right)^{1/p'} \\
&\le C \|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \|\varphi_2\|^{-1}_{L_{q,\gamma}(E(0,t))}.
\end{aligned}$$

By the inequality (6), we obtain

$$\begin{aligned}
\left\| I_{\alpha,\gamma} f_1 \right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} &\le C \|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \\
&\le C \|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \frac{\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}}{\omega_1(t)\|\varphi_1\|_{L_{p,\gamma}(E(0,t))}} \int\limits_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}} \frac{dr}{r} \\
&\le C \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t) \|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.
\end{aligned}$$

Hence, we have

$$\left\| I_{\alpha,\gamma} f_1 \right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t) \|\varphi_2\|_{L_{q,\gamma}(E(0,t))}. \tag{8}$$

Now we estimate $I_{\alpha,\gamma} f_2(x)$. By using the Hölder's inequality, we get

$$\begin{aligned}
|I_{\alpha,\gamma} f_2(x)| &\le \int\limits_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y |f(x)| |y|^{\alpha-Q} (y')^\gamma dy \\
&\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-Q} \int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y |f(x)| (y')^\gamma dy \\
&\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-Q} \left(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} \varphi_1^{-p'}(y) (y')^\gamma dy \right)^{1/p'} \\
&\times \left(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y\, |f(x)|^p\, \varphi_1^p(y) (y')^\gamma dy \right)^{1/p} \\
&\le C \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \int\limits_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}} \frac{dr}{r}.
\end{aligned}$$

Thus, from the inequality (6), we get

$$|I_{\alpha,\gamma} f_2(x)| \le C \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t). \tag{9}$$

So, from (8) and (9), we have

$$\begin{aligned}
\left\| I_{\alpha,\gamma} f \right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} &\le \left\| I_{\alpha,\gamma} f_1 \right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} + \left\| I_{\alpha,\gamma} f_2 \right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \\
&\le C \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t) \|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.
\end{aligned}$$

Finally $I_{\alpha,\gamma} f \in \mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\| I_{\alpha,\gamma} f \right\|_{\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}} \leq C \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}.$$

□

Corollary 1. Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$. The operator $I_{\alpha,\gamma}$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n)$ to $L_{q,\varphi_2,\gamma}(\mathbb{R}^n)$.

Corollary 2. Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$. The operator $M_{\alpha,\gamma}$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n)$ to $L_{q,\varphi_2,\gamma}(\mathbb{R}^n)$.

4 | COMMUTATORS OF B-RIESZ POTENTIAL IN B-MORREY SPACES WITH TWO WEIGHTS

In this section, we consider commutators of the B-Riesz potential defined as the following equality

$$[b, I_{\alpha,\gamma}]f(x) = \int_{\mathbb{R}^n_{k,+}} (b(x) - b(y))|y|^{\alpha-Q} T^y f(x)(y')^\gamma dy, \quad 0 < \alpha < Q.$$

Given a measurable function b the operator $|b, I_{\alpha,\gamma}|$ is defined by

$$|b, I_{\alpha,\gamma}|f(x) = \int_{\mathbb{R}^n_{k,+}} |b(x) - b(y)||y|^{\alpha-Q} T^y f(x)(y')^\gamma dy, \quad 0 < \alpha < Q.$$

Theorem 9. Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition (6). Then $|b, I_{\alpha,\gamma}|$ is bounded from $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}$.

Proof. Let $f \in \mathcal{M}_{p,\omega_1,\varphi_1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then

$$\begin{aligned} |b, I_{\alpha,\gamma}|f(x) &= \left(\int_{E(0,t)} + \int_{\mathbb{R}^n_{k,+} \setminus E(0,t)} \right) T^y |[b - b(x)] f(x)||y|^{\alpha-Q}(y')^\gamma dy \\ &\equiv F_1(x,t) + F_2(x,t). \end{aligned}$$

Firstly, we estimate $F_1(x,t)$. By using the Hölder's inequality, we have

$$\begin{aligned} F_1(x,t) &= \int_{E(0,t)} T^y |[b - b(x)] f(x)||y|^{\alpha-Q}(y')^\gamma dy \\ &\leq \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y |[b - b(x)] f(x)|(y')^\gamma dy \\ &\leq \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} |T^y b(x) - b|^{p'} \varphi_1^{-1}(y)(y')^\gamma dy \right)^{1/p'} \\ &\times \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y \, |f(x)|^p \, \varphi_1(y)(y')^\gamma dy \right)^{1/p} \\ &\leq C \, \|b\|_{BMO_\gamma} \, \|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \|\varphi_2\|^{-1}_{L_{q,\gamma}(E(0,t))}. \end{aligned}$$

By the inequality (6), we obtain

$$\|F_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C\, \|b\|_{BMO_\gamma}\, \|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))}$$
$$\le C\, \|b\|_{BMO_\gamma}\, \|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \frac{\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}}{\omega_1(t)\|\varphi_1\|_{L_{p,\gamma}(E(0,t))}} \int_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}} \frac{dr}{r}$$
$$\le C\, \|b\|_{BMO_\gamma}\, \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.$$

Hence we have

$$\|F_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C\, \|b\|_{BMO_\gamma}\, \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}. \tag{10}$$

Now we estimate $F_2(x,t)$. By using the Hölder's inequality, we get

$$F_2(x,t) \le \int_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y(|b-b(x)||f(x)|)|y|^{\alpha-Q}(y')^\gamma dy$$
$$\le \sum_{j=0}^\infty \left(2^j t\right)^{\alpha-Q} \int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y(|b-b(x)||f(x)|)(y')^\gamma dy$$
$$\le \sum_{j=0}^\infty \left(2^j t\right)^{\alpha-Q} \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} |T^y b(x)-b|^{p'} \varphi_1^{-1}(y)(y')^\gamma dy\right)^{1/p'}$$
$$\times \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y\,|f(x)|^p\, \varphi_1(y)(y')^\gamma dy\right)^{1/p}$$
$$\le C\, \|b\|_{BMO_\gamma}\, \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \int_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}} \frac{dr}{r}.$$

Thus y the inequality (6), we get

$$F_2(x,t) \le C\, \|b\|_{BMO_\gamma}\, \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t). \tag{11}$$

Therefore, from (10) and (11), we have

$$\left\||b, I_{\alpha,\gamma}|f\right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le \left\|F_1\right\|_{L_{q,\varphi_2,\gamma}(E(0,t))} + \left\|F_2\right\|_{L_{q,\varphi_2,\gamma}(E(0,t))}$$
$$\le C\, \|b\|_{BMO_\gamma}\, \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.$$

Finally, we get $|b, I_{\alpha,\gamma}|f \in \mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\left\||b, I_{\alpha,\gamma}|f\right\|_{\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}} \le C\, \|b\|_{BMO_\gamma}\, \|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}.$$

□

Corollary 3. Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$ and $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$. The operator $|b, I_{\alpha,\gamma}|$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n)$ to $L_{q,\varphi_2,\gamma}(\mathbb{R}^n)$.

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