Qualitative Analysis of Stochastically Perturbed HIV Model

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Abstract

The dynamical behavior of a perturbed Human Immunodeficiency Virus (HIV) model is investigated in this paper. We first determine a positively invariant set in which the perturbed system admits a unique, positive, global solution. Following that, we discuss the stability of infection-free equilibrium of the deterministic model. We also obtain the conditions required for the p^{th}-moment exponential stability, for the perturbed system. Later we show that if $R_0 > 1$, for smaller intensity of noise, the solution of stochastic system oscillates around E^*. Our results demonstrate that a large value of noise suppresses the disease from persistence exponentially. We also derive the condition for the persistence of the disease. Finally, comparison of our analytical results with simulations is to be done.

Keywords: Stochastic perturbation, Endemic equilibrium, Exponentially stable in mean square, Extinction, Persistence.

1 Introduction

Epidemiology is a branch of medical science that studies infectious diseases in populations and is concerned with all aspects of an epidemic, such as its transmission, control, vaccination strategy. Many models addressing the transmission of infectious diseases are based on the famous SIR model of Kermack and Mckendrick[6].

AIDS, which is caused by the HIV virus, is one of the life threatening disease. However, there is a global commitment to reducing new HIV infections and providing treatment to all HIV-positive persons.

Mathematical modelling is a common technique for understanding and researching infectious disease dynamics, as well as suggesting disease outbreak mitigation strategies.

In 2013, Swarnali Sharma and G.P. Samanta[13] developed a five compartmental HIV model:

$$\begin{cases} \frac{dS}{dt} = \Lambda - (\beta_1 I_a + \beta_2 I_s)S - \mu S \\ \frac{dI_a}{dt} = (\beta_1 I_a + \beta_2 I_s)S + \eta(I_a + I_s) - (\delta + \mu)I_a \\ \frac{dI_s}{dt} = \delta I_a - (\sigma + \kappa + \mu)I_s \\ \frac{dT}{dt} = \kappa I_s - (\gamma + \mu)T \\ \frac{dA}{dt} = \gamma T + \sigma I_s - (d + \mu)A \end{cases} \tag{1.1}$$

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Here, the entire population is split up into five groups: Susceptible population $S(t)$, infective population without symptoms $I_a(t)$, infective population with symptoms $I_s(t)$, infected population under treatment $T(t)$ and full-blown AIDS group $A(t)$; the key parameters are: Λ, the recruitment rate of S; β_1, horizontal transmission rate ofI_a; β_2, horizontal transmission rate of I_s; η, recruitment rate of new borne infected children into I_a; δ, progression rate from I_a to I_s; κ, the proportion of I_s who enter into T; σ, progression rate to A from I_s; γ, transfer rate from T to A; d, death rate of A due to the disease; μ, the natural death rate.

In [13], the authors introduced a single discrete time delay on the sub model:

$$\begin{cases} \frac{dS}{dt} = \Lambda - (\beta_1 I_a + \beta_2 I_s)S - \mu S \\ \frac{dI_a}{dt} = (\beta_1 I_a + \beta_2 I_s)S + \eta(I_a + I_s) - (\delta + \mu)I_a \\ \frac{dI_s}{dt} = \delta I_a - (\sigma + \kappa + \mu)I_s \end{cases} \tag{1.2}$$

with initial conditions:

$$S(0) > 0, \quad I_a(0) > 0, \quad I_s(0) \geq 0.$$

In deterministic system (1.2), the threshold parameter is $R_0 = \frac{\Lambda}{\mu}\frac{\beta_1(\kappa+\sigma+\mu)+\delta\beta_2}{(\delta+\mu-\eta)(\kappa+\sigma+\mu)-\delta\eta}$. They[13] showed that if $R_0 < 1$, the disease-free equilibrium of the system is asymptotically stable, and the disease disappears in the population and if $R_0 > 1$, the infection persists because the system (1.2) possess a unique endemic equilibrium $E^* = (S^*, I_a^*, I_s^*)$ that is asymptotically stable.

Environmental noise should be incorporated in every dynamic population model since all biological populations display some sort of stochastic behaviour. A brownian motion process is commonly used to introduce such noise. Deterministic elements serve to make the model of the response variable predictable from the initial conditions, whereas stochastic elements can be attributed to a variety of sources, including demographic stochasticity, environmental stochasticity, mensuration stochasticity, and informational stochasticity[9]. In reality environmental stochasticity[9] refers to the impact on local and global populations of factors such as weather, major accidents, epidemics,natural disasters, agriculture failures and international dislocations. These factors are expected to work independently in the long run. Random variations in the environment influence the whole population in the case of environmental stochasticity. With this motivation, we have study the long term behavior of deterministic model (1.2) with environmental stochasticity.

Stochasticity may be included into an epidemic model in a variety of ways. Many authors included environmental stochasticity into model parameters and developed numerous characteristics on the behaviour of perturbed models[2, 5, 10, 11, 15, 16]. Another way to introduce stochasticity into deterministic models is telegraph noise where the parameter switch from one set to another according to a Markov switching process[14]. Greenhalgh et.al.[3] studied SIS epidemic model with demographic stochasticity. However in this paper we include environmental stochasticity into the horizontal transmission coefficient parameter β_1.

We can improve the accuracy of our estimations or predictions of epidemic diseases by including random phenomena in our model. This need for greater accuracy has prompted the use of stochastic process models, and these types of models can be solved via stochastic differential equations, that is the sample path of the process can be described by SDE.

In epidemic models, the disease transmission rate is the key parameter for disease transmission. In this article, we include environmental variations in the transmission coefficient rate, making it more biologically plausible for the HIV dynamics transmission in a homogeneously mixed population of varied sizes. To account for the impact of a randomly changing environment in system (1.2), white noise is introduced in the parameter β_1,

$$\beta_1 \to \beta_1 + \lambda \dot{B}(t),$$

where $B(t)$ denotes one dimensional standard Brownian motion with the property $B(0) = 0$, λ represents the noise intensity.

We then get the following stochastic epidemic HIV model of (1.2) to be studied in this paper:

$$\begin{cases} dS = [\Lambda - (\beta_1 I_a + \beta_2 I_s)S - \mu S]\, dt - \lambda I_a S dB(t) \\ dI_a = [(\beta_1 I_a + \beta_2 I_s)S + \eta(I_a + I_s) - (\delta + \mu)I_a]\, dt + \lambda I_a S dB(t) \\ dI_s = [\delta I_a - (\sigma + \kappa + \mu)I_s]\, dt \end{cases} \tag{1.3}$$

The main purpose of our work is to analyze the stochastic version of deterministic model (1.2) and examine the long term behavior of the system (1.3).

The paper is structured as follows. We will establish the existence of a unique positive solution in the next Section. The asymptotic behavior of the solution to the stochastic model around the disease-free equilibrium of the underlying model is to study in Section 3. We examine the stability of the stochastic model near the endemic equilibrium point of the deterministic model in Section 4. Further more we examine the p^{th} moment exponential stability of E_0. We will investigate two of the most prominent components of every biological system, namely, the conditions necessary for extinction and persistence in Sections 6 and 7. Finally, to validate the analytical conclusions, we will do numerical simulations with realistic parameter values.

Throughout the article, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filteration $\{\mathcal{F}_t\}_{t\geq 0}$ that meets the standard criteria (i.e., it is right continuous and $\mathcal{F}_0$ includes all P-null sets)[7].

Denote $R^n_+ = \{x \in R^n : x_i > 0, \text{ for all } 1 \leq i \leq n\}$, $\overline{R^n_+} = \{x \in R^n : x_i \geq 0, \text{ for all } 1 \leq i \leq n\}$.

A general d-dimensional stochastic differential equation takes the form[7]

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \text{ on } t \geq t_0 \tag{1.4}$$

with initial value $x(t_0) = x_0 \in R^d$, where $B(t)$ denotes d-dimensional standard Brownian motion defined on the above probability space.

2 Existence and Uniqueness of Global Positive Solution

In this section, we show that the system (1.3) has a unique, positive and global solution using Lyapunov analysis method[2].

Theorem 2.1. *For any initial value $(S(0), I_1(0), I_2(0)) \in R^3_+$, the system (1.3) admits a unique solution $(S(t), I_a(t), I_s(t))$ on $t \geq 0$, and the solution will remain in R^3_+ with probability 1, (i.e.), $(S(t), I_a(t), I_s(t)) \in R^3_+$ for all $t \geq 0$ almost surely.*

Proof: Note that the coefficient of the equations in system (1.3) are locally Lipschitz continuous for any given initial value $(S(0), I_1(0), I_2(0)) \in R^3_+$. Therefore, the system possesses a unique local solution $(S(t), I_a(t), I_s(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time[1]. To demonstrate that this solution is global, just we want to show that $\tau_e = \infty$

a.s. Allow $k^{'} \geq 1$ to be sufficiently large enough that all $(S(0), I_1(0), I_2(0))$ lie inside the interval $[1/k^{'}, k^{'}]$. For each integer $n \geq k^{'}$, let us define the stopping time as

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I_a(t), I_s(t)\} \leq \frac{1}{n} \quad \text{or} \quad \max\{S(t), I_a(t), I_s(t)\} \geq n\right\},$$

where, throughout the paper, we assume $\inf \oslash = \infty$ (as usual $\oslash$ denotes the empty set). Clearly, τ_n is increasing when $n \to \infty$. Denote $\tau_\infty = \lim_{n\to\infty} \tau_n$, when $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., consequently $\tau_e = \infty$ and $(S(t), I_a(t), I_s(t)) \in R^3_+$ a.s. for all $t \geq 0$. To put it in other words, we just need to claim that $\tau_\infty = \infty$ a.s.
If it is not, we can find some $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon.$$

Thus there is an integer $k_1 \geq k^{'}$ such that

$$P\{\tau_n \leq T\} \geq \epsilon \quad \text{for all}\, n \geq k_1. \tag{2.1}$$

Meanwhile, for $t \leq \tau_n$,

$$\begin{aligned} d(S + I_a + I_s) &= [\Lambda - \mu S + \eta(I_a + I_s) - \mu I_a - \sigma I_s - \kappa I_s - \mu I_s]dt \\ &\leq [\Lambda - (\mu - \eta)(S + I_a + I_s)]dt \end{aligned}$$

and

$$S(t) + I_a(t) + I_s(t) \leq \begin{cases} \frac{\Lambda}{\mu-\eta}, & \text{when} \quad S(0) + I_1(0) + I_2(0) \leq \frac{\Lambda}{\mu-\eta} \\ S(0) + I_1(0) + I_2(0), & \text{when} \quad S(0) + I_1(0) + I_2(0) > \frac{\Lambda}{\mu-\eta} \end{cases} : K$$

Now a C^2 function $V_1 : R^3_+ \to \overline{R}_+$ is defined by

$$V_1(S, I_a, I_s) = (S - 1 - \log S) + (I_a - 1 - \log I_a) + (I_s - 1 - \log I_s),$$

which is non-negative, since $v - 1 - \log v \geq 0, \quad \forall\, v \geq 0$.
It$\hat{o}$'s formula yields,

$$\begin{aligned} dV_1 &= \left(1 - \frac{1}{S}\right)[(\Lambda - \beta_1 I_a S - \beta_2 I_s S - \mu S)]\, dt \\ &\quad + \left(1 - \frac{1}{I_a}\right)[(\beta_1 I_a S + \beta_2 I_s S + \eta(I_a + I_s) - (\delta + \mu) I_a)]\, dt \\ &\quad + \left(1 - \frac{1}{I_s}\right)[\delta I_a - (\sigma + \kappa + \mu) I_s]\, dt + \left[\frac{1}{2}\lambda^2 I_a^2 + \frac{1}{2}\lambda^2 S^2\right] dt + \lambda(I_a - S)\, dB(t) \\ &\leq [\Lambda + 3\mu + \delta + \sigma + \kappa + (\eta + \beta_1 + \beta_2)K + \frac{1}{2}\lambda^2 K^2]\, dt + \lambda(I_a - S)\, dB(t) \\ &= \widetilde{K} dt + \lambda(I_a - S)\, dB(t). \end{aligned}$$

The remainder of the proof follows that in [8] and hence is omitted here.

Theorem 2.2. *For any initial value $(S(0), I_1(0), I_2(0)) \in \overline{R}^3_+$, the solution of system (1.3) will remain in $\overline{R}^3_+$ with probability 1.*

Proof: Clearly, First equation of (1.3) gives

$$\begin{aligned} S(t) &= e^{-\mu t-\int\limits_0^t[\beta_1 I_a(u)+\beta_2 I_s(u)+\frac{\lambda^2}{2}I_a^2(u)]du-\lambda\int\limits_0^t I_a(u)dB(u)} \\ &\left[S(0)+\Lambda\int\limits_0^t e^{\mu u+\int\limits_0^t[\beta_1 I_a(u)+\beta_2 I_s(u)+\frac{\lambda^2}{2}I_a^2(u)]du+\lambda\int\limits_0^t I_a(u)dB(u)}du\right] \end{aligned}$$

Then $S(t) > 0$ if $S(0) > 0$ or $S(0) = 0$.
Next, we consider the infective population without symptoms $I_a(t)$.

$$\begin{aligned} I_a(t) &= e^{-(\delta+\mu-\eta)t+\int\limits_0^t[\beta_1 S(u)-\frac{\lambda^2}{2}S^2(u)]du+\lambda\int\limits_0^t S(u)dB(u)} \\ &\left[I_a(0)+\int\limits_0^t(\beta_2 I_s(u)S(u)+\eta I_s(u))e^{(\delta+\mu-\eta)u-\int\limits_0^t[\beta_1 S(u)-\frac{\lambda^2}{2}S^2(u)]du-\lambda\int\limits_0^t S(u)dB(u)}du\right] \end{aligned}$$

Obviously, $I_a(t) > 0$ no matter $I_a(0) > 0$ or $I_a(0) = 0$.
Third equation of (1.3) yields,

$$\begin{aligned} \frac{dI_s}{dt} &\geq -(\kappa+\sigma+\mu)I_s \\ \Longrightarrow I_s(t) &\geq I_s(0)e^{-(\kappa+\sigma+\mu)t} \geq 0. \end{aligned}$$

($i.e.$) We can conclude that the variables $S(t) > 0, \quad I_a(t) > 0 \quad$ and $\quad I_s(t) \geq 0$.

Remark 2.3. From Theorem 2.1 and 2.2, we see that that for any initial value $(S(0), I_1(0), I_2(0)) \in \overline{R}_+^3$, the system(1.3) admits a unique, global solution $(S(t), I_a(t), I_s(t)) \in \overline{R}_+^3$ almost surely. Therefore,

$$d(S+I_a+I_s) \leq \left[\Lambda-(\mu-\eta)(S+I_a+I_s)\right]dt$$

and

$$S(t)+I_a(t)+I_s(t) \leq \frac{\Lambda}{\mu-\eta}+e^{-(\mu-\eta)t}\left[S(0)+I_1(0)+I_2(0)-\frac{\Lambda}{\mu-\eta}\right]$$

Obviously, $S(t)+I_a(t)+I_s(t) \leq \frac{\Lambda}{\mu-\eta}$, when $S(0)+I_1(0)+I_2(0) \leq \frac{\Lambda}{\mu-\eta}$.

$\therefore$ The region defined by

$$\Omega^* = \left\{(S, I_a, I_s) : S > 0, I_a > 0, I_s \geq 0, \ S+I_a+I_s \leq \frac{\Lambda}{\mu-\eta}\right\}$$

is a positively invariant set of the stochastic system (1.3).
Hereafter, we assume that any initial solution $(S(0), I_1(0), I_2(0)) \in \Omega^*$.

3 Asymptotic Behavior around Disease-Free Equilibrium

Theorem 2.1 shows that the perturbed system (1.3) will remain to have a global positive solution in $\bar{R}_+^3$. In the sequel we therefore only need to consider how the solution behaves in $\bar{R}^3_+$. Because there is no explicit solution to the model (1.3), asymptotic behavior must be studied.

In this section, we primarily use the stochastic Lyapunov function[7] to determine the stability of the disease free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ of the deterministic model.

Lemma 3.1. [7] *If there exists a positive-definite decrescent radially unbounded function $V(x,t) \in C^{2,1}(R^d \times [t_0, \infty); \overline{R}_+)$ such that $LV(x,t)$ is negative-definite, then the trivial solution of Equation (1.4) is stochastically asymptotically stable in the large.*

In this section, we present the main theorem based on the aforementioned lemma. Define

$$R_1 = \frac{\frac{\Lambda}{\mu}[\beta_1(\kappa+\sigma+\mu) + \beta_2(\delta+\eta)] + \frac{\lambda^2}{2}(\frac{\Lambda}{\mu})^2(\sigma+\kappa+\mu) + \eta^2}{(\delta+\mu-\eta)(\sigma+\kappa+\mu) - \delta\eta}$$

Theorem 3.2. *If $R_1 < 1$, $(\delta+\mu-\eta)^2 < 1$ and $\frac{\mu(\delta+\eta)(\eta+\beta_2\frac{\Lambda}{\mu})(\kappa+\sigma+\mu)}{(\delta+\mu-\eta)(\kappa+\sigma+\mu)-\delta\eta} \leq 1 - R_1$, then the disease-free equilibrium $(\frac{\Lambda}{\mu}, 0, 0)$ of the system (1.3) is stochastically asymptotically stable in the large.*

Proof. Let $u = S - \frac{\Lambda}{\mu}$, $v = I_a$, $w = I_s$.
Then $u \leq 0$, $v > 0$, $w \geq 0$ and system (1.3) takes the form

$$\begin{aligned}
du(t) &= \left[\Lambda - \beta_1 v(u+\frac{\Lambda}{\mu}) - \beta_2 w(u+\frac{\Lambda}{\mu}) - \mu(u+\frac{\Lambda}{\mu})\right]dt - \lambda v\left(u+\frac{\Lambda}{\mu}\right)dB(t)\\
dv(t) &= \left[\beta_1 v(u+\frac{\Lambda}{\mu}) + \beta_2 w(u+\frac{\Lambda}{\mu}) + \eta(v+w) - (\delta+\mu)v\right]dt + \lambda v\left(u+\frac{\Lambda}{\mu}\right)dB(t) \qquad (3.1)\\
dw(t) &= \left[\delta v - (\kappa+\sigma+\mu)w\right]dt
\end{aligned}$$

Define the stochastic Lyapunov function $R^3 \to \bar{R}_+$:

$$V_2(u,v,w) = (u+v)^2 + cv^2 + w^2,$$

where $c > 0$, to be found later.

It is easily check that $\frac{c}{2+c}u^2 + \frac{c}{2}v^2 + w^2 \leq V_2(u,v,w) \leq 2u^2 + (2+c)v^2 + w^2$.

Thus $V_2(u,v,w)$ is positive-definite, decrescent and radially unbounded.
Let L be the generating operator of system (1.3). Then

$$\begin{aligned}
LV_2 &= 2(u+v)[-\mu u - (\delta+\mu-\eta)v + \eta w] + 2cv\left[\beta_1 uv + \beta_2 uw + \left(\beta_1\frac{\Lambda}{\mu} + \eta - \delta - \mu\right)v\right.\\
&\quad \left. + \left(\beta_2\frac{\Lambda}{\mu} + \eta\right)w\right] + c\lambda^2\left(u+\frac{\Lambda}{\mu}\right)^2 v^2 + 2w[\delta v - (\sigma+\kappa+\mu)w]\\
&= -2\mu u^2 - 2uv(\delta+2\mu-\eta) - 2(1+c)(\delta+\mu-\eta)v^2 + 2cv\beta_1 uv^2 + 2c\frac{\Lambda}{\mu}\beta_1 v^2\\
&\quad + c\lambda^2\left(u+\frac{\Lambda}{\mu}\right)^2 v^2 + 2\eta vw(1+c) + 2\eta uw + 2c\beta_2 vw + 2cvw\beta_2\frac{\Lambda}{\mu}\\
&\quad + 2\delta vw - 2(\sigma+\kappa+\mu)w^2 \qquad (3.2)
\end{aligned}$$

Note that $u \leq 0$ and $c\lambda^2 v^2 \left(u + \frac{\Lambda}{\mu}\right)^2 \leq c\lambda^2 v^2 (\frac{\Lambda}{\mu})^2$. Then

$$\begin{aligned} LV_2 &\leq -2\mu u^2 - 2(\delta + \mu - \eta + \mu)uv + [2c\beta_1 \frac{\Lambda}{\mu} + c\lambda^2 (\frac{\Lambda}{\mu})^2 - 2(1+c)(\delta + \mu - \eta)]v^2 \\ &\quad + 2\left[(\delta + \eta) + c(\eta + \beta_2 \frac{\Lambda}{\mu})\right] vw - 2(\kappa + \sigma + \mu)w^2 \end{aligned}$$

With the identity $a + b \leq 2ab$, we get

$$\begin{aligned} LV &\leq -2\mu u^2 + 2(2(\delta + \mu - \eta)\mu)uv + 2(2(\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})c)vw \\ &\quad + [2c\beta_1 \frac{\Lambda}{\mu} + c\lambda^2 (\frac{\Lambda}{\mu})^2 - 2(1+c)(\delta + \mu - \eta)]v^2 - 2(\kappa + \sigma + \mu)w^2 \end{aligned}$$

Again applying the identity $2ab \leq a^2 + b^2$,

$$\begin{aligned} LV &\leq -2\mu u^2 + 2(\delta + \mu - \eta)^2 \mu u^2 + 2\mu v^2 + 2\frac{(\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})c}{\kappa + \sigma + \mu} v^2 \\ &\quad + 2(\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})(\kappa + \sigma + \mu)cw^2 \\ &\quad + [2c\beta_1 \frac{\Lambda}{\mu} + c\lambda^2 (\frac{\Lambda}{\mu})^2 - 2(1+c)(\delta + \mu - \eta)]v^2 - 2(\kappa + \sigma + \mu)w^2 \end{aligned}$$

$$\begin{aligned} LV &\leq -2\mu(1 - (\delta + \mu - \eta)^2)u^2 \\ &\quad + 2\left[\mu + c\left\{\frac{(\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})}{\kappa + \sigma + \mu} + \beta_1 \frac{\Lambda}{\mu} + \lambda^2 (\frac{\Lambda}{\mu})^2 - (\delta + \mu - \eta)\right\} - (\delta + \mu - \eta)\right] v^2 \\ &\quad - 2\left[(\kappa + \sigma + \mu) - (\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})(\kappa + \sigma + \mu)c\right] w^2 \\ &\leq -2\mu(1 - (\delta + \mu - \eta)^2)u^2 - 2(\delta + \mu - \eta)v^2 \\ &\quad + \frac{2}{\kappa + \sigma + \mu}\left[\mu(\kappa + \sigma + \mu) + c\{\beta_1 \frac{\Lambda}{\mu}(\kappa + \sigma + \mu) + (\delta + \mu)\beta_2 \frac{\Lambda}{\mu} + \eta^2 + \lambda^2 (\frac{\Lambda}{\mu})^2 (\kappa + \sigma + \mu)\right. \\ &\quad \left. - [(\delta + \mu - \eta)(\kappa + \sigma + \mu) - \delta\eta]\}\right] v^2 - 2\left[(\kappa + \sigma + \mu) - (\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})(\kappa + \sigma + \mu)c\right] w^2 \\ &\leq -2\mu(1 - (\delta + \mu - \eta)^2)u^2 - 2(\delta + \mu - \eta)v^2 \\ &\quad + \frac{2}{\kappa + \sigma + \mu}\{\mu(\kappa + \sigma + \mu) + c(R_1 - 1)[(\delta + \mu - \eta)(\kappa + \sigma + \mu) - \delta\eta]\}v^2 \\ &\quad - 2\left[(\kappa + \sigma + \mu) - (\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})(\kappa + \sigma + \mu)c\right] w^2 \end{aligned}$$

Choose $c = \frac{\mu(\kappa+\sigma+\mu)}{(1-R_1)[(\delta+\mu-\eta)(\kappa+\sigma+\mu)-\delta\eta]}$ so that

$$\mu(\kappa + \sigma + \mu) + c(R_1 - 1)[(\delta + \mu - \eta)(\kappa + \sigma + \mu) - \delta\eta] = 0$$

Then

$$LV_2 \leq -2\mu(1-(\delta+\mu-\eta)^2)u^2 - 2(\delta+\mu-\eta)v^2 - 2\left[(\kappa + \sigma + \mu) - (\delta + \eta)(\eta + \beta_2 \frac{\Lambda}{\mu})(\kappa + \sigma + \mu)c\right] w^2 \tag{3.3}$$

which is negative-definite.

(i.e.) V_2 satisfies the conditions in Lemma 3.1. Thus we conclude that when $R_1 < 1$with additional conditions, the disease-free equilibrium $(\frac{\Lambda}{\mu}, 0, 0)$ of the system (1.3) is stochastically asymptotically stable in the large.

4 Moment Exponential Stability

In this section, we investigate the p^{th}-moment exponential stability, for the stochastic system and we shall assume $p > 0$.

Theorem 4.1. [12] *Let $p, d_1, d_2, d_3 > 0$. If there exists a function $W(t, y) \in C^{2,1}(R_+ \times R^d, R_+)$, satisfying the property*

$$d_1 \mid y \mid^p \leq W(t, y) \leq d_2 \mid y \mid^p \quad \text{and} \quad W(t, y) \leq -d_3 \mid y \mid^p, t \geq 0, \tag{4.1}$$

then the equilibrium of stochastic system (1.3) is p^{th}-moment exponentially stable. Furthermore When $p = 2$, it is said to be exponentially stable in mean square and the equilibrium point $y = 0$ is globally asymptotically stable.

As a result of Young's inequality, (i.e.), $mn \leq \frac{m^p}{p} + \frac{n^p}{p}$ for $m, n > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we obtain the following.

Lemma 4.2. [12] Set $p \geq 2$ and $\epsilon,\ m,\ n > 0$. Then

$$\begin{aligned} m^{p-1}n &\leq \frac{(p-1)\epsilon}{p}m^p + \frac{1}{p\epsilon^{p-1}}n^p \\ \text{and} \quad m^{p-2}n^2 &\leq \frac{(p-2)\epsilon}{p}m^p + \frac{2}{p\epsilon^{(\frac{p-2}{2})}}n^p \end{aligned}$$

The following theorem is proved using the aforementioned lemma.

Theorem 4.3. Assume $p \geq 2$. If the conditions $\Lambda = \mu$ and $\beta_1 \frac{\Lambda}{\mu - \eta} + \frac{p-1}{2}(\frac{\Lambda}{\mu-\eta})^2 \lambda^2 < (\delta + \mu - \eta)$ hold, then the solution E_0 of the stochastically perturbed system (1.3) is p^{th} -moment exponentially stable in Ω^*.

Proof:

Define $V_3 = a_1(1 - S)^p + a_2 \frac{1}{p} I_1^p + a_3 \frac{1}{p} I_2^p$, where a_1, a_2, a_3 are real positive constants.
By Itô's formula,

$$\begin{aligned} LV_3 &= -a_1 p(1 - S)^{p-1}[\Lambda - (\beta_1 I_1 + \beta_2 I_s)S - \mu S] + \frac{a_1}{2}p(p-1)(1 - S)^{p-2}\lambda^2 I_a^2 S^2 \\ &\quad + a_2 I_a^{p-1}[(\beta_1 I_a + \beta_2 I_s)S + \eta(I_a + I_s) - (\delta + \mu)I_a] + \frac{a_2}{2}(p-1)I_a^{p-2}\lambda^2 I_a^2 S^2 \\ &\quad + a_3 I_s^{p-1}[\delta I_a - (\kappa + \sigma + \mu)I_s] \end{aligned}$$

In Ω^*, we have $max\{S, I_a, I_s\} \leq \frac{\Lambda}{\mu - \eta}$ and hence

$$\begin{aligned} LV_3 \leq\ & -a_1 p\mu(1 - S)^p + a_2 \left[\beta_1 \frac{\Lambda}{\mu - \eta} - (\delta + \mu - \eta) + \frac{\lambda^2}{2}(\frac{\Lambda}{\mu - \eta})^2 (p-1)\right] I_a^p - a_3(\kappa + \sigma + \mu)I_s^p \\ & + a_1 p\beta_1 \frac{\Lambda}{\mu - \eta}(1 - S)^{p-1}I_a + a_1 p\beta_2 \frac{\Lambda}{\mu - \eta}(1 - S)^{p-1}I_s + a_2(\eta + \beta_2 \frac{\Lambda}{\mu - \eta})I_a^{p-1}I_s \\ & + a_3\delta I_s^{p-1}I_a + a_1 \frac{\lambda^2}{2}p(p-1)(\frac{\Lambda}{\mu - \eta})^2(1 - S)^{p-2}I_a^2 \end{aligned} \tag{4.2}$$

Applying Lemma (4.2) for any $\epsilon > 0$, we get

$$\begin{aligned} (1 - S)^{p-1}I_a &\leq \frac{(p-1)\epsilon}{p}(1 - S)^p + \frac{1}{p\epsilon^{p-1}}I_a^p \\ (1 - S)^{p-1}I_s &\leq \frac{(p-1)\epsilon}{p}(1 - S)^p + \frac{1}{p\epsilon^{p-1}}I_s^p \\ (1 - S)^{p-2}I_1^2 &\leq \frac{(p-2)\epsilon}{p}(1 - S)^p + \frac{2}{p\epsilon^{(\frac{p-2}{2})}}I_a^p \\ I_a^{p-1}I_s &\leq \frac{(p-1)\epsilon}{p}I_a^p + \frac{1}{p\epsilon^{p-1}}I_s^p \\ I_a I_s^{p-1} &\leq \frac{(p-1)\epsilon}{p}I_s^p + \frac{1}{p\epsilon^{p-1}}I_a^p \end{aligned}$$

Substituting the above inequalities into (4.2), we have

$$
\begin{aligned}
LV_3 \;\leq\; & -a_1 p\mu(1-S)^p + a_2\left[\beta_1\frac{\Lambda}{\mu-\eta} - (\delta+\mu-\eta) + \frac{\lambda^2}{2}(\frac{\Lambda}{\mu-\eta})^2(p-1)\right] I_a^p \\
& -a_3(\kappa+\sigma+\mu)I_s^p + a_1 p\beta_1\frac{\Lambda}{\mu-\eta}\left(\frac{(p-1)\epsilon}{p}(1-S)^p + \frac{1}{p\epsilon^{p-1}}I_a^p\right) \\
& +a_1 p\beta_2\frac{\Lambda}{\mu-\eta}\left(\frac{(p-1)\epsilon}{p}(1-S)^p + \frac{1}{p\epsilon^{p-1}}I_s^p\right) \\
& +a_2(\eta+\beta_2\frac{\Lambda}{\mu-\eta})\left(\frac{(p-1)\epsilon}{p}I_a^p + \frac{1}{p\epsilon^{p-1}}I_s^p\right) + a_3\delta\left(\frac{(p-1)\epsilon}{p}I_s^p + \frac{1}{p\epsilon^{p-1}}I_a^p\right) \\
& +a_1\frac{\lambda^2}{2}p(p-1)(\frac{\Lambda}{\mu-\eta})^2\left[\frac{(p-2)\epsilon}{p}(1-S)^p + \frac{2}{p\epsilon^{\frac{p-2}{2}}}I_a^p\right]
\end{aligned}
$$

$$
\begin{aligned}
(i.e.)\quad LV_3 \;\leq\; & a_1\left[-p\mu + \beta_1\frac{\Lambda}{\mu-\eta}(p-1)\epsilon + \beta_2\frac{\Lambda}{\mu-\eta}(p-1)\epsilon\right. \\
& \left.+\frac{\lambda^2}{2}(p-1)(p-2)\left(\frac{\Lambda}{\mu-\eta}\right)^2\epsilon\right](1-S)^p \\
& +\left[a_2\left\{\beta_1\frac{\Lambda}{\mu-\eta} - (\delta+\mu-\eta) + \frac{p-1}{2}\left(\frac{\Lambda}{\mu-\eta}\right)^2\lambda^2 + (\eta+\beta_2\frac{\Lambda}{\mu-\eta})\frac{p-1}{p}\epsilon\right\}\right. \\
& \left.+a_1\left\{\beta_1\frac{\Lambda}{\mu-\eta}\frac{1}{\epsilon^{p-1}} + \lambda^2(p-1)\left(\frac{\Lambda}{\mu-\eta}\right)^2\frac{1}{\epsilon^{\frac{p-2}{2}}}\right\} + a_3\frac{\delta}{p\epsilon^{p-1}}\right] I_a^p \\
& +\left[-a_3(\kappa+\sigma+\mu) + a_1\left(\beta_2\frac{\Lambda}{\mu-\eta}\frac{1}{\epsilon^{p-1}} + (\eta+\beta_2\frac{\Lambda}{\mu-\eta})\frac{1}{p\epsilon^{p-1}}\right) + a_3\frac{\delta(p-1)}{p}\epsilon\right] I_s^p.
\end{aligned}
$$

As $\beta_1\frac{\Lambda}{\mu-\eta} + \frac{p-1}{2}(\frac{\Lambda}{\mu-\eta})^2\lambda^2 < (\delta+\mu-\eta)$, we choose ϵ sufficiently small as well as the constants $a_i, (i=1,2,3)$ such that the coefficients of $(1-S)^p, I_a^p$ and I_s^p are negative.

Thus the p^{th} moment of E_0 approaches to 0 exponentially fast.

Remark 4.4. From Lemma 4.2, Theorem 4.3 and for $p=2$, E_0 is exponentially stable in mean square when $\Lambda = \mu$ and $\beta_1\frac{\Lambda}{\mu-\eta} + \frac{1}{2}(\frac{\Lambda}{\mu-\eta})^2\lambda^2 < (\delta+\mu-\eta)$.

5 Asymptotic Behavior Near the Endemic Equilibrium of the Deterministic System

In this section, we look at how the solution of the stochastic system (1.3) behaves towards the endemic equilibrium of the deterministic system E^*, to see whether the disease will prevail.

Theorem 5.1. *Let $(S(t), I_a(t), I_s(t))$ be the solution of system (1.3) with any initial value $(S(0), I_1(0), I_2(0)) \in R_+^3$. If $R_0 > 1$, then we have*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t [(S(s)-S^*)^2 + (I_a(s)-I_a^*)^2 + (I_s(s)-I_s^*)^2]ds \leq K_1\lambda^2 \quad a.s.,$$

where (S^*, I_a^*, I_s^*) is the endemic equilibrium of system (1.2),

$$K_1 = \frac{(2+c_1)\left(\frac{\Lambda}{\mu-\eta}\right)^4}{\min\{2(\mu-\eta), (\delta+\mu), 2c_2(\kappa+\sigma+\mu) - (\eta(1+c_1)+\delta c_2) - 2\eta\}}$$

$$\textit{where } c_1 = \frac{(\delta+2\mu-\eta)(\mu-\eta)}{\Lambda(\beta_1+\beta_2)} \textit{ and } c_2 = \frac{(1+2c_1)(\delta+\mu-\eta)-\eta c_1+2c_1S^*(\beta_1+\beta_2)\dfrac{\Lambda}{\mu-\eta}}{\delta}$$

Proof: Define

$$V_4 = (S-S^*+I_a-I_a^*)^2 + c_1(I_a-I_a^*)^2 + c_2(I_s-I_s^*)^2$$

where c_1, c_2 are positive constants to be found later.
By Itô's formula,
$dV_4 = LV_4 dt + 2\lambda c_1 S I_a(I_a - I_a^*)dB(t)$,
where

$$\begin{aligned}
LV_4 \leq\ & 2(S-S*+I_a-I_a^*)\left[(\delta+\mu)I_a^* - \eta(I_a^*+I_s^*) + \mu S^* - \mu S + \eta(I_a+I_s) - (\delta+\mu)I_a\right] \\
& +2c_1(I_a-I_a^*)\{(\beta_1+\beta_2)\frac{\Lambda}{\mu-\eta}(S-S^*) + (\beta_1+\beta_2)\frac{\Lambda}{\mu-\eta}S^* + \eta(I_a+I_s) \\
& -(\delta+\mu)(I_a-I_a^*) - \eta(I_a^*+I_s^*) - (\beta_1 I_a^* + \beta_2 I_s^*)S^*\} \\
& +2c_2(I_s-I_s^*)\left[\delta(I_a-I_a^*) - (\kappa+\sigma+\mu)(I_s-I_s^*)\right] + (2+c_1)\lambda^2 I_a^2 S^2 \\
\therefore LV_4 \leq\ & -2\mu(S-S^*)^2 - 2(1+c_1)(\delta+\mu-\eta)\ (I_a-I_a^*)^2 - 2c_2(\kappa+\sigma+\mu)\ (I_s-I_s^*)^2 \\
& +\ (S-S^*)\ (I_a-I_a^*)\left\{-2(\delta+2\mu-\eta) + 2c_1(\beta_1+\beta_2)\frac{\Lambda}{\mu-\eta}\right\} \\
& +\ (I_a-I_a^*)\ (I_s-I_s^*)\{2(1+c_1)\eta + 2c_2\delta\} + 2\eta\ (I_s-I_s^*)\ (S-S^*) \\
& +2c_1S^*\left\{(\beta_1+\beta_2)\frac{\Lambda}{\mu-\eta} - (\beta_1 I_a^* + \beta_2 I_s^*)\right\}\ (I_a-I_a^*)^2 + (2+c_1)\lambda^2 I_a^2 S^2
\end{aligned}$$

Choose $c_1 = \frac{(\delta+\mu-\eta)(\mu-\eta)}{\Lambda(\beta_1+\beta_2)}$ so that $\frac{2c_1\Lambda(\beta_1+\beta_2)}{\mu-\eta} - 2(\delta+\mu-\eta) = 0$.

$$\begin{aligned}
\therefore LV_4 \leq\ & -2\mu\ (S-S^*)^2 \\
& -\left\{2(1+c_1)(\delta+\mu-\eta) - (\eta(1+c_1)+\delta c_2) + 2c_1S^*(\beta_1+\beta_2)\frac{\Lambda}{\mu-\eta}\right\}\ (I_a-I_a^*)^2 \\
& -\{2c_2(\kappa+\sigma+\mu) - (\eta(1+c_1)+\delta c_2) - 2\eta\}\ (I_s-I_s^*)^2 + (2+c_1)\lambda^2 I_a^2 S^2
\end{aligned}$$

Choose $c_2 = \frac{(1+2c_1)(\delta+\mu-\eta)-\eta c_1++2c_1S^*(\beta_1+\beta_2)\frac{\Lambda}{\mu-\eta}}{\delta}$ such that
$2(1+c_1)(\delta+\mu-\eta) - (\eta c_1+\delta c_2) + 2c_1S^*(\beta_1+\beta_2)\dfrac{\Lambda}{\mu-\eta} - (\delta+\mu-\eta) = 0.$

$$\begin{aligned}
\therefore LV_4 \leq\ & -2(\mu-\eta)\ (S-S^*)^2 - (\delta+\mu)\ (I_a-I_a^*)^2 \\
& -\{2c_2(\kappa+\sigma+\mu) - (\eta(1+c_1)+\delta c_2) - 2\eta\}\ (I_s-I_s^*)^2 + (2+c_1)\lambda^2 I_a^2 S^2 \\
\leq\ & -\left\{m_1\ (S-S^*)^2 + m_2\ (I_a-I_a^*)^2 + m_3\ (I_s-I_s^*)^2\right\} + (2+c_1)\lambda^2 I_a^2 S^2 \\
\therefore dV_4 \leq\ & \Big\{-min\{m_1,m_2,m_3\}\left[\ (S-S^*)^2 +\ (I_a-I_a^*)^2 +\ (I_s-I_s^*)^2\right] \\
& +(2+c_1)\lambda^2\left(\frac{\Lambda}{\mu-\eta}\right)^4 t\Big\}\ dt + 2\lambda c_1 S I_a\ (I_a-I_a^*)dB(t)
\end{aligned}$$

Integrating it from 0 to t gives,

$$\begin{aligned}
V_4(t) - V_4(0) \leq\ & -min\{m_1,m_2,m_3\}\int_0^t \left[\ (S-S^*)^2 +\ (I_a-I_a^*)^2 +\ (I_s-I_s^*)^2\right] ds \\
& +(2+c_1)\lambda^2\left(\frac{\Lambda}{\mu-\eta}\right)^4 t + 2\lambda c_1\int_0^t S(s)I_a(s)(I_a(s)-I_a^*)dB(s)
\end{aligned}$$

From this,

$$\int_0^t \left[(S(s)-S^*)^2+(I_a(s)-I_a^*)^2+(I_s(s)-I_s^*)^2\right]ds$$

$$\leq \frac{V_4(0)-V_4(t)}{\min\{m_1,m_2,m_3\}}+\frac{(2+c_1)\lambda^2\left(\frac{\Lambda}{\mu-\eta}\right)^4}{\min\{m_1,m_2,m_3\}}t$$

$$+\frac{2c_1\lambda}{\min\{m_1,m_2,m_3\}}\int_0^t S(s)I_a(s)(I_a(s)-I_a^*)dB(s) \tag{5.1}$$

Let $M(t)=\int_0^t S(s)I_a(s)(I_a(s)-I_a^*)dB(s)$.
Clearly $M(t)$ is continuous, local martingale and also $M(0)=0$.
By the strong law of large numbers[7], we can easily prove that

$$\lim_{t\to\infty}\frac{M(t)}{t}=\frac{\int_0^t S(u)I_a(u)(I_a(u)-I_a^*)dB(u)}{t}=0 \quad \text{a.s.} \tag{5.2}$$

It therefore follows from (5.1) that

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t [(S(s)-S^*)^2+(I_a(s)-I_a^*)^2+(I_s(s)-I_s^*)^2\,ds \quad &\leq \quad \frac{(2+c_1)\lambda^2\left(\frac{\Lambda}{\mu-\eta}\right)^4}{\min\{m_1,m_2,m_3\}}\\ &\leq \quad K_1\lambda^2 \quad \text{a.s.}\end{aligned}$$

This completes the proof of Theorem 5.1.

Remark 5.2. Theorem 5.1 shows that, if $R_0>1$, the solution of the system (1.3) oscillates around the endemic equilibrium E^*, for a long time while the intensity of the white noise is small.

6 Extinction

In this section, we develop a criteria for the case that the infected populations eventually die out.

For this, let us define

$\langle y(t)\rangle=\frac{1}{t}\int_0^t y(u)\,du$ and as in the proof in [15], we can conclude the following lemma.

Lemma 6.1. *Let $(S(t),I_a(t),I_s(t))$ be the solution of the system (1.3) with any given initial value $(S(0),I_a(0),I_s(0))\in R_+^3$. Then $\lim_{t\to\infty}\frac{S(t)+I_a(t)+I_s(t)}{t}=0$ a.s.*
Furthermore

$$\lim_{t\to\infty}\frac{S(t)}{t}=0,\quad \lim_{t\to\infty}\frac{I_a(t)}{t}=0,\quad \lim_{t\to\infty}\frac{I_s(t)}{t}=0 \quad a.s.$$

Define $\tilde{R_1}=\frac{\Lambda}{\mu}\frac{\beta_1(\kappa+\sigma+\mu)+\delta\beta_2}{\mu-\eta}$.

Theorem 6.2. *Let $(S(t), I_a(t), I_s(t))$ be the solution of the system (1.3) with any given initial value $(S(0), I_a(0), I_s(0)) \in R^3_+$. If $\tilde{R}_1 < 1$, the model (1.3) has the following property,*

$$\lim_{t\to\infty} \langle S(t)\rangle = \frac{\Lambda}{\mu},\ \lim_{t\to\infty} \langle I_a(t)\rangle = 0,\ \lim_{t\to\infty} \langle I_s(t)\rangle = 0 \quad a.s.$$

In other words, $I_a(t)$ and $I_s(t)$ will go to zero exponentially with probability one.

Proof: we compute that

$$d(S + I_a) = (\Lambda - \mu S - (\delta + \mu - \eta)I_a + \eta I_s)\, dt$$

Integrating from 0 to t and dividing t on both sides, we get

$$\frac{S(t) + I_a(t)}{t} - \frac{S(0) + I_a(0)}{t} = \Lambda - \mu\langle S(t)\rangle - (\delta + \mu - \eta)\langle I_a(t)\rangle + \eta\langle I_s(t)\rangle$$

From this,

$$\langle S(t)\rangle = \frac{\Lambda}{\mu} - \frac{(\delta + \mu - \eta)}{\mu}\langle I_a(t)\rangle + \frac{\eta}{\mu}\langle I_s(t)\rangle + \frac{1}{\mu}\phi(t) \tag{6.1}$$

where $\phi(t) = \frac{S(0)+I_a(0)}{t} - \frac{S(t)+I_a(t)}{t}$.
From Lemma 6.1, we can obtain that $\lim_{t\to\infty} \phi(t) = 0. \quad a.s.$

If we consider the equation $dI_s = \delta I_a - (\kappa + \sigma + \mu)$, and integrating from 0 to t and dividing t on both sides, we get

$$\langle I_s(t)\rangle \ = \ \frac{\delta}{\kappa + \sigma + \mu}\langle I_a(t)\rangle - \frac{1}{\kappa + \sigma + \mu}\phi_1(t) \tag{6.2}$$

where $\phi_1(t) = \frac{I_s(t)-I_s(0)}{t}$.
Again from Lemma 6.1, $\lim_{t\to\infty} \phi_1(t) = 0 \quad a.s.$
Now Choose $V_5 = \ln(I_a(t) + I_s(t))$.
By Itô's formula,

$$\begin{aligned} dV_5 &= \frac{1}{I_a + I_s}\left\{(\beta_1 I_a + \beta_2 I_s)S + \eta(I_a + I_s) - (\delta + \mu)I_a + \delta I_a - (\kappa + \sigma + \mu)I_s\right\} dt \\ &\quad - \frac{1}{(I_a + I_s)^2}\lambda^2 I_a^2 S^2\, dt \\ &\leq \frac{1}{I_a + I_s}\left\{(\beta_1 I_a + \beta_2 I_s)S + \eta(I_a + I_s) - \mu(I_a + I_s) - (\kappa + \sigma)(I_a + I_s) + (\kappa + \sigma)I_a\right\} dt \\ &\leq [(\beta_1 + \beta_2)S - (\mu - \eta)]dt \end{aligned}$$

Integrating on both sides from 0 to t yields,

$$\begin{aligned} \ln(I_a(t) + I_s(t)) &\leq \ln(I_a(0) + I_s(0)) + (\beta_1 + \beta_2)\int_0^t S(u)du - (\mu - \eta)t \\ (i.e.) \quad \frac{\ln(I_a(t) + I_s(t))}{t} &\leq \frac{\beta_1 + \beta_2}{t}\langle S(t)\rangle - (\mu - \eta) + \frac{\ln(I_a(0) + I_s(0))}{t} \end{aligned}$$

Using (6.1), we have

$$\begin{aligned}\frac{\ln(I_a(t)+I_s(t))}{t} &\leq \left[(\beta_1(\kappa+\sigma+\mu)+\delta\beta_2)\frac{\Lambda}{\mu}-(\mu-\eta)\right] \\ &\quad -(\beta_1+\beta_2)\left[\frac{\delta+\mu-\eta}{\mu}-\frac{\delta\eta}{\mu(\kappa+\sigma+\mu)}\right]\langle I_a(t)\rangle \\ &\quad -\frac{\eta}{\mu(\kappa+\sigma+\mu)}\frac{I_s(t)-I_s(0)}{t}+\frac{\beta_1+\beta_2}{\mu}\phi(t) \\ &\leq -(\mu-\eta)[1-\tilde{R_1}]-\frac{\eta}{\mu(\kappa+\sigma+\mu)}\frac{I_s(t)-I_s(0)}{t}+\frac{\beta_1+\beta_2}{\mu}\phi(t)\end{aligned}$$

$\therefore \limsup\limits_{t\to\infty} \frac{\ln(I_a(t)+I_s(t))}{t} \leq 0 \quad a.s.$

That is $\lim\limits_{t\to\infty}(I_a(t)+I_s(t)) = 0 \quad a.s.$

Hence $\lim\limits_{t\to\infty} I_a(t) = 0 \quad a.s.$ and $\lim\limits_{t\to\infty} I_s(t) = 0 \quad a.s.$
(i.e.) The disease will extinct with probability one.
Note that, from (6.1), $\lim\limits_{t\to\infty}\langle S(t)\rangle = \frac{\Lambda}{\mu} \quad a.s.$

As a result, the infective populations extinct over time but, the susceptible population stabilizes at $\frac{\Lambda}{\mu}$.

7 Persistence in Mean

This section establishes a criterion for persistence in the mean of the disease.

We begin by recalling the notion of persistence in mean.

Definition 7.1. *The stochastic system (1.3) is said to be persistence in mean, if*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_a(u)du > 0, \quad \liminf_{t\to\infty}\frac{1}{t}\int_0^t I_s(u)du > 0 \quad a.s.$$

Let $\bar{\lambda} = \frac{1}{2}\lambda^2\left(\frac{\Lambda}{\mu-\eta}\right)^2$.
Define $\tilde{R_2} = \dfrac{\Lambda\delta\beta_2}{(\delta+\mu+\bar{\lambda})(\kappa+\sigma+\mu)(\mu+\bar{\lambda})}$ and $k_1 = (\mu+\bar{\lambda})(\tilde{R_2}-1)$.

Theorem 7.2. *Let $(S(t), I_a(t), I_s(t))$ be the solution of the system (1.3) with any given initial solution $(S(0), I_a(0), I_s(0)) \in R^3_+$. If $\tilde{R_2} > 1$ holds, then $I_a(t)$ and $I_s(t)$ persistence in mean. Also*

$$\liminf_{t\to\infty}\langle I_s(t)\rangle \geq \frac{k_1\delta}{\beta_1(\kappa+\sigma+\mu)+\delta\beta_2}, \quad \liminf_{t\to\infty}\langle I_a(t)\rangle \geq \frac{k_1(\kappa+\sigma+\mu)}{\beta_1(\kappa+\sigma+\mu)+\delta\beta_2} \quad a.s.$$

Proof:
Consider the function $V_6(S, I_a, I_s) = -\ln S - b_1 \ln I_a - b_2 \ln I_s$, where b_1 and b_2 are positive constants to be found later.

By Ito's formula,

$$\begin{aligned}
dV_6 \quad &\leq \quad \left[-\frac{\Lambda}{S} + (\beta_1 I_a + \beta_2 I_s) + \mu - \frac{b_1 \beta I_s S}{I_a} - \eta b_1 \frac{I_s}{I_a} - \delta b_2 \frac{I_s}{I_a} + \frac{1}{2}\lambda^2 I_a^2 + b_1(\delta + \mu + \frac{1}{2}\lambda^2 S^2)\right. \\
&\qquad \left. + b_2(\kappa + \sigma + \mu)\right] dt + \lambda(I_a - b_1 S) dB(t) \\
&\leq \quad \left[-\frac{\Lambda}{S} + (\beta_1 I_a + \beta_2 I_s) + \mu - \frac{b_1 \beta I_s S}{I_a} - \eta b_1 \frac{I_s}{I_a} + \frac{1}{2}\lambda^2 \left(\frac{\Lambda}{\mu - \eta}\right)^2\right. \\
&\qquad \left. + b_1 \left(\delta + \mu + \frac{1}{2}\lambda^2 \left(\frac{\Lambda}{\mu - \eta}\right)^2\right) + b_2(\kappa + \sigma + \mu)\right] dt + \lambda(I_a - b_1 S) dB(t)
\end{aligned}$$

Thus we have

$$dV_6 \leq [-3\sqrt[3]{\Lambda \delta \beta_2 b_1 b_2} + \mu + (\beta_1 I_a + \beta_2 I_s) + \bar{\lambda} + b_1[\delta + \mu + \bar{\lambda}] + b_2(\kappa + \sigma + \mu)]dt + \lambda(I_a - b_1 S)dB(t)$$

Choose $b_1[\delta + \mu + \bar{\lambda}] = \frac{\Lambda \delta \beta_2}{(\delta + \mu + \bar{\lambda})(\kappa + \sigma + \mu)} = b_2(\kappa + \sigma + \mu)$.

$$\therefore b_1 = \frac{\Lambda \delta \beta_2}{(\delta + \mu + \bar{\lambda})^2 (\kappa + \sigma + \mu)} \quad \text{and} \quad b_2 = \frac{\Lambda \delta \beta_2}{(\delta + \mu + \bar{\lambda})(\kappa + \sigma + \mu)^2}$$

So,

$$\begin{aligned}
dV_6 \quad &\leq \quad \left[-\frac{\Lambda \delta \beta_2}{(\delta + \mu + \bar{\lambda})(\kappa + \sigma + \mu)} + \mu + \bar{\lambda} + \beta_1 I_a + \beta_2 I_s + \frac{1}{2}\lambda^2 \left(\frac{\Lambda}{\mu - \eta}\right)^2\right] dt + \lambda(I_a - b_1 S) dB(t) \\
&= \left[-(\mu + \bar{\lambda}) \left[\frac{\Lambda \delta \beta_2}{(\delta + \mu + \bar{\lambda})(\kappa + \sigma + \mu)(\mu + \bar{\lambda})} - 1\right] + \beta_1 I_a + \beta_2 I_s\right] dt + \lambda(I_a - b_1 S) dB(t) \\
&\leq \quad [-k_1 + \beta_1 I_a + \beta_2 I_s] dt + \lambda(I_a - b_1 S) dB(t)
\end{aligned}$$

Integrating from 0 to t gives,

$$\begin{aligned}
V(S(t), I_a(t), I_s(t)) - V(S(0), I_a(0), I_s(0)) \quad &\leq \quad -k_1 t + \beta_1 \int_0^t I_a(u) du + \beta_2 \int_0^t I_s(u) du \\
&\qquad + \lambda \int_0^t (I_a(u) - b_1 S(u)) dB(u)
\end{aligned}$$

Dividing t on both sides, we get

$$\begin{aligned}
\frac{\ln(S(0) - \ln S(t))}{t} + b_1 \frac{\ln(I_a(0) - \ln I_a(t))}{t} + b_2 \frac{\ln(I_s(0) - \ln I_s(t))}{t} \quad &\leq \quad -k_1 + \beta_1 \langle I_a(t) \rangle + \beta_2 \langle I_s(t) \rangle \\
&\qquad + \frac{\lambda}{t} \int_0^t (I_a(u) - b_1 S(u)) dB(u)
\end{aligned}$$

From (6.2), We have

$$\begin{aligned}
&\frac{\ln(S(0) - \ln S(t))}{t} + b_1 \frac{(\ln I_a(0) - \ln I_a(t))}{t} + b_2 \frac{\ln(I_s(0) - \ln I_s(t))}{t} \\
&\leq \quad -k_1 + \beta_1 \left[\frac{\kappa + \sigma + \mu}{\delta} \langle I_s(t) \rangle + \frac{1}{\delta}\phi_1(t)\right] + \beta_2 \langle I_s(t) \rangle + \frac{\lambda}{t} \int_0^t (I_a(u) - b_1 S(u)) dB(u) \\
&= -k_1 + \left[\frac{\beta_1(\kappa + \sigma + \mu)}{\delta} + \beta_2\right] \langle I_s(t) \rangle + \frac{\beta_1}{\delta}\phi_1(t) + \frac{\lambda}{t} M_1(t), \qquad (7.1)
\end{aligned}$$

where $M_1(t) = \lambda \int\limits_0^t (I_a(u) - b_1 S(u))dB(u)$ which is continuous local martingale and $M(0) = 0$, and its quadratic variation

$$\langle M_1, M_1 \rangle_t = \int\limits_0^t (I_a(u) - b_1 S(u))^2 du \leq (1 + b_1^2) \left(\frac{\Lambda}{\mu - \eta} \right)^2 t$$

$$\text{Thus} \quad \limsup_{t\to\infty} \frac{\langle M_1, M_1 \rangle_t}{t} \leq (1 + b_1^2) \left(\frac{\Lambda}{\mu - \eta} \right)^2 < \infty \quad a.s.$$

Again by the strong law of large numbers given in [7],

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = \frac{1}{t} \int\limits_0^t (I_a(u) - b_1 S(u))dB(u) = 0 \quad a.s.$$

Letting $t \to \infty$ in (7.1), we have

$$0 \leq -k_1 + \left[\frac{\beta_1(\kappa + \sigma + \mu) + \delta\beta_2}{\delta} \right] \lim_{t\to\infty} \langle I_s(t) \rangle.$$

Hence

$$\lim_{t\to\infty} \langle I_s(t) \rangle \geq \frac{k_1 \delta}{\beta_1(\kappa + \sigma + \mu) + \delta\beta_2}, \quad \text{so} \quad \liminf_{t\to\infty} \langle I_s(t) \rangle \geq \frac{k_1 \delta}{\beta_1(\kappa + \sigma + \mu) + \delta\beta_2}.$$

Also from (6.2), we can see that

$$\liminf_{t\to\infty} \langle I_a(t) \rangle \geq \frac{k_1(\kappa + \sigma + \mu)}{\beta_1(\kappa + \sigma + \mu) + \delta\beta_2}$$

This show that the disease persists in mean.

8 Numerical Simulation

This section deals with the numerical simulations to support our analytical results, using Euler's Higher Order method[4] with initial value $(x_1, x_2, x_3) = (1, 1, 0.5)$ and time step $\sqrt{\Delta t}$=0.004.

The discretization equations are:

$$\begin{aligned}
x_1{}^{(j+1)} &= x_1{}^{(j)} + (\Lambda - \beta_1 x_2{}^{(j)} x_1{}^{(j)} - \beta_2 x_1{}^{(j)} x_2{}^{(j)} - \mu x_1{}^{(j)})\Delta t - (\lambda x_1{}^{(j)} x_2{}^{(j)})\sqrt{\Delta t}; \\
x_2{}^{(j+1)} &= x_2{}^{(j)} + (\beta_1 x_2{}^{(j)} x_1{}^{(j)} + \beta_2 x_1{}^{(j)} x_2{}^{(j)} + \eta(x_2{}^{(j)} + x_3{}^{(j)}) - (\delta + \mu) x_2{}^{(j)})\Delta t + (\lambda x_1{}^{(j)} x_2{}^{(j)})\sqrt{\Delta t}; \\
x_3{}^{(j+1)} &= x_3{}^{(j)} + (\delta x_2{}^{(j)} - (\mu + \sigma + \kappa) x_3{}^{(j)})\Delta t;
\end{aligned}$$

Swarnali et.al.[13] shown that the disease free equilibrium E_0 of the deterministic system (1.2) is globally stable when $R_0 < 1$. We are now performing numerical simulations on the perturbed system (1.3).

Case(i): In Figure 1, the parameters are: $\Lambda = 1$; $\mu = 0.7$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; $\lambda = 0.5$ resulting $R_1 = 0.8832 < 1$ and $R_0 = 0.5656 < 1$. Therefore the condition of Theorem 3.3 is met. The disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ of the stochastic system (1.3) is globally asymptotically stable in Ω^*, as shown in Theorem 3.3. This was verified by Figure 1.

The deterministic and the stochastic systems have comparable characteristics. Both solutions of the system tend to the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0) = (1.4246, 0, 0)$.

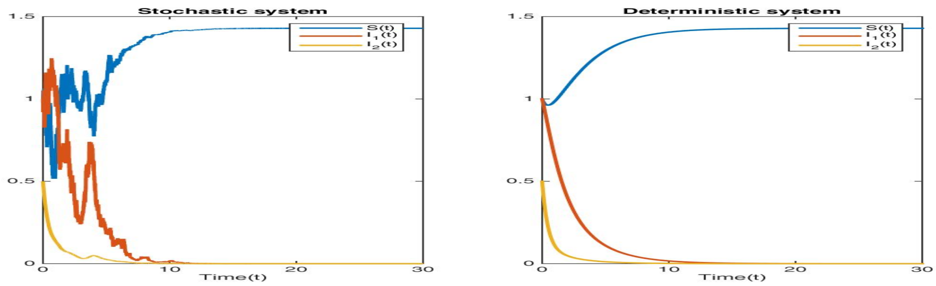

Figure 1: The trajectories of the stochastic model (1.3) and the deterministic model (1.2) for parameter values $\Lambda = 1$; $\mu = 0.7$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; $\lambda = 0.5$ such that $R_1 = 0.8832 < 1$ and $R_0 = 0.5656 < 1$.

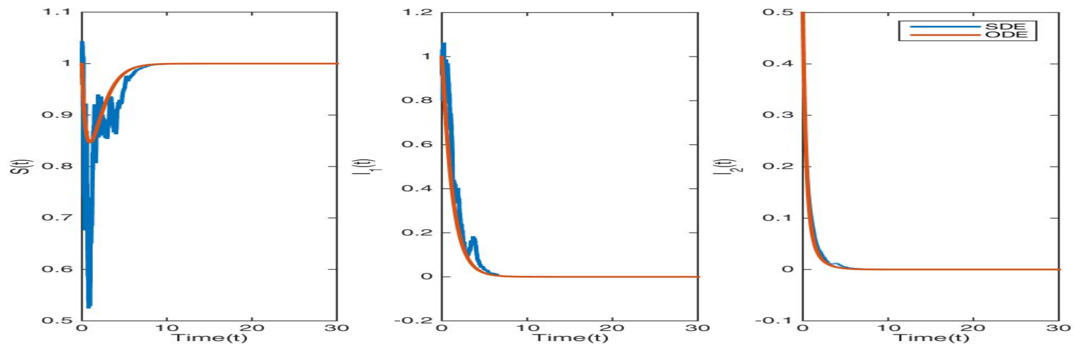

Figure 2: The trajectories of system (1.3) for parameter values $\Lambda = 1$; $\mu = 0.5$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; with different noise intensities $\lambda = 0, 0.1, 0.05, 0.01$.

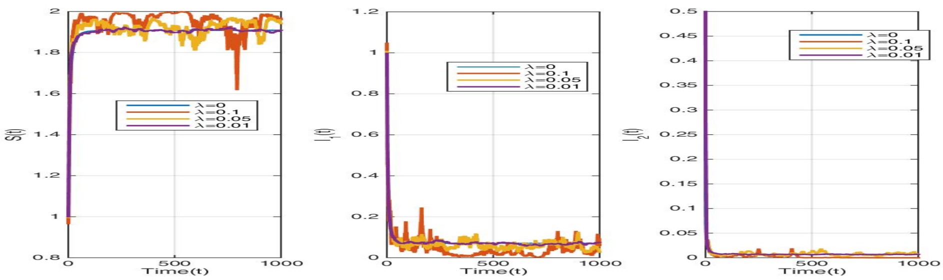

Figure 3: The trajectories of system (1.3) for parameter values $\Lambda = 1$; $\mu = 0.5$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; with different noise intensities $\lambda = 0, 0.1, 0.05, 0.01$.

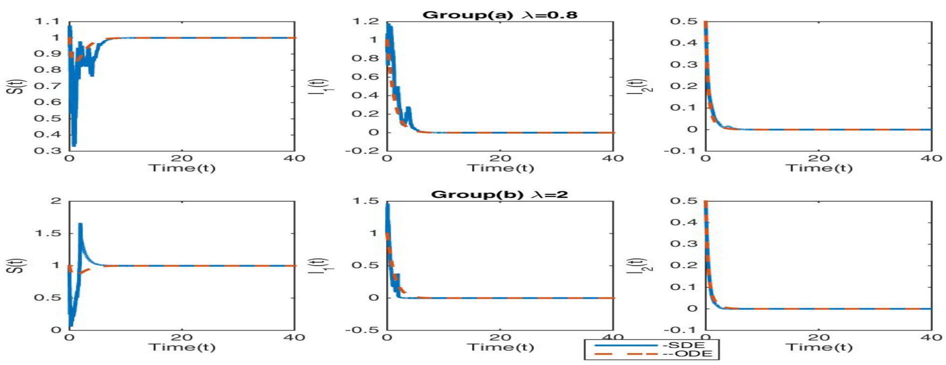

Figure 4: The trajectories of system (1.3) and system (1.2) for parameter values $\Lambda = 1$; $\mu = 1$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; for noise level $\lambda = 0.8$ and $\lambda = 2$.

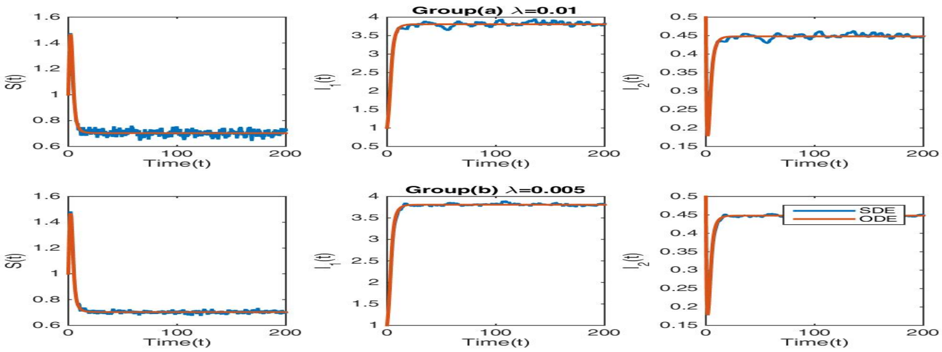

Figure 5: The trajectories of system (1.3) and system (1.2) for parameter values $\Lambda = 1$; $\mu = 0.1$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; for noise level $\lambda = 0.01$ and $\lambda = 0.005$.

Case(iv): The parameters can be fixed as in Figure 4: $\Lambda = 1$; $\mu = 1$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$, with intensity of large white noise λ=0.8 and $\lambda = 2$. With these parameter values $\tilde{R}_1 = \dfrac{\Lambda}{\mu}\dfrac{\beta_1(\kappa+\sigma+\mu)+\delta\beta_2}{\mu-\eta} < 1$. Theorem 6.2 shows that a large noise disturbance causes the disease to go extinct.

Case(v): According to Theorem 7.2, as the intensity of white noise decreases, the disease might become persistent. The parameters in Figure 5 are taken as follows: $\Lambda = 1$; $\mu = 0.1$; $\eta = 0.05$; $\delta = 0.2$; $\sigma = 0.7$; $\beta_1 = 0.3$; $\beta_2 = 0.4$; $\kappa = 0.9$; $\lambda = 0.01$. For these values, $\tilde{R}_2 = \frac{\Lambda\delta\beta_2}{(\delta+\mu+\lambda)(\kappa+\sigma+\mu)(\mu+\lambda)} = 1.2255 > 1$ and $k_1 = 0.0271$. Moreover if we choose $\lambda = 0.005$, the small intensity of the white noise yields $\tilde{R}_2 = 1.4694$ and $k_1 = 0.0493$. Thus infectious disease is persistent supporting the conclusion in Theorem 7.2.

9 Conclusion

We proposed a stochastic HIV epidemic model involving the transmission of HIV disease between susceptible, infected without symptoms, infected with symptoms. We first demonstrated that positivity of the solution of the stochastic model (1.3) using Lyapunov analysis method. We investigated the stability properties of the equilibrium points in order to understand the behavior of the system (1.3) as well as we derived the conditions under which the disease might be eliminated or be endemic. Furthermore, we demonstrated that if $R_1 < 1$, then the disease free equilibrium $(\frac{\Lambda}{\mu}, 0, 0)$ of the system (1.3) is stochastically asymptotically stable in the large and if $R_1 > 1$ the solution of stochastic system (1.3) is fluctuates around the solution of deterministic system (1.2) for a long time, and the fluctuation becomes smaller as the value of λ decreases. We also proved that the disease free equilibrium E_0 of the stochastic system (1.3) is exponentially stable in mean square when $\beta_1\frac{\Lambda}{\mu-\eta} + \frac{1}{2}\lambda^2(\frac{\Lambda}{\mu-\eta})^2 < \delta + \mu - \eta$. Figure 2 shows that how $I_a(t)$ and $I_s(t)$ approach to zero exponentially very fast.

In Section 6, we showed that if the threshold parameter $\tilde{R}_1 < 1$, the disease dies out. We determined that the persistence of the infective populations depend on the stochastic fluctuation intensity of the noise from the expression of $\tilde{R}_2$. We inferred from Theorem 7.2 that the disease will exist when the perturbations are weak. Finally, Our numerical simulations showed that both deterministic and stochastic systems have comparable characteristics and they are consistent. Hence our study on parameter perturbation is reliable. In order to control the disease, future research will focus on incorporating environmental stochasticity into the parameter η, the vertical transmission rate.

Availability of data and materials Data sharing is not applicable to this article since no datasets were generated or analyzed during the current study.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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